Chu-Harrington Limit of Electrically Small Antennas

Convention: $i \to -j$; $e^{ikr} \to e^{-jkr}$; $h^{(1)} \to \underbrace{h^{(2)}}_{\text{outgoing.}}$

> For ESA, $\exists$ a fundamental bound on the bandwidth efficiency product.

> Assumptions: LTI, passive.

> We start with a lossless antenna & we will introduce loss later.

$$BW < BW_{max}$$

$$\iff Q > \underbrace{Q_{min}}_{\text{Show this.}}$$

$$Q = \frac{\text{Stored energy}}{\text{Power dissipated per period.}}$$

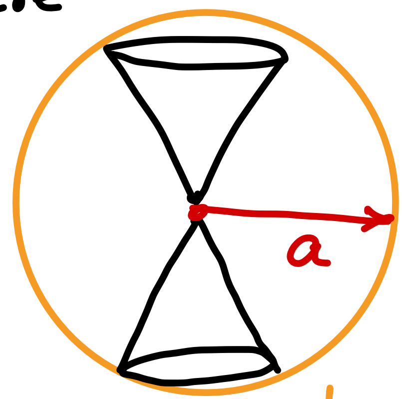
$$Q = \frac{\omega W}{P_r}$$

$W \to$ 2 Max (W_m, W_e).

$P_r \to$ Radiated power

> Fields outside the min. sphere satisfy FSME.

> SWF using a generating function approach.



> Caution: Minimum sphere must enclose all radiating elements (GND plane, cables, etc.)

FSME

$$\nabla \times \vec{E} = -j\omega\mu \vec{H}$$
$$\nabla \times \vec{H} = j\omega\epsilon \vec{E}$$
$$\nabla \cdot \vec{E} = 0$$
$$\nabla \cdot \vec{H} = 0$$

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0$$
$$\nabla^2 \vec{H} + k^2 \vec{H} = 0$$

$$\nabla^2 \vec{F} = \nabla(\nabla \cdot \vec{F}) - \nabla \times (\nabla \times \vec{F})$$

Claim: Solutions to the vector wave eqn.

$\nabla^2 \vec{F} + k^2 \vec{F} = 0$ are:

$\vec{F}_1 = \nabla \times (F\vec{a})$ $\vec{F}_2 = k^{-1} \nabla \times \vec{F}_1$, where $\vec{a}$ is a constant vector. F is a scalar function that satisfies the scalar wave eq.

$F \rightarrow$ Generating function.

Proof:

$$\nabla^2 \vec{F}_1 = \nabla^2 (\nabla \times (F\vec{a})) = \nabla \times (\nabla^2 (F\vec{a}))$$

$$= \nabla \times (\vec{a} (\nabla^2 F)) = -k^2 (\nabla \times (F\vec{a}))$$

$$= -k^2 (\vec{F}_1)$$

$$\nabla^2 \vec{F}_2 = k^{-1} (\nabla^2 (\nabla \times F_1)) = k^{-1} (\nabla \times (-k^2 \vec{F}_1))$$

$$= -k \nabla \times \vec{F}_1 = -k^2 \vec{F}_2 .$$

$$\left. \vec{H} = \frac{1}{\mu} \nabla \times \vec{A} ; \; \vec{E} = \frac{1}{j\omega\epsilon} \nabla \times \vec{H} \right\} \text{TM modes.}$$

$$\vec{E} = \frac{1}{\epsilon}\nabla\times\vec{A}_m\,; \quad \vec{H} = \frac{-1}{j\omega\mu}\nabla\times\vec{E} \Big\}$$ TE modes.

$\vec{F}_1 \Leftrightarrow \vec{H}$ & $\vec{F}_2 \Leftrightarrow \vec{E}$ } TM modes

$\vec{F}_1 \Leftrightarrow \vec{E}$ & $\vec{F}_2 \Leftrightarrow \vec{H}$ } TE modes.

