Oscillator Theory - State Transition Matrix & Floquet Theorem

Pre-requisites

i) **Linear Algebra** - Eigenvalues, eigenvectors.
- Linear transformation.
- Matrix decompositions. (Eigendecomp, SVD)

(3Blue 1 Brown's series on lin. alg.)

ii) **Differential Eqns.** - Vector calculus in $\mathbb{R}^n$.

Note: Not discussing proofs.

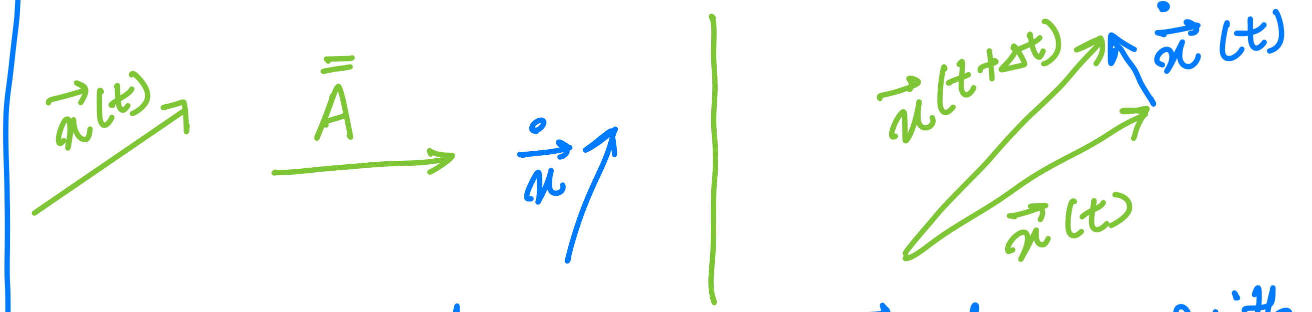
(I) Linear Homogeneous Differential Eqns.

Time varying model.

$$\boxed{\dot{\vec{x}} = \bar{\bar{A}}(t)\,\vec{x}} \quad (1)$$

$$\begin{bmatrix} \dfrac{\partial V}{\partial t} \\[2ex] \dfrac{\partial I}{\partial t} \end{bmatrix} = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix} \begin{bmatrix} V \\ I \end{bmatrix}$$

→ What is this saying?



→ Describing how $\underbrace{\text{a vector } \vec{x}}_{\mathbb{R}^n}$ changes with time.

Here, $\vec{x} \in \mathbb{R}^n$ & $\vec{x}(t_0) = \vec{x}_0$.

> Under certain (general) conditions on $A(t)$, it can be shown that the solutions of Eq ① $\boxed{\text{span } \mathbb{R}^n.}$

(Eq ① can be interpreted as a generating eqn. for $\dot{\vec{x}}$)

"Why do we care? Oscillators obey 2^{nd} order NDE. So it is important to study the dynamics of such systems in the   of differential equations"

> Let $\vec{\phi}(t, t_0, \vec{x}_0)$ be the set of solutions of Eq. ①.

$\Rightarrow \phi(t, t_0, \vec{x}_0)$ spans $\mathbb{R}^n$.

> If $\{\vec{x}_i\}$ is a basis for $\mathbb{R}^n$. $\Rightarrow \vec{\phi}(t, t_0, \vec{x}_i)$ is also a basis for $\mathbb{R}^n$ $\forall t$.

$$\psi_i(t) = \vec{\phi}(t, t_0, \vec{x}_i)$$

Def: Fundamental Matrix

$$\bar{\bar{X}} = \left[\psi_1(t) \quad \psi_2(t) \quad \cdots \quad \psi_n(t) \right]$$

Def: State Transition Matrix

If $\psi_i(t) = \phi(t, t_0, \hat{x}_i)$, then $\bar{\bar{X}}$ is the STM.

where $\hat{x}_i = [0, 0, \ldots, 0, \underset{i^{th}\ \text{position}}{1}, 0, \ldots, 0]$ → canonical basis vectors.

STM: Matrix whose columns have evolved from the canonical basis vectors as $t_0 \to t$.

$$\Phi(t, t_0) = \left[\phi(t, t_0, \hat{x}_1) \quad \phi(t, t_0, \hat{x}_2) \cdots \right]$$

> It is the fundamental characterization of how Eq ① "warps" $\mathbb{R}^n$ with time.

> Therefore, to know where an arbitrary vector $\vec{x}_0$ went from $t_0 \to t$, simply do

$$\boxed{\vec{\phi}(t, t_0, \vec{x}_0) = \Phi(t, t_0)\,\vec{x}_0}$$

(II) Linear Inhomogeneous DE

$$\underbrace{\dot{\vec{x}}(t) = \bar{\bar{A}}(t)\,\vec{x}(t)} + \vec{b}(t) \quad ; \quad \vec{x}(t_0) = \vec{x}_0 .$$

Soln. is given by

$$\vec{\phi}(t, t_0, \vec{x}_0) = \bar{\bar{\Phi}}(t, t_0)\,\vec{x}_0 + \int_{t_0}^{t} \bar{\bar{\Phi}}(t, \tau)\,\vec{b}(\tau)\,d\tau .$$

(III) Linear Diff. Eqns. with Periodic Coeffs.

$$\boxed{\dot{\vec{x}} = \bar{\bar{A}}(t)\,\vec{x}(t)} \; ; \quad \begin{array}{l} \vec{x}(t_0) = \vec{x} \;\; \& \\ \bar{\bar{A}}(t+T) = \bar{\bar{A}}(t). \end{array}$$

Observations

> Is $\vec{x}(t)$ periodic? No (in general)

If $A(t)$ is scaling by 2 in every period.

> $\bar{\bar{\Phi}}(t+T, t_0) = \bar{\bar{\Phi}}(t, t_0)$? No (in general)

> What can we say about the periodic properties of $\bar{\bar{\Phi}}$?

Floquet Theorem

$$\boxed{\Phi(t,s) = U(t)\, D(t-s)\, V(s)} \quad (2)$$

where $U(t)$ & $V(s)$ are T-periodic and they satisfy $U(t) = V^{-1}(t)$ and

$$D(t-s) = \mathrm{diag}\left[\exp(\mu_1(t-s)), \ldots, \exp(\mu_n(t-s))\right]$$

$\{\mu_i\}$ are called Floquet (characteristic) exponents; $\lambda_i = \exp(\mu_i T)$ are called Floquet (characteristic) multipliers.

Expanding Eq (2)

$$\boxed{\phi(t,s) = \sum_{i=1}^{n} \exp(\mu_i(t-s))\, u_i(t)\, v_i^T(s)} \quad (3)$$

where $u_i(t)$ are columns of U & $v_i^T(s)$ are rows of V.

Furthermore, $\{u_i\}$ & $\{v_i\}$ each span $\mathbb{R}^n$ & satisfy

$$\boxed{v_i^T(t)\, u_j(t) = \delta_{ij}} \quad (4)$$

Biorthogonality Relation