# EM17 - Plane Waves

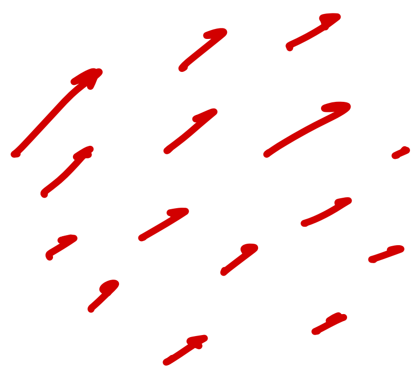
## Homogeneous medium

$$\nabla\times\vec{E} = i\omega\mu\vec{H} \qquad \nabla\cdot\vec{E}=0$$
$$\nabla\times\vec{H} = -i\omega\epsilon\vec{E} \qquad \nabla\cdot\vec{H}=0$$

Source free THME

$$\Rightarrow \underbrace{\nabla\times\nabla\times\vec{E}}_{\cancelto{0}{\nabla\nabla\cdot\vec{E}}\;-\nabla^2\vec{E}} = i\omega\mu(-i\omega\epsilon\vec{E})$$
$$= \omega^2\mu\epsilon\vec{E}$$
$$= k^2\vec{E}$$

$$\Rightarrow \boxed{\nabla^2\vec{E} + k^2\vec{E} = 0}$$



$$\vec{E} = \vec{E}_0\,\psi(\vec{r})$$

$$\Rightarrow \vec{E}_0\left(\nabla^2\psi(\vec{r}) + k^2\psi(\vec{r})\right) = 0$$

## Seperation of variables

$$\psi(\vec{r}) = X(x)\,Y(y)\,Z(z)$$

$$\nabla^2 XYZ + k^2 XYZ = 0$$

$\div XYZ$

$$\frac{1}{X}\frac{\partial^2}{\partial x^2}X + \frac{1}{Y}\frac{\partial^2}{\partial y^2}Y + \frac{1}{Z}\frac{\partial^2}{\partial z^2}Z + k^2 = 0$$

$$\left.\begin{aligned} \frac{1}{X}\frac{\partial^2}{\partial x^2}X &= -k_x^2 \\ \frac{1}{Y}\frac{\partial^2}{\partial y^2}Y &= -k_y^2 \\ \frac{1}{Z}\frac{\partial^2}{\partial z^2}Z &= k_z^2 \end{aligned}\right\} \quad \boxed{k_x^2 + k_y^2 + k_z^2 = k^2}$$

DISPERSION RELATION.

$$X = e^{ik_x x} \qquad Y = e^{ik_y y} \qquad Z = e^{ik_z z}$$

$$\vec{E}(\vec{r}) = \vec{E}_0 e^{i(k_x x + k_y y + k_z z)}$$

$$\boxed{\vec{E}(\vec{r}) = \vec{E}_0 e^{i\vec{k}\cdot\vec{r}}}$$

PLANE WAVE

$$\vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$$

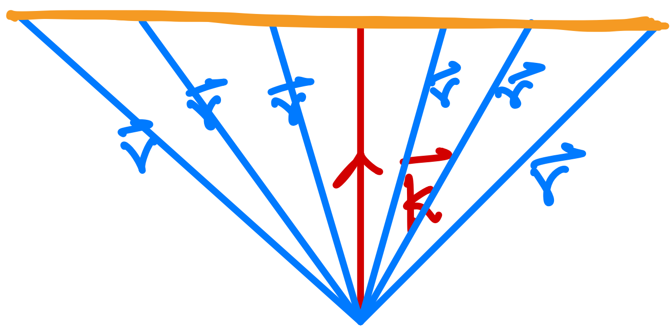
$$|\vec{k}| = k$$

↳ wave vector.