Spherical coordinates

$$F_{mn} = h_n^{(2)}(kr)\, P_n^m(\cos\theta)\, e^{jm\phi}$$

$$\vec{F}_{1mn} = \nabla\times(F_{mn}\hat{r}) = \nabla F_{mn}\times\hat{r}$$

$$\vec{F}_{2mn} = k^{-1}(\nabla\times\vec{F}_{1mn})$$

$$\vec{E} = \sum_{m,n} C^e_{nm}\,\vec{F}_{1mn} + \sum_{m,n} D^e_{nm}\,\vec{F}_{2mn}$$

$$\vec{H} = \sum_{n,m} C^h_{nm}\,\vec{F}_{1mn} + \sum_{n,m} D^h_{nm}\,\vec{F}_{2mn}$$

$\underbrace{\text{TM}}_{\text{TM}_{mn}}$ & TE are duals of each other.

> Furthermore, it can be shown that energy in a mode is independent of m.

$\Rightarrow$ TM_{0n} mode is sufficient.

TM0n modes are given by:

$$E_\theta = \cancel{C_n^h} A_n \frac{-k_0 \sin\theta}{j\omega\epsilon_0} \frac{d P_n(\cos\theta)}{d(\cos\theta)} \frac{d(k_0 r h_n^{(2)}(k_0 r))}{d(k_0 r)}$$

$$E_r = A_n \frac{n(n+1)}{j\omega\epsilon_0 r^2} P_n(\cos\theta)\left[k_0 r h_n^{(2)}(k_0 r)\right]$$

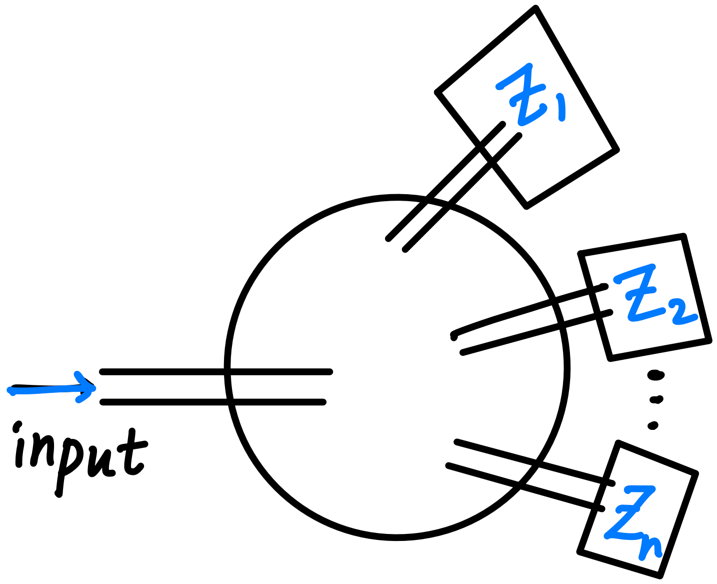
$$H_\phi = A_n \frac{\sin\theta}{r} \frac{d P_n(\cos\theta)}{d(\cos\theta)}\left[k_0 r h_n^{(2)}(k_0 r)\right]$$

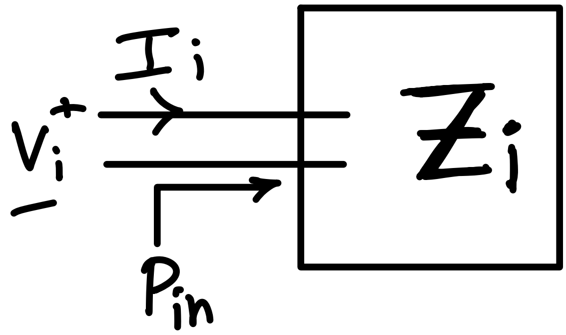
At $r=a$ we define $\rho = k_0 a$; $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$.

$$Z_n \triangleq \frac{E_\theta}{H_\phi} = j Z_0 \frac{d(\rho h_n^{(2)}(\rho))}{d(\rho)} \cdot \frac{1}{\rho h_n^{(2)}(\rho)}$$

$$\frac{Z_n}{Z_0} = j \cdot \frac{d(\rho h_n^{(2)}(\rho))}{d\rho} \cdot \frac{1}{\rho h_n^{(2)}(\rho)}$$

$$S_r = \frac{1}{2} \mathrm{Re}\{\vec{E} \times \vec{H}^*\} = \frac{k_0 \pi}{\omega \mu} \frac{2n(n+1)}{2n+1} |A_n|^2$$





