X Method of Stationary Phase (MOSP)

> Solve 2-D FT of the radar Green's function.

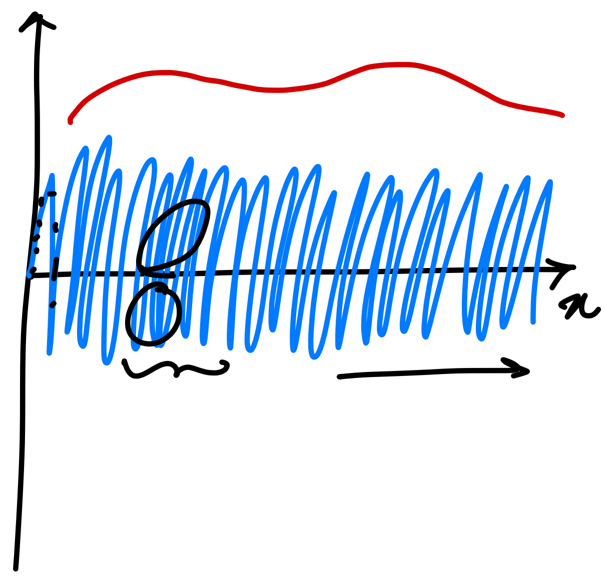
$$\frac{e^{-jkr}}{4\pi r} \rightarrow$$ MIMO one way

> Papoulis (Proof)
"Asymptotic Techniques"

$$\frac{e^{-2jkr}}{4\pi r^2} \rightarrow$$ SAR two way

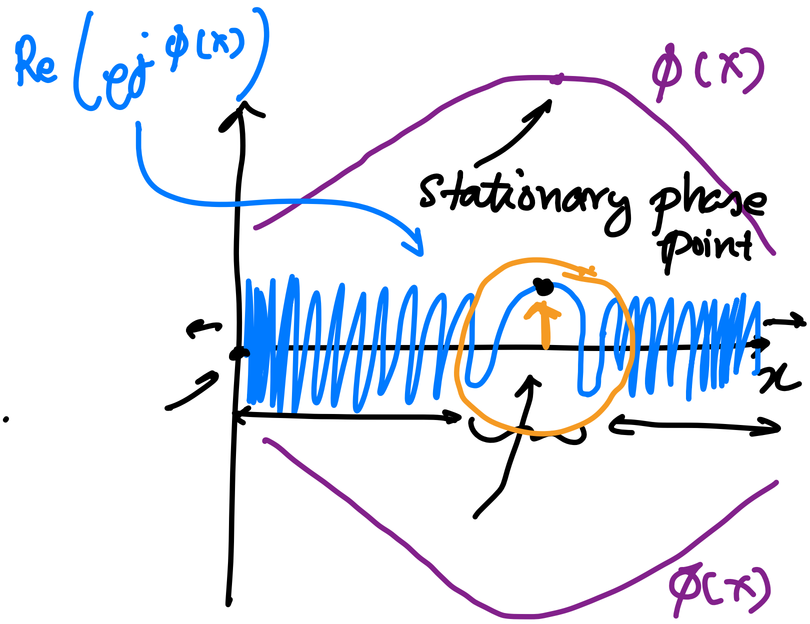
10.1 Intuition

$$\int_x \underbrace{f(x)}_{\text{slow}} \underbrace{e^{j\phi(x)}}_{\text{fast}} dx$$



↳ Numerically unstable.

$$\sum_{x_n} f[x_n]\, e^{j\phi[x_n]}$$



> If $\phi(x)$ has a stationary phase pt. i.e. $\phi'(x)=0$, then $\int$ can be computed in the neighbourhood of x_0, (i.e soln. to $\phi'(x)=0$).

10.2 2-D MOSP

Thm

$$\iint_{x\,y} f(x,y)\, e^{j\phi(x,y)}\, dx\,dy \stackrel{\sim}{=} \frac{2\pi j\, f(x_0,y_0)}{\sqrt{\phi_{xx}\phi_{yy} - \phi_{xy}^2}}\, e^{j\phi(x_0,y_0)}$$

where (x_0,y_0) is the only singular extremum of $\phi(x,y)$; $(x_0,y_0) \rightarrow \frac{\partial\phi}{\partial x} = 0$ & $\frac{\partial\phi}{\partial y} = 0$.

$$\phi_{xx} = \left.\frac{\partial^2\phi}{\partial x^2}\right|_{x_0,y_0} ; \quad \phi_{yy} = \left.\frac{\partial^2\phi}{\partial y^2}\right|_{x_0,y_0} ; \quad \phi_{xy} = \left.\frac{\partial^2\phi}{\partial x\partial y}\right|_{x_0,y_0}$$

$f(x,y)$ is continuous at (x_0,y_0) &

$\phi_{xx}\phi_{yy} - \phi_{xy}^2 \neq 0$; $\phi_{yy} \neq 0$.

