EM23 - Cylindrical Wave Functions

$$\nabla^2\psi + k^2\psi = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = 0$$

$$\psi = R(\rho)\,\Phi(\phi)\,Z(z)$$

Sub & divide by ψ.

$$\frac{1}{\rho R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} + \underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{\text{only term with } z \text{ dep.}} + k^2 = 0$$

$$\frac{1}{Z}\frac{\partial^2 Z(z)}{\partial z^2} = -k_z^2 = -\beta^2$$

$$\Rightarrow Z(z) = Ae^{ik_z z} + Be^{-ik_z z}$$

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \overbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}}^{-\nu^2} + k_\rho^2\rho^2 = 0$$

$$k_\rho^2 = k^2 - k_z^2$$

$$\Phi(\phi) = C e^{i\nu\phi} + D e^{-i\nu\phi}$$

$$\rho \frac{d}{d\rho}\left[\rho \frac{dR}{d\rho}\right] + \left[(k_\rho \rho)^2 - \nu^2\right] R = 0$$

$R \rightarrow R_\nu$ & apply product rule & multiply & divide by k_ρ^2

$$(k_\rho \rho)^2 \frac{d^2 R_\nu}{d(k_\rho \rho)^2} + (k_\rho \rho) \frac{dR_\nu}{d(k_\rho \rho)} + \left[(k_\rho \rho)^2 - \nu^2\right] R_\nu = 0$$

$$\tilde{\rho} = k_\rho \rho$$

$$\Rightarrow \boxed{\frac{d^2 R_\nu}{d\tilde{\rho}^2} + \frac{1}{\tilde{\rho}} \frac{dR_\nu}{d\tilde{\rho}} + \left[1 - \left(\frac{\nu}{\tilde{\rho}}\right)^2\right] R_\nu = 0}$$

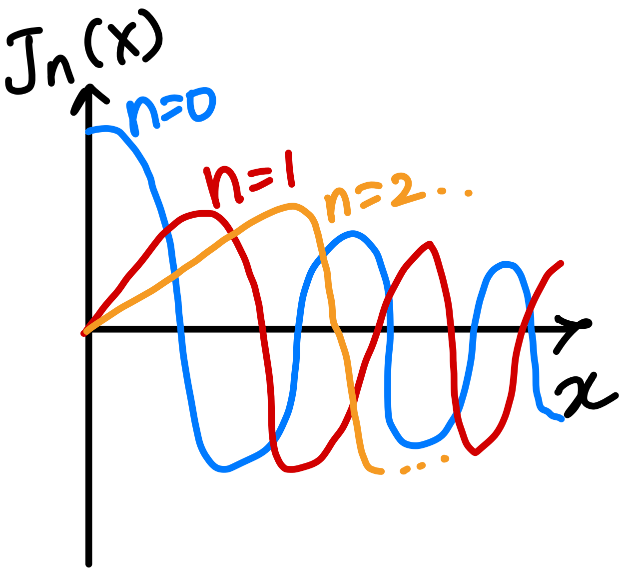
BESSEL DIFF. EQUATION

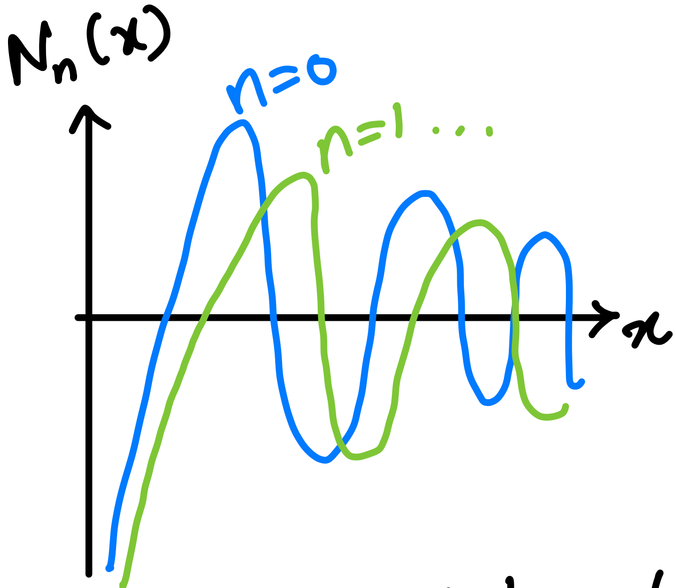
Solutions are

$$\underbrace{R_\nu(k_\rho \rho)}_{\text{Standing waves}} = A\,\underbrace{J_\nu(k_\rho \rho)}_{\text{Bessel fn. first kind order }\nu} + B\,\underbrace{N_\nu(k_\rho \rho)}_{\text{Bessel fn. of second kind order }\nu}$$

