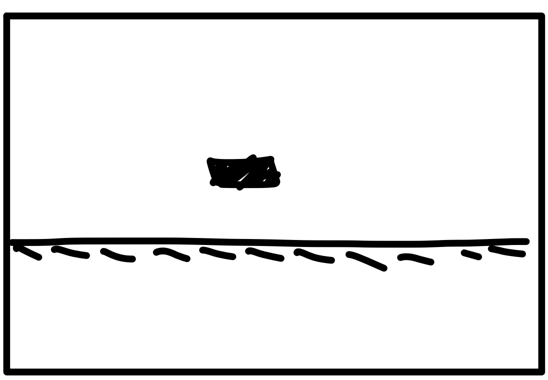
EM22B - Finite Element Methods

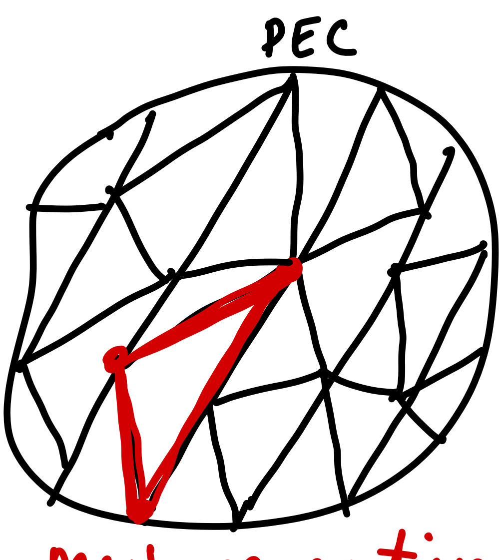
$$F(\psi) = \frac{\iint_S \nabla\psi \cdot \nabla\psi \, ds}{\iint_S \psi^2 \, ds} = k_c^2 \text{ when}$$

ψ is a wavefunction & $F(\psi)$ is minimum.

FEM → Variational (Rayleigh Ritz)
FEM → Differential (Galerkin's).

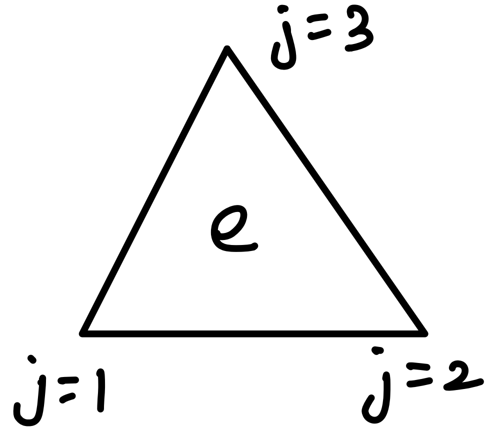
> Discretizing the 2D geometry into a triangular mesh.





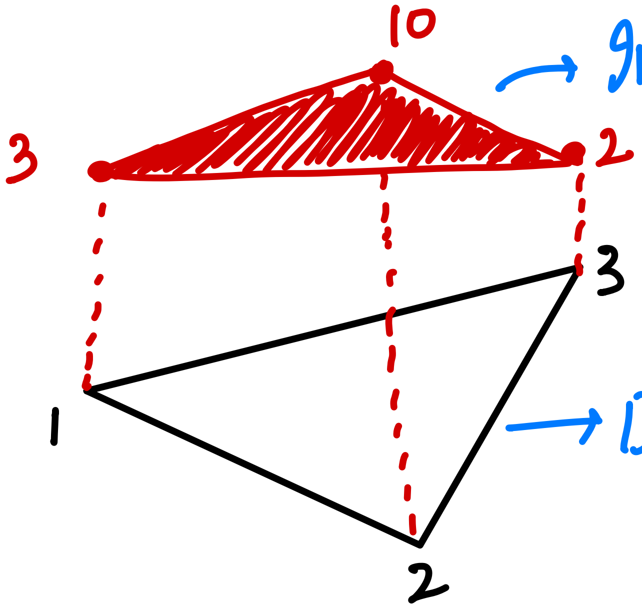
Mesh generation.

