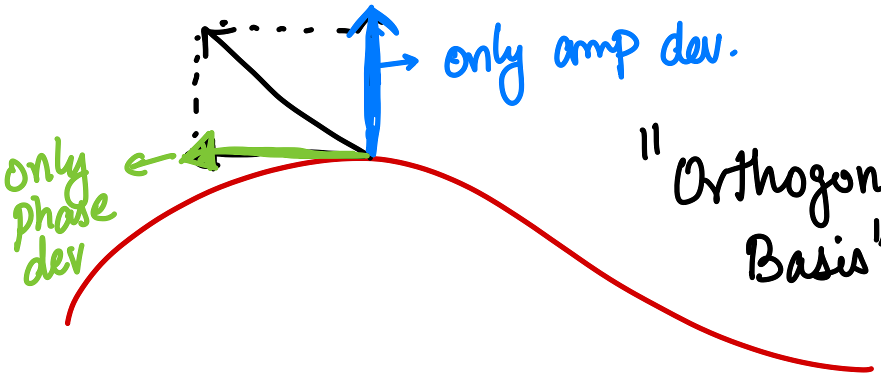
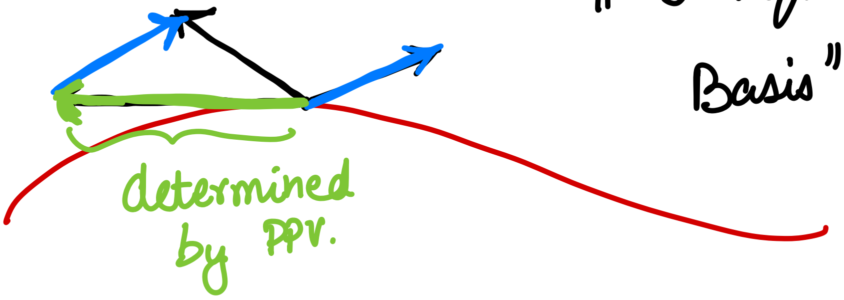
Perturbative Projection Vector Theory



"Orthogonal Basis"



"Oblique Basis"

ISF model

- "Simple" & intuitive.
- Good enough for design intuition.
  {Co ⟷ Flicker upconversion}

PPV model

- Rigorous.
- Fast computation.
- Deeper insight

Prerequisites MMIC 04, MMIC 14, MMIC 15*, Lin Alg., DE.

Perturbative Projection Vector Theory

Recap

$$\dot{\vec{x}} = A(t)\vec{x}$$

$$\vec{x}_H(t) = \underbrace{\Phi(t,0)}_{\text{STM}}\vec{x}_0$$

Homogeneous ①

$$\dot{\vec{x}} = A(t)\vec{x} + b(t)$$

$$\vec{x}_{IH}(t) = \Phi(t,0)\vec{x}_0 + \int_0^t \Phi(t,s)\, b(s)\, ds$$

Inhomogeneous ②

If $A(t+T) = A(t)$,

STM $\Phi(t,s) = \sum_{i=1}^{n} \exp(\mu_i(t-s))\, u_i(t)\, v_i^T(s)$

where $\boxed{v_i^T(t)\, u_j(s) = \delta_{ij}}$ $\forall t$ Biorthogonality conditions!

Floquet ③

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③ → ① gives

$$\vec{x}_H(t) = \sum_{i=1}^{n} \exp(\mu_i t)\, \vec{u}_i(t)\, \vec{v}_i^T(0)\, \vec{x}(0)$$

since $s = t_0 = 0$

③ → ② gives

$$\vec{x}_{IH}(t) = \vec{x}_H(t) + \int_0^t \sum_{i=1}^{n} \exp(\mu_i(t-s))\, \vec{u}_i(t)\, \vec{v}_i^T(s)\, b(s)\, ds$$

$$= \vec{x}_H(t) + \sum_{i=1}^{n} \vec{u}_i(t) \int_0^t \exp(\mu_i(t-s))\, \vec{v}_i^T(s)\, b(s)\, ds.$$

Oscillator Dynamics & Floquet Theory

> $\dot{\vec{x}} = f(\vec{x}) \rightarrow$ General dynamical system (MMIC 14) (for example)

$\vec{x} \in \mathbb{R}^n$ & $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$

> Let $\vec{x}_s(t)$ be a limit cycle solution (MMIC 04).

$$\Rightarrow \quad \dot{\vec{x}}_s(t) = f(\vec{x}_s(t))$$

$$\Rightarrow \quad \ddot{\vec{x}}_s(t) = \left.\frac{\partial f}{\partial \vec{x}}\right|_{\vec{x}=\vec{x}_s} \cdot \frac{\partial \vec{x}}{\partial t} = A(t)\, \dot{\vec{x}}_s(t)$$

$A(t)$ not a fn. of $\vec{x}$ since $\vec{x} = \vec{x}_s$. So it is only a fn. of time.

