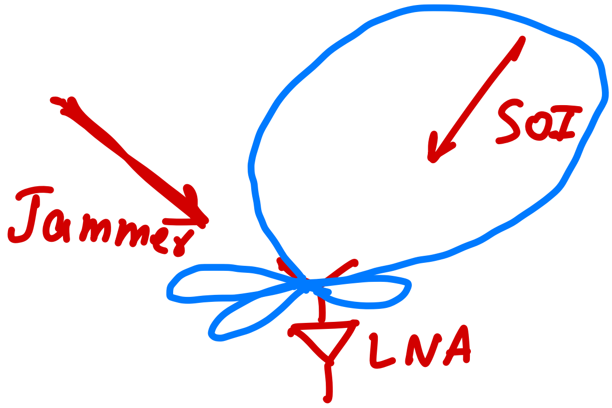
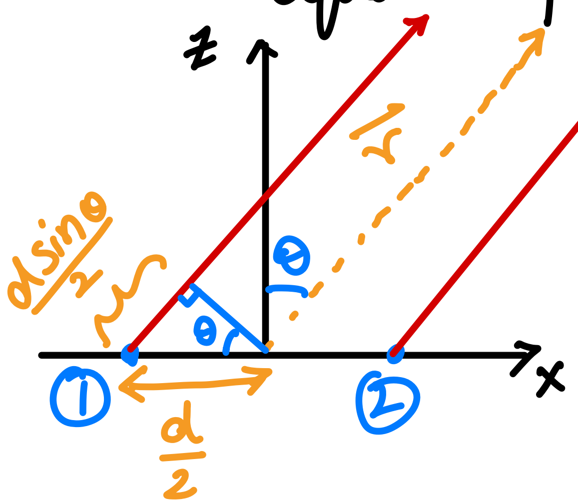
EM II - Arrays

- More gain.
- Beam steering.
- Null placement.



Case 1) Two isotropic point sources with equal amplitude & phase.



$$E = E_0 \frac{e^{ikr}}{4\pi r}$$

$$E_1 = E_0 \frac{e^{ikr}}{4\pi r} e^{ik\frac{d\sin\theta}{2}}$$

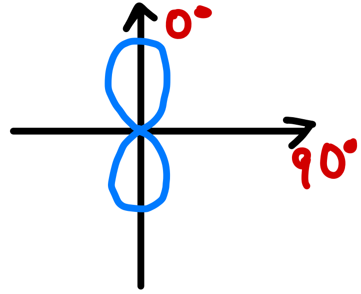
$$E_2 = E_0 \frac{e^{ikr}}{4\pi r} e^{-ik\frac{d\sin\theta}{2}}$$

Let $\psi \triangleq kd\sin\theta$.

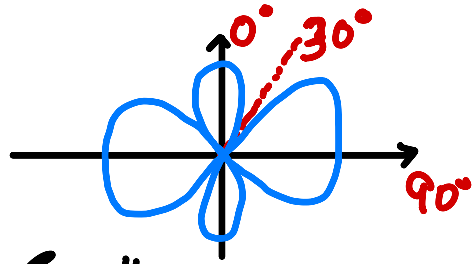
$$E = E_1 + E_2 = 2E_0 \frac{e^{ikr}}{4\pi r}\left[\frac{e^{\frac{i\psi}{2}} + e^{-\frac{i\psi}{2}}}{2}\right]$$

$$E_N = \frac{E}{2E_0 \frac{e^{ikr}}{4\pi r}} = \cos\left(\frac{\psi}{2}\right) = \cos\left(\frac{kd\sin\theta}{2}\right)$$

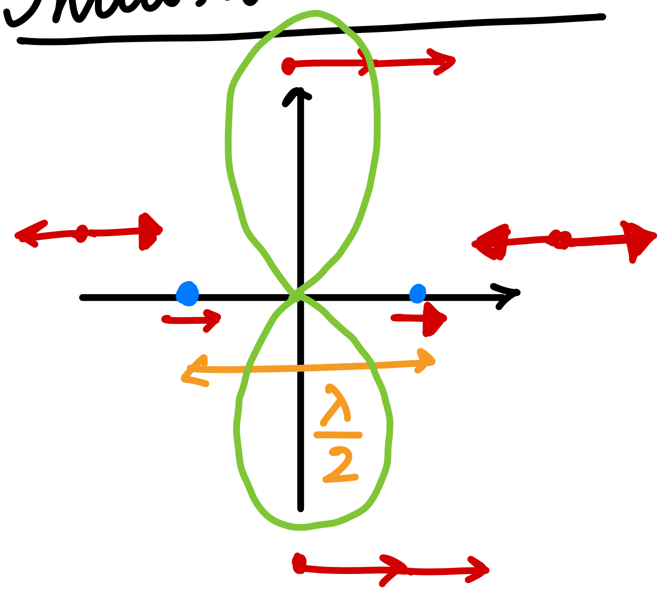
$$d = \frac{\lambda}{2} \Rightarrow E_N = \cos\left(\frac{\pi}{2}\sin\theta\right)$$



