EM12 - Green's Functions

$$D(f(\vec{r})) = F(\vec{r})$$

$$D(g(\vec{r})) = \delta(\vec{r})$$

$$\nabla^2 g(\vec{r}) + k^2 g(\vec{r}) = -\delta(\vec{r})$$

<u>Recall:</u> $\delta(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)$

$$= \frac{1}{\rho}\,\delta(\rho)\,\delta(\phi)\,\delta(z)$$

$$= \frac{1}{r^2\sin\theta}\,\delta(r)\,\delta(\theta)\,\delta(\phi)$$

From sph. symmetry

$$\delta(\vec{r}) = \frac{\delta(r)}{4\pi r^2} \qquad \left(\int_0^R \int_0^{2\pi} \int_0^{\pi} \frac{\delta(r)}{4\pi r^2}\, r^2 \sin\theta \, d\theta \, d\phi \, dr = 1 \right)$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial g(r)}{\partial r}\right) + k^2 g(r) = -\frac{\delta(r)}{4\pi r^2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial g(r)}{\partial r}\right) = \frac{d^2}{dr^2}\left(r\,g(r)\right)$$

$$\Rightarrow \frac{d^2}{dr^2}\left(r\,g(r)\right) + k^2\left(r\,g(r)\right) = \frac{-\delta(r)}{4\pi r}$$

$r > 0$

$$g(r) = A_1 \underbrace{\frac{e^{ikr}}{r}}_{\text{outgoing wave.}} + \cancel{A_2}^{\,0} \underbrace{\frac{e^{-ikr}}{r}}_{\text{incoming wave}}$$

$$\boxed{g(r) = A_1\frac{e^{ikr}}{r}}$$

What is A_1?

$$\int_V \nabla\cdot\nabla g(r)\,dv + k^2\int_V g(r)\,dv = -1$$

$$\oint\!\!\!\oint_S \nabla g(r)\cdot\vec{ds} + k^2\int_V g(r)\,dv = -1$$

$$g(r) = A_1 \frac{e^{ikr}}{r} \Rightarrow \nabla g(r) = \frac{\partial}{\partial r}\left(A_1 \frac{e^{ikr}}{r}\right)\hat{r}$$

S is a sphere of radius R

$$\oiint_S \frac{\partial}{\partial R}\left(A_1 \frac{e^{ikR}}{R}\right) ds + k^2 \int_V A_1 \frac{e^{ikr}}{r} dv = -1$$

$$A_1\left(\frac{\partial}{\partial R}\left(\frac{e^{ikR}}{R}\right)\right)(4\pi R^2) + k^2 4\pi A_1 \int_0^R r e^{ikr} dr = -1$$

$$-\cancel{A_1 e^{ikR}(4\pi)} + A_1 R ik e^{ikR}(4\pi) + k^2 4\pi A_1 \frac{e^{ikR}}{ik} R$$

$$+ \cancel{4\pi A_1 e^{ikR}} - 4\pi A_1 = -1$$

$$\lim_{R \to 0} \Rightarrow A_1 = \frac{1}{4\pi}$$

$$\boxed{g(r) = \frac{e^{ikr}}{4\pi r}}$$

$$\boxed{g(\vec{r}, \vec{r}\,') = \frac{e^{ik|\vec{r} - \vec{r}\,'|}}{4\pi |\vec{r} - \vec{r}\,'|}}$$

$g(\vec{r}, \vec{r}\,') = g(\vec{r}\,', \vec{r})$

Green's Theorem

$\phi, \psi \rightarrow$ scalar fields, whose 1^{st} & 2^{nd} derivatives are continuous in V, & on S.

$$\int_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi)\, dv = \oiint_S \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n}\right) ds$$

Proof :- $\displaystyle\int_V \nabla \cdot (\psi \nabla \phi)\, dv = \oiint_S (\psi \nabla \phi) \cdot d\vec{S}$

$\nabla \phi \cdot \hat{n} = \dfrac{\partial \phi}{\partial n}$ & $\nabla \cdot (\psi \nabla \phi) = \psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi$

$$\int_V \nabla \psi \cdot \nabla \phi \, dv + \int_V \psi \nabla^2 \phi \, dv = \oiint_S \psi \nabla \phi \cdot d\vec{S}$$