$A(t)$ is also periodic since $\vec{x} = \vec{x}_s$ & $\vec{x}_s$ is T-periodic

$\Rightarrow \dot{\vec{x}}_s(t)$ satisfies the Homogeneous ODE

velocity of the limit cycle.

$$\dot{\vec{x}}(t) = A(t)\, \vec{x}(t)$$

} Homogeneous $\Rightarrow$ Has a STM $\underline{\Phi}(t,s)$

$A(t)$ periodic & $\dot{\vec{x}}_s$ is a sol.

$$\Rightarrow \quad \dot{\vec{x}}_s(t) = \sum_{i=1}^{n} \exp(\mu_i t)\, \vec{u}_i(t)\, \vec{v}_i^T(0)\, \dot{\vec{x}}_s(0) \quad \text{from Floquet.}$$

$$= \sum_{i=1}^{n} \underbrace{\left[\exp(\mu_i t)\, \vec{v}_i^T(0)\, \dot{\vec{x}}_s(0)\right]}_{\text{Scalar}} \underbrace{\vec{u}_i(t)}_{\text{basis vectors.}}$$

For some value of i, say $i=1$ WLOG, we see that

$\boxed{\vec{u}_1(t) = \dot{\vec{x}}_s(t)}$ satisfies the above eqn.

$\left\{ \vec{v}_i^T(0)\, \vec{u}_1(0) = 0 \;\forall\, i \neq 1 \right.$ & $\mu_1 = 0 \Rightarrow$ LHS = RHS $\}$

$\Rightarrow \lambda_1 = \exp(\mu_1 T) = 1$

Lemma: If $\lambda_1 = 1$ & $|\lambda_i| < 1$ $\forall$ $i = 2, 3 \ldots, n$; Then $x_s(t)$ is a limit cycle soln. of $\dot{\vec{x}} = f(\vec{x})$

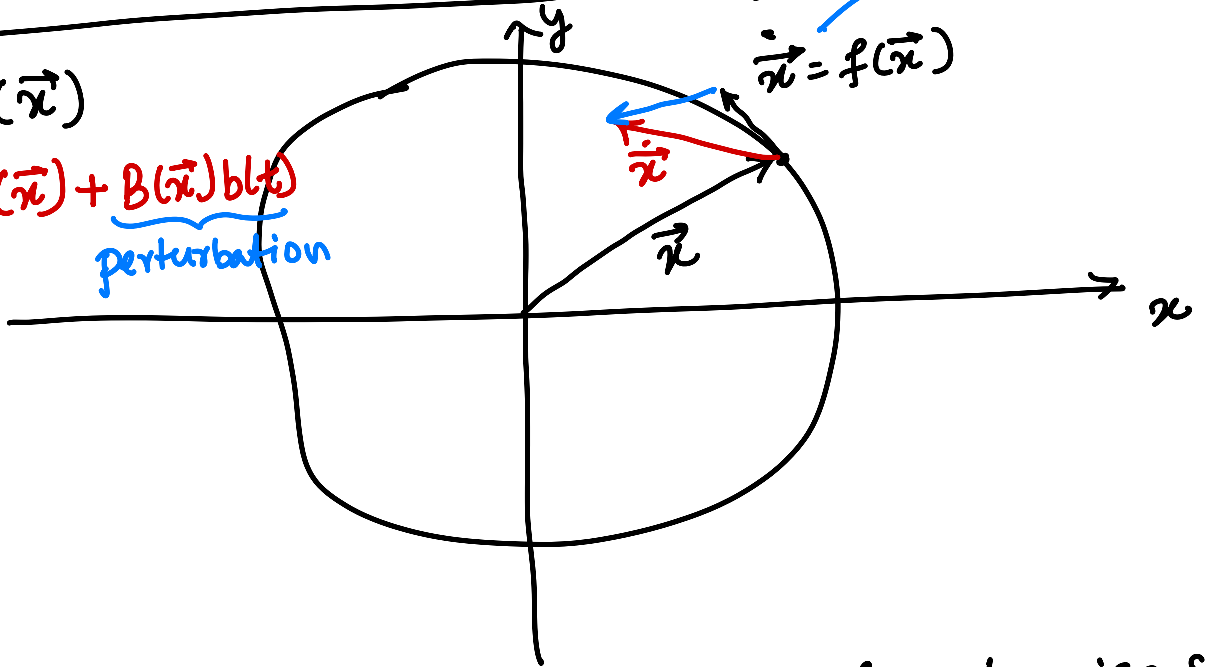
Since $x_s(t)$ is a limit cycle & $\lambda_1 = 1$, we assume $|\lambda_i| < 1$, (i.e., stable system).

