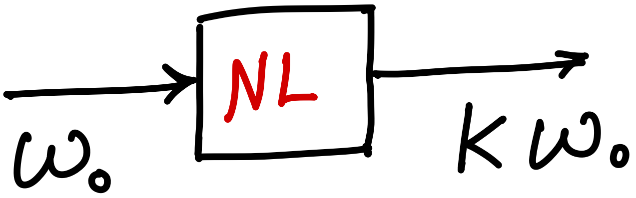
Frequency Multipliers

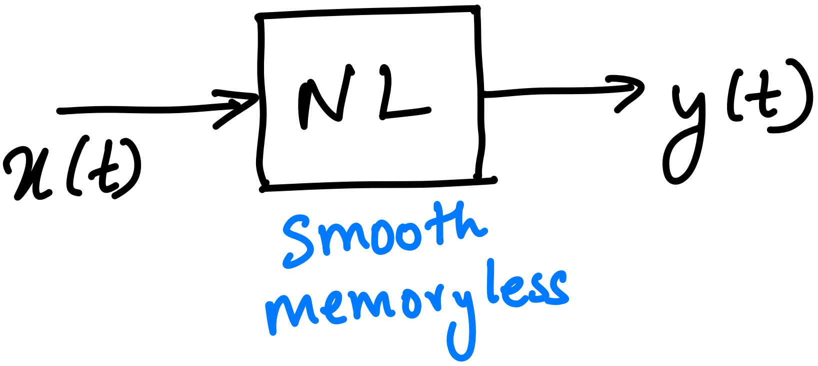


where $K \geq 2$

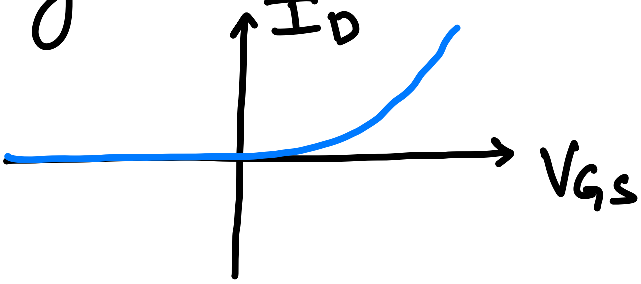
> LTI cannot work.

- Nonlinear element ✓
- Time variation.

Non-linear multiplier



$$y(t) = a_o + a_1 x(t) + a_2 x^2(t) + \dots$$



$$x(t) = A\cos(\omega_0 t)$$

$$y(t) = a_0 + a_1 A\cos(\omega_0 t) + a_2 A^2 \cos^2(\omega_0 t) + \dots$$

$$\Rightarrow y(t) = b_0 + b_1 \cos(\omega_0 t) + b_2 \cos(2\omega_0 t) + \dots$$

$$b_0 = a_0 + \frac{a_2 A^2}{2} + \frac{3 a_4 A^4}{8} + \dots$$

$$b_1 = a_1 A + \frac{3 a_3 A^3}{4} + \dots$$

$$b_2 = \frac{a_2 A^2}{2} + \frac{a_4 A^4}{2} + \dots$$

$$b_3 = \frac{a_3 A^3}{4} + \dots$$

> $a_x \rightarrow$ Taylor/Polynomial Coeff.

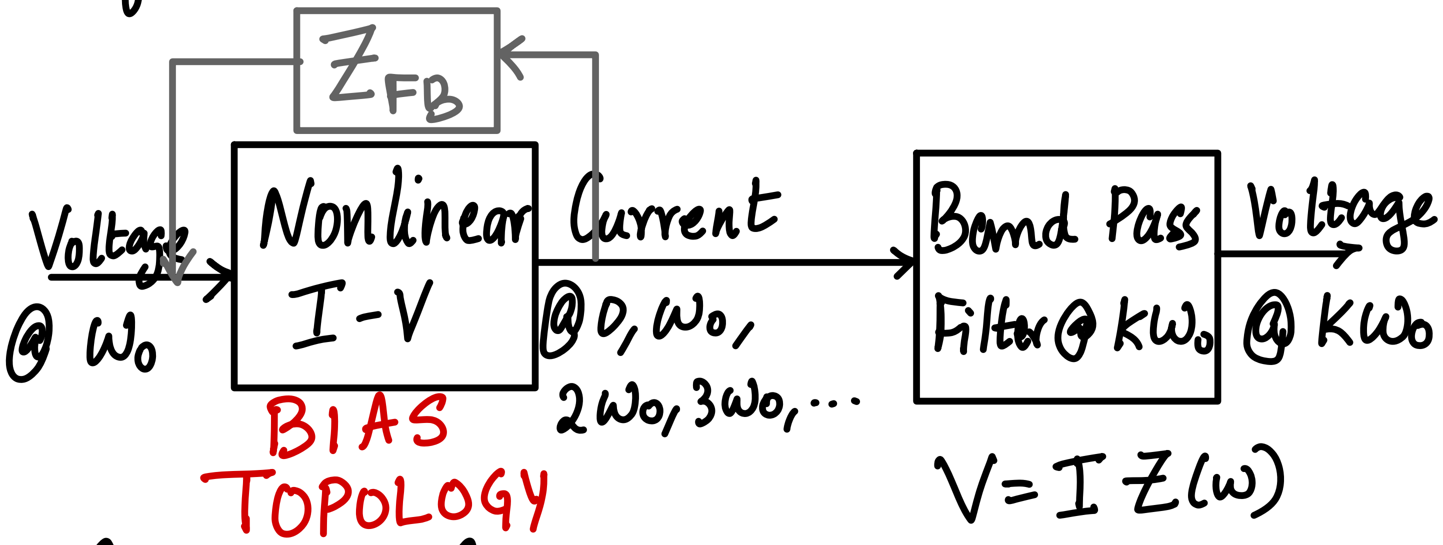
$b_x \rightarrow$ Harmonic coeffs.

