Babinet's Theorem

Lemma 1 :

H_{tan} in the plane containing $\vec{J}_{surf} = 0$.

Proof:

$$\vec{A}(\vec{r}) = \mu \iint \vec{J}(\vec{r}')\, g(\vec{r},\vec{r}')\, ds'$$

where $g(\vec{r},\vec{r}') = \dfrac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}$ → Green's fn.

$$\vec{H}(\vec{r}) = \frac{1}{\mu}\nabla\times\vec{A} = \iint_S \underbrace{\nabla\times\left[\left(\vec{J}(\vec{r}')\, g(\vec{r},\vec{r}')\right)\right]}_{\nabla g(\vec{r},\vec{r}')\times\vec{J}(\vec{r}')} ds$$

$$\nabla g = \left(ik - \frac{1}{|\vec{r}-\vec{r}'|}\right)\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|^2}(\vec{r}-\vec{r}')$$

$\Rightarrow$ $\vec{H}(\vec{r})$ is $\perp$ to the surface S.

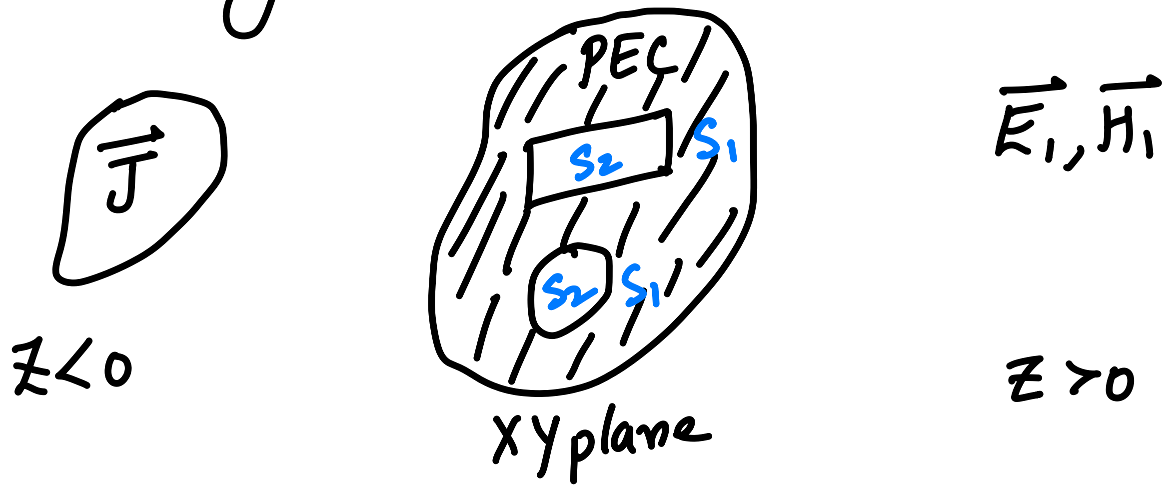
$\Rightarrow$ $\vec{H}_{tan} = 0$ on S.

Lemma: $\vec{E}_{tan}$ due to $\vec{J}_{ms}$ is zero on S.

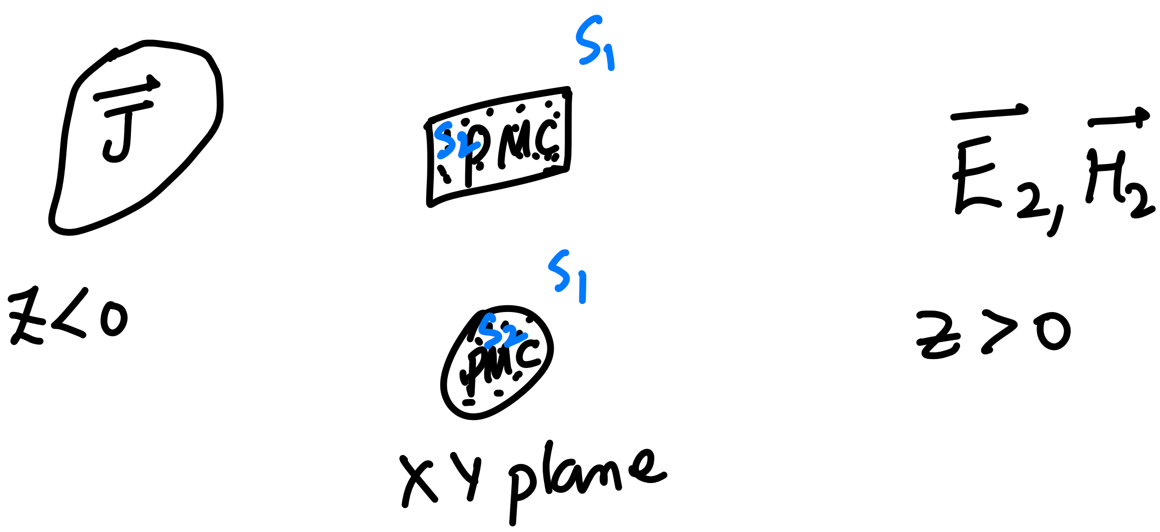
Proof: Duality.

