EM14 - Dyadic Green's Functions

> Solving Maxwell's Equations without potentials.

$$\nabla \times \vec{E} = i\omega\mu \vec{H}$$
$$\nabla \times \vec{H} = -i\omega\epsilon \vec{E} + \vec{J}$$

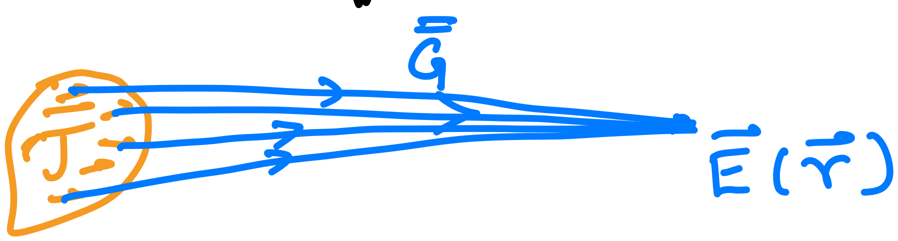
$$\boxed{\nabla \times \nabla \times \vec{E} - k^2 \vec{E} = i\omega\mu \vec{J}}$$ Vector Wave Equation.

To go from $\vec{J} \xrightarrow{\bar{\bar{D}}} \vec{E}$ we need a tensor or matrix or dyad.

$$\bar{\bar{D}} = \vec{A}\vec{B} = \vec{A}\,\vec{B}^T = |A\rangle\langle B| = \begin{bmatrix} A \end{bmatrix}\begin{bmatrix} B \end{bmatrix}$$

$\bar{\bar{G}} : \vec{J} \rightarrow \vec{E}$ through a convolution integral.

$$\vec{E}(\vec{r}) = i\omega\mu \int_V \bar{\bar{G}}(\vec{r}, \vec{r}\,') \cdot \vec{J}(\vec{r}\,')\, d\vec{r}\,'$$



Claim: $\bar{\bar{G}}(\bar{r},\bar{r}') = \left[\bar{\bar{I}} + \frac{1}{K^2}\nabla\nabla^T\right] \underbrace{g(\bar{r},\bar{r}')}_{\frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}}$.

$$\begin{bmatrix}\partial_x\\ \partial_y\\ \partial_z\end{bmatrix}\begin{bmatrix}\partial_x & \partial_y & \partial_z\end{bmatrix} = \begin{bmatrix}\partial_{xx} & \partial_{xy} & \partial_{xz}\\ \partial_{yx} & \partial_{yy} & \partial_{yz}\\ \partial_{zx} & \partial_{zy} & \partial_{zz}\end{bmatrix}$$

Proof: $g(\bar{r},\bar{r}')$ satisfies $\nabla^2 g + k^2 g = -\delta(\bar{r}-\bar{r}')$.

$$\vec{E}(\bar{r}) = i\omega\mu \int_V \bar{\bar{G}}(\bar{r},\bar{r}') \cdot \bar{J}(\bar{r}')\, d\bar{r}'$$

$$\vec{J}(\vec{r}) = \int_V \delta(\bar{r}-\bar{r}')\, \bar{\bar{I}} \cdot \vec{J}(\vec{r}')\, d\bar{r}'$$

$$\Rightarrow \nabla\times\nabla\times\left[i\omega\mu\int_V \bar{\bar{G}}\cdot\bar{J}\, d\bar{r}'\right] - K^2\left[i\omega\mu\int_V \bar{\bar{G}}\cdot\bar{J}\, d\bar{r}'\right]$$
$$= i\omega\mu\int_V \delta(\bar{r}-\bar{r}')\,\bar{\bar{I}}\cdot\bar{J}\, d\bar{r}'$$