> Leading term in b_x is $a_x A^x$.

> Even order $b_x \longleftrightarrow$ Even order a_x.

$$G_{C,K} \triangleq \frac{P_{out}(K\omega_0)}{P_{in}(\omega_0)}$$. If $G_{C,K} > 1 \Rightarrow$ Active.

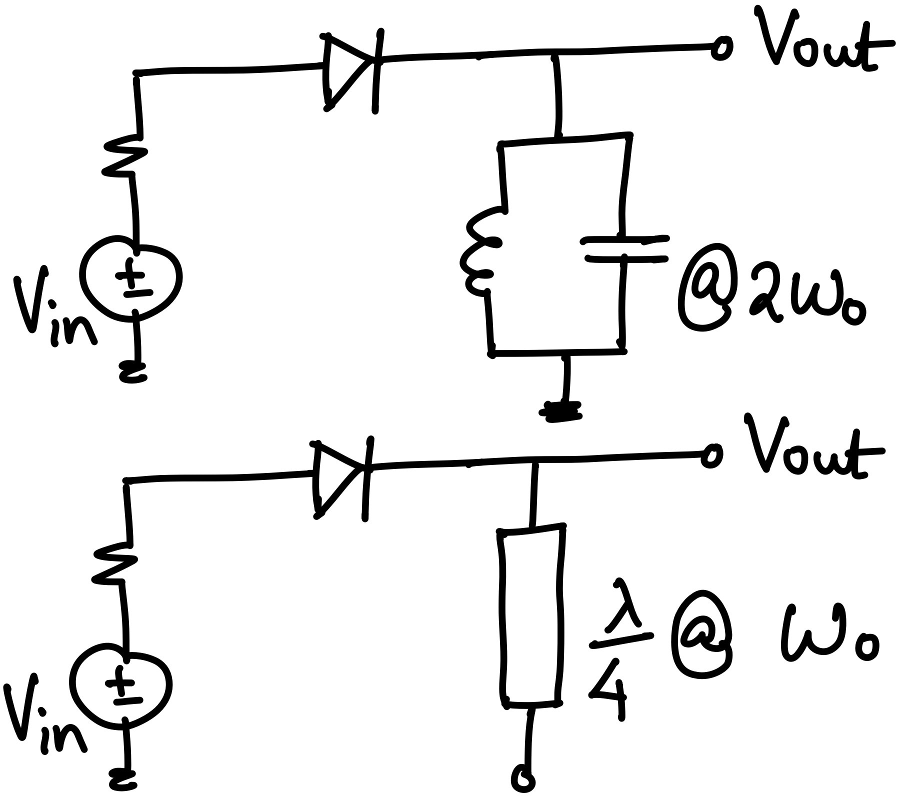
System/Circuit Level View

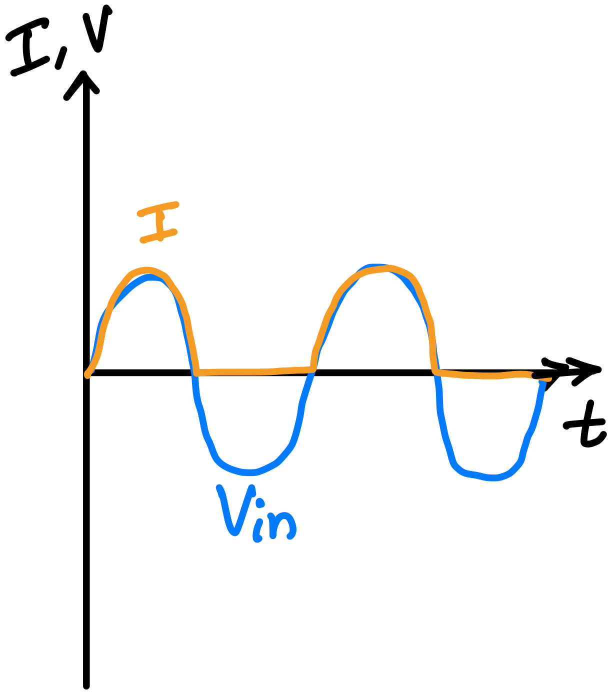


$$V = I\,Z(\omega)$$

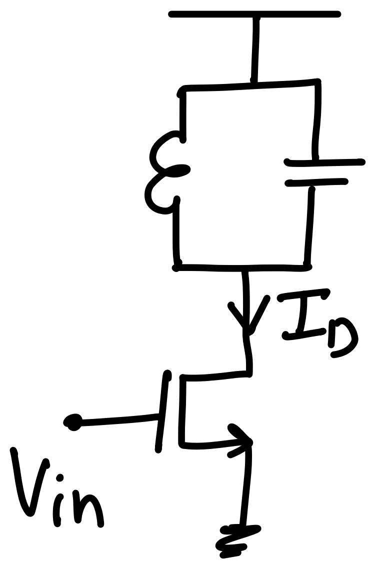
