Adler's Equation

Definitions

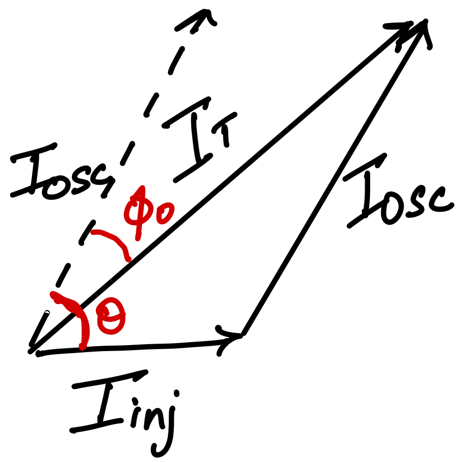
$\omega \rightarrow$ Instantaneous freq. of osc. after injection.

$\omega_{inj} \rightarrow$ Freq. of injected signal.

$\omega_0 \rightarrow$ Free-running freq. (before injection)

$\phi_0 \rightarrow$ Phase diff. b/w $\vec{I_T}$ & $\vec{I_{osc}}$ & it is also the phase shift of the tank.

$\theta \rightarrow$ Phase diff. b/w $\vec{I_{inj}}$ & $\vec{I_{osc}}$



Derivation

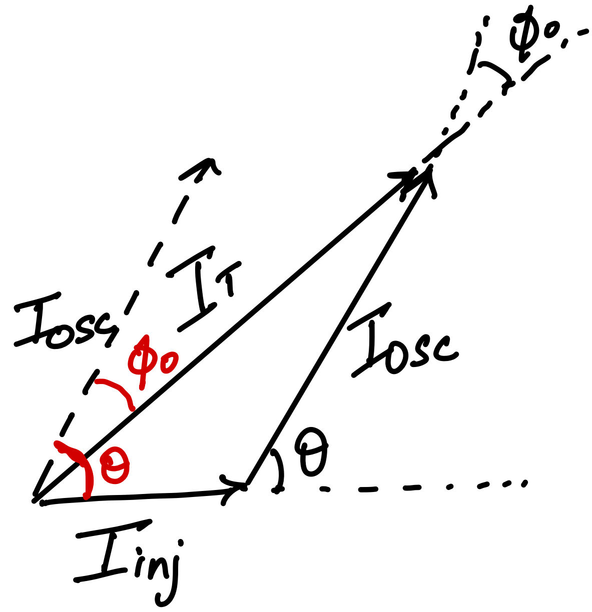
Step ①

$$\theta = \phi_{osc} - \phi_{inj}$$

$$\Rightarrow \boxed{\frac{d\theta}{dt} = \omega - \omega_{inj}} \text{ — ①}$$

Step ②

$$\frac{I_{inj}}{\sin\phi_0} = \frac{I_T}{\sin(180^\circ - \theta)} = \frac{I_T}{\sin\theta}$$



$$\Rightarrow \sin\phi_0 = \frac{I_{inj}}{I_T}\sin\theta$$

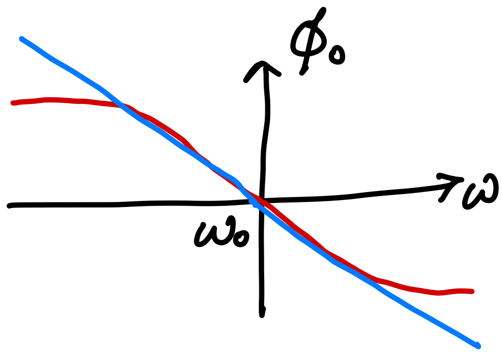
$$\sin\phi_0 = \frac{I_{inj}\sin\theta}{\sqrt{I_{osc}^2 + I_{inj}^2 - 2I_{osc}I_{inj}\cos\theta}}$$

Assume $I_{inj} \ll I_{osc} \Rightarrow \vec{I_T} \approx \vec{I}_{osc} \Rightarrow \sin\phi_0 \simeq \phi_0$

& denominator $\approx I_{osc}$.

$$\boxed{\phi_0 = \frac{I_{inj}}{I_{osc}} \sin\theta} \quad (2)$$

2^{nd} order parallel LC tank



$$\phi_0 = \tan^{-1}\left(\underbrace{\frac{R}{\omega L}}_{Q} \cdot \frac{\omega_0^2 - \omega^2}{\omega_0^2}\right)$$

if ω is close to ω_0

$$\phi_0 \simeq Q \cdot \frac{\overset{\simeq 2\omega_0}{\cancel{(\omega_0+\omega)}}(\omega_0-\omega)}{\omega_0^{\cancel{2}}}$$

$$\boxed{\phi_0 = \frac{2Q}{\omega_0}(\omega_0-\omega)} \quad (3)$$

Equating (2) & (3)

$$\frac{I_{inj}}{I_{osc}} \sin\theta = \frac{2Q}{\omega_0}(\omega_0-\omega)$$

$$= \frac{2Q}{\omega_0}\left((\omega_0-\omega_{inj}) - \underbrace{(\omega-\omega_{inj})}_{\frac{d\theta}{dt}}\right)$$

$$\frac{I_{inj}}{I_{osc}} \sin\theta = \frac{2Q}{\omega_0}\left[(\omega_0 - \omega_{inj}) - \frac{d\theta}{dt}\right]$$

$$\boxed{\frac{d\theta}{dt} = (\omega_0 - \omega_{inj}) - \frac{\omega_0}{2Q}\frac{I_{inj}}{I_{osc}} \cdot \sin\theta}$$

Adler's Equation!

↳ Differential Eqn. describing the dynamics of θ as a fn. of time.

Insights

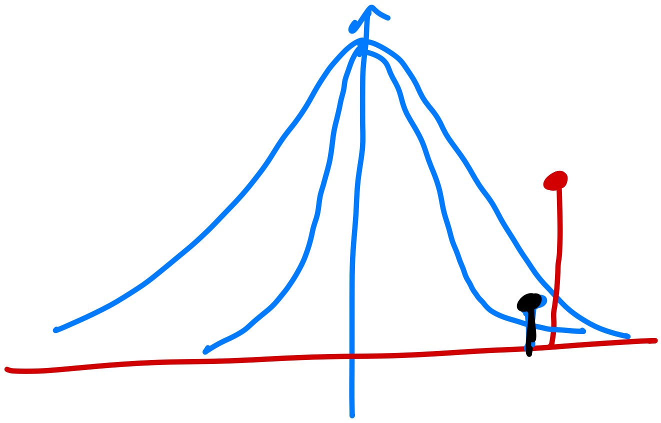
1) When locked, θ is constant. $\Rightarrow \frac{d\theta}{dt} = 0$.

$$\boxed{\sin\theta = \frac{2Q}{\omega_0} \cdot \frac{I_{osc}}{I_{inj}}(\omega_0 - \omega_{inj})}$$

2) $\sin\theta \in [-1, 1] \Rightarrow$ Locking can only occur with a range.

$$(\omega_{inj} - \omega_0) = \omega_L^+ = \frac{\omega_0 I_{inj}}{2Q I_{osc}} = -\omega_L^-$$

$$\Rightarrow \quad \boxed{\frac{\omega_L^{DS}}{\omega_0} = \frac{I_{inj}}{Q\, I_{osc}}}$$



Limitation of Adler's model

1) Weak injection!
2) Limited to LC oscillators!
3) Predicts symmetric locking range.
4) Assume sinusoidal injection.
5) Q? I_{osc}?