Spherical Wave Functions

$$\vec{\Pi} = \psi(x,y)\, e^{i\beta z}\, \hat{z} = \psi(\rho,\phi)\, e^{i\beta z}\, \hat{z}.$$

$$\nabla \times \nabla \times \vec{A} - k^2 \vec{A} - i\omega\mu\epsilon \nabla \Phi = \mu \vec{J}$$

$$\nabla^2 \phi - i\omega \nabla \cdot \vec{A} = \frac{-\rho}{\epsilon}$$

$\vec{A} = A_r \hat{r}$ & $\vec{J} = J_r \hat{r}$ $\longrightarrow$ Assume.

Sub & simplify,

$$\left[\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial\phi^2} + k^2\right] A_r$$

$$= -i\omega\mu\epsilon \frac{d\Phi}{\partial r} - \mu J_r$$

$$\frac{1}{r} \frac{\partial}{\partial\theta}\left(\frac{\partial A_r}{\partial r} - i\omega\mu\epsilon \Phi\right) = 0$$

$$\frac{1}{r\sin\theta} \frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial r} A_r - i\omega\mu\epsilon \Phi\right) = 0$$

Sph. Gauge: $\frac{\partial A_r}{\partial r} = i\omega\mu\epsilon\,\Phi$.

$$\Rightarrow \left[\frac{\partial}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + k^2\right]A_r = -\mu J_r$$

$$\Rightarrow \quad (\nabla^2 + k^2)\frac{A_r}{r} = -\mu\frac{J_r}{r}$$

Duality: $(\nabla^2 + k^2)\frac{A_{mr}}{r} = -\epsilon\frac{J_{mr}}{r}$

<u>Fields</u>

$$E_r = -\frac{1}{i\omega\mu\epsilon}\left(\frac{\partial^2}{\partial r^2} + k^2\right)A_r \qquad H_\theta = \frac{1}{\mu r\sin\theta}\frac{\partial A_r}{\partial\phi}$$

$$E_\theta = -\frac{1}{i\omega\mu\epsilon r}\left(\frac{\partial^2}{\partial r\partial\theta}\right)A_r \qquad H_\phi = -\frac{1}{\mu r}\frac{\partial A_r}{\partial\theta}$$

$$E_\phi = -\frac{1}{i\omega\mu\epsilon}\frac{1}{r\sin\theta}\frac{\partial^2}{\partial r\partial\phi}A_r \qquad H_r = 0$$

$A_r \rightarrow TM_r$

$$\psi = \frac{A_r}{r} \Rightarrow \boxed{(\nabla^2 + k^2)\,\psi(r,\theta,\phi) = 0}$$

in source free Homogeneous space.

<u>SOV</u>: $\psi(r,\theta,\phi) = R(r)\,\Theta(\theta)\,F(\phi)$.

Sub & dividing by $\frac{R\Theta F}{r^2\sin^2\theta}$

$$\Rightarrow \frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$

$$+ \frac{1}{F}\frac{d^2F}{d\phi^2} + k^2r^2\sin^2\theta = 0.$$

Sub & divide by $\sin^2\theta$

$$\Rightarrow \frac{1}{F}\frac{d^2F}{d\phi^2} = -\mu^2$$

$$\Rightarrow \underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2r^2}_{\nu(\nu+1)} + \overbrace{\frac{1}{\sin\theta}\frac{1}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{\mu^2}{\sin^2\theta}}^{-\nu(\nu+1)} = 0$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[(kr)^2 - \nu(\nu+1)\right]R = 0 \longrightarrow 1a$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(\nu(\nu+1) - \frac{\mu^2}{\sin^2\theta}\right)\Theta = 0$$

$\longrightarrow 1b$

$$\frac{d^2F}{d\phi^2} + \mu^2 F = 0 \longrightarrow 1c$$

$$1c \Rightarrow \quad F = Ae^{i\mu\phi} + Be^{-i\mu\phi}$$

& μ is an integer if $\phi \in [0, 2\pi]$.

$$e^{i\mu\phi} = e^{i\mu(\phi+2\pi)} = e^{i\mu\phi}\, e^{i2\mu\pi}$$

> (1a) is similar to Bessel D.E. with known solutions

$$R_n(kr) = \left(\frac{\pi}{2kr}\right)^{1/2} Z_{n+\frac{1}{2}}(kr)$$

$R_n \rightarrow \underbrace{j, n, h^{(1)}, h^{(2)}}_{\text{Spherical Bessel \& Hankel!}}$

$Z \rightarrow \underbrace{J, N}_{\text{Bessel}}, \underbrace{H^{(1)}, H^{(2)}}_{\text{Hankel}}$

Properties: Large arg.