We want V_i & I_i such that

$$\frac{1}{2} \mathrm{Re}\{V_i I_i^*\} = P_r \quad \& \quad Z_n = \frac{V_i}{I_i}$$

$$V_i = 4 \sqrt{\frac{\mu}{\epsilon}} \frac{A_n}{k} \sqrt{\frac{4\pi n(n+1)}{2n+1}}\, j \frac{d(\rho h_n^{(2)}(\rho))}{d\rho} \sqrt{Z_0}$$

$$I_i = 4 \sqrt{\frac{\mu}{\epsilon}} \frac{A_n}{k} \sqrt{\frac{4\pi n(n+1)}{2n+1}}\, \rho\, h_n^{(2)}(\rho) \frac{1}{\sqrt{Z_0}}$$

$$P_n = \frac{1}{2} \mathrm{Re}\{V_i I_i^*\}$$

$$= \frac{1}{2}\left[4\sqrt{\frac{\mu}{\epsilon}}\,\frac{A_n}{k}\sqrt{\frac{4\pi n(n+1)}{2n+1}}\right]^2 \underbrace{\mathrm{Re}\left\{j\cdot\frac{d(\rho h_n^{(2)})}{d\rho}\cdot \rho h_n^{(2)*}\right\}}_{\frac{1}{2}\text{ (Wronskian)}}$$

$$= 4\frac{\mu}{\epsilon}\frac{|A_n|^2}{k^2}\,\frac{4\pi n(n+1)}{2n+1} \;//$$

$$\frac{Z_n}{Z_0} = \frac{j}{\rho h_n^{(2)}(\rho)}\frac{d}{d\rho}\left(\rho h_n^{(2)}(\rho)\right)$$

$$= \frac{j}{\rho} + \frac{j}{h_n^{(2)}(\rho)}\cdot\frac{dh_n^{(2)}}{d\rho}$$

<u>Key Idea</u>: Build a circuit that has this impedance (Circuit Synthesis).

Recurrence relation: $f_n = \frac{2n-1}{\rho} f_{n-1} - f_{n-2}$

Derivative: $\frac{df_n}{d\rho} = f_{n-1} - \frac{n+1}{\rho} f_n.$

$$\frac{Z_n}{Z_0} = \frac{j}{\rho} + \frac{j}{h_n^{(2)}}\left[h_{n-1}^{(2)} - \frac{n+1}{\rho} h_n^{(2)}\right]$$

$$= \cancel{\frac{j}{\rho}} + \frac{j\,h_{n-1}^{(2)}}{h_n^{(2)}} - \frac{jn}{\rho} - \cancel{\frac{j}{\rho}}$$

$$= \frac{n}{j\rho} + \frac{1}{\dfrac{h_n^{(2)}}{j\,h_{n-1}^{(2)}}}$$

$$= \frac{n}{j\rho} + \frac{1}{\dfrac{1}{j\,h_{n-1}^{(2)}}\left[\dfrac{2n-1}{\rho} h_{n-1}^{(2)} - h_{n-2}^{(2)}\right]}$$

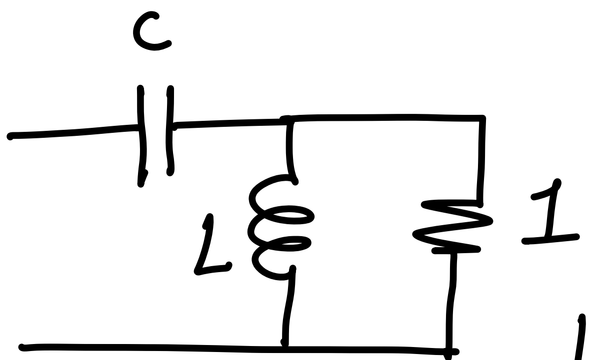
$$= \frac{n}{j\rho} + \frac{1}{\dfrac{2n-1}{j\rho} + \dfrac{1}{\dfrac{h_{n-1}^{(2)}}{j\,h_{n-2}^{(2)}}}}$$

$$= \frac{n}{j\rho} + \cfrac{1}{\cfrac{2n-1}{j\rho} + \cfrac{1}{\cfrac{2n-3}{j\rho} + \cdots + \cfrac{1}{\cfrac{3}{j\rho} + \cfrac{1}{\cfrac{1}{j\rho} + 1}}}}$$

⇒ Ladder network!