Circuit Implementation

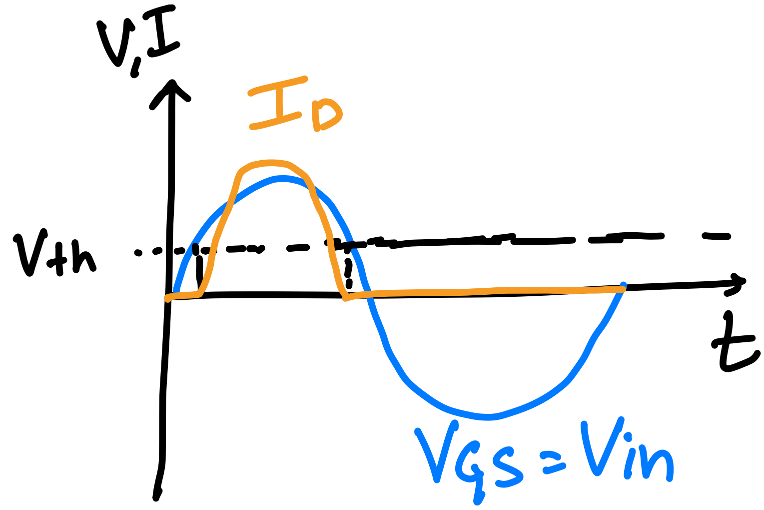
Diode





Transistor

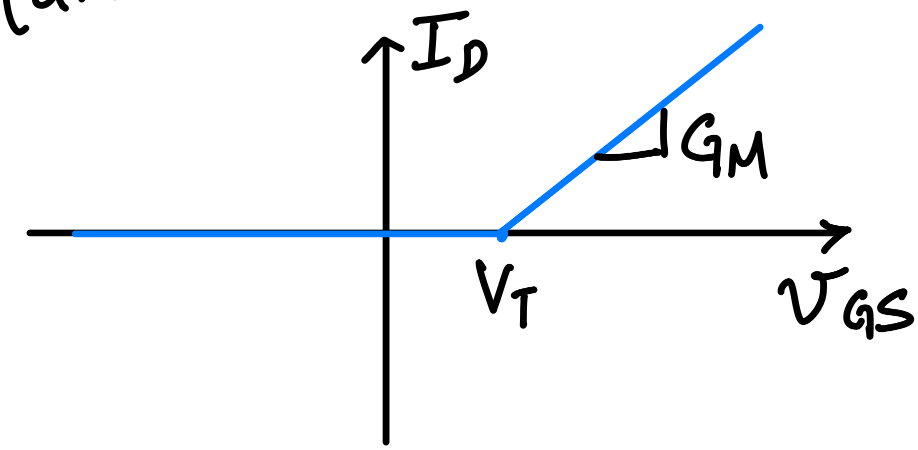




BIAS POINT SELECTION

- Given a NL IV device, what should be the bias to maximize power at $K\omega_0$.

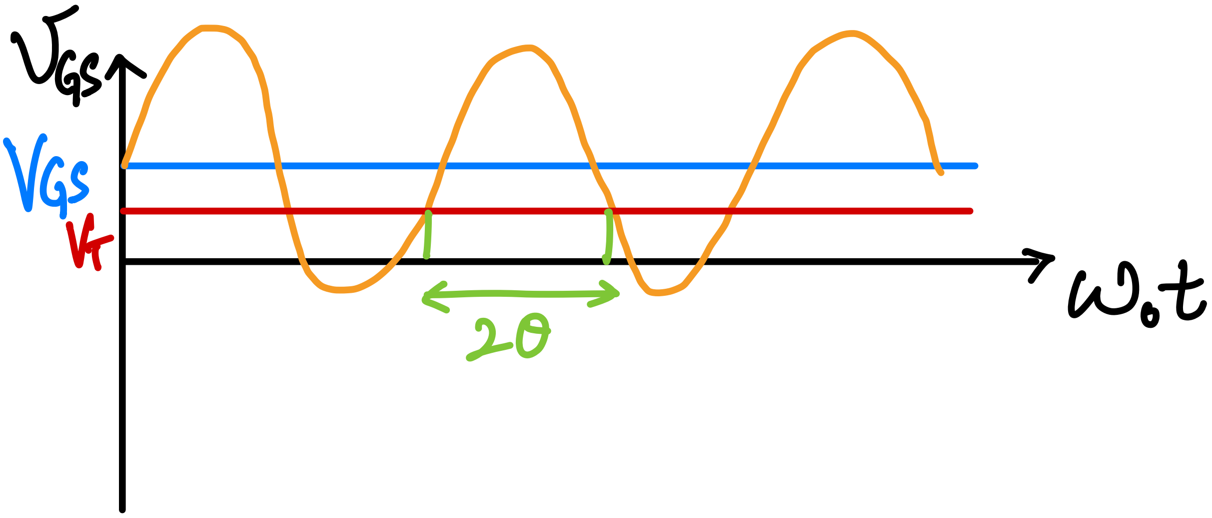
Assumption: IV is Piecewise Linear (aka Truncated Cosine Model)