$$\underbrace{R_\nu(k_\rho \rho)}_{\text{Propagating waves}} = A\,\underbrace{H_\nu^{(1)}(k_\rho \rho)} + B\,\underbrace{H_\nu^{(2)}}(k_\rho \rho)$$

Cylindrical Hankel functions of the first & second kind & order ν.





> If $\phi \in [0, 2\pi] \Rightarrow \nu$ becomes $n \rightarrow$ integer!

$$\Phi(\phi) = A e^{i\nu\phi} + B e^{-i\nu\phi}$$

$$\phi + 2\pi \Longleftrightarrow \phi$$

$$e^{i2\pi\nu} e^{i2\pi\phi} = e^{i2\pi\phi}$$

$$\Rightarrow \nu = n \in \mathbb{Z}$$

$$H_\nu^{(1)}(k_\rho \rho) = J_\nu(k_\rho \rho) + i N_\nu(k_\rho \rho)$$

$$H_\nu^{(2)}(k_\rho \rho) = J_\nu(k_\rho \rho) - i N_\nu(k_\rho \rho)$$

Asymptotic Forms

$$\lim_{k_\rho \rho \to \infty} J_n(k_\rho \rho) \simeq \sqrt{\frac{2}{\pi k_\rho \rho}} \cos\left(k_\rho \rho - \frac{2n+1}{4}\pi\right)$$

$$\lim_{k_\rho \rho \to \infty} N_n(k_\rho \rho) \simeq \sqrt{\frac{2}{\pi k_\rho \rho}} \sin\left(k_\rho \rho - \frac{2n+1}{4}\pi\right)$$

$$\lim_{k_\rho \rho \to \infty} H_n^{(1),(2)}(k_\rho \rho) \simeq \sqrt{\frac{2}{\pi k_\rho \rho}}\, e^{\pm i\left(k_\rho \rho - \frac{2n+1}{4}\pi\right)}$$

$> k_\rho = \sqrt{k^2 - k_z^2}$ & when $|k_z| > k$

$k_\rho \rightarrow$ imaginary $\longrightarrow$ evanescent wave

$$k_\rho = i\alpha$$

$$\Rightarrow \rho \frac{d}{d\rho}\left[\rho \frac{dR}{d\rho}\right] - \left[(\alpha\rho)^2 + \nu^2\right] R = 0$$

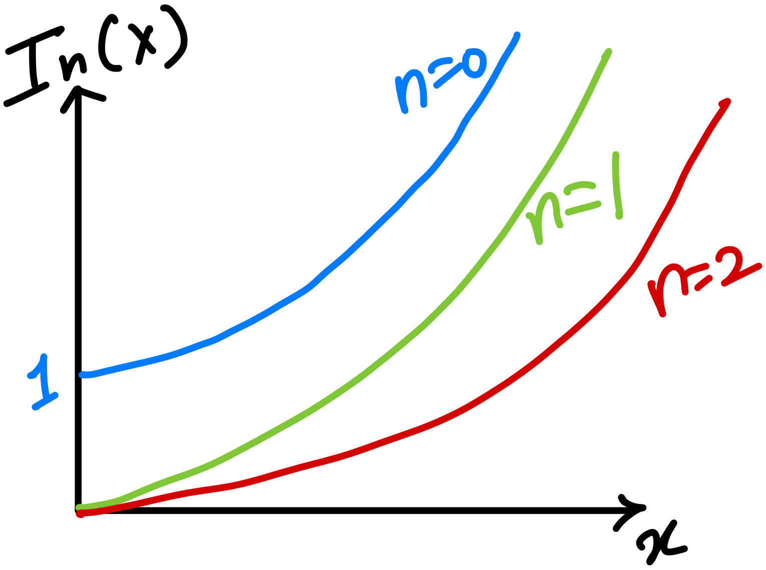
$\Rightarrow \tilde{\rho} = \alpha\rho \;, \; R \rightarrow R_\nu$

