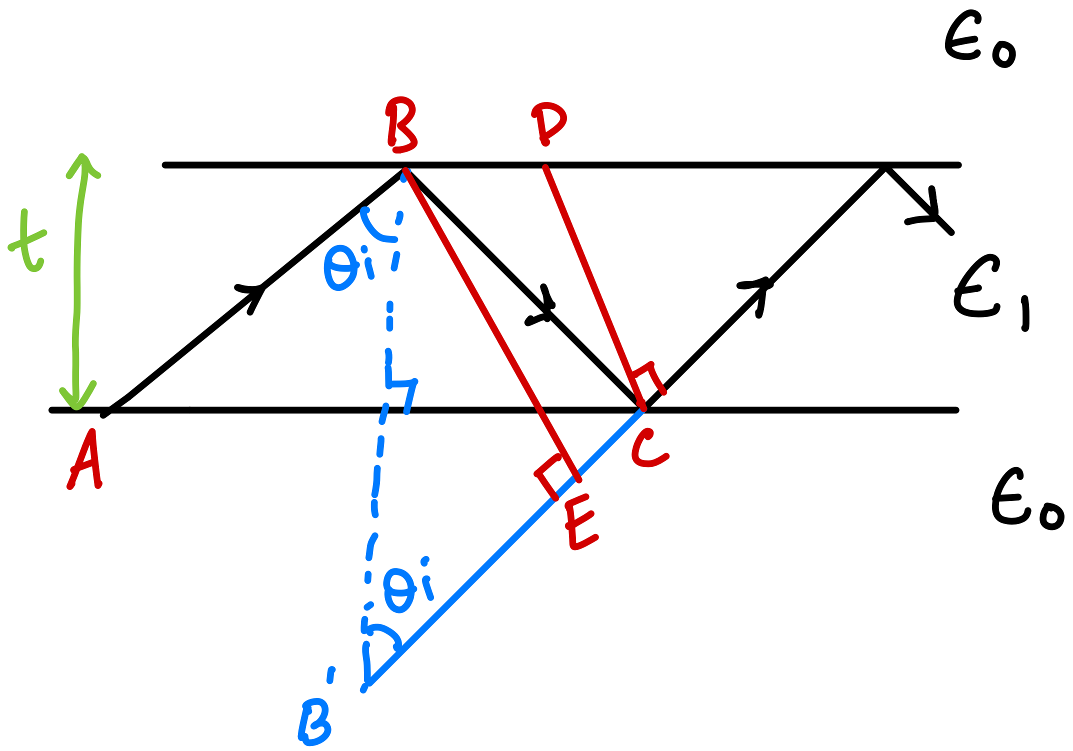
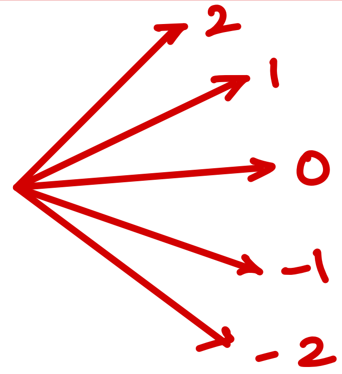
EM20 - Waveguides



$$k(BC - EC) = 2m\pi$$

$$\Rightarrow kB'E = 2m\pi$$

$$\Rightarrow 2tk\cos\theta_i = 2m\pi \Rightarrow \boxed{\theta_i = \cos^{-1}\left(\frac{m\pi}{tk}\right)}$$



TE/TM Field Solutions

Recall

$$\left.\begin{aligned}\nabla^2\overline{\Pi} + k^2\overline{\Pi} &= 0\\ \nabla^2\overline{\Pi}_m + k^2\overline{\Pi}_m &= 0\end{aligned}\right\}\ \text{Source free.}$$

$$\vec{E} = \nabla\nabla\cdot\overline{\Pi} + k^2\overline{\Pi} + i\omega\mu\nabla\times\overline{\Pi}_m$$

$$\vec{H} = -i\omega\epsilon\nabla\times\overline{\Pi} + \nabla\nabla\cdot\overline{\Pi}_m + k^2\overline{\Pi}_m$$

TE - Transverse Electric

$\hat{k} \perp \vec{E}$

$\hat{k} \not\perp \vec{H}$

$\overline{\Pi} = \Pi_z\hat{z} \longrightarrow$ TM

$\overline{\Pi}_m = \Pi_{mz}\hat{z} \longrightarrow$ TE

$\overline{\Pi}_m = 0$ for TM solution. $\overline{\Pi} = 0$ for TE soln.

Say $\hat{k}$ is $\hat{z}$.