10.3 One way Free space Green's Function FT

$$FT_{2D} \text{ of } \frac{e^{-jk\sqrt{x^2+y^2+z_0^2}}}{\sqrt{x^2+y^2+z_0^2}}$$

$$= \iint_{x\,y} \frac{e^{-jk\sqrt{x^2+y^2+z_0^2} - jk_x x - jk_y y}}{\sqrt{x^2+y^2+z_0^2}}\, dx\,dy$$

$$f(x,y) = \frac{1}{\sqrt{x^2+y^2+z_0^2}} \; ; \; \phi(x,y) = -k\sqrt{x^2+y^2+z_0^2} - k_x x - k_y y$$

$$\phi_x = \frac{-xk}{\sqrt{x^2+y^2+z_0^2}} - k_x \; ; \; \phi_y = \frac{-yk}{\sqrt{x^2+y^2+z_0^2}} - k_y$$

$\phi_x = 0$ & $\phi_y = 0 \rightarrow$ find soln

$$x_0 = \frac{\pm z_0 k_x}{\sqrt{k^2-k_x^2-k_y^2}} \; ; \; y_0 = \frac{\pm z_0 k_y}{\sqrt{k^2-k_x^2-k_y^2}}$$

Susbstituting reveals only -ve soln. satisfies $\phi_x = \phi_y = 0$

$$\phi_{xx}(x_0,y_0) = \frac{(k^2-k_x^2)\sqrt{k^2-k_x^2-k_y^2}}{k^2 z_0}$$

$$\phi_{yy}(x_0,y_0) = \frac{(k^2-k_y^2)\sqrt{k^2-k_x^2-k_y^2}}{k^2 z_0}$$

$$\phi_{xy}(x_0,y_0) = \frac{-k_x k_y\sqrt{k^2-k_x^2-k_y^2}}{k^2 z_0}$$

$$\phi(x_0, y_0) = -z_0\sqrt{k^2 - k_x^2 - k_y^2}$$

$$f(x_0, y_0) = \frac{\sqrt{k^2 - k_x^2 - k_y^2}}{z_0 k}$$

$$\sqrt{\phi_{xx}\phi_{yy} - \phi_{xy}^2} = \frac{k^2 - k_x^2 - k_y^2}{k z_0}$$

$$FT_{2D} = \frac{2\pi j \, e^{-j z_0 \underbrace{\sqrt{k^2 - k_x^2 - k_y^2}}_{k_z}}}{\underbrace{\sqrt{k^2 - k_x^2 - k_y^2}}_{k_z}}$$

$$= \frac{2\pi j \, e^{-j k_z z_0}}{k_z}$$

$$FT_{2D}\left\{ \frac{e^{-jk\sqrt{(x-x')^2 + (y-y')^2 + z^2}}}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}} \right\}$$

$$= \frac{2\pi j}{k_z} e^{-j k_z z_0 - j k_x x' - j k_y y'} \Bigg\}$$

singularity when $k_z = 0$

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

10.4 2-way free space GF F.T

$$\iint_{xy} \frac{e^{-j2k\sqrt{x^2+y^2+z_0^2}}}{x^2+y^2+z_0^2} e^{-jk_x x - jk_y y} \, dx dy$$

$$f(x,y) = \frac{1}{x^2+y^2+z_0^2} \qquad \phi(x,y) = -2k\sqrt{x^2+y^2+z_0^2} - k_x x - k_y y$$

$$\phi(x_0, y_0) = -z_0\sqrt{4k^2-k_x^2-k_y^2}$$

$$\sqrt{\phi_{xx}\phi_{yy} - \phi_{xy}^2} = \frac{4k^2-k_x^2-k_y^2}{2kz_0}$$

$$f(x_0,y_0) = \frac{4k^2-k_x^2-k_y^2}{4k^2z_0^2}$$

$$FT = \frac{\pi j}{kz_0} e^{-jz\sqrt{4k^2-k_x^2-k_y^2}}$$

$$FT_{2D}\left\{ \frac{e^{-2jk\sqrt{(x-x')^2+(y-y')^2+z_0^2}}}{(x-x')^2+(y-y')^2+z_0^2} \right\}$$

$$= \frac{\pi j}{k z_0} e^{-j z_0 \sqrt{4k^2 - k_x^2 - k_y^2} - j k_x x' - j k_y y'}$$