$\psi \leftrightarrow \phi$ & subtract $\blacksquare$

Formal Soln. of HH Eqn.

$$\nabla^2 g(\vec{r},\vec{r}') + k^2 g(\vec{r},\vec{r}') = -\delta(\vec{r}-\vec{r}')$$

$$\nabla^2 \phi(\vec{r}) + k^2 \phi(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon}$$

We want to find a solution for a bounded region.

$$\text{Green's Thm} \Rightarrow \int_V g\nabla^2\phi - \phi\nabla^2 g \, dv = \oint_S g\frac{\partial\phi}{\partial n} - \phi\frac{\partial g}{\partial n}\, ds$$

LHS

$$= \int_V \left(-k^2 \cancel{g}\phi - g\frac{\rho}{\epsilon} + k^2 \cancel{\phi} g + \underbrace{\phi\delta}_{\int_V \phi(\vec{r})\,\delta(\vec{r}-\vec{r}')\,dv \,=\, \phi(\vec{r}')}\right) dv = \text{RHS}.$$

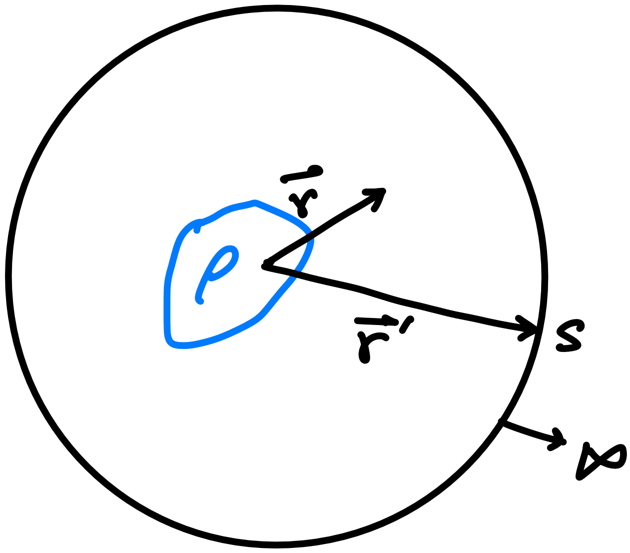
$$\phi(\vec{r}') - \frac{1}{\epsilon}\int_V \rho(\vec{r})\, g(\vec{r},\vec{r}')\, dv = \text{RHS}.$$

$$\phi(\vec{r}') = \frac{1}{\epsilon}\int_V \rho(\vec{r})\, g(\vec{r},\vec{r}')\, dv + \oint_S \left(g\frac{\partial\phi}{\partial n} - \phi\frac{\partial g}{\partial n}\right) ds$$

$$\vec{r} \longleftrightarrow \vec{r}'$$

$$\phi(\vec{r}) = \frac{1}{\epsilon}\int_V \rho(\vec{r}\,')\, g(\vec{r},\vec{r}\,')\, dv' + \oint\!\!\oint_S \left(g(\vec{r},\vec{r}\,') \frac{\partial \phi(\vec{r}\,')}{\partial n'} - \phi(\vec{r}\,') \frac{\partial g(\vec{r},\vec{r}\,')}{\partial n'} \right) ds'$$

$$I = \oint\!\!\oint_S \left(g \frac{\partial \phi}{\partial n'} - \phi \frac{\partial g}{\partial n'} \right) ds$$



$$g(\vec{r},\vec{r}\,') \simeq \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}\,'|}$$

$$\hat{n} = \hat{r}' \qquad n' = r'$$

$$\frac{\partial g}{\partial n'} = \frac{\partial}{\partial r'}\left(\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{4\pi r'} \right) = -\left(\frac{-ik}{r'} + \frac{1}{r'^2} \right) \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{4\pi}$$