<u>n = 1</u>

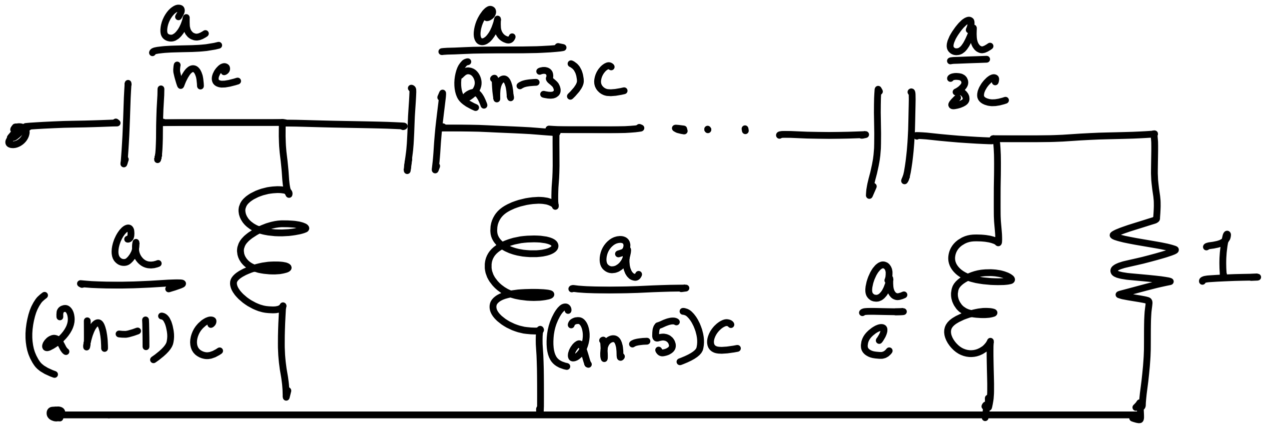
$$\frac{Z_n}{Z_0} = \frac{1}{j\rho} + \cfrac{1}{1 + \cfrac{1}{j\rho}}$$



$$\left(\frac{1}{1} + \frac{1}{j\omega L}\right)^{-1} + \frac{1}{j\omega C}$$

$$L = \frac{\rho}{\omega} = \frac{a}{c}$$

$$C = \frac{a}{c} = \frac{\rho}{\omega}$$



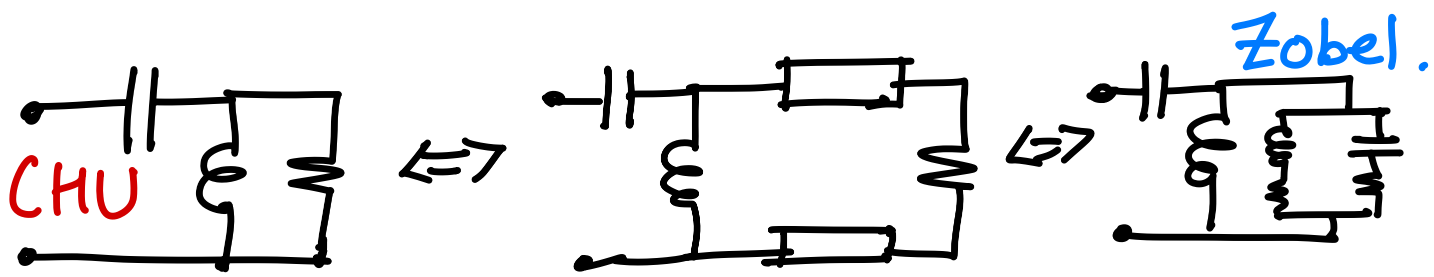
Interpretation

1) $1\,\Omega$ represents radiated power to ∞.