> Each triangle → element "e" has 3 nodes "j"



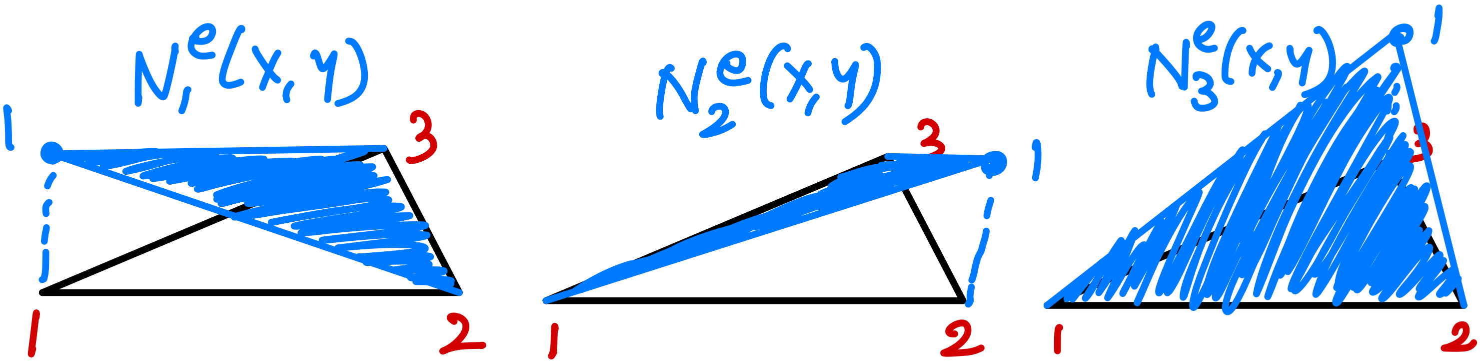
> ψ is discretized into a set of unknowns at each node of the mesh.

> We use a "linear interpolation".



$$\psi^e(x,y) = \sum_{j=1}^{3} \underbrace{N_j^e(x,y)}_{\text{interpolation functions.}} \psi_j^e$$

↳ Value at nodes



$$N_j^e = \frac{1}{2\Delta^e}\left(a_j^e + b_j^e x + c_j^e y\right)$$

$$a_1^e = x_2^e y_3^e - y_2^e x_3^e \,;\; b_1^e = y_2^e - y_3^e \,;\; c_1^e = x_3^e - x_2^e$$

$a_2^e, a_3^e, b_2^e, b_3^e, c_2^e, c_3^e \rightarrow$ cycling through indeces.

$$\Delta^e = \text{area of } \Delta^e = \frac{1}{2}\left(b_1^e c_2^e - b_2^e c_1^e\right)$$

$$\psi(x,y) = \sum_{e=1}^{N} \psi^e(x,y) = \sum_{e=1}^{N}\sum_{j=1}^{3} N_j^e(x,y)\,\psi_j^e$$

entirely determined by mesh geometry.

Unknowns

$$F(\psi) = \frac{\iint_S \nabla\left(\sum_{e=1}^{N}\psi^e(x,y)\right)\cdot\nabla\left(\sum_{e=1}^{N}\psi^e(x,y)\right)ds}{\iint_S \left(\sum_{e=1}^{N}\psi^e(x,y)\right)^2 ds}$$

$$\frac{\partial F(\psi)}{\partial \psi_j^e} = 0$$ gives $\psi_j^e \to$ vector of nodal quantities which closely approximates the eigenfn.

$\downarrow$ Long derivation.

$$\boxed{\tilde{A}\,\tilde{\Psi} = k_c^2\,\tilde{B}\,\tilde{\psi}} \to \text{Finite Element Equation}$$

Stiffness matrix ($\tilde{A}$)

Mass matrix ($\tilde{B}$)

$$\tilde{A}_{ij}^e = \frac{1}{4\Delta^e}\left(b_i^e b_j^e + c_i^e c_j^e\right) \longrightarrow 3N \times 3N$$

$$\tilde{B}_{ij}^e = \frac{\Delta^e}{12}\left(1 + \underbrace{\delta_{ij}}_{\text{Kronecker } \delta.}\right) \longrightarrow 3N \times 3N$$

> $\vec{\tilde{\psi}}$ has many repeated elements so we need to reduce this to remove repeated nodes.

$$\overline{\tilde{\Psi}} = \bar{\bar{C}}\, \overline{\Psi}$$

$$\boxed{\bar{\bar{A}}\, \overline{\Psi} = K_c^2\, \bar{\bar{B}}\, \overline{\Psi}}$$

where $\bar{\bar{A}} = \bar{\bar{C}}^t \tilde{\bar{\bar{A}}}\, \bar{\bar{C}}$

$\bar{\bar{B}} = \bar{\bar{C}}^t \tilde{\bar{\bar{B}}}\, \bar{\bar{C}}$

Generalized Eigenvalue Problem.

