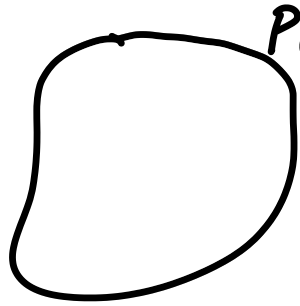
Lec 21 - Rectangular Waveguides



PEC

TM

$$E_z = k_c^2 \psi(x,y) e^{i\beta z}$$

$E_z = 0$ on PEC $\Rightarrow$ $\boxed{\psi(x,y) = 0}$ Dirichlet BC (DBC)

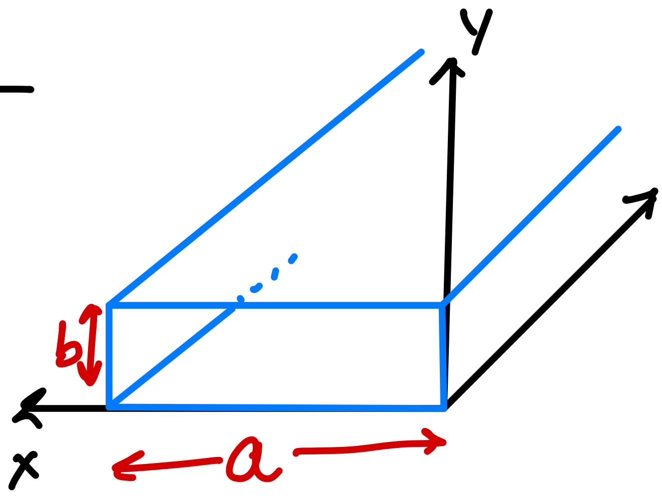
TE

$$\hat{n} \times \vec{E} = i\omega\mu \, \hat{n} \times (\nabla_t \psi_m \times \hat{z}) e^{i\beta z}$$

$$= -i\omega\mu \frac{\partial \psi_m}{\partial n} e^{i\beta z} \hat{z}$$

$\Rightarrow$ $\boxed{\dfrac{\partial \psi_m}{\partial n} = 0}$ Neumann BC (NBC)

Rectangular Waveguide



$$(\nabla_t^2 + k_c^2)\psi(x,y) = 0$$

DBC: $\psi(0,y) = \psi(a,y) = 0$

$\psi(x,0) = \psi(x,b) = 0$

SOV: $\psi(x,y) = X(x)\,Y(y)$

$$\Rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + k_c^2 = 0$$

X, Y independant $\Rightarrow \dfrac{X''(x)}{X(x)} = -k_x^2$; $\dfrac{Y''}{Y} = -k_y^2$

$$\Rightarrow k_x^2 + k_y^2 = k_c^2$$

$X(x) = A\sin(k_x x) + B\cos(k_x x)$

$Y(y) = C\sin(k_y y) + D\cos(k_y y)$

DBC $\Rightarrow X(0) = X(a) = 0$; $Y(0) = Y(b) = 0$

$\Rightarrow B, D = 0 \quad \& \quad \sin(k_x a) = 0 \ \& \ \sin(k_y b) = 0$

$$\Rightarrow k_x = \frac{m\pi}{a} \qquad k_y = \frac{n\pi}{b} \qquad m, n \in \mathbb{N}.$$

$$\Rightarrow \boxed{\psi_{mn}(x, y) = E_0 \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right)}$$

$$\beta_{mn} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$\lambda_g = \frac{2\pi}{\beta_{mn}}$$

$\hookrightarrow$ Note that diff. modes have diff β & λ.

Fields

$$E_x = i E_0 \beta_{mn} \left(\frac{m\pi}{a}\right) \cos\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) e^{i\beta_{mn} z}$$

$$E_y = i E_0 \beta_{mn} \left(\frac{n\pi}{b}\right) \sin\left(\frac{m\pi}{a} x\right) \cos\left(\frac{n\pi}{b} y\right) e^{i\beta_{mn} z}$$

$$E_z = E_0 \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) e^{i\beta_{mn} z}$$

$H_x \ldots, \ H_y \ldots$

$$Z_{TM} = \frac{|\hat{z} \times \vec{E}|}{|\vec{H}|} = \frac{E_x}{H_y} = \frac{E_y}{-H_x} = \frac{\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}{\omega \epsilon}$$

$$\beta_{mn} = 0 \Rightarrow f_c^{mn} = \frac{1}{2\sqrt{\mu \epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

TM_{11} is the lowest order TM mode.