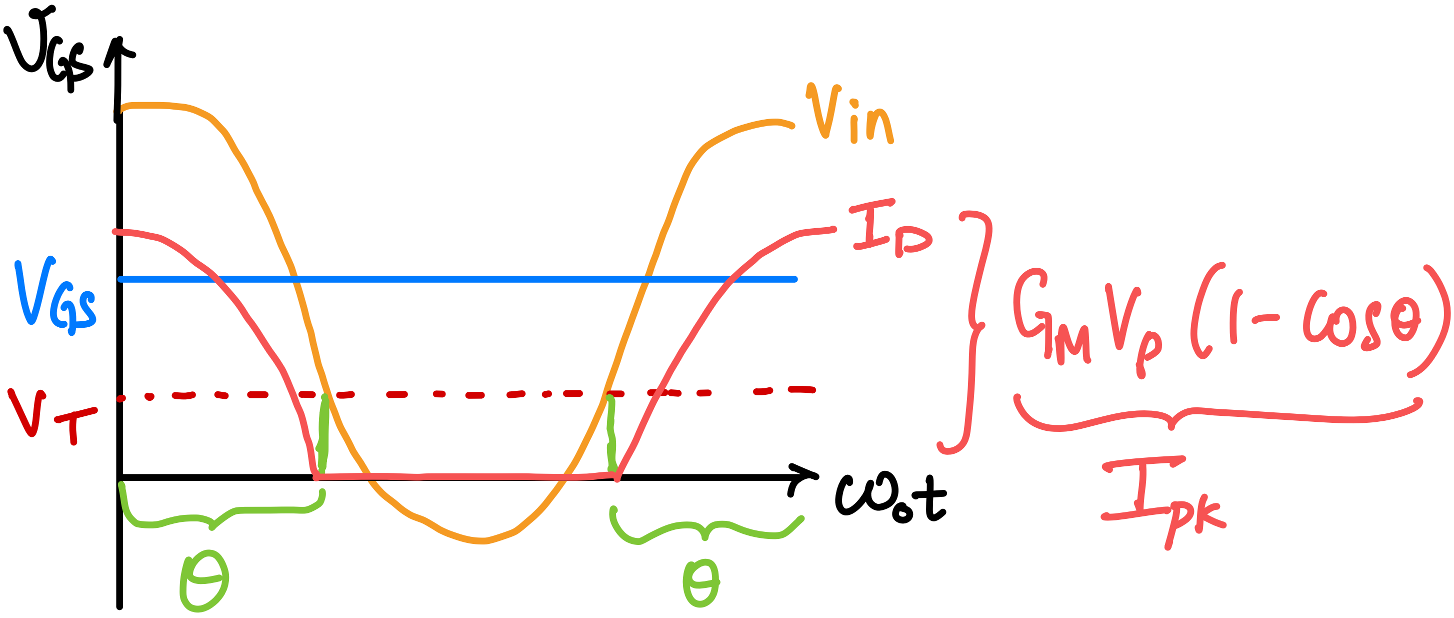


- Good model for short Channel FETs.

$$I_D = \begin{cases} G_M(V_{GS} - V_T) & V_{GS} \geqslant V_T \\ 0 & V_{GS} < V_T \end{cases}$$

$$\boxed{v_{GS} = V_{GS} + V_p\cos(\omega_0 t)}$$





2θ → Conduction Angle.

$$V_{GS} + V_p\cos\theta = V_T$$

$$\boxed{\cos\theta = \frac{V_T - V_{GS}}{V_P}}$$

θ depends on both V_{GS} & V_P.

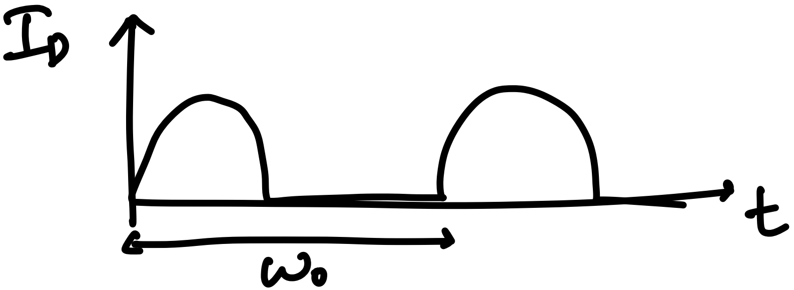
$$I_D = G_M\left(v_{GS} - V_T\right) \qquad V_{GS} > V_T$$

$$= G_M\left(V_{GS} - V_T + V_P\cos(\omega_0 t)\right)$$

$$= G_M\left(-V_P\cos\theta + V_P\cos(\omega_0 t)\right)$$

$$I_D = \begin{cases} G_M V_P\left(\cos(\omega_0 t) - \cos\theta\right) & |\omega_0 t| < \theta \\ 0 & |\omega_0 t| > \theta \end{cases}$$

$$\boxed{I_{pk} = G_M V_P\left(1 - \cos\theta\right)}$$



$$I_D(t) = I_0 + \sum_{n=1}^{\infty} I_n \cos(n\omega_0 t)$$

$$I_0 = \frac{G_m V_p}{2\pi}\left(2\sin\theta - 2\theta\cos\theta\right)$$

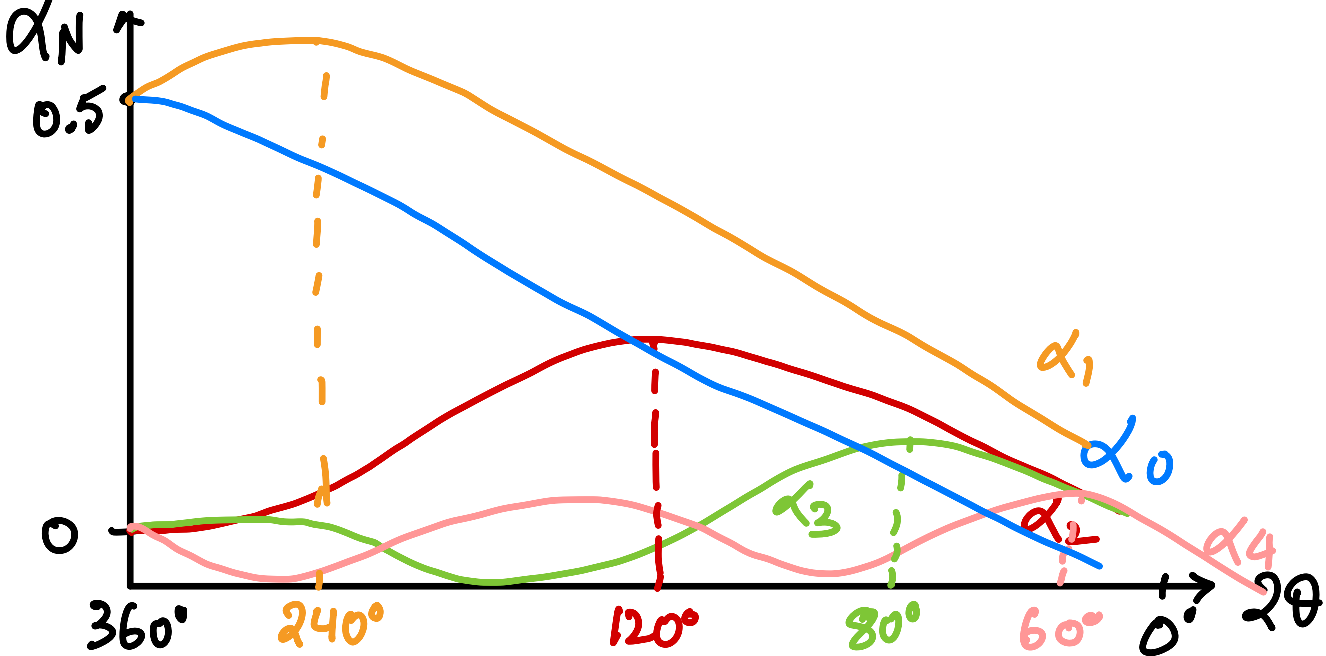
$$I_1 = \frac{G_m V_p}{2\pi}\left(2\theta - \sin(2\theta)\right)$$

$$I_N = \frac{G_m V_p}{\pi}\left[\frac{2\{\sin(N\theta)\cos\theta - N\cos(N\theta)\sin\theta\}}{N(N^2-1)}\right]$$