When $\vec{r} \neq \vec{r}'$,

$$\nabla\times\nabla\times\bar{\bar{G}}(\bar{r},\bar{r}') - k^2\bar{\bar{G}}(\bar{r},\bar{r}') = \bar{\bar{I}}\,\delta(\bar{r}-\bar{r}')$$

$$\nabla^2 g + k^2 g = -\delta(\bar{r}-\bar{r}').$$

$$\Rightarrow \bar{\bar{I}}\,\nabla^2 g + \bar{\bar{I}}\,k^2 g = -\bar{\bar{I}}\,\delta(\bar{r}-\bar{r}').$$

$$\Rightarrow \nabla^2 \bar{\bar{I}} g + k^2 \bar{\bar{I}} g = -\bar{\bar{I}}\,\delta(\bar{r}-\bar{r}').$$

$$\Rightarrow \nabla\nabla g - \nabla^2 \bar{\bar{I}} g - k^2 \bar{\bar{I}} g - \nabla\nabla g = \bar{\bar{I}}\,\delta(\bar{r}-\bar{r}').$$

$$\Rightarrow \nabla\nabla\cdot\bar{\bar{I}} g - \nabla^2 \bar{\bar{I}} g - k^2 \bar{\bar{I}} g - \nabla\nabla g = \bar{\bar{I}}\,\delta(\bar{r}-\bar{r}').$$

[Recall

$$\nabla\times\nabla\times\vec{A} = \nabla\nabla\cdot\vec{A} - \nabla^2\vec{A}$$

$$\nabla\times\nabla\times\bar{\bar{A}} = \nabla\nabla\cdot\bar{\bar{A}} - \nabla^2\bar{\bar{A}}$$]

$$\nabla\times = \begin{bmatrix} 0 & -\partial_x & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{bmatrix}$$

$$\Rightarrow \nabla\times\nabla\times(\bar{\bar{I}} g) - k^2 \bar{\bar{I}} g - \nabla\nabla g = \bar{\bar{I}}\,\delta(\bar{r}-\bar{r}').$$

$$\Rightarrow \nabla\times\nabla\times(\bar{\bar{I}} g) + \frac{1}{k^2}\nabla\times\nabla\times\nabla\nabla g - k^2 \bar{\bar{I}} g - \nabla\nabla g = \bar{\bar{I}}\,\delta(\bar{r}-\bar{r}')$$

$$\Rightarrow \nabla\times\nabla\times\left[\bar{\bar{I}} + \frac{1}{k^2}\nabla\nabla\right] g - k^2\left[\bar{\bar{I}} + \frac{1}{k^2}\nabla\nabla\right] g = \bar{\bar{I}}\,\delta \;//$$

∎

$$\bar{\bar{G}} = \left[\bar{\bar{I}} + \frac{1}{k^2}\nabla\nabla\right] g(\bar{r}, \bar{r}')$$

$$\vec{E} = i\omega\mu \int_V \bar{\bar{G}} \cdot \vec{J} \, d\bar{r}'.$$

$$\vec{H} = \frac{1}{i\omega\mu} \nabla \times \vec{E} = \int_V \nabla_r \times \bar{\bar{G}}(\bar{r}, \bar{r}') \cdot \vec{J}(\bar{r}') \, d\bar{r}'.$$

We assumed that $\vec{J}_m = 0$.

If $\vec{J}_m \neq 0$, $\vec{J} = 0$

$$\vec{H} = i\omega\epsilon \int_V \bar{\bar{G}}(\bar{r}, \bar{r}') \cdot \vec{J}(\bar{r}') \, d\bar{r}'$$

$$\vec{E} = \frac{1}{-i\omega\epsilon} \nabla \times \vec{H} = -\int_V \nabla_r \times \bar{\bar{G}}(\bar{r}, \bar{r}') \cdot \vec{J}_m(\bar{r}') \, d\bar{r}'.$$