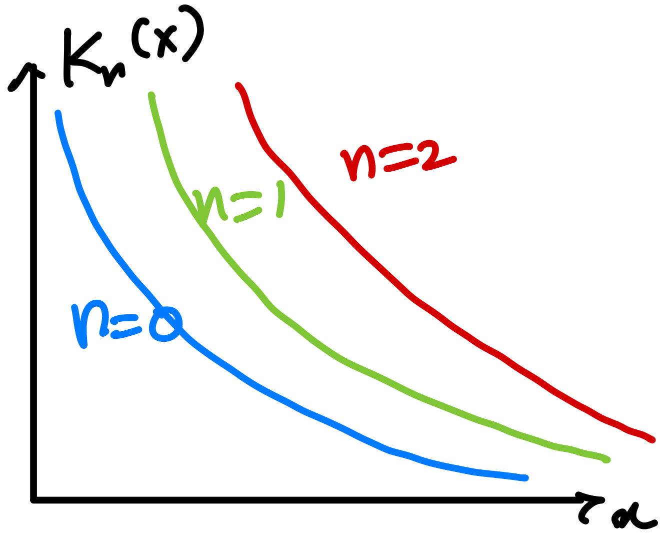
$$\boxed{\frac{d^2 R_\nu}{d\tilde{\rho}^2} + \frac{1}{\tilde{\rho}}\frac{dR_\nu}{d\tilde{\rho}} - \left[1 + \left(\frac{\nu}{\tilde{\rho}}\right)^2\right] R_\nu = 0}$$

Modified Bessel Equation.

Solutions: $I_n(\alpha\rho)$ & $K_n(\alpha\rho)$ are the modified Bessel functions.

$\longrightarrow$ Evanescent waves!





$$J_n(i\alpha\rho) = (i)^n I_n(\alpha\rho)$$

$$H_n^{(1)}(i\alpha\rho) = \frac{2}{\pi}(i)^{n+1} K_n(\alpha\rho)$$

$$\lim_{\alpha\rho\to\infty} I_n(\alpha\rho) \simeq \frac{e^{\alpha\rho}}{\sqrt{2\pi\alpha\rho}}$$

$$\lim_{\alpha\rho\to\infty} K_n(\alpha\rho) \simeq \sqrt{\frac{\pi}{2\alpha\rho}}\, e^{-\alpha\rho}$$

Useful Properties

Recurrence relations

$$R_{\nu-1} + R_{\nu+1} = \frac{2\nu}{\rho} R_\nu \qquad R_\nu \to J_\nu, N_\nu, H_\nu^{(1)}, H_\nu^{(2)}$$

$$I_{\nu-1} - I_{\nu+1} = \frac{2\nu}{\rho} I_\nu$$

$$K_{\nu-1} - K_{\nu+1} = \frac{-2\nu}{\rho} K_\nu$$

Derivatives

$$\frac{dR_\nu}{d\rho} = \frac{1}{2}\left[R_{\nu-1} - R_{\nu+1}\right]$$

$$\frac{d}{d\rho}\left[\rho^\nu R_\nu(\rho)\right] = \rho^\nu R_{\nu-1}$$

$$\frac{d}{d\rho}\left[\rho^\nu R_\nu(\rho)\right] = -\rho^{-\nu} R_{\nu+1}$$

$$\frac{dI_\nu}{d\rho} = \frac{1}{2}\left[I_{\nu-1} + I_{\nu+1}\right]$$

$$\frac{dK_\nu}{d\rho} = -\frac{1}{2}\left[K_{\nu-1} + K_{\nu+1}\right]$$

$$\frac{d}{d\rho}\left[\rho^\nu \underbrace{I_\nu(\rho)}_{K_\nu(\rho)}\right] = \underbrace{\rho^\nu}_{-\rho^\nu} \underbrace{I_{\nu-1}}_{K_{\nu-1}}(\rho)$$

Wronskian Relation

derivative

$$J_\nu(\rho) N_\nu'(\rho) - J_\nu'(\rho) N_\nu(\rho) = \frac{2}{\pi \rho}$$

$$J_\nu(\rho) H_\nu^{(1)'}(\rho) - J_\nu'(\rho) H_\nu^{(1)}(\rho) = \frac{i2}{\pi \rho}$$

