EM 22 - Calculus of Variation for EM Waveguides

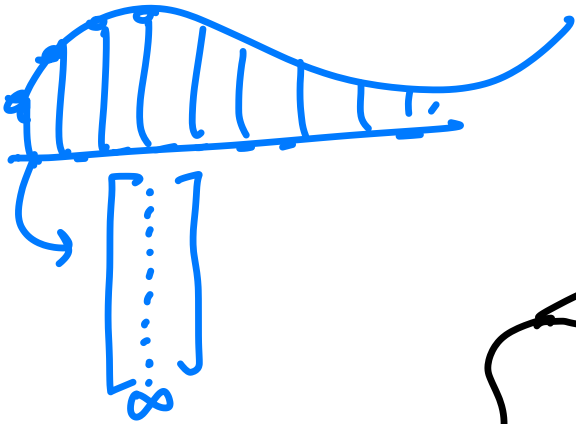
$$\nabla_t^2 \psi + k_c^2 \psi = 0$$

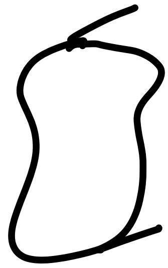
eigen~~value~~

$$\nabla_t^2 \psi = -k_c^2 \psi \rightarrow \text{Eigenvalue.}$$

Operator   eigen~~vector~~

eigenfunctions





<u>Claim:</u> k_c can be derived from ψ.

$$\iint_S \left(\nabla_t \psi_{mn} \cdot \nabla_t \psi_{mn} + \psi_{mn} \underbrace{\nabla_t^2 \psi_{mn}}_{-k_c^2 \psi_{mn}} \right) ds = \underbrace{\oint_C \psi_{mn} \frac{\partial \psi_{mn}}{\partial n} dl}_{\substack{0 \\ \text{DBC, NBC}}}$$

$$\Rightarrow \boxed{k_c^2 = \frac{\iint_S \nabla_t \psi_{mn} \cdot \nabla_t \psi_{mn} \, ds}{\iint_S \psi_{mn}^2 \, ds}}$$