<u>Full soln:</u>

$$\vec{E} = i\omega\mu \int_V \bar{\bar{G}} \cdot \vec{J} \, d\bar{r}' - \int_V \nabla \times \bar{\bar{G}} \cdot \vec{J} \, d\bar{r}'$$

$$\vec{H} = i\omega\epsilon \int_V \bar{\bar{G}} \cdot \vec{J}_m \, d\bar{r}' + \int_V \nabla \times \bar{\bar{G}} \cdot \vec{J} \, d\bar{r}'.$$

Explicit Forms of $\overline{\overline{G}}$

$$\overline{\overline{G}}(R) = \left[\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla\right]\frac{e^{ikR}}{4\pi R} \; ; \; R = |\bar{r} - \bar{r}'|.$$

$$\nabla \frac{e^{ikR}}{4\pi R} = \frac{R\nabla e^{ikR} - e^{ikR}\nabla R}{4\pi R^2}$$

$$= \frac{ikRe^{ikR}\nabla R - e^{ikR}\nabla R}{4\pi R^2}$$

$$= \left(\frac{ik}{R} - \frac{1}{R^2}\right)\frac{e^{ikR}}{4\pi}\nabla R$$

$$\nabla\nabla \frac{e^{ikR}}{4\pi R} = \left(-\frac{ik}{R^2} + \frac{2}{R^3}\right)\frac{e^{ikR}}{4\pi}\nabla R\nabla R$$

$$+ \left(\frac{-k^2}{R} - \frac{ik}{R^2}\right)\frac{e^{ikR}}{4\pi}\nabla R\nabla R$$

$$+ \left(\frac{ik}{R} - \frac{1}{R^2}\right)\frac{e^{ikR}}{4\pi}\nabla\nabla R$$

$$= \frac{e^{ikR}}{4\pi R}\left[\left(\frac{2 - 2ikR - k^2R^2}{R^2}\right)\nabla R\nabla R + \left(\frac{ikR^2 - R}{R^2}\right)\nabla\nabla R\right]$$

$$\nabla R = \nabla\sqrt{x^2+y^2+z^2} = \frac{\bar{R}}{R} = \hat{R} \qquad \begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}(x\ \ y\ \ z)$$

$$\nabla\nabla R = \nabla\left(\frac{\bar{R}}{R}\right) = \frac{R\nabla\bar{R} - \bar{R}\nabla R}{R^2} = \frac{1}{R}\Big(\underbrace{\nabla\bar{R}}_{\bar{\bar{I}}} - \hat{R}\nabla R\Big)$$

$$= \frac{1}{R}\left(\bar{\bar{I}} - \hat{R}\hat{R}\right)$$

$$\Rightarrow \nabla\nabla\frac{e^{ikR}}{4\pi R} = \frac{e^{ikR}}{4\pi R}\left(\left(\frac{2-2ikR-k^2R^2}{R^2}\right)\hat{R}\hat{R} + \left(\frac{ikR^2-R}{R^2}\right)\frac{\bar{\bar{I}}-\hat{R}\hat{R}}{R}\right)$$

$$\boxed{\bar{\bar{G}}(R) = \left[\left(\frac{3}{k^2R^2} - \frac{3i}{kR} - 1\right)\hat{R}\hat{R} + \left(1+\frac{i}{kR} - \frac{1}{k^2R^2}\right)\bar{\bar{I}}\right]\frac{e^{ikR}}{4\pi R}}$$

$$\nabla\times\bar{\bar{G}} = \nabla\times\left[\bar{\bar{I}} + \cancelto{0}{\frac{1}{k^2}\nabla\nabla}\right]g = \nabla\times\bar{\bar{I}}g$$

$$= \nabla g\times\bar{\bar{I}} + g(\cancelto{0}{\nabla\times\bar{\bar{I}}})$$

$$= \nabla g\times\bar{\bar{I}}$$

$$\boxed{\nabla\times\bar{\bar{G}} = \left(ik - \frac{1}{R}\right)\frac{e^{ikR}}{4\pi R}\hat{R}\times\bar{\bar{I}}}$$

$$\vec{E}(\vec{r}) = i\omega\mu \int_V \left[a(R)(\hat{R}\cdot\vec{J})\hat{R} + b(R)\vec{J} \right] \frac{e^{ikR}}{4\pi R}\, d\vec{r}\,'$$