Nonlinear Perturbation Analysis

Satisfies a HLDE

$\dot{\vec{x}} = f(\vec{x})$

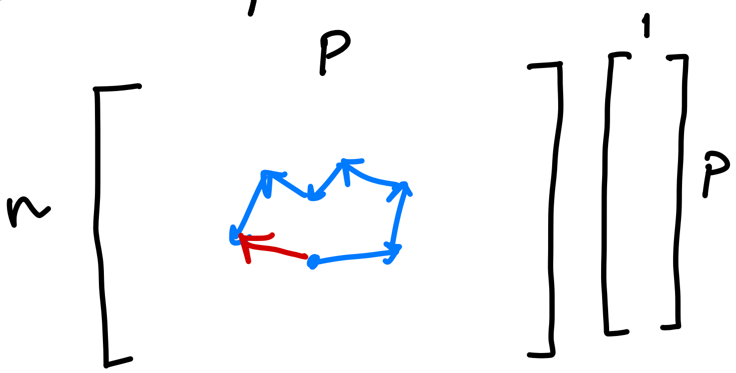
$\dot{\vec{x}} = f(\vec{x}) + \underbrace{B(\vec{x})b(t)}_{\text{perturbation}}$



$b(t): \mathbb{R} \to \mathbb{R}^p$ } At time t, p different noise sources produce a $p \times 1$ noise vector.

$B(\vec{x}): \mathbb{R}^n \to \mathbb{R}^{n \times p}$ } At location $\vec{x}$, the response of the "state" is broken up into a response due to the p different noise sources.

$\underbrace{B(\vec{x})\, b(t)}$ } Note that this is not an impulse, it is an arbitrary "small" perturbation.

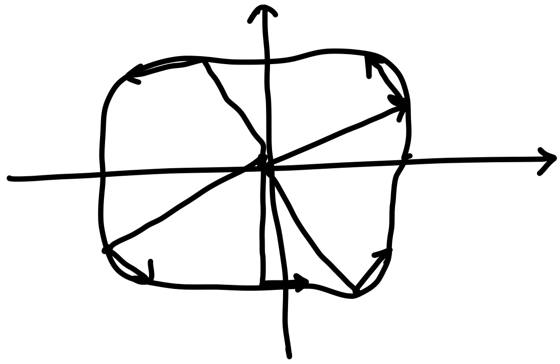


Lemma: $\exists\ \vec{b}_1(x,t) = B(\vec{x})\,\vec{b}(t)$ that only produces a phase shift in $\vec{x}_s(t)$. In other words,

$$\dot{\vec{x}} = f(\vec{x}) + \vec{b}_1(\vec{x},t)$$

is solved by

$$\vec{x}_p(t) = \vec{x}_s\big(t + \underbrace{\alpha(t)}_{\text{phase deviation.}}\big)$$



Proof:

$$\frac{\partial}{\partial t}\Big(\vec{x}_s(t+\alpha(t))\Big) = f\big(\vec{x}_s(t+\alpha(t))\big) + \vec{b}_1\big(\vec{x}_s(t+\alpha(t)),t\big)$$

$$\Rightarrow \dot{\vec{x}}_s(t+\alpha(t))\,(1+\dot{\alpha}(t)) = f\big(\vec{x}_s(t+\alpha(t))\big) + \vec{b}_1\big(\vec{x}_s(t+\alpha(t)),t\big)$$

Choose $\vec{b}_1(x,t) = c_1(\vec{x},\alpha(t),t)\,u_1(t+\alpha(t))$

i.e. we choose $\vec{b}_1$ to be tangential to the limit cycle at $t+\alpha(t)$ with a scaling factor c_1.