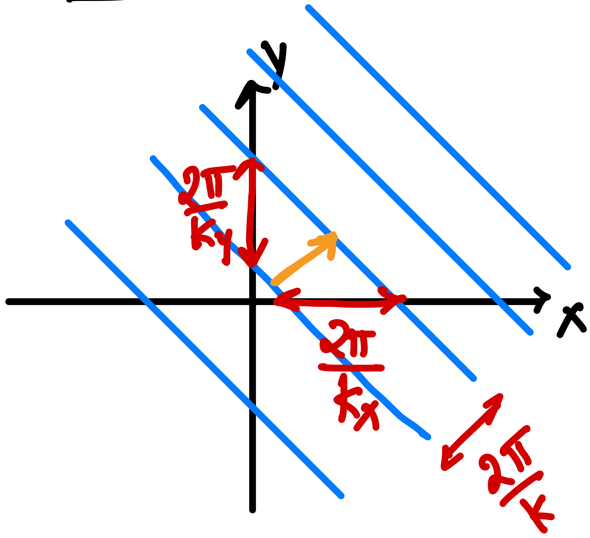
$k = \frac{2\pi}{\lambda}$ → spatial frequency. $\boxed{k = \frac{\omega}{v}}$

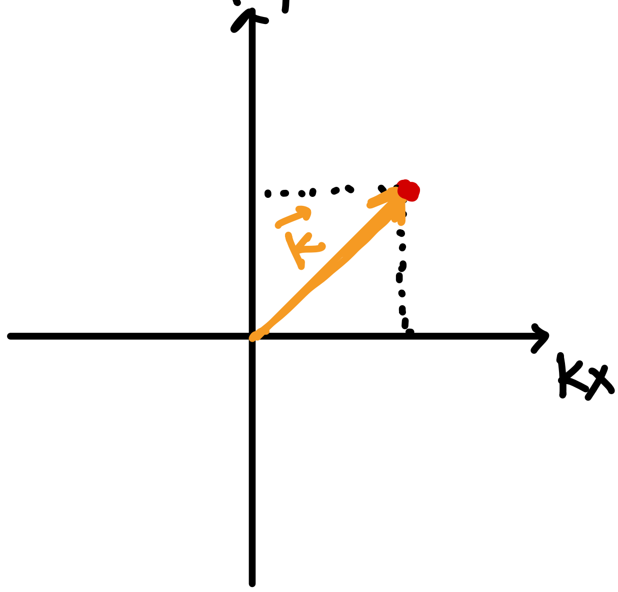
$$\omega = \frac{d\phi}{dt} \qquad \vec{k} = \nabla(\vec{k}\cdot\vec{r})$$

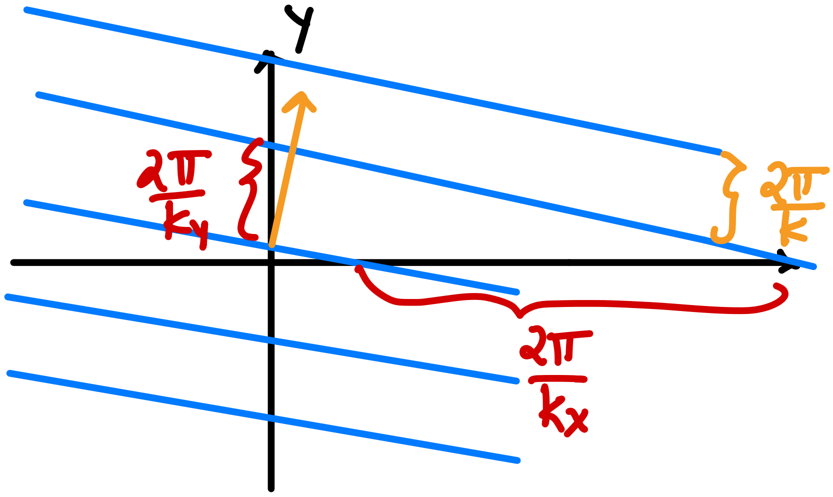
Surfaces of constant phase → <u>PLANES</u>
normal to $\vec{k}$.