TM solution

$$\vec{E} = \nabla\left(\frac{\partial\Pi_z}{\partial z}\right) + k^2\Pi_z\hat{z}$$

Let $\nabla = \nabla_t + \frac{\partial}{\partial z}\hat{z}$

$$\vec{E} = \frac{\partial}{\partial z}\nabla_t \pi_z + \left\{\left(\frac{\partial^2 \pi_z}{\partial z^2}\right) + k^2 \pi_z\right\}\hat{z}$$

$$\vec{H} = -i\omega\epsilon \nabla_t \pi_z \times \hat{z}$$

Wave Eq: $\nabla_t^2 \pi_z + \frac{\partial^2}{\partial z^2}\pi_z + k^2 \pi_z = 0$

$$\pi_z = \psi(x,y)\, e^{\pm i\beta z}$$

$$\boxed{\nabla_t^2 \psi + k_c^2 \psi = 0} \rightarrow \text{2D Wave Eqn.}$$

$$\underbrace{k_c^2}_{\text{transverse wave number}} = \underbrace{k^2}_{\omega^2\mu\epsilon} - \underbrace{\beta^2}_{\text{wave number along } z}$$

$$\left.\begin{aligned}\vec{E} &= \left(\pm i\beta \nabla_t \psi + k_c^2 \psi \hat{z}\right) e^{\pm i\beta z} \\ \vec{H} &= -i\omega\epsilon\left(\nabla_t \psi \times \hat{z}\right) e^{\pm i\beta z}\end{aligned}\right\} \underline{\underline{\text{TM}}}$$

$$Z_{TM} = \frac{\hat{z} \times \vec{E}}{\vec{H}} = \frac{\beta}{\omega \epsilon} \rightarrow \text{TM wave impedance.}$$

<u>TE</u>

$$\vec{\Pi}_m = \Pi_{mz} \hat{z}$$

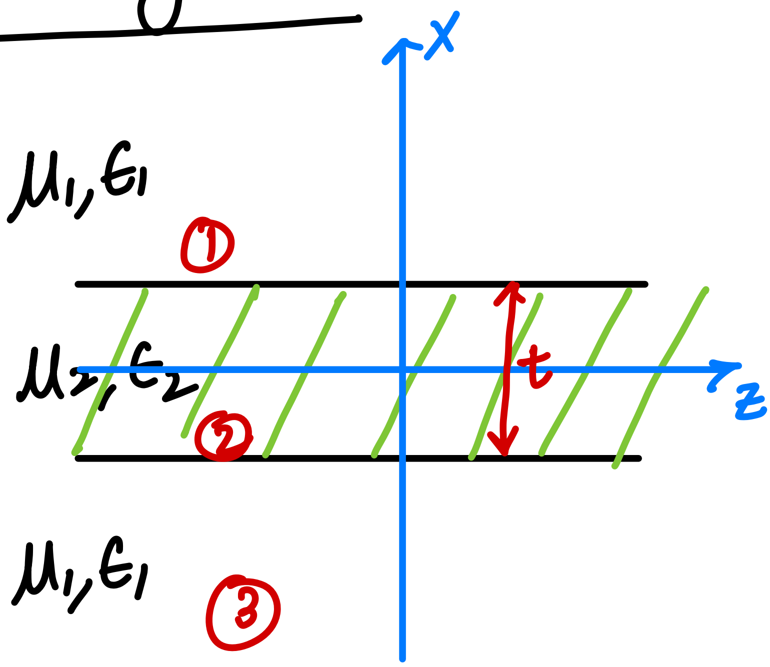
$$\left.\begin{aligned} \vec{H} &= \left(\pm i\beta \nabla_t \psi_m + k_c^2 \psi_m \hat{z} \right) e^{\pm i\beta z} \\ \vec{E} &= i\omega\mu \left(\nabla_t \psi_m \times \hat{z} \right) e^{\pm i\beta z} \end{aligned}\right\} \text{Duality.}$$

$$Y_{TE} = \frac{-\hat{z} \times \vec{H}}{\vec{E}} = \frac{\beta}{\omega \mu} \rightarrow \text{TE Admittance.}$$

Dielectric Slab Waveguide

TM case

$\Pi_{lz}(x,z) = \psi_l(x) e^{i\beta z}$



$l = 1, 2, 3.$

$\psi_l(x)$ satisfies $\dfrac{d^2\psi_l}{dx^2} + K_{lc}^2 \psi_l = 0 \longrightarrow *$

$K_{lc}^2 = k_l^2 - \beta^2$ & $\beta_l = \beta$ for all $l = 1, 2, 3$

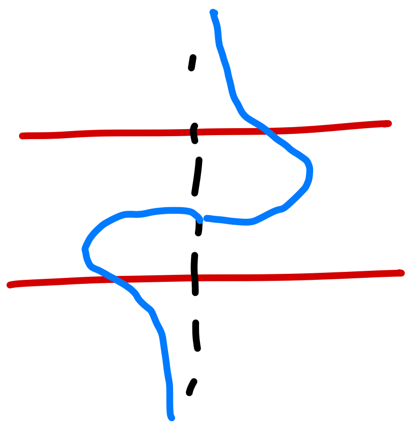
$K_{1c} = K_{3c} = i\nu \longrightarrow$ confined mode.