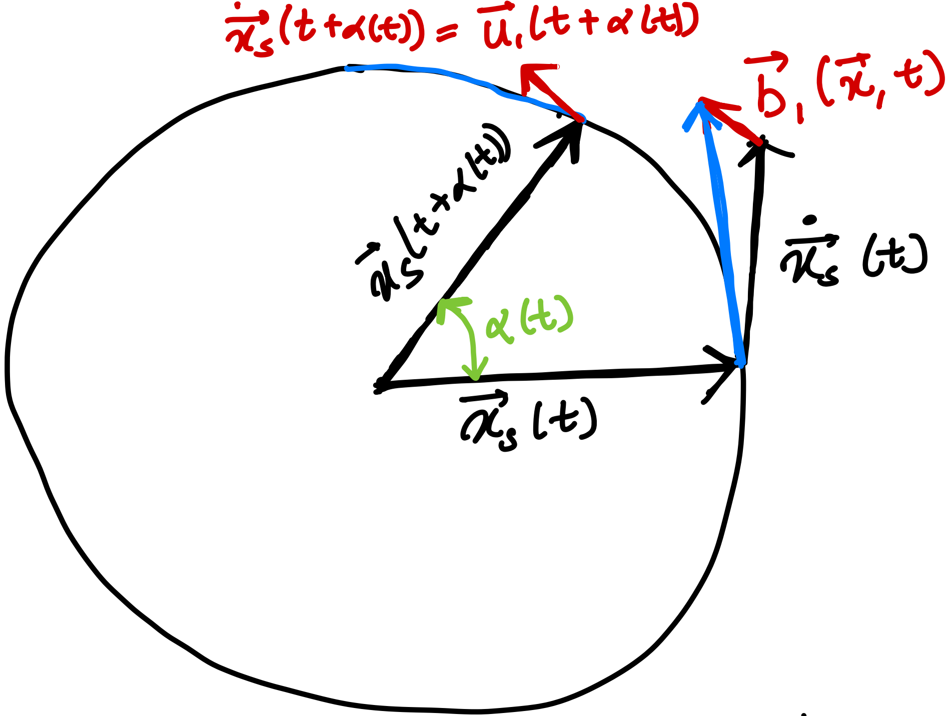
$$\Rightarrow \underbrace{\dot{\vec{x}}_s(t+\alpha(t))}_{u_1(t+\alpha(t))}\,(\cancel{1}+\dot{\alpha}(t)) = \cancel{\overbrace{f\big(\vec{x}_s(t+\alpha(t))\big)}^{\dot{\vec{x}}_s(t+\alpha(t))}} + c_1\big(\vec{x}_s(t+\alpha(t)),\alpha(t),t\big)\,u_1(t+\alpha(t))$$

$$\Rightarrow \boxed{c_1\big(\vec{x}_s(t+\alpha(t)),\alpha(t),t\big) = \dot{\alpha}(t)}$$

$\{$ Since $u_1 \neq 0\ \forall t$ $\}$

∎

Intuition



Key Insights ① A perturbation to the system changes the velocity & not the instantaneous state of the system.

② $\vec{u}_1$ vector determines the phase deviation.

③ $\vec{u}_1 \longleftrightarrow \alpha(t) \longleftrightarrow c_1$ are all related.

④* $\vec{b}_1(x,t)$ is the perturbation that "pushes" or "pulls" the state along the limit cycle. THE STATE NEVER LEAVES THE LIMIT CYCLE when $\vec{b}_1$ is applied.

What is c_1 & hence $\alpha(t)$?

> In general $B(\vec{x})\vec{b}(t)$ produces phase deviations & orbital deviations.

> Let $\tilde{b}(x,t) = B(\vec{x})\vec{b}(t) - \vec{b}_1(x,t)$ be the "rest" of the perturbation.

> Since $\vec{u}_1(t)$ forms a basis for $\mathbb{R}^n$, we can expand $B(\vec{x})\,\vec{b}(t)$ in it as

$$B(\vec{x})\,\vec{b}(t) = \sum_{i=1}^{n} \underbrace{c_i(x,\alpha(t),t)}_{\text{scalar}}\, u_i(t+\alpha(t))$$

From biorthogonality, c_i is given by $\langle v_i^T, B(\vec{x})\vec{b}(t)\rangle$

$$\Rightarrow \boxed{c_i(x,\alpha(t),t) = v_i^T(t+\alpha(t))\, B(x)\,\vec{b}(t)}$$

$\Rightarrow$ since $\vec{b}_1(x,t) = c_1(x,\alpha(t),t)\, u_1(t+\alpha(t))$

$$\tilde{b}(x,t) = \sum_{i=2}^{n} c_i(x,\alpha(t),t)\, u_i(t+\alpha(t))$$

Also, $\dot{\alpha}(t) = c_1$

Note that it is nonlinear.

$$\Rightarrow \boxed{\frac{d\alpha(t)}{dt} = \underbrace{v_1^T(t+\alpha(t))}_{\text{PPV}}\ \underbrace{B(x)\,\vec{b}(t)}_{\text{perturbation}}} \quad \& \quad \alpha(0)=0$$

<u>Thm</u>: $\tilde{b}(x,t)$ produces amplitude deviations that are bounded. In other words, $x_s(t+\alpha(t)) + \vec{z}(t)$ solves $\dot{\vec{x}} = f(x) + b_1(x,t) + \tilde{b}(x,t) = f(x) + B(\vec{x})\vec{b}(t)$ where $\vec{z}(t) \to 0$ as $t \to 0$ if $b(t)=0$ after $t = t_c$.