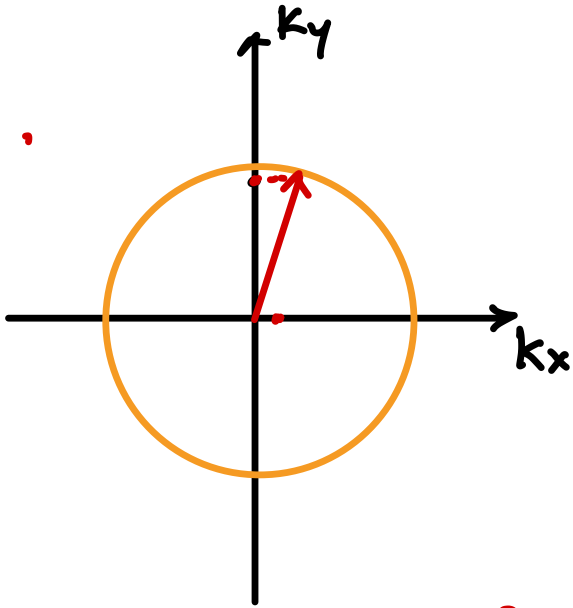



<u>Examples</u>






$$k_x^2 + k_y^2 + k_z^2 = k^2$$

$$k_x^2 + k_y^2 > k^2 \Rightarrow k_z = i\sqrt{k_x^2 + k_y^2 - k^2}$$

$$\vec{E} = \vec{E}_0 e^{i(k_x x + k_y y)} e^{-k_z'' z}$$ where $k_z = i k_z''$

$$\boxed{\vec{E} = E_0 e^{-k_z'' z} e^{i(k_x x + k_y y)} \hat{E}_0}$$

EVANESCENT WAVES

$$\nabla \cdot \vec{E} = 0 \Rightarrow \vec{E}_0 \cdot \nabla e^{i\vec{k}\cdot\vec{r}} = \vec{E}_0 \cdot i\vec{k} e^{i\vec{k}\cdot\vec{r}} = 0$$

$$\Rightarrow \vec{E}_0 \perp \vec{k}$$

$$\vec{H} = \frac{-i}{k\eta} \nabla \times (\vec{E}_0 e^{i\vec{k}\cdot\vec{r}}) = \frac{-i}{k\eta}(i\vec{k} \times \vec{E}_0) e^{i\vec{k}\cdot\vec{r}}$$

$$= \frac{\hat{k} \times \vec{E}_0}{\eta} e^{i\vec{k}\cdot\vec{r}}$$

wave imp. of free space.

$$\Rightarrow \boxed{\vec{H} \perp \vec{E} \perp \hat{k}} \quad \& \quad \frac{|E|}{|H|} = \eta = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377$$

for free space.

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## Plane Wave Basis

$$\vec{E}(\vec{r}) = \iiint_{k_x k_y k_z} \tilde{\vec{E}}(k_x, k_y, k_z)\, e^{ik_x x + ik_y y + ik_z z}\, dk_x dk_y dk_z$$

$$\vec{E}(\vec{r}) = \int_{\vec{k}} \tilde{\vec{E}}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}\, d\vec{k}$$

This is for an arbitrary vector field. But $\vec{E}$ must satisfy the wave equation.

$$k_x^2 + k_y^2 + k_z^2 = k^2$$

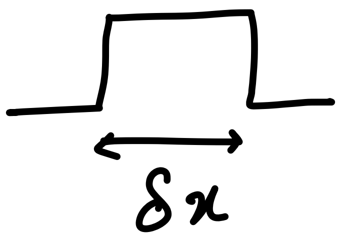
$$\Rightarrow k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

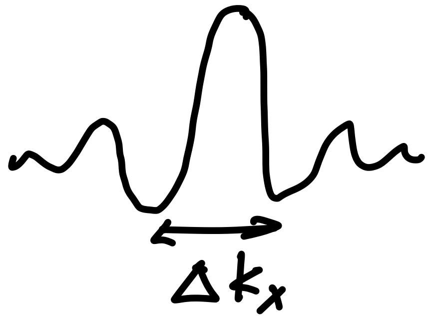
$$\vec{E}(x,y,z) = \int_{k_x}\int_{k_y} \Big(\underbrace{\tilde{\vec{E}}(k_x,k_y)}\, e^{ik_z z}\Big) e^{ik_x x} e^{ik_y y}\, dk_x\, dk_y$$

Angular Spectrum.
K-space spectrum.