2) HOM have higher stored energy!
$\Rightarrow$ Q is higher!

3) Lowest Q is associated with TM_{01} mode, which is just the dipole mode.

Caution: There are other circuits with same Z_n but higher stored energy.



> Chu circuits are minimum energy circuits (maximum recoverable energy).

> From circuit model we can compute Q!

For $n=1$, $Q = \frac{1}{\rho^3} + \frac{1}{\rho}$

Any antenna always has

$$Q \geq \left[\frac{1}{(k_0 a)^3} + \frac{1}{k_0 a}\right]$$

Caveats

1) This is a conservative estimate because it ignores energy store inside the minimum sphere.

2) Electric Dipole + Mag Dipole

$$Q_{min} = \left[\frac{1}{2(k_o a)^3} + \frac{1}{k_o a}\right]$$

Also true for circular pol.

Bandwidth Efficiency Limit

$$BW = \frac{1}{Q}\left(\frac{S-1}{\sqrt{S}}\right)$$

$S = VSWR$.

$$S = \frac{1+|\Gamma|}{1-|\Gamma|}$$

$S = 2$

$$\Rightarrow \underbrace{BW}_{\text{fractional BW}} \cong \frac{1}{\sqrt{2}\,Q}$$

Only true for a single tuned MN.

$S = 2 \Rightarrow 90\%$ power is accepted

$\Rightarrow |\Gamma| = -10dB.$

$$\frac{\Delta f}{f_o} \simeq \frac{1}{\sqrt{2}\,Q}$$

$$BW_{max} \simeq (k_0 a)^3 \quad (\text{when } ka \text{ is small } ka \ll 1)$$

$$\frac{2\pi}{\lambda} \cdot \frac{\lambda}{2} = \pi$$

<u>Reinbroducing loss</u>

$$\eta = \frac{P_r}{P_r + P_l} \; ; \; Q_r = \omega \frac{W_{stored}}{P_r}$$

$$Q_t = \omega \frac{W_{stored}}{P_r + P_l}$$

$$\Rightarrow Q_t = \eta \, Q_r$$

$$\Rightarrow Q_t \geq \frac{\eta}{(ka)^3}$$

$$\Rightarrow \boxed{\eta . BW \leq \frac{(ka)^3}{\sqrt{2}}}$$

$$\boxed{\eta \cdot BW \leq \frac{1}{\sqrt{2}}\left(\frac{1}{ka} + \frac{1}{n(ka)^3}\right)^{-1}}$$

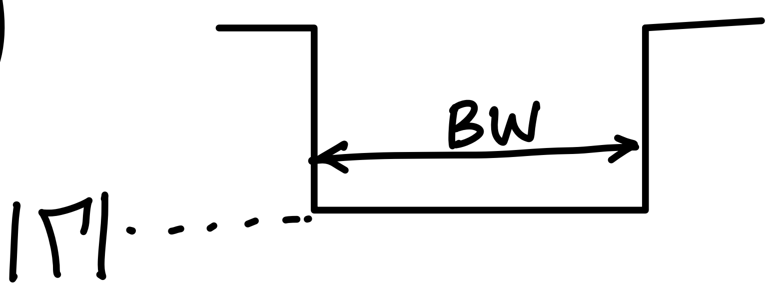
$n=1 \rightarrow$ single mode or LP

$n=2 \rightarrow$ dual mode or CP.

Valid for single tuned matching network!

Multituned Matching Network

$$BW = \frac{1}{Q}\frac{\pi}{\ln\left(\frac{1}{|\Gamma|}\right)} \rightarrow \text{Bodé Fano Limit}$$



$$VSWR = 2 \Rightarrow |\Gamma| = \frac{1}{3} \Rightarrow \ln\left(\frac{1}{|\Gamma|}\right) = 1.099$$

$$\Rightarrow BW = \frac{2.86}{Q}$$

$\Rightarrow$ 4 times more BW! (Best case scenario.)

> Two tuned circuit ⇒ 2.3x

Practical Limit

$$\eta\, BW \leq \frac{1}{\sqrt{2}}\left(\frac{1}{ka} + \frac{1}{n(ka)^3}\right)^{-1}$$