$N \geq 2$

$$\boxed{\alpha_N(\theta) = \frac{I_N(\theta)}{I_{pk}}}$$

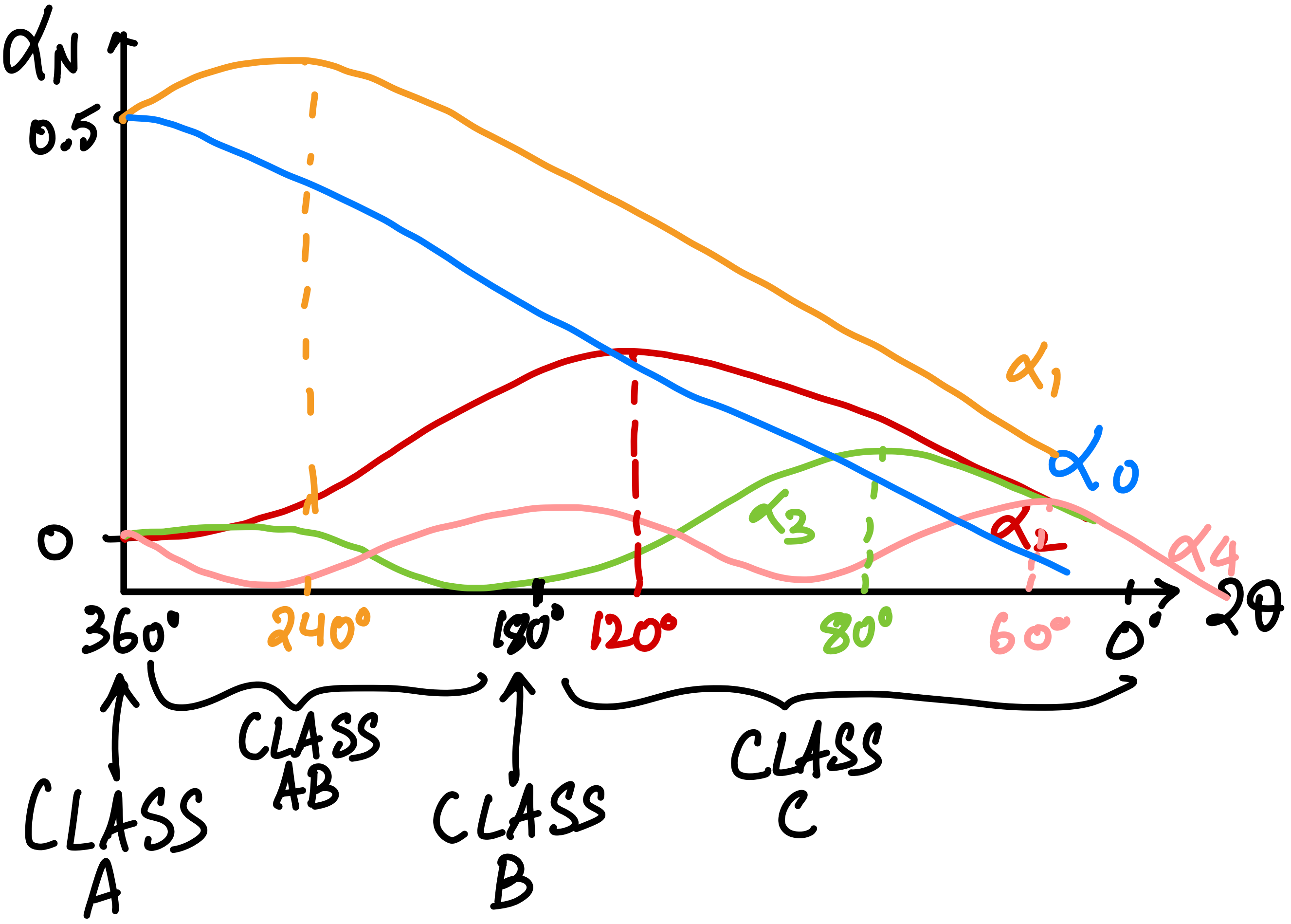
→ Measures how much power goes into Nth harmonic.



$$\boxed{2\theta_{opt} \simeq \frac{240^\circ}{N}} \rightarrow \text{Optimal Conduction Angle.}$$

> $\frac{\alpha_N}{\alpha_0} = \frac{I_N}{I_0}$ → At optimum, this is almost constant.

$\eta_{opt} \simeq 63\%$



Caution: This is valid under the truncated cosine model.

Doubler X2 vs Quadrupler

$$I_2(\theta_{opt}) = G_m V_p (0.1378)$$

$$I_4(\theta_{opt}) = G_m V_p (0.01857)$$

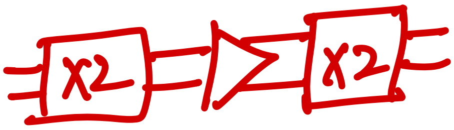
$$V_{p_2} = G_m V_p (0.1378) R_{out}$$

$$I_{2\times2}(\theta_{opt}^{2\times2}) = G_m^2 V_p [0.1378^2] R_{out}$$

$$\boxed{\begin{aligned} I_{2\times2}(\theta_{opt}^{2\times2}) &= G_m V_p [0.0189] A_v \\ I_4(\theta_{opt}^{4}) &= G_m V_p [0.01857]. \end{aligned}}$$

Caveats

> Truncated Cosine Assumption. Waveform Shaping to beat this.