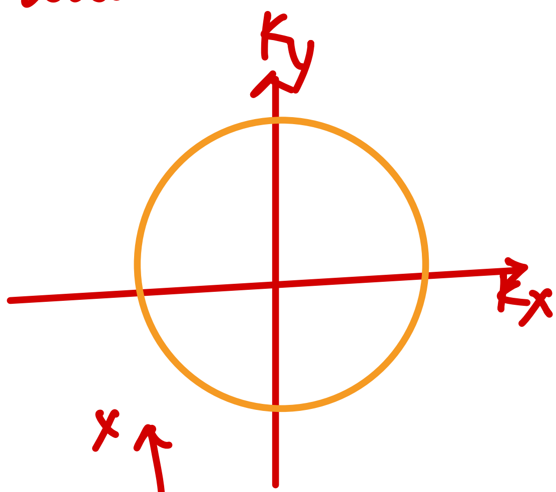
## Diffraction Limit (Abbé)

$$\delta x \, \Delta k_x = 2\pi$$



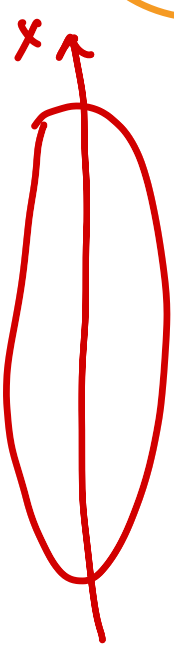

If you have propagating waves.




$$\max\{k_x\} = 2k$$

$$\min\{\delta x\} = \frac{2\pi}{\max\{k_x\}}$$

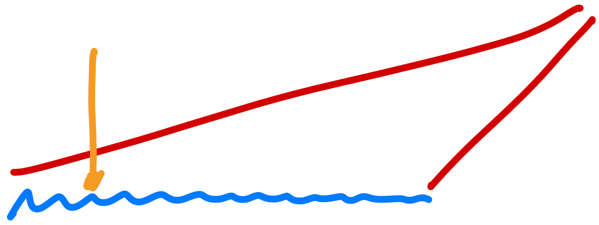
$$\boxed{\delta x_{min} = \frac{\lambda}{2}}$$



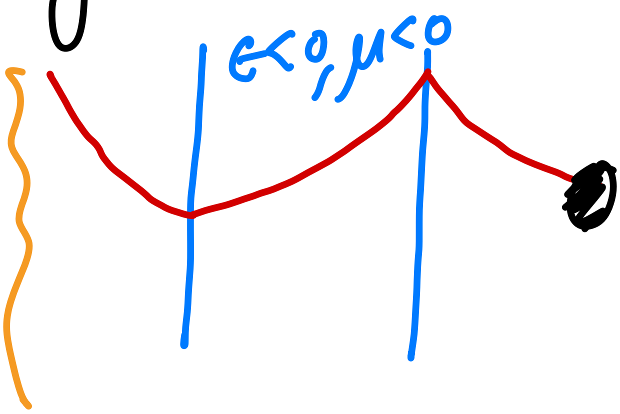
# Breaking the diffraction limit

1) Access the evanescent spectrum.

- Near-field microscopy.



- Negative index lens.




$ik_z z$

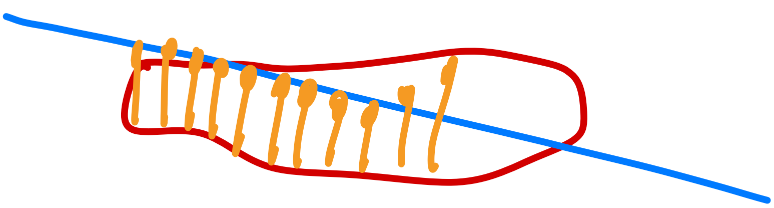
$i(-ik_z'')z$

$= +k_z'' z$

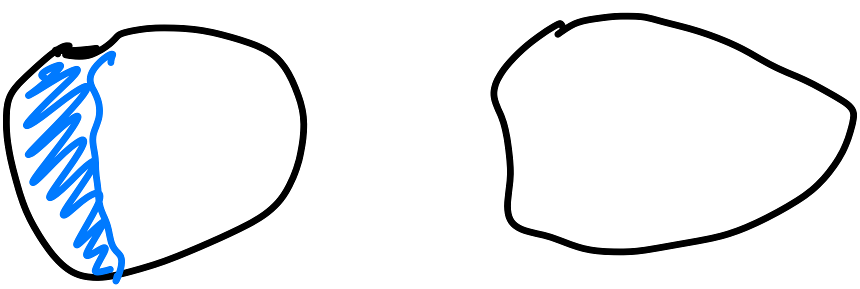
## 2) Additional Information of Target

- MRI (2003)

1 GHz ⇒ 15 cm



- Fluorescence Microscopy (2014)



- Spectral Estimation (MUSIC, ESPRIT, Compressed Sensing, Deconvolution).