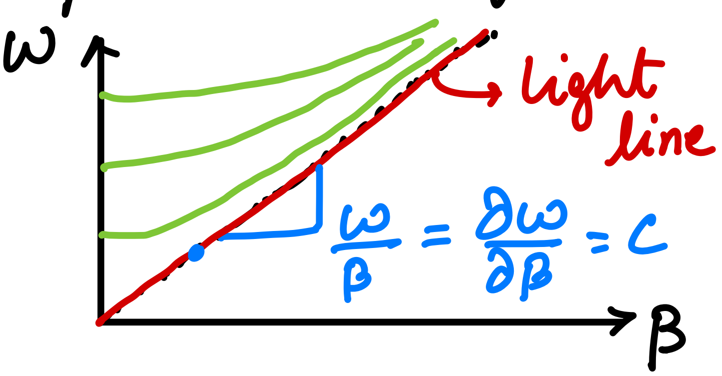
TE

$$\boxed{\psi_{mn}(x, y) = H_0 \cos\left(\frac{m\pi}{a} x\right) \cos\left(\frac{n\pi}{b} y\right)}$$

Lowest order mode for TE is 10 or TE_{10}

$$f_c^{10} = \frac{1}{2a\sqrt{\mu \epsilon}}$$

Dispersion Diagram



$$\frac{\omega}{\beta} = \frac{\partial \omega}{\partial \beta} = c$$

$$\underbrace{k^2 - \beta^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}_{\text{Hyperbola}}$$

Phase Velocity

$$\frac{d}{dt}\left(\omega t - \beta z = \text{phase}\right)$$

$$u_p = \frac{dz}{dt}$$

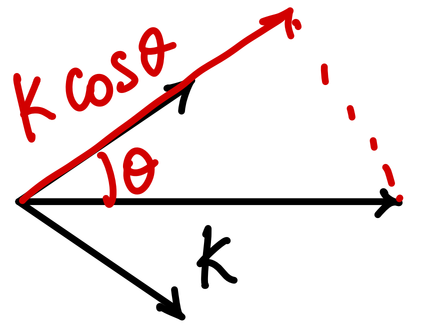
$$\omega - \beta \frac{\partial z}{\partial t} = 0$$

$$\boxed{u_p = \frac{\omega}{\beta}}$$

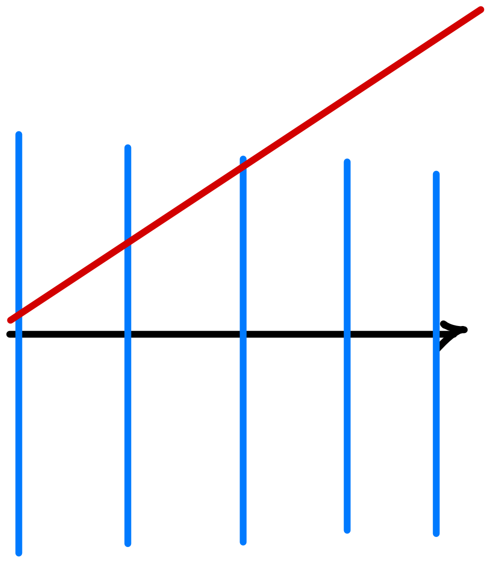
$$\Rightarrow \boxed{\frac{\partial z}{\partial t} = \frac{\omega}{\beta}}$$

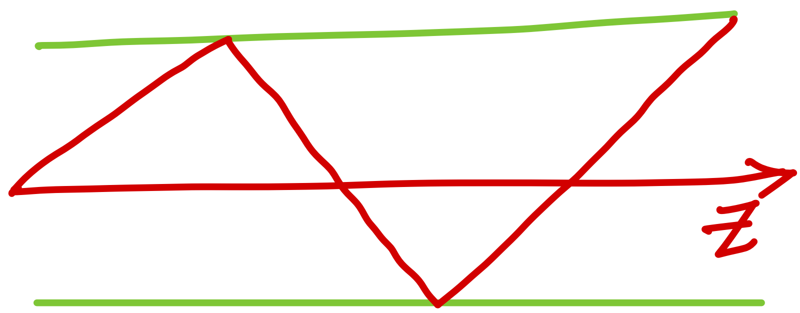
Phase velocity

$u_p > c$ $\forall$ modes at all frequencies.