<u>Proof</u>: $$\frac{\partial}{\partial t}\left[\vec{x}_s(t+\alpha(t)) + \vec{z}(t)\right] = f(\vec{x}_s(t+\alpha(t)) + \vec{z}(t)) + b_1(\vec{x}_s + \vec{z}) + \tilde{b}(\vec{x}_s + \vec{z})$$

$$f(\vec{x}_s + \vec{z}) = f(\vec{x}_s) + \left(\frac{\partial f}{\partial x}\right)^T \vec{z}$$ when $\vec{z}$ is small.

$$\Rightarrow \quad \dot{\vec{x}}_s \cancel{(1+\dot{\alpha})} + \dot{\vec{z}} = \cancel{\dot{\vec{x}}_s} + A(t)\vec{z} + \cancel{b_1} + \tilde{b}$$

$$\Rightarrow \quad \dot{\vec{z}} = \underbrace{A(t)}_{\text{same} \Rightarrow \text{same STM}} \vec{z} + \tilde{b}(\vec{x}_s + \vec{z}) \rightarrow \text{IHDE}$$

$$\Rightarrow \quad \vec{z} = \cancelto{0}{\sum_{i=1}^{n} \exp(\mu_i t)\, u_i v_i^T \vec{z}(0)} + \sum_{i=1}^{n} u_i(t) \int_0^t \exp(\mu_i(t-s))\, v_i^T \tilde{b}\, ds$$

But $\tilde{b} = \sum_{i=2}^{n} c_i u_i$ $\Rightarrow$ Summation only fires for $i = 2 \rightarrow n$.

$$\Rightarrow \vec{z}(t) = \sum_{i=2}^{n} u_i(t) \int_0^t \exp(\mu_i(t-s))\, v_i^T(s)\, \tilde{b}(\vec{x}_s, t)\, ds$$

since $i \geq 2$
$b_1 \leftrightarrow 0$

$$= \sum_{i=2}^{n} u_i(t) \int_0^t \exp(\mu_i(t-s))\, v_i^T(s)\, B(x_s(s))\, b(s)\, ds$$

Since $|\exp(\mu_i T)| < 1$ for $i \geq 2$, $\vec{z}(t)$ is within a constant factor of $b(t)$ & hence bounded! ∎

Impulse Perturbations

$$B(\vec{x})\,\vec{b}(t) = \bar{b}\,\delta(t) \quad \text{where } \vec{b} \in \mathbb{R}^n.$$

$$\bar{b}\,\delta(t) = b_1\delta(t) + \tilde{b}\,\delta(t) = c_1 u_1(0)\,\delta(t) + \tilde{b}\,\delta(t)$$

$$\frac{d\alpha(t)}{dt} = c_1(x_s(t), \alpha(t), t) = c_1\,\delta(t)$$

$$\Rightarrow \quad \alpha(t) = \begin{cases} 0 & \text{if } t = 0 \\ c_1 & \text{if } t > 0 \end{cases}$$

Similar to ISF

Also, $x_s(t + \alpha(t)) = x_s(t + c_1)$ solves

$$\dot{x} = f(\vec{x}) + b_1\delta(t) = f(\vec{x}) + c_1 u_1(0)\,\delta(t)$$
$$= f(\vec{x}) + c_1\,\dot{x}_s(0)\,\delta(t).$$

& $c_1 = v_1^T(0)\,\bar{b} \quad \Rightarrow \quad \boxed{\alpha(t) = v_1^T(0)\,\bar{b}}$

Similar to ISF

From Thm, $\tilde{b}$ does not produce any deviation as $t \to \infty$,

$\Rightarrow$ asymptotic phase is given by $\underbrace{v_1^T(0)}_{\text{PPV}}\,\bar{b}$ //

τ for $\delta(t-\tau)$

Isochrones

Set of perturbations that do **not** contribute to the asymptotic phase of the oscillator.