$$d = \lambda \Rightarrow E_N = \cos(\pi \sin\theta)$$

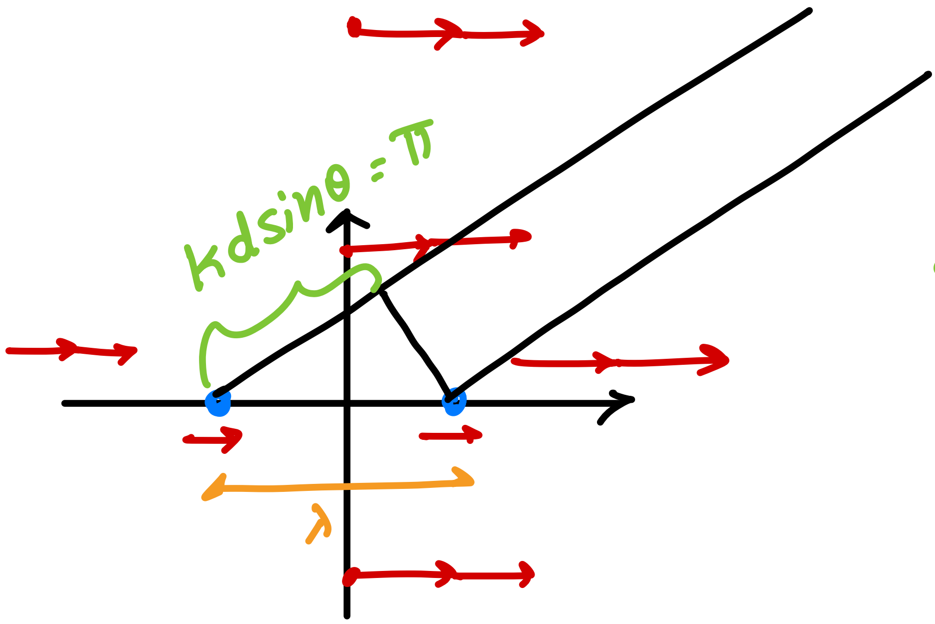


Intuitive Picture: "Phasor Surfing"



$$e^{ikr}$$

$$\mathrm{Re}\{e^{i(kr-\omega t)}\}$$



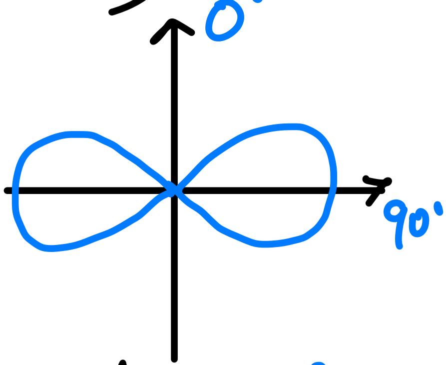
$$\frac{2\pi}{\cancel{\lambda}} \cdot \cancel{\lambda} \sin\theta = \cancel{\pi}$$

$$\Rightarrow \theta = 30^\circ$$

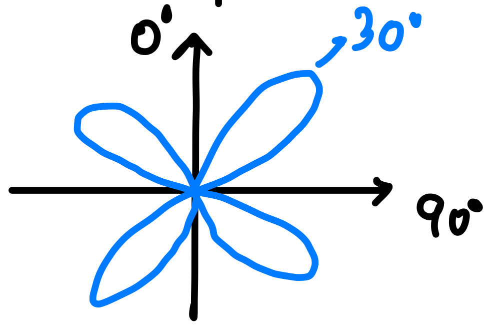
Case 2 Equal amp. opposite phase

$$E_N = \frac{e^{i\frac{\psi}{2}} - e^{-i\frac{\psi}{2}}}{2} = \cancel{i}\sin\left(\frac{kd\sin\theta}{2}\right)$$