> If you can amplify 

is better.

Takeaways

> RF frequencies (100 GHz & below)
Cascaded doublers are better.

> THz frequencies (Above 100 GHz),
Quadruplers are better.

> Waveform engineering/shaping changes this dogma.

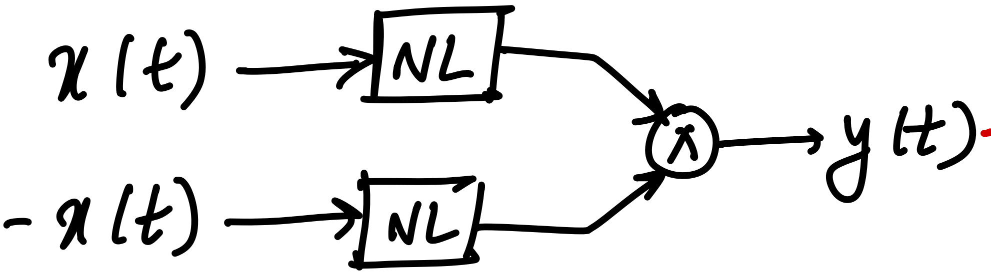
TOPOLOGY SELECTION

Balanced Multipliers

$$y_p(t) = f(x(t)) = b_0 + b_1\cos(\omega_0 t) + b_2\cos^2(\omega_0 t) + \ldots$$

$$y_n(t) = f(-x(t)) = b_0 - b_1\cos(\omega_0 t) + b_2\cos^2 - b_3\cos^3$$

$$y_p(t) + y_n(t) = 2\left[b_0 + b_2\cos^2(2\omega_0 t) + b_4\cos^4 + \ldots\right]$$



→ Only even harmonics!

Intuitive Viewpoint

$$\cos(\omega_0 t) \quad + \quad \cos(\omega_0 t + 180^\circ) = 0$$

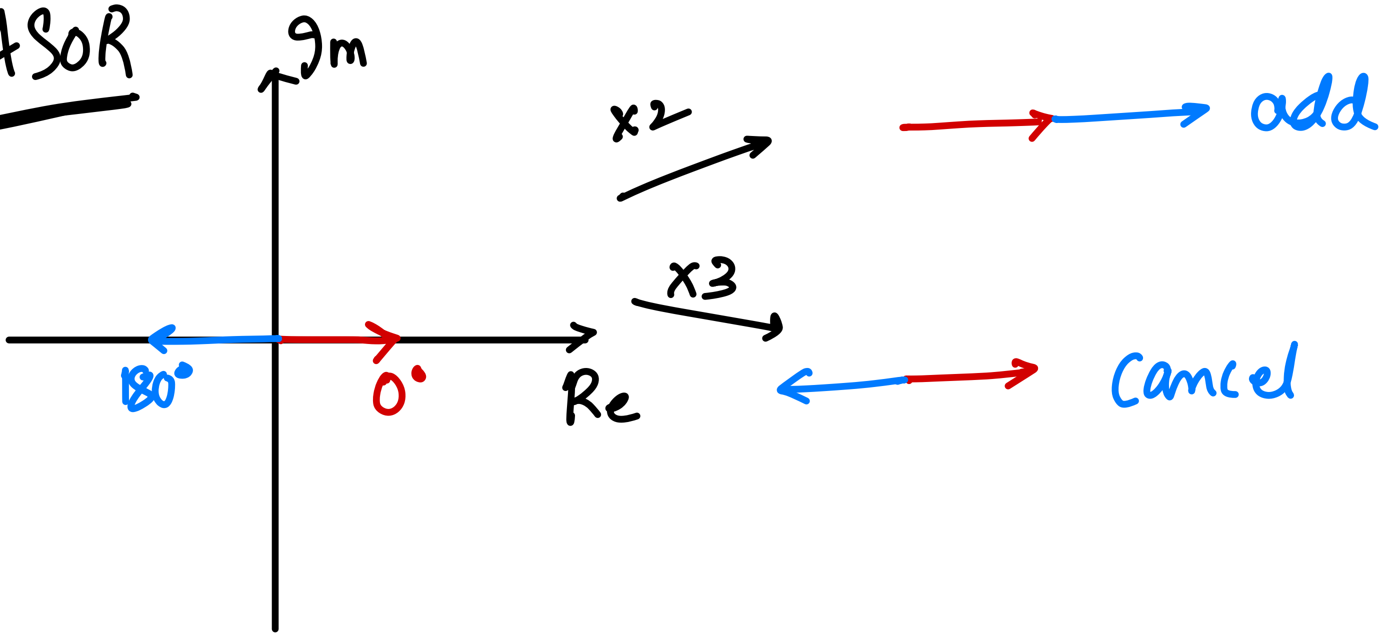
$\downarrow 2x$ $\qquad\qquad$ $\downarrow 2x$