Orthogonality Relations

$$\int_0^a J_\nu\left(\frac{x_{\nu p}}{a}\rho\right) J_\nu\left(\frac{x_{\nu q}}{a}\rho\right)\rho\, d\rho = 0$$

$$\int_0^a J_\nu\left(\frac{x_{\nu p}'}{a}\rho\right) J_\nu\left(\frac{x_{\nu q}'}{a}\rho\right)\rho\, d\rho = 0$$

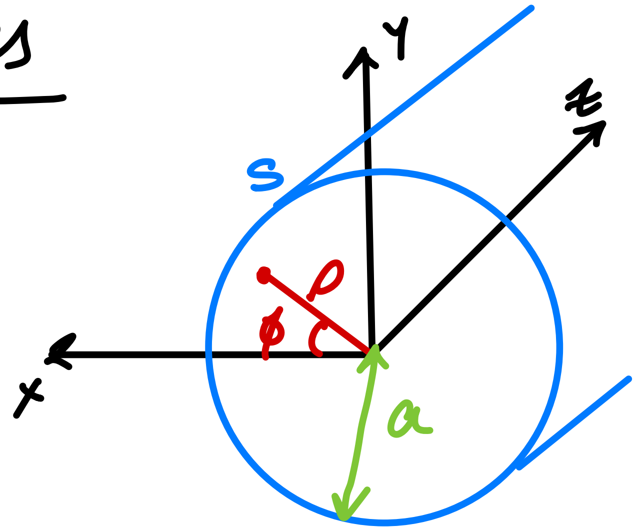
$x_{\nu p}$, $x_{\nu q}$ are zeroes of $J_\nu(x)$; $p \neq q$

$x_{\nu p}'$, $x_{\nu q}'$ are zeroes of $J_\nu'(x)$; $p \neq q$

$$\int_0^a J_\nu^2\left(\frac{x_{\nu p}}{a}\rho\right)\rho\, d\rho = \frac{a^2}{2}\left[J_{\nu+1}(x_{\nu p})\right]^2$$

$$\int_0^a J_\nu^2\left(\frac{x_{\nu p}'}{a}\rho\right)\rho\, d\rho = \frac{a^2}{2}\left[1-\left(\frac{\nu^2}{x_{\nu p}'^2}\right)\right]\left[J_\nu(x_{\nu p}')\right]^2$$

Cylindrical Waveguides



$\psi(\vec{r}) = 0 \Rightarrow$ TM

$\frac{\partial \psi(\vec{r})}{\partial \rho} = 0 \Rightarrow$ TE

$$\Pi = \psi(x,y)\, e^{\pm i\beta z} = \psi(\rho,\phi)\, e^{\pm i\beta z}$$

$$\psi(\rho,\phi) = A_n J_n(k_\rho \rho) + \cancelto{0}{B_n} N_n(k_\rho \rho) \underbrace{\begin{cases} \sin(n\phi) \\ \cos(n\phi) \end{cases}}_{\text{degenerate}}$$

$$= A_n J_n(k_\rho \rho) \cos(n\phi)$$

TM BC $\Rightarrow J_n(k_\rho a) = 0$

$$\Rightarrow k_\rho a = x_{np}$$

$$\Rightarrow k_\rho = \frac{x_{np}}{a} \qquad K_c = \frac{m\pi}{a}$$

$$\Rightarrow \beta = \sqrt{k^2 - \left(\frac{x_{np}}{a}\right)^2} \quad //$$

$$K_{cutoff} = \frac{x_{np}}{a} \Rightarrow f_c^{TM\,np} = \frac{x_{np}}{2\pi a\sqrt{\mu\epsilon}}$$

TE

$$J_n'(k_\rho a) = 0$$

$$\Rightarrow K_\rho^{TE} = \frac{x_{np}'}{a} \Rightarrow f_c^{TE\,np} = \frac{x_{np}'}{2\pi a\sqrt{\mu\epsilon}}$$

Fields (TM)

$$E_\rho = E_0\, i\beta \left(\frac{x_{np}}{a}\right) J_n'\left(\frac{x_{np}}{a}\rho\right) \cos(n\phi)\, e^{i\beta z}$$

$\vdots$