Computing $\dfrac{\partial F(\overline{\Psi})}{\partial \overline{\Psi}} = 0$

$$N(\psi) = \iint_S \nabla\psi \cdot \nabla\psi \, ds$$

$$= \iint_S \nabla\Big(\sum_e \sum_j \psi_j^e N_j^e(x,y)\Big) \cdot \nabla\Big(\sum_e \sum_j \psi_j^e N_j^e(x,y)\Big)$$

$$\Rightarrow N(\psi) = \sum_{e=1}^{N} N^e(\psi^e)$$

$$\mathbb{N}^e(\psi^e) = \iint_{\Delta^e} \Big(\sum_j \psi_j^e \nabla N_j^e\Big) \cdot \Big(\sum_i \psi_i^e \nabla N_i^e\Big) ds$$

∇N_i^e & ∇N_j^e are constants!

$$\Rightarrow \mathbb{N}^e(\psi^e) = \sum_i \sum_j \psi_i^e \psi_j^e \left(\nabla N_i^e \cdot \nabla N_j^e\right) \underbrace{\iint_{\Delta^e} ds}_{\Delta^e}$$

$$\nabla N_i^e \cdot \nabla N_j^e = \frac{1}{4\Delta^{e^2}} \left(b_i^e b_j^e + c_i^e c_j^e\right)$$

$$\mathbb{N}^e(\psi^e) = \sum_i \sum_j \psi_i^e \psi_j^e A_{ij}^e$$

where $A_{ij}^e = \frac{1}{4\Delta^e} \left(b_i^e b_j^e + c_i^e c_j^e\right)$

$$\mathbb{N}^e(\psi^e) = \overrightarrow{\psi_e}^T \bar{\bar{A}}_e \overrightarrow{\psi_e}$$

$$\mathbb{D}^e(\psi^e) = \int_{\Delta^e} \psi^{e^2} ds = \int_{\Delta^e} \Big(\sum_i \psi_i^e N_i\Big)\Big(\sum_j \psi_j^e N_j\Big) ds$$

$$= \sum_i \sum_j \psi_i^e \psi_j^e \iint_{\Delta^e} N_i N_j \, ds$$

$$\iint_{\Delta^e} N_i N_j \, ds = \begin{cases} \frac{\Delta^e}{6} & i=j \\ \frac{\Delta^e}{12} & i \neq j \end{cases} = \frac{\Delta^e}{12}(1+\delta_{ij})$$

$$\Rightarrow \mathbb{D}^e(\psi^e) = \sum_i \sum_j \psi_i^e \psi_j^e B_{ij}^e$$

$$B_{ij}^e = \frac{\Delta^e}{12}(1+\delta_{ij})$$

$$\Rightarrow \mathbb{D}^e(\psi^e) = \vec{\psi_e}^T \bar{\bar{B_e}} \vec{\psi_e}$$

$$F(\vec{\psi^e}) = \frac{\vec{\psi_e}^T \bar{\bar{A_e}} \vec{\psi_e}}{\vec{\psi_e}^T \bar{\bar{B_e}} \vec{\psi_e}}$$ where $\vec{\psi_e}$ is for each element.

If $\vec{\psi}$ as the stacked version of $\vec{\psi_e}$, $\bar{\bar{A}}$ & $\bar{\bar{B}}$

are the block diagonal forms of $\bar{A}_e$ & $\bar{\bar{B}}_e$,

Then $F(\underset{3N\times 1}{\vec{\psi}}) = \dfrac{\vec{\psi}^T \bar{\bar{A}} \vec{\psi}}{\vec{\psi}^T \bar{\bar{B}} \vec{\psi}}$

Note that $\alpha \vec{\psi}$ also minimizes F if $\vec{\psi}$ minimizes F. So we normalize $\vec{\psi}$ such that $\vec{\psi}^T \bar{\bar{B}} \vec{\psi} = 1$.

$\Rightarrow$ minimizing F $\Longleftrightarrow$ minimizing Num subject to Den = 1.