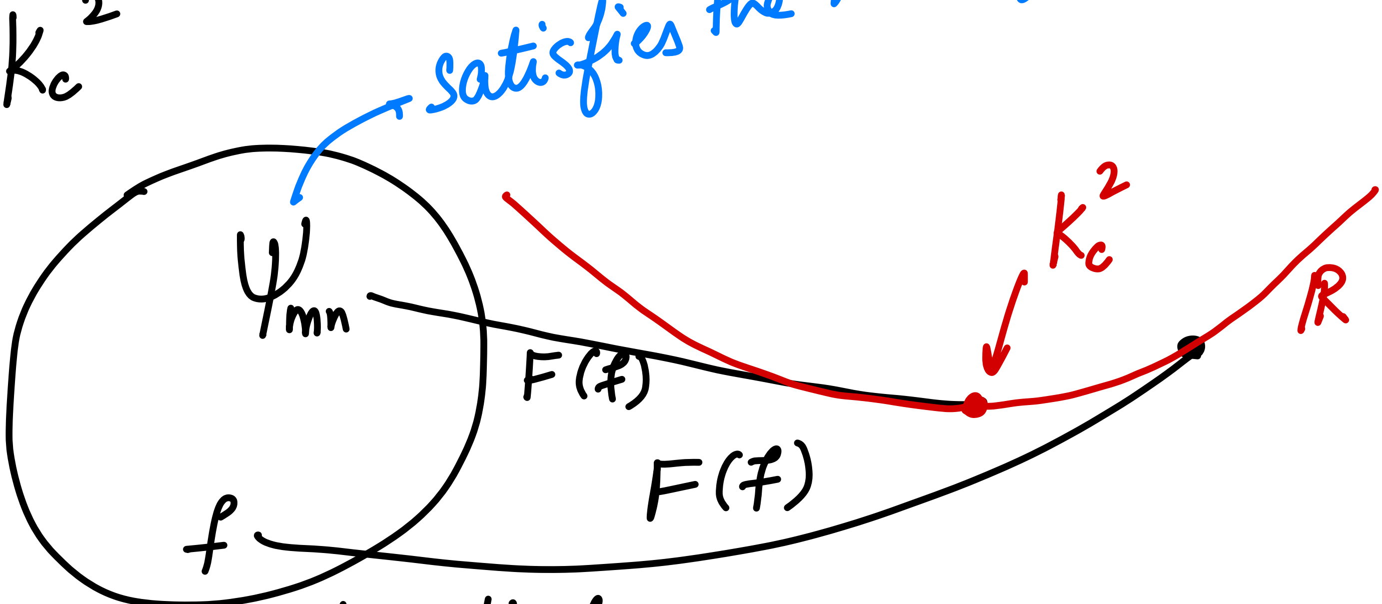
Calculus of Variational

Define a functional

$$F(f) = \frac{N(f)}{D(f)} = \frac{\iint_S \nabla_t f \cdot \nabla_t f \, ds}{\iint_S f^2 \, ds}$$

$$F(f) : f \xrightarrow{F} \mathbb{R}$$

Claim: The function that minimizes this functional F is an eigenfunction of the wave equation, with eigenvalue K_c^2



$S \to$ space of admissible functions

Proof Suppose ψ_1 minimizes $F(f)$ & has eigenvalue k_{c_1}.

$$\Rightarrow \nabla_t^2 \psi_1 + k_{c_1}^2 \psi_1 = 0$$

$$\frac{N(\psi_1)}{D(\psi_1)} = k_{c_1}^2$$

$$\Rightarrow N(\psi_1) = k_{c_1}^2 D(\psi_1).$$

$$N(\psi_1 + \delta g) - k_{c_1}^2 D(\psi_1 + \delta g) > 0$$

$$N(f) = \iint_S \nabla_t f \cdot \nabla_t f \, ds$$

$$N(f,g) \triangleq \iint_S \nabla_t f \cdot \nabla_t g \, ds$$

$$N(f+g) = \iint_S \nabla_t (f+g) \cdot \nabla_t (f+g) \, ds$$

$$= \iint_S \nabla_t f \cdot \nabla_t f + \nabla_t g \cdot \nabla_t g + 2(\nabla_t f \cdot \nabla_t g)$$

$$= N(f) + N(g) + 2N(f,g)$$

$$N(\psi_1 + \delta g) = N(\psi_1) + \delta^2 N(g) + 2\delta N(\psi_1, g)$$

sub

$$\left[D(\psi_1+\delta g) = D(\psi_1) + \delta^2 D(g) + 2\delta D(\psi_1, g)\right] \times K_{c_1}^2$$

$$\Rightarrow 2\delta\left[N(\psi_1, g) - K_{c_1}^2 D(\psi_1, g)\right] + \delta^2\left[N(g) - K_{c_1}^2 D(g)\right]$$

$$+ \underbrace{\left[N(\psi_1) - K_{c_1}^2 D(\psi_1)\right]}_{0} = \underbrace{\left[N(\psi_1+\delta g) - K_{c_1}^2 D(\psi_1 \delta g)\right]}_{>0}$$

$$\Rightarrow 2\delta \underbrace{\left[N(\psi_1, g) - K_{c_1}^2 D(\psi_1, g)\right]}_{\Delta = 0 \,\to\, \text{Claim}} + \delta^2 \underbrace{\left[N(g) - K_{c_1}^2 D(g)\right]}_{>0}$$

Proof of Claim

$\Delta > 0 \Rightarrow$ Choose $\delta < 0$ & small enough.
Such that LHS $< 0 \longrightarrow$ contradiction.

$\Delta < 0 \Rightarrow$ Choose $\delta > 0$ & small enough
Such that LHS $< 0 \longrightarrow$ Contradiction.

$\Rightarrow \Delta = 0$ //

$$\Rightarrow \boxed{N(\psi_1, g) - k_{c_1}^2 D(\psi_1, g) = 0}$$

$$\Rightarrow \iint_S \nabla_t \psi . \nabla_t g \, ds - k_{c_1}^2 \iint_S \psi_1 g \, ds = 0$$

$$\text{GFI} \Rightarrow \iint_S (\nabla_t \psi . \nabla_t g + g \nabla_t^2 \psi_1) ds$$
$$= \oint_C g \frac{\partial \psi_1}{\partial n} dl$$

$$\Rightarrow \iint_S g(\nabla_t^2 \psi_1 + k_{c_1}^2 \psi_1) ds = \oint_C g \frac{\partial \psi_1}{\partial n} dl.$$

<u>Claim</u>: RHS $= 0$

TE $\rightarrow \frac{\partial \psi_1}{\partial n} = 0$ from NBC.