(Sec 8-1) recursion.

Wronskian.

Derivative.

Sometimes: $\hat{Z}_n(kr) = kr\, Z_n(kr)$

Schelkunoff BF. ($A_r = r\psi$)

$$h_0^{(1)} = \frac{\sin(kr)}{kr} + i\left(-\frac{\cos(kr)}{kr}\right) = \underbrace{\frac{e^{ikr}}{ikr}}_{\text{Green's Fn.}}$$

(1b): Generalized Legendre's Eq. & sol are the Associated Legendre's Functions:

$$\underbrace{P_\nu^\mu(\cos\theta)}_{\text{first kind}}, \quad \underbrace{Q_\nu^\mu(\cos\theta)}_{\text{second kind}}$$

$\mu \rightarrow$ order (related to ϕ)

$\nu \rightarrow$ degree (related to θ)

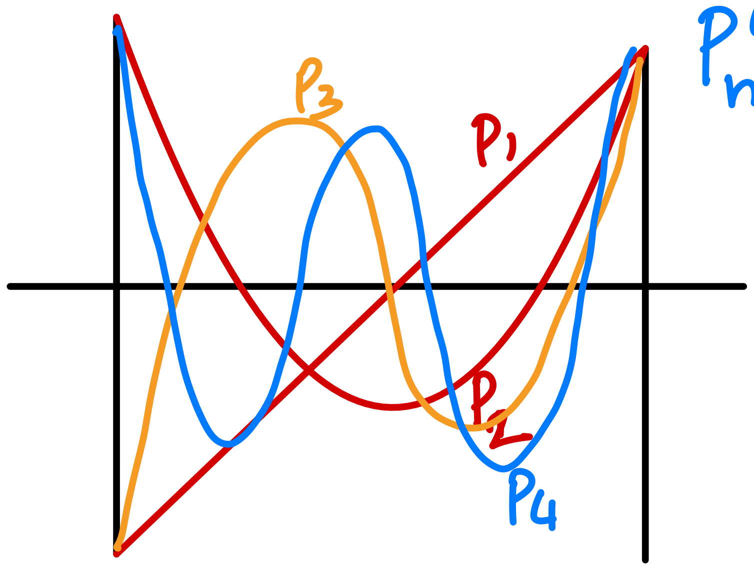
> ν must be an integer if $\theta \in [0,\pi]$.

ν is not an integer then, P becomes multiple valued (branch cut at $(-1,1]$).

> Also Q cannot exist, because $Q(\pm 1) \to \infty$.

$\theta \in [0,\pi] \Rightarrow P_n^m(\cos\theta)$ is the only admissible solution.

> Also $m \leq n$ because $P_n^m(x) = 0$ for $m > n$.



$P_n^0(x) \to$ Legendre Polynomials.

Spherical Wave Expansion:

$$\psi(r,\theta,\phi) = \sum_n \sum_m \left[a_n j_n(kr) + b_n n_n(kr)\right] \cdot \underbrace{P_n^m(\cos\theta)\, e^{im\phi}}_{\text{Spherical harmonics } Y_n^m(\theta,\phi)}$$