TM Odd modes

$\Pi_{2z}^{o}(x,z) = A \sin(k_{2c} x) e^{i\beta z}$

$\Pi_{1z}^{o}(x,z) = B e^{-\nu x} e^{i\beta z}$

$\Pi_{3z}^{o}(x,z) = -B e^{\nu x} e^{i\beta z}$



$k_{2c}^2 = k^2 - \beta^2 = \omega^2 \mu_2 \epsilon_2 - \beta^2$ & $\nu^2 = \beta^2 - k^2 = \beta^2 - \omega^2 \mu_1 \epsilon_1$

$$E_{2z} = A\, k_{2c}^2 \sin(k_{2c} x) e^{i\beta z}$$

$$E_{1z} = -B \nu^2 e^{-\nu z} e^{i\beta z}$$

$$E_{3z} = B\nu^2 e^{\nu x} e^{i\beta z}$$

$$H_{2y} = iA\omega \epsilon_2 k_{2c} \cos(k_{2c} x) e^{i\beta z}$$

$$H_{1y} = -iB\omega\epsilon_1 \nu e^{-\nu x} e^{i\beta z}$$

$$H_{3y} = iB\omega\epsilon_1 \nu e^{\nu x} e^{i\beta z}$$

BC on E & H at $x = \pm \frac{t}{2}$,

$$A k_{2c}^2 \sin\left(\frac{k_{2c} t}{2}\right) = -B\nu^2 e^{-\frac{\nu t}{2}}$$

$$iA\omega\epsilon_2 k_{2c} \cos\left(\frac{k_{2c} t}{2}\right) = -iB\omega\epsilon_1 \nu e^{-\frac{\nu t}{2}}$$

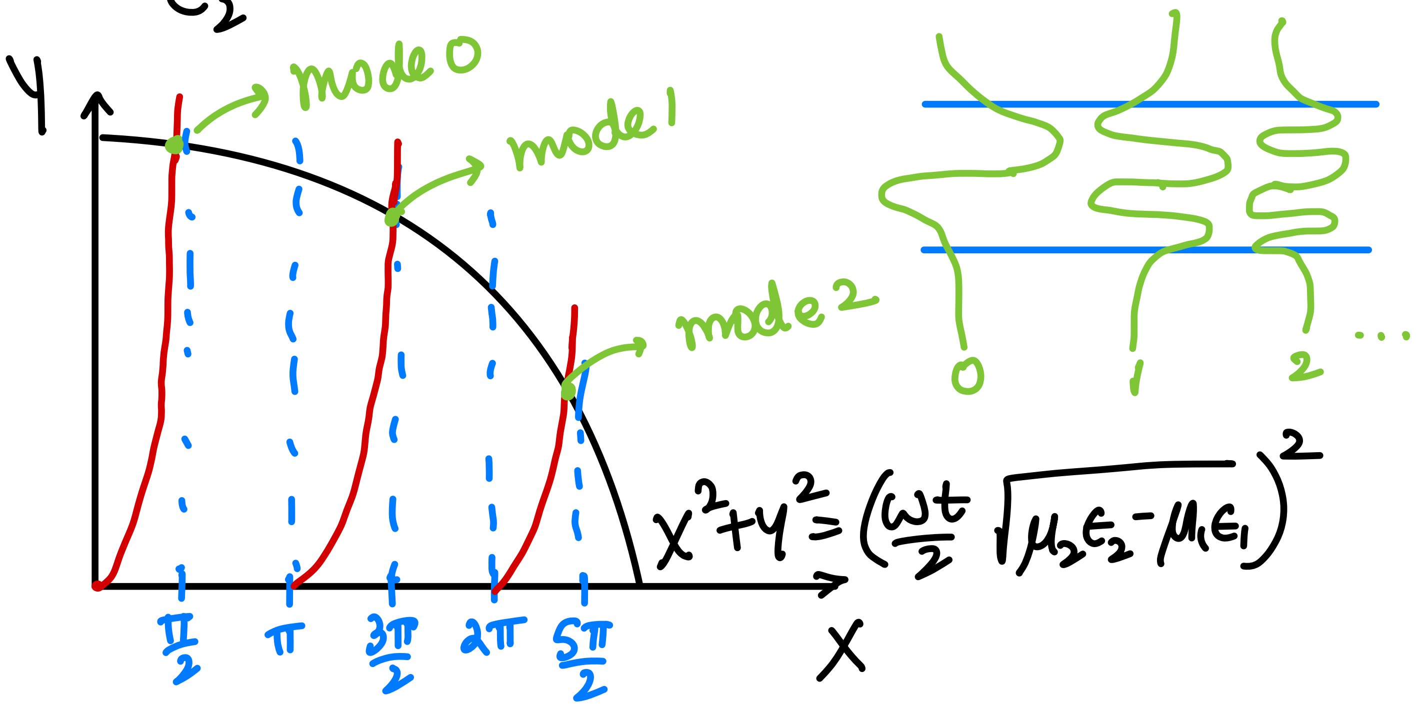
Ratio:

$$\frac{k_{2c}}{\epsilon_2} \tan\left(k_{2c} \frac{t}{2}\right) = \frac{\nu}{\epsilon_1}$$

Sum of DR: $k_{2c}^2 + \nu^2 = \omega^2(\mu_2\epsilon_2 - \mu_1\epsilon_1)$

$$X = \frac{k_{2c} t}{2} \qquad Y = \frac{\nu t}{2}$$

$$\Rightarrow \quad \frac{\epsilon_1}{\epsilon_2} X \tan X = Y \;\;\&\;\; X^2 + Y^2 = \left(\frac{\omega t}{2}\sqrt{\mu_2\epsilon_2 - \mu_1\epsilon_1}\right)^2$$



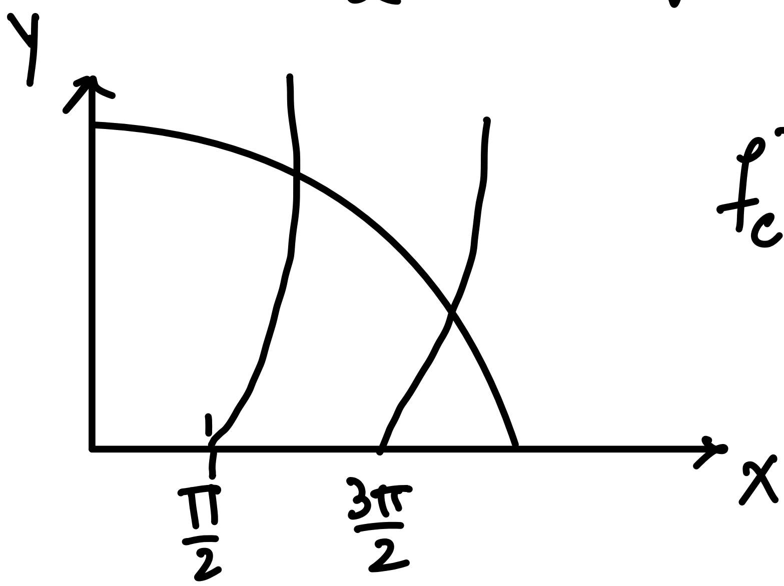
$$\frac{k_{2c} t}{2} = n\pi = \frac{\omega t}{2}\sqrt{\mu_2\epsilon_2 - \mu_1\epsilon_1} \qquad n \in \{0, 1, 2, \dots\}$$

$$f_c^{TM_n^o} = \frac{n}{t\sqrt{\mu_2\epsilon_2 - \mu_1\epsilon_1}} \Rightarrow n = 0 \text{ has no cut off.}$$

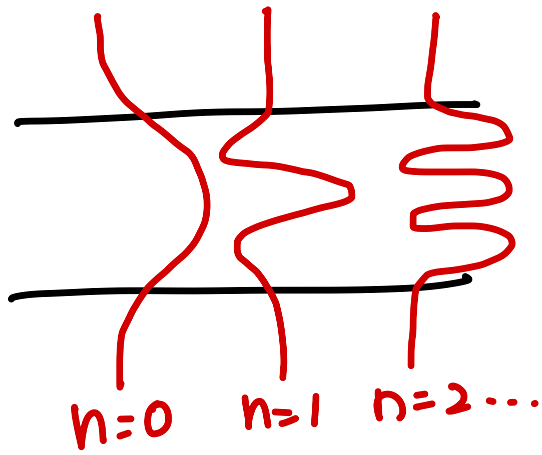
TM Even modes

$$\Pi_{2z}^e(x,z) = A\cos(k_{2c}x)\, e^{i\beta z}$$

$$B \cdot C \Rightarrow -\frac{k_{2c} t}{2} \cot\left(\frac{k_{2c} t}{2}\right) = \frac{\epsilon_2}{\epsilon_1} \frac{\nu t}{2}$$



$$f_c^{TM_n^e} = \frac{2n+1}{2t\sqrt{\mu_2\epsilon_2 - \mu_1\epsilon_1}}.$$



TE → Duality