TM $\rightarrow g$ is from TM space of solutions $\Rightarrow$ DBC
$g = 0$

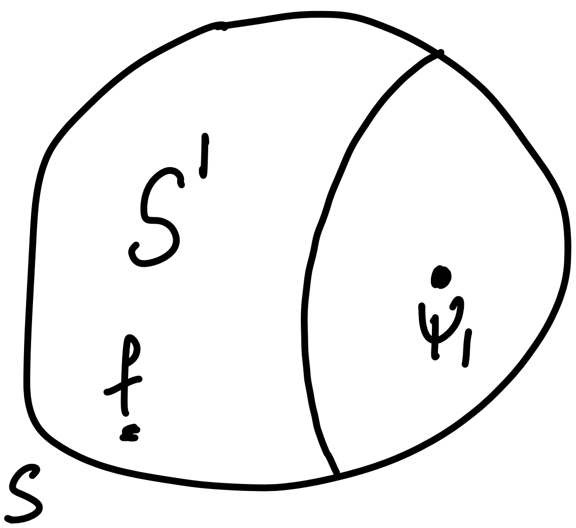
$$\Rightarrow \iint_S g\left(\nabla_t^2 \psi_1 + k_{c_1}^2 \psi_1\right) ds = 0$$

Since is arbitrary

$$\Rightarrow \boxed{\nabla_t^2 \psi_1 + k_{c_1}^2 \psi_1 = 0}$$

To find other (higher order) modes we use orthogonality.

Where S' is the orthogonal subspace to ψ_1.



$$\iint_S f \psi_1 \, ds = 0$$

$$\Rightarrow D(f, \psi_1) = 0$$

$k_{c_2}^2$ is obtained by minimizing $\frac{N(f)}{D(f)}$ under this constraint.

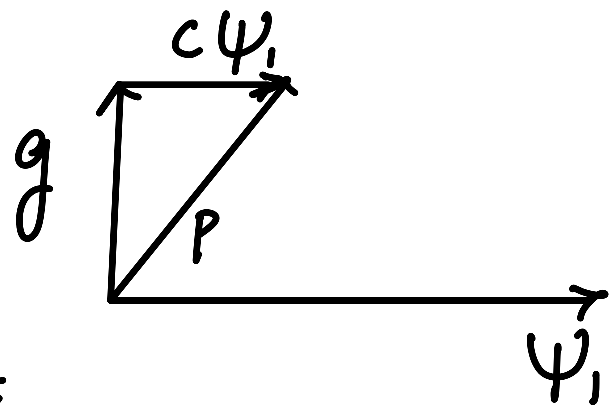
$$k_{c_2}^2 > k_{c_1}^2 \ , \ \psi_2$$

Proof: $N(\psi_2 + \delta g) > K_{c_2}^2 D(\psi_2 + \delta g)$

$D(\psi_1, g) = 0$ Since $g \in S'$

Let $p \in S$ such that

$$\left.\begin{aligned} g &= p - c\,\psi_1 \\ c &= \frac{D(\psi_1, p)}{D(\psi_1)} \end{aligned}\right\} \text{Gram Schmidt procedure}$$



$$N(\psi_2 + \delta p - \delta c \psi_1) > K_{c_2}^2 D(\psi_2 + \delta p - \delta c \psi_1)$$

Expanding & setting coeff. of $\delta = 0$

$$N(\psi_2, p) - K_{c_2}^2 D(\psi_2 - p)$$
$$- c\left[N(\cancelto{0}{\psi_1, \psi_2}) - K_{c_2}^2 D(\cancelto{0}{\psi_1, \psi_2})\right] = 0$$

$$N(\psi_1, p) = K_{c_1}^2 D(\psi_1, p)$$

$$\Rightarrow N(\psi_1, \psi_2) = K_{c_1}^2 \overbrace{D(\cancel{\psi_1, \psi_2})}^{0} = 0$$

$$N(\psi_2, p) - K_{c_2}^2 D(\psi_2, p) = 0$$

GFI & RHS = 0

$$\Rightarrow \boxed{\nabla_t^2 \psi_2 + K_{c_2}^2 \psi_2 = 0}$$

Rayleigh Ritz Method
Galerkin's Method } Finite Element Methods (FEM)

WAVEPORTS