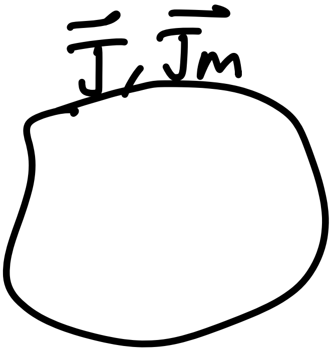
$$- \int_V \left(ik - \frac{1}{R}\right) \frac{e^{ikR}}{4\pi R} (\hat{R} \times \vec{J}_m)\, d\vec{r}\,'$$

$$\vec{H}(\vec{r}) = i\omega\epsilon \int_V \left[a(R)(\hat{R}\cdot\vec{J}_m)\hat{R} + b(R)\vec{J}_m \right] \frac{e^{ikR}}{4\pi R}\, d\vec{r}\,'$$

$$+ \int_V \left(ik - \frac{1}{R}\right) \frac{e^{ikR}}{4\pi R} (\hat{R} \times \vec{J})\, d\vec{r}\,'$$

where $a(R) = \dfrac{3}{k^2R^2} - \dfrac{3i}{kR} - 1$

$$b(R) = 1 + \frac{i}{kR} - \frac{1}{k^2R^2}$$



<u>FEP:</u> $\vec{J} = \hat{n} \times \vec{H}$

$\vec{J}_m = -\hat{n} \times \vec{E}$

<u>Eq Curr.</u>: $\vec{J}_p = -i\omega\epsilon_0(\epsilon_r - 1)\vec{E}(\vec{r})$

$\vec{J}_{mp} = -i\omega\mu_0(\mu_r - 1)\vec{H}(\vec{r})$

Far-Field

$$\bar{\bar{G}} \simeq \left[\bar{\bar{I}} - \hat{r}\hat{r}\right] \frac{e^{ikr}}{4\pi r} e^{-ik\vec{r}'\cdot\hat{r}}$$

$$a(R) = -1 \; ; \; b(R) = 1 \; ; \; \hat{R} = \hat{r}$$

$$R = r - \vec{r}'\cdot\hat{r}$$

$$\vec{E} = i\omega\mu \int_V \Big[\underbrace{\vec{J} - (\hat{r}\cdot\vec{J})\hat{r}}\Big] \frac{e^{ik(r - \vec{r}'\cdot\hat{r})}}{4\pi r} d\vec{r}' \quad + \vec{J}_m \text{ term.}$$

$$\left[\begin{aligned} &\vec{J}(\hat{r}\cdot\hat{r}) - (\hat{r}\cdot\vec{J})\hat{r} \\ &= -(\vec{J}\times\hat{r})\times\hat{r} \end{aligned}\right]$$

$$\vec{E}(\vec{r}) = -i\omega\mu \frac{e^{ikr}}{4\pi r} \int_V (\vec{J}\times\hat{r})\times\hat{r}\, e^{-ik\vec{r}'\cdot\hat{r}} d\vec{r}'$$

$$+ ik \frac{e^{ikr}}{4\pi r} \int_V (\vec{J}_m \times\hat{r}) e^{-ik\vec{r}'\cdot\hat{r}} d\vec{r}'$$

$$\vec{H}(\vec{r}) = -i\omega\epsilon \frac{e^{ikr}}{4\pi r} \int_V (\vec{J}_m\times\hat{r})\times\hat{r}\, e^{-ik\vec{r}'\cdot\hat{r}} d\vec{r}'$$

$$- ik \frac{e^{ikr}}{4\pi r} \int_V (\vec{J}\times\hat{r}) e^{-ik\vec{r}'\cdot\hat{r}} d\vec{r}' .$$