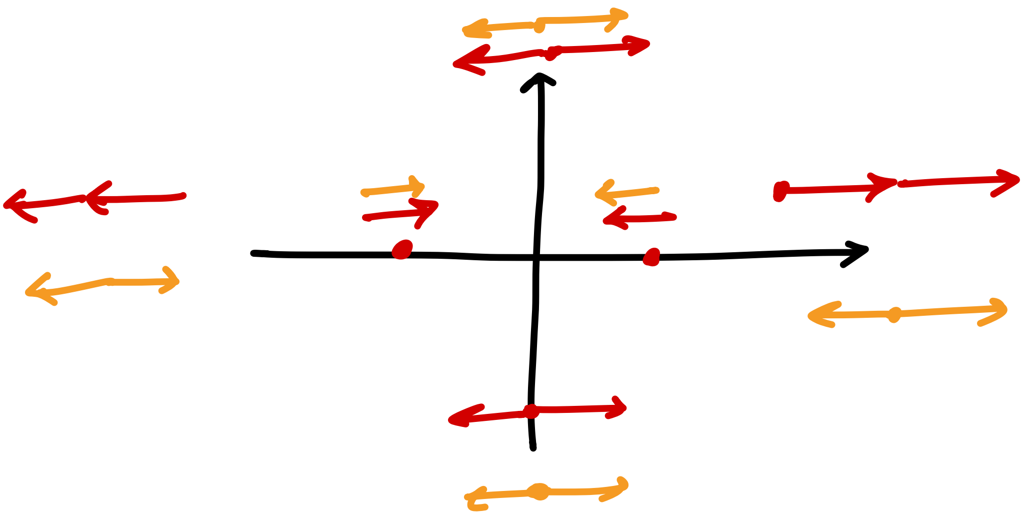
ignore



$$d = \frac{\lambda}{2} \Rightarrow E_N = \sin\left(\frac{\pi}{2}\sin\theta\right)$$



$$d = \lambda \Rightarrow E_N = \sin(\pi\sin\theta)$$



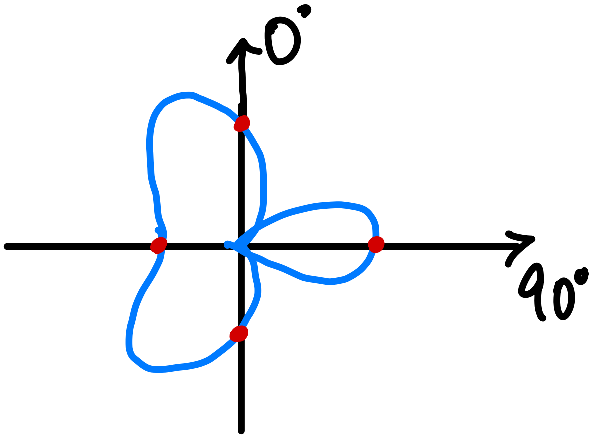
Case 3 Phase Quadrature

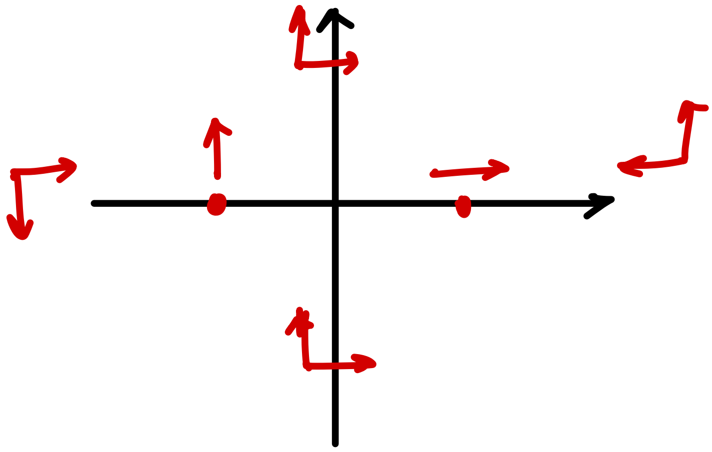
$$E_N = \frac{e^{i\left(\frac{\psi}{2}+\frac{\pi}{4}\right)} + e^{-i\left(\frac{\psi}{2}+\frac{\pi}{4}\right)}}{2}$$