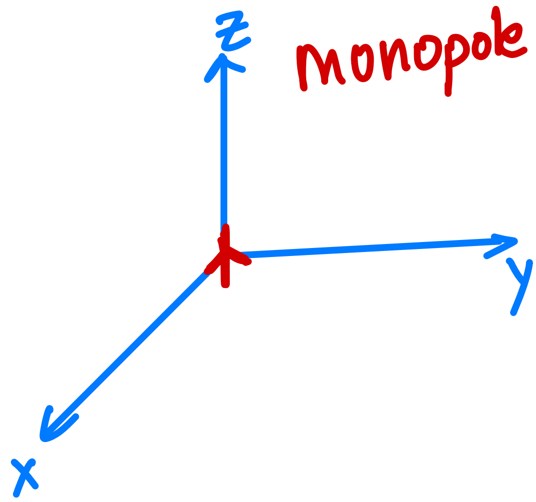
For example: $U(\theta, \phi) = \sum_n \sum_m a_{nm} P_n^m e^{im\phi}$

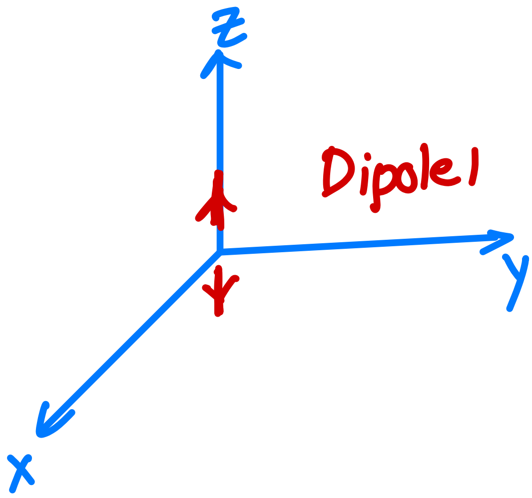
$$\frac{\delta(\theta)\delta(\phi)}{\sin\theta} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \alpha_m \frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!} P_n^m(1)$$

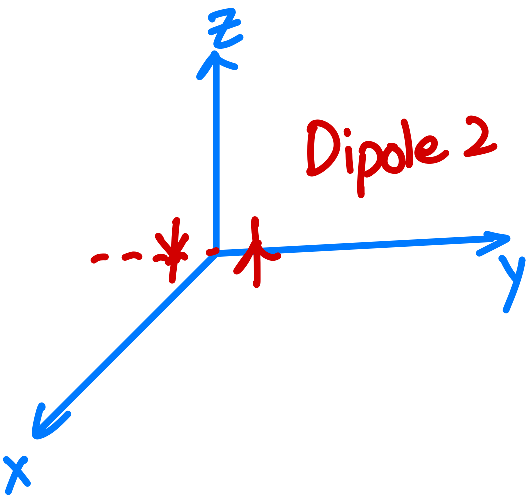
$$\alpha_m = \begin{cases} 1 & m=0 \\ 2 & m>0 \end{cases}$$

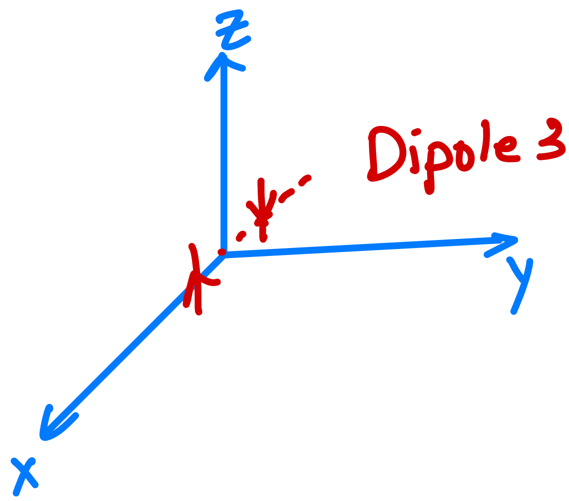
Ex: 8-1

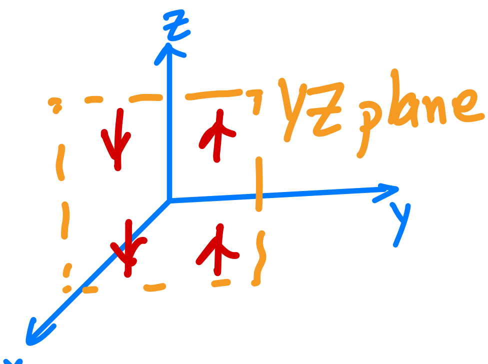
Multipole Representation of SWF











Quadrupole ...