$$\cos(2\omega_0 t) \quad + \quad \cos(2\omega_0 t) = 2\cos(2\omega_0 t)$$

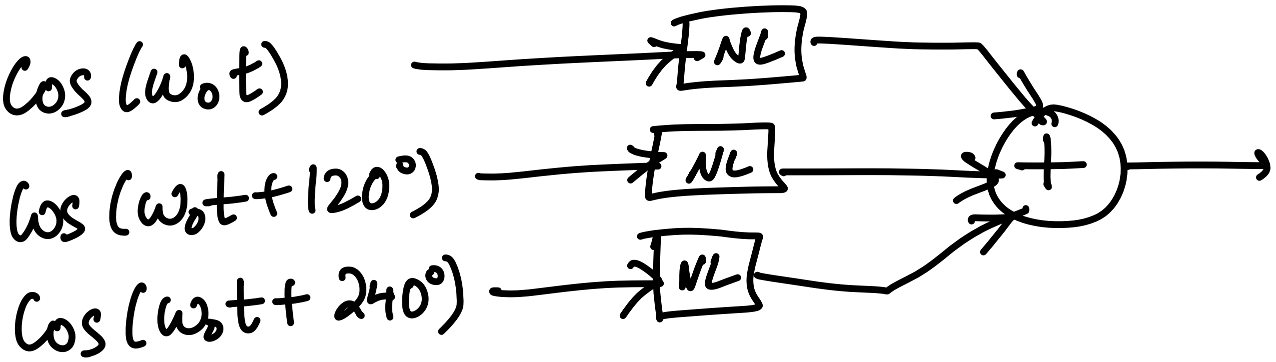
3x

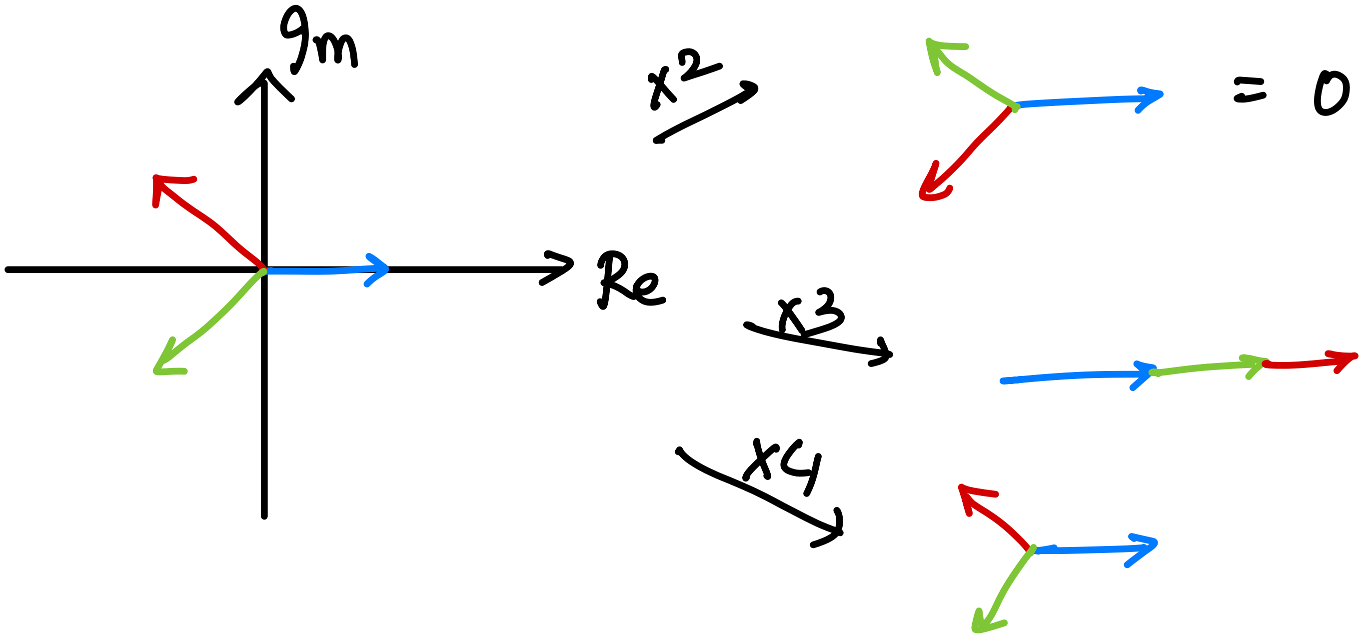
$$\cos(3\omega_0 t) \quad + \quad \cos(3\omega_0 t + 180) = 0$$

PHASOR



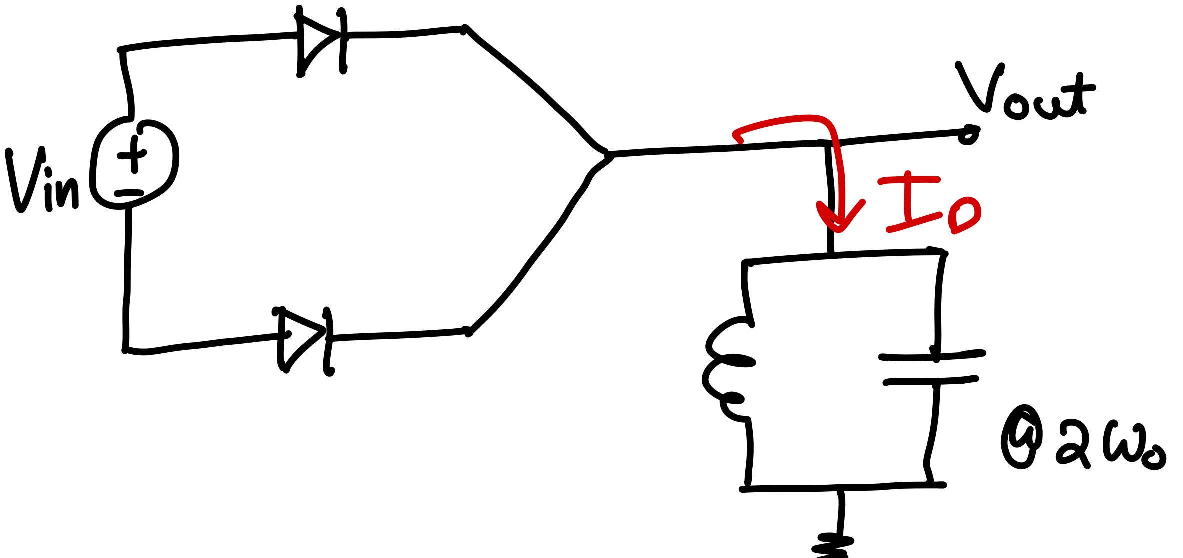
Tripler ⇒ Three Phase Signals.