$$E_N = \cos\left(\frac{\pi}{4} + \frac{kd\sin\theta}{2}\right)$$

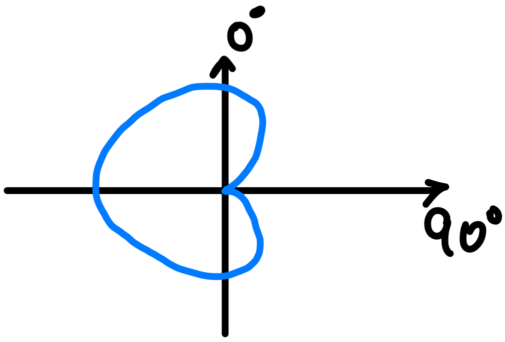
δ	$0°$
$90°$	$0°$
$45°$	$-45°$
$\frac{\delta}{2}$	$-\frac{\delta}{2}$

$$d = \frac{\lambda}{2} \Rightarrow \quad E_N = \cos\left(\frac{\pi}{4} + \frac{\pi}{2}\sin\theta\right)$$

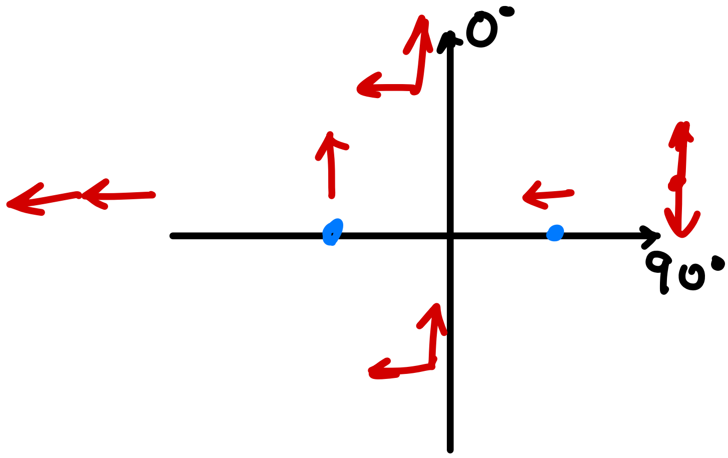




$$d = \frac{\lambda}{4} \Rightarrow \quad E_N = \cos\left(\frac{\pi}{4} + \frac{\pi}{4}\sin\theta\right)$$



ENDFIRE



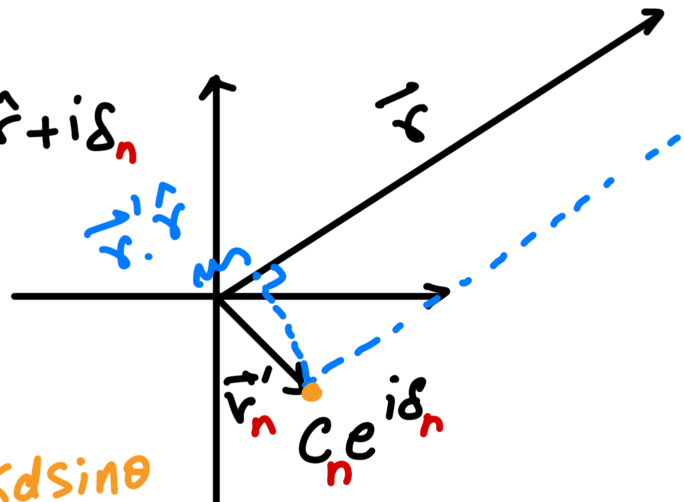
Case 4) General phase diff.