Proof: $\underbrace{A_z^{I}(\vec{r})}_{\text{Lorentz Gauge}} = \frac{\mu I\,dl}{4\pi}\frac{e^{ikr}}{r}$

$$= \frac{i\mu k I\,dl}{4\pi} h_0^{(1)}(kr)\, P_0(\cos\theta)$$

$$\vec{H} = \frac{k^2 I\,dl}{4\pi}\left(-i+\frac{1}{kr}\right)\frac{e^{ikr}}{r}\sin\theta\,\hat{\phi}$$

$$H_\phi(\vec{r}) = -\frac{1}{\mu r}\frac{\partial A_r}{\partial\theta}$$ spherical gauge.

$$\Rightarrow A_r(\vec{r}) = \frac{\mu k^2 I\,dl}{4\pi}\left(-i+\frac{1}{kr}\right)e^{ikr}\cos\theta.$$

Note: $\hat{h}_1^{(1)}(kr) = -\left(1+\frac{i}{kr}\right)e^{ikr}$

$$P_1^0(\cos\theta) = \cos\theta$$

$$\Rightarrow A_r(\vec{r}) = \frac{i\mu k^2 I\,dl}{4\pi}\hat{h}_1^{(1)}(kr)\,P_1^0(\cos\theta)$$

In Sph Gauge: a_{10} multipole.

In Lorentz Gauge: a_{00} multipole. → monople.

Case 2: Dipole

$$A_z^{2z} = A_z^1\left(x, y, z - \frac{\delta}{2}\right) - A_z^1\left(x, y, z + \frac{\delta}{2}\right)$$

$$\simeq \delta \frac{\partial A_z^1}{\partial z} = -i \frac{\mu k I dl \delta}{4\pi} \frac{\partial}{\partial z} h_0^{(1)}(kr)$$

From derivative properties

$$= \frac{i\mu k I dl \delta}{4\pi} h_1^{(1)}(kr) P_1(\cos\theta)$$

$n=1, \; m=0.$

$$\underbrace{A_z^{2(x,y)}}_{-\delta \frac{\partial A_z^1}{\partial x, y}} = \frac{i\mu k I dl \delta}{4\pi} h_1^{(1)}(kr) P_1'(\cos\theta) \begin{cases} \cos\phi \to x \\ \sin\phi \to y \end{cases}$$

$n=1, \; m=1$

$$A_z^{4yz} = \delta_1 \delta_2 \frac{\partial^2 A_z'(\vec{r})}{\partial y\, \partial z} = \frac{-j\mu k^2 I dl\, \delta_1 \delta_2}{12\pi}$$

$$h_2^{(1)}(kr) P_2^1(\cos\theta) \sin\phi.$$

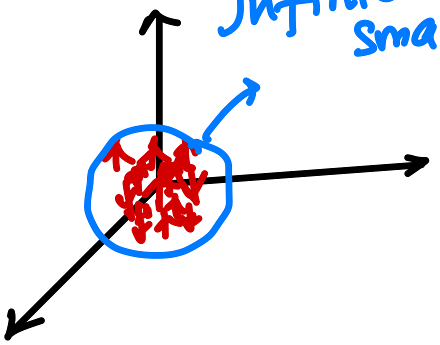
$$n=2, \; m=1.$$

Superdirective Antenna

> A highly directive radiation pattern $\left(\text{like } \frac{\delta(\theta)\,\delta(\phi)}{\sin\theta}\right) \rightarrow$ Decompose into SWE

$\rightarrow$ Realize SWE with multipoles.

Infinitesimally small.



In condradiction? No.

$\xrightarrow{FT?}$ $a\,| + b e^{ikr} + c e^{i2kr} + \cdots\cdot$

Note: In practice Directivity improvement is modest! $\left\{ \begin{array}{l} \text{Mutual coupling.} \\ \text{High currents.} \\ \text{Efficiency} \leftrightarrow \text{Bandwidth} \end{array} \right\}$