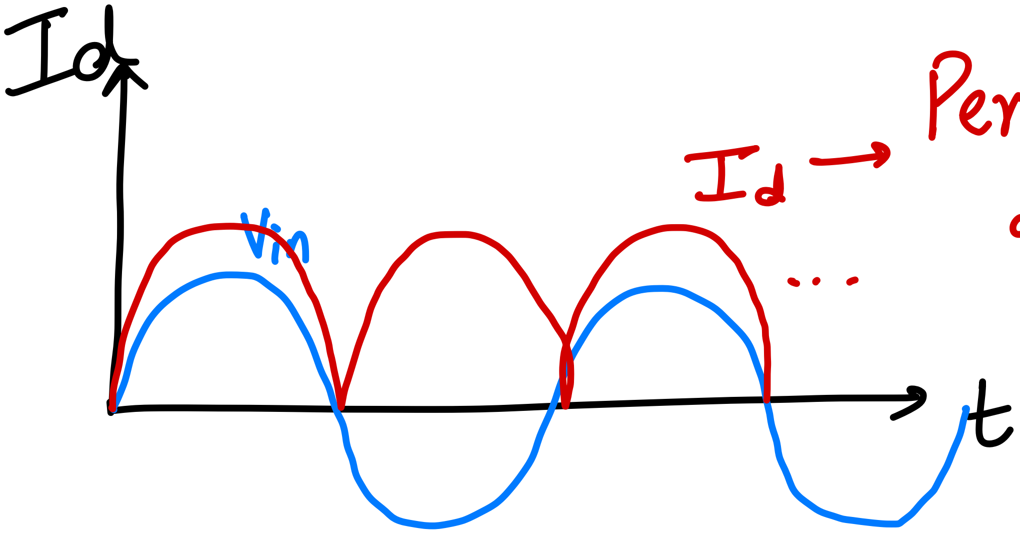


Circuit Implementation

Diodes



Rectifier.

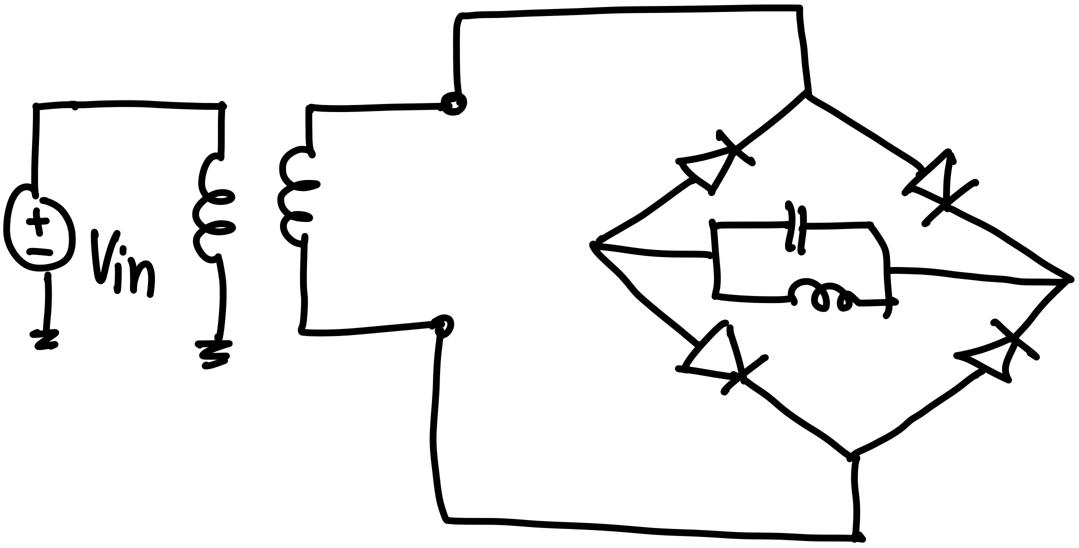


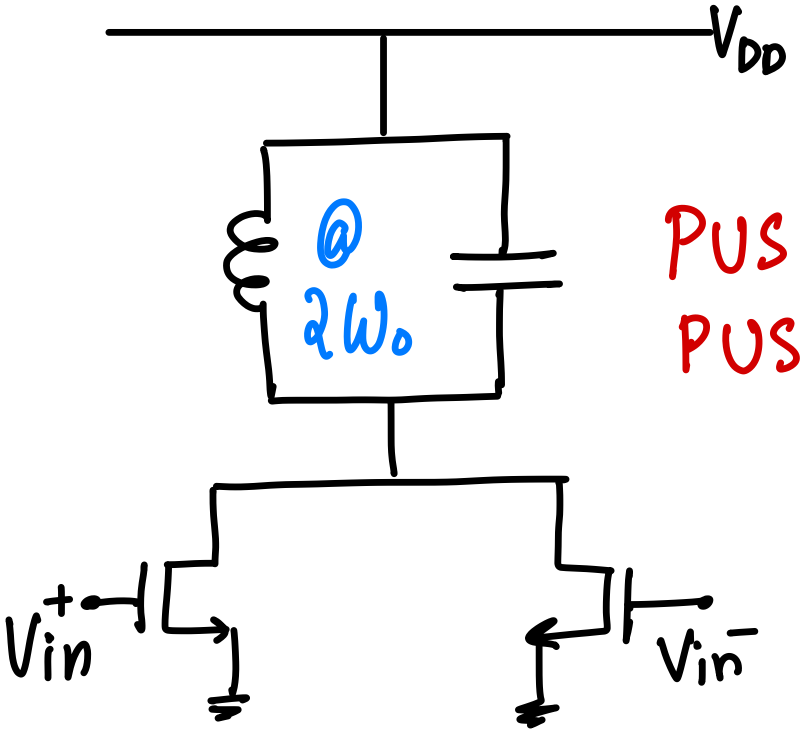
$I_d \rightarrow$ Periodicity

$2\omega_0$

$\downarrow$

$4\omega_0, 6\omega_0 \ldots$





PUSH PUSH