$$\psi = kd\sin\theta + \delta$$

$$E_N = \cos\left(\frac{kd\sin\theta + \delta}{2}\right)$$

N-element array

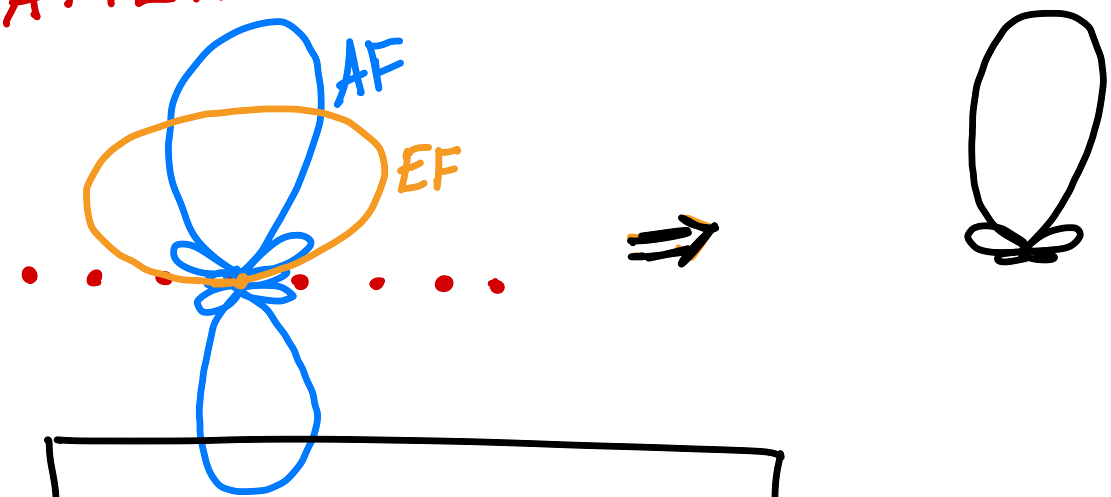
$$E(\vec{r}) = C_n f(\theta,\phi)\frac{e^{ikr}}{4\pi r}e^{-ik\vec{r}_n'\cdot\hat{r}+i\delta_n}$$

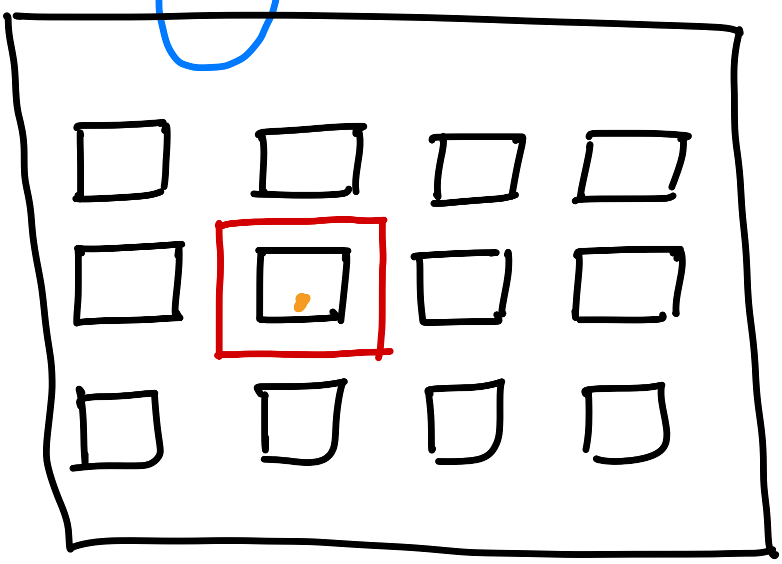


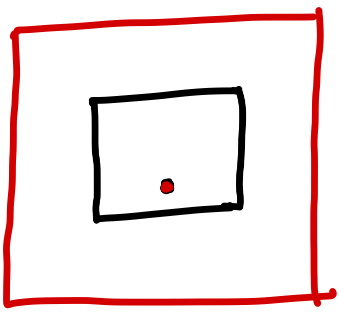
$$E_{tot}(\vec{r})$$

$$= \underbrace{f(\theta,\phi)\frac{e^{ikr}}{4\pi r}}_{\text{Element Factor}} \underbrace{\sum_{n=1}^{N} C_n e^{-i\overbrace{k\vec{r}_n'\cdot\hat{r}}^{kd\sin\theta}+i\delta_n}}_{\text{Array Factor}}$$

PATTERN MULTIPLICATION!

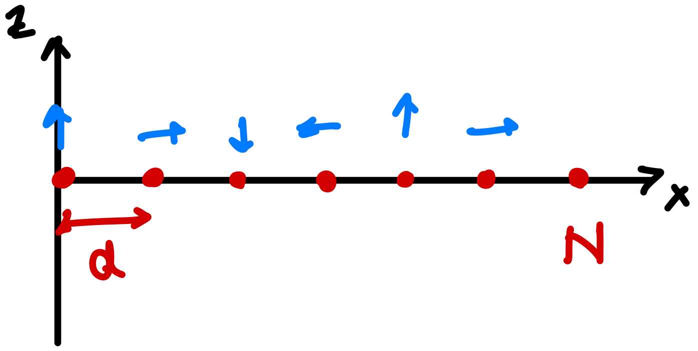






Element Factor!