$\Rightarrow$ minimize $\vec{\psi}^T \bar{\bar{A}} \vec{\psi}$ subject to $\vec{\psi}^T \bar{\bar{B}} \vec{\psi} - 1 = 0$ $\rightarrow$ Constrained opt. problem. $\rightarrow$ Method of Lagrange multipliers.

$$\mathcal{L}(\vec{\psi}) = N(\vec{\psi}) - \lambda\left(D(\vec{\psi}) - 1\right)$$

& $\nabla_{\vec{\psi}} \mathcal{L}(\vec{\psi}) = 0$ //

$$\nabla_{\vec{\psi}}\left(\vec{\psi}^{T}\bar{\bar{A}}\vec{\psi}\right) = 2\bar{\bar{A}}\vec{\psi} \text{ if } A \text{ is a}$$

symmetric matrix.

$$\frac{\partial}{\partial \psi_k}\sum_{ij}\psi_i A_{ij}\psi_j = \sum_i \psi_i A_{ik} + \sum_j A_{kj}\psi_j$$

$$= 2\sum_j A_{kj}\psi_j = 2\bar{\bar{A}}\vec{\psi}$$

$$\Rightarrow \nabla \mathcal{L}(\vec{\psi}) = 0 \Rightarrow 2\bar{\bar{A}}\vec{\psi} - \lambda 2B\vec{\psi} = 0$$

$$\Rightarrow \tilde{A}\vec{\tilde{\psi}} = \lambda \tilde{B}\vec{\tilde{\psi}}$$

$$\vec{\psi}^{T}A\vec{\psi} = \lambda \vec{\psi}^{T}B\vec{\psi}$$

$$\Rightarrow \lambda = \frac{\psi^T A \psi}{\psi^T B \psi} = k_c^2$$

$$\Rightarrow \boxed{\tilde{A}\vec{\tilde{\psi}} = k_c^2 \tilde{B}\vec{\tilde{\psi}}}$$

Ritz Method

FEM (Galerkin's Method) → Diff. Eqn.

$\mathcal{L}(\phi) = f$

f → excitation { wave port, lumped port }

ϕ → unknown field.

$\mathcal{L}$ → Differential operator { HH $\nabla^2 - k^2$, WE $\nabla^2 + \frac{\partial^2}{\partial t^2}$ }

Approximate ϕ in a function basis.

$$\phi \simeq \sum_{j=1}^{N} c_j v_j$$

c_j → unknown

v_j → basis functions

$$\Rightarrow \tilde{\phi} = \sum_{j=1}^{N} c_j v_j = \vec{v}^T \vec{c}$$

Let $\mathcal{L}(\tilde{\phi}) - f = r \neq 0$

r → residual

Weighted residual $R_i = \int_\Omega w_i \, r \, d\Omega$

w_i → weighting functions.

$R_i = 0 \longrightarrow N$ equations & N unknowns which are c_j

$\Rightarrow$ Matrix equations.

<u>Galerkin's Method</u> $\left(w_i = v_i \quad \forall\, i = 1, 2, \ldots N \right)$

$$R_i = \int_{\Omega} \left(v_i \mathcal{L}(\vec{v}^T \vec{c}) - v_i f \right) d\Omega = 0$$

$$= \int_{\Omega} \sum_j v_i \mathcal{L}(v_j c_j)\, d\Omega - \int_{\Omega} v_i f\, d\Omega = 0$$

$$= \sum_j c_j \int_{\Omega} v_i \mathcal{L}(v_j)\, d\Omega - \int_{\Omega} v_i f\, d\Omega = 0$$

$$\Rightarrow \boxed{\bar{\bar{S}}\, \vec{c} - \vec{b} = 0}$$

where $\bar{\bar{S}}_{ij} = \int_{\Omega} v_i \mathcal{L}(v_j)\, d\Omega$

$\bar{b}_i = \int_{\Omega} v_i f \, d\Omega$

<u>HFSS:</u>

1) $\nabla \times (\mu^{-1} \nabla \times \vec{E}) - \omega^2 \epsilon \vec{E} = -j\omega \vec{J}$

2) $\underbrace{\text{Vector basis functions}}_{\to \text{(Nédélec basis fns)}}$ with a tetrahedral mesh.

Jian Ming Jin → FEM in EM.