$$\Rightarrow I \simeq \oint\!\!\oint_{S'} \left(\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{4\pi r'} \frac{\partial \phi(\vec{r}\,')}{\partial r'} + \phi(\vec{r}\,') \left(\frac{-ik}{r'} + \overbrace{\cancel{\frac{1}{r'^2}}}^{0} \right) \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{4\pi} \right) \underbrace{ds'}_{r'^2 \sin\theta'\, d\theta'\, d\phi'}$$

$$= r'^2\, \Omega'$$

$$I \approx \oint\!\!\oint_{S'} \left(r' \underbrace{\left(\frac{\partial \phi}{\partial r'} - ik\phi \right)}_{0} \frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi} \right) d\Omega'$$

$$\Rightarrow \boxed{\lim_{r' \to \infty} r' \left(\frac{\partial \phi(\bar{r}')}{\partial r'} - ik\phi(\bar{r}') \right) = 0}$$

Sommerfeld Radiation Condition.

$$\lim_{r' \to \infty} r' \left(\frac{\partial \vec{A}(\vec{r}')}{\partial r'} - ik\vec{A}(\vec{r}') \right) = 0$$

$$\phi = \tilde{\phi}(\theta, \phi) \frac{e^{ikr}}{r} \quad \& \quad \vec{A} = \tilde{\vec{A}}(\theta, \phi) \frac{e^{ikr}}{r}$$

$$\boxed{\begin{gathered} \lim_{r \to \infty} r \left[\vec{E}(\vec{r}) + \hat{r} \times \eta \vec{H}(\vec{r}) \right] = 0 \\ \lim_{r \to \infty} r \left[\vec{H}(\vec{r}) - \hat{r} \times \frac{\vec{E}(\vec{r})}{\eta} \right] = 0 \end{gathered}}$$

Silver Müller Boundary Condition. <u>TEM.</u>

Assumptions: ① $s \to \infty$

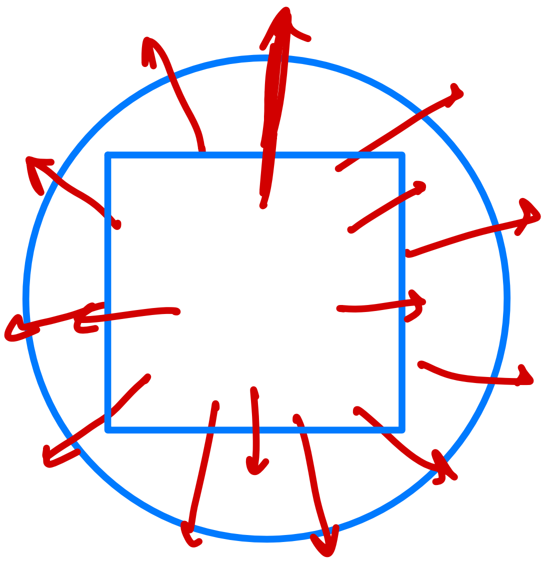
② Spherical.

$$\vec{H} = \frac{1}{i\omega\mu}\nabla \times \vec{E}$$

$$(\nabla \times \vec{E})_{tan} = ik\vec{E}_{tan}$$

→ First order Absorbing Boundary Condition.

$$(\nabla \times \vec{E})_{tan} = ik\vec{E}_{tan} \underbrace{- \frac{i}{k}\nabla_t \times (\nabla_t \times \vec{E}_{tan}) + \frac{i}{k}\nabla_t(\nabla_t \cdot \vec{E})}$$

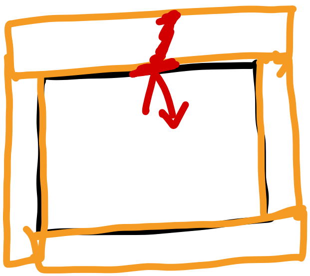


> They account for oblique incidence.

> S can be $\frac{\lambda}{2}$ away from source.

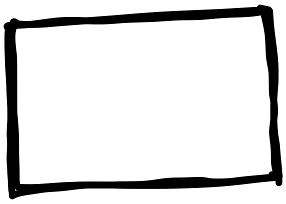
$$\nabla_t = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}.$$

PML - Perfectly Matched Layer.



>PML for antennas.

FEBI - Finite Element Boundary Integral.



FEM & MoM.