Uniform Linear Array (ULA)



$$\tilde{AF} = 1 + e^{-i\psi} + e^{-2i\psi} + \dots + e^{-(N-1)i\psi}$$

$$\psi = kd\sin\theta + \delta$$

$$\tilde{AF}\left(e^{-i\psi}\right) = e^{-i\psi} + \dots + e^{-Ni\psi}$$

$$AF = \sum_{n=0}^{N-1} e^{-in\psi}$$

$$\tilde{AF}\left(1 - e^{-i\psi}\right) = 1 - e^{-iN\psi}$$

$$\Rightarrow \tilde{AF} = \frac{1 - e^{-iN\psi}}{1 - e^{-i\psi}} = \frac{e^{-i\frac{N\psi}{2}}}{e^{-i\frac{\psi}{2}}} \cdot \frac{e^{i\frac{N\psi}{2}} - e^{-i\frac{N\psi}{2}}}{e^{i\frac{\psi}{2}} - e^{-i\frac{\psi}{2}}}$$

$$\tilde{AF} = e^{-i\frac{\psi}{2}(N-1)} \left[\frac{\sin\left(\frac{N\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)}\right]$$

Moving phase ref to array center, normalising,

$$\boxed{AF = \frac{1}{N}\frac{\sin\left(\frac{N\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)}}$$

AF is max when $\psi = 2m\pi$, $m \in \mathbb{Z}$

$$\boxed{SLL_{max} \simeq -13.4\,\text{dB}}$$

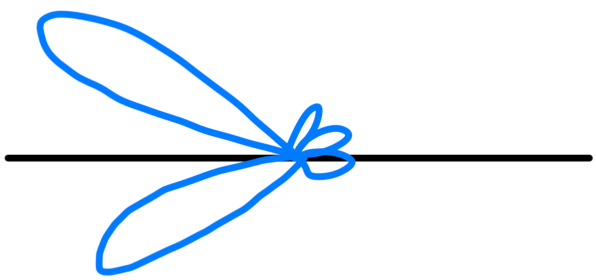
Broadside: $kd\sin\cancel{\theta} + \delta = 0$

0

$\Rightarrow \delta = 0$

Endfire: $d = \frac{\lambda}{4}$, $\delta = \frac{\pi}{2}$.

Phased array: $d = \frac{\lambda}{2}$ $\Rightarrow$ $\boxed{\delta = \pi \sin\theta}$

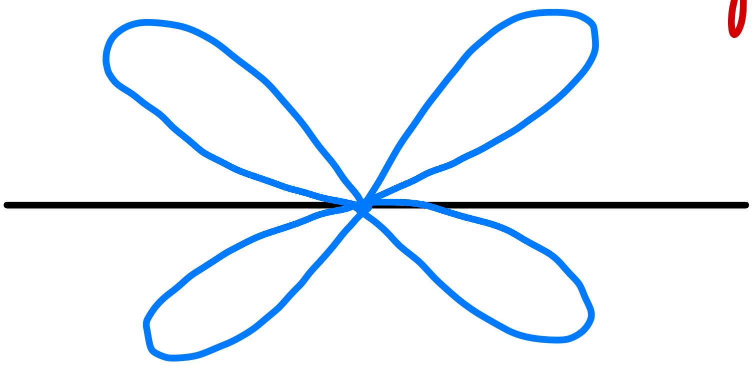


Grating lobes

$\psi = 2m\pi$

$$\Rightarrow \sin\theta = \frac{2m\pi - \delta}{kd} = \frac{\lambda}{d}\left(m - \frac{\delta}{2\pi}\right)$$

When $d > \frac{\lambda}{2} \Rightarrow$ multiple global maxima (for θ).



Uniform Rectangular Array

$$AF = AF_x AF_y$$

$$= \left[\frac{1}{M} \frac{\sin\left(\frac{M}{2}\psi_x\right)}{\sin\left(\frac{\psi_x}{2}\right)}\right]\left[\frac{1}{N} \frac{\sin\left(\frac{N}{2}\psi_y\right)}{\sin\left(\frac{\psi_y}{2}\right)}\right]$$

$$\psi_x = k d_x \sin\theta \cos\phi + \delta_x$$

$$\psi_y = k d_y \sin\theta \sin\phi + \delta_y$$