$$v = \frac{\omega}{K\cos\theta} = \frac{c}{\cos\theta} \geq c$$





Group Velocity

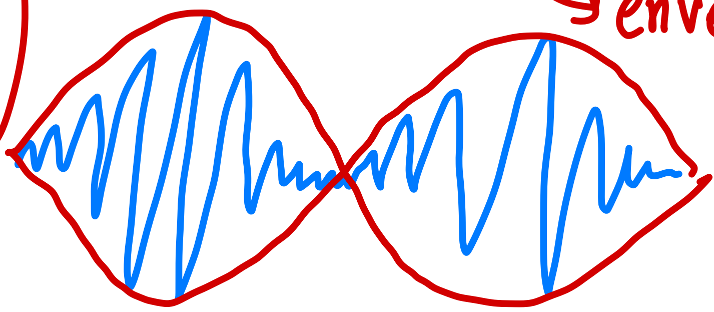
- Take two tones $\omega_1 = \omega_0 + \Delta\omega$;
  $\omega_2 = \omega_0 - \Delta\omega$.

$$\Pi_z(r,t) = \psi(x,y)\left\{\cos\left[(\omega_0+\Delta\omega)t - (\beta_0+\Delta\beta)z\right] + \cos\left[(\omega_0-\Delta\omega)t - (\beta_0-\Delta\beta)z\right]\right\}$$

$$= 2\psi(x,y)\,\underbrace{\cos(\omega_0 t - \beta z)}_{\text{carrier}}\;\underbrace{\cos(\Delta\omega t - \Delta\beta z)}_{\text{envelope}}$$

$$U_g = \frac{dz}{dt} = \frac{d}{dt}\left(\frac{\Delta\omega t}{\Delta\beta}\right)$$

$$= \frac{\Delta\omega}{\Delta\beta}$$



$$\boxed{U_g = \frac{d\omega}{d\beta}}$$

$$\beta^2 = k^2 - k_c^2 = \frac{\omega^2}{c^2} - k_c^2$$

$$\Rightarrow \frac{2\omega}{c^2}\frac{d\omega}{d\beta} = 2\beta$$

$$\Rightarrow \frac{\omega}{\beta} \cdot \frac{d\omega}{d\beta} = c^2 \Rightarrow \boxed{u_p u_g = c^2}$$

Orthogonality & Completeness

> ψ_{mn} , $\psi_{m'n'}$ they form a basis in the fields.

> $$(\nabla_t^2 + k_{mn}^2)\psi_{mn} = 0 \quad \times \psi_{m'n'}$$

$$(\nabla_t^2 + k_{m'n'}^2)\psi_{m'n'} = 0 \quad \times \psi_{mn}$$

sub

$$\psi_{m'n'} \nabla_t^2 \psi_{mn} - \psi_{mn} \nabla_t^2 \psi_{m'n'} = (k_{m'n'}^2 - k_{mn}^2) \times \psi_{mn} \psi_{m'n'}$$

$\iint_S$ & apply GSI

$$\oint_C \left(\psi_{m'n'} \frac{\partial \psi_{mn}}{\partial n} - \psi_{mn} \frac{\partial \psi_{m'n'}}{\partial n}\right) dl = 0$$

$$\left(k_{m'n'}^2 - k_{mn}^2\right)\underbrace{\iint_S \psi_{mn}\,\psi_{m'n'}\,ds = 0}_{\text{Orthogonality}}$$

<u>GFI</u>

$$\iint_S \left(\nabla_t \psi_{mn}\cdot\nabla_t \psi_{m'n'} + \psi_{mn}\cancelto{0}{\overbrace{\nabla_t^2 \psi_{m'n'}}^{-k_{m'n'}^2\,\psi_{m'n'}}}\right)ds$$

$$= \underbrace{\oint \psi_{mn}\,\frac{\partial \psi_{m'n'}}{\partial n}\,dl}_{0 \text{ from DBC, NBC}}$$

$$\Rightarrow \iint_S \nabla_t \psi_{mn}\cdot\nabla_t \psi_{m'n'}\,ds = 0 \qquad \blacksquare$$

See Sec 6.8.1

$$a_{mn} = \frac{\displaystyle\iint_S f(x,y)\,\psi_{mn}(x,y)\,ds}{\displaystyle\iint_S \psi_{mn}^2(x,y)\,ds}$$

$$f(x, y) = \sum_m \sum_n a_{mn} \psi_{mn}(x, y)$$