Complementary Theorem

a)



b)



c)

$\vec{E}_i = \vec{E}_1 + \vec{E}_2$ $\qquad$ $\vec{H}_i = \vec{H}_1 + \vec{H}_2$

<u>Proof</u>:

On S_1, $\hat{n} \times \vec{E}_1 = 0$

On S_2, $\hat{n} \times \vec{H}_2 = 0$

In case a), $\vec{H}_i$ produces $\vec{J}_s$ on S_1 which produces no $\vec{H}_{tan}$ on S_2.

$$\Rightarrow \hat{n} \times \vec{H}_1 = \hat{n} \times \vec{H}_i \text{ on } S_2$$

Similarly in case b)

$$\hat{n} \times \vec{E}_2 = \hat{n} \times \vec{E}_i \text{ on } S_1$$

$\Rightarrow$ on S_1, $\hat{n} \times (\vec{E}_1 + \vec{E}_2) = \hat{n} \times \vec{E}_i$

on S_2, $\hat{n} \times (\vec{H}_1 + \vec{H}_2) = \hat{n} \times \vec{H}_i$

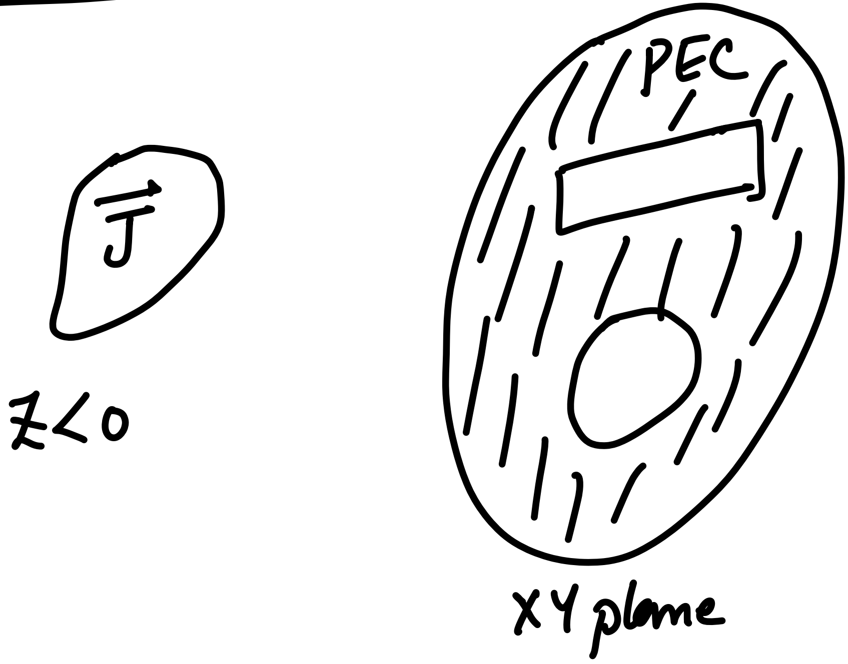
$\Rightarrow$ From uniqueness,

$$\left.\begin{aligned} \vec{E}_1 + \vec{E}_2 &= \vec{E}_i \\ \vec{H}_1 + \vec{H}_2 &= \vec{H}_i \end{aligned}\right\} \text{ in } z>0 \text{ space!}$$

$\blacksquare$

Babinet's Theorem

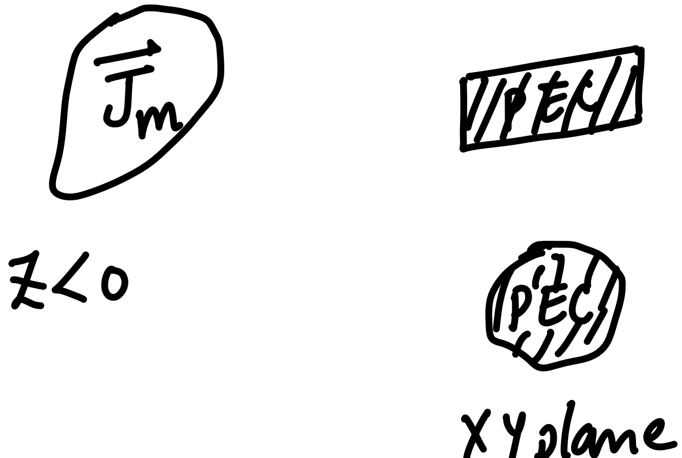
a)



$$\vec{E}_1^e = \vec{E}_i^e + \vec{E}_s^e$$
$$\vec{H}_1^e = \vec{H}_i^e + \vec{H}_s^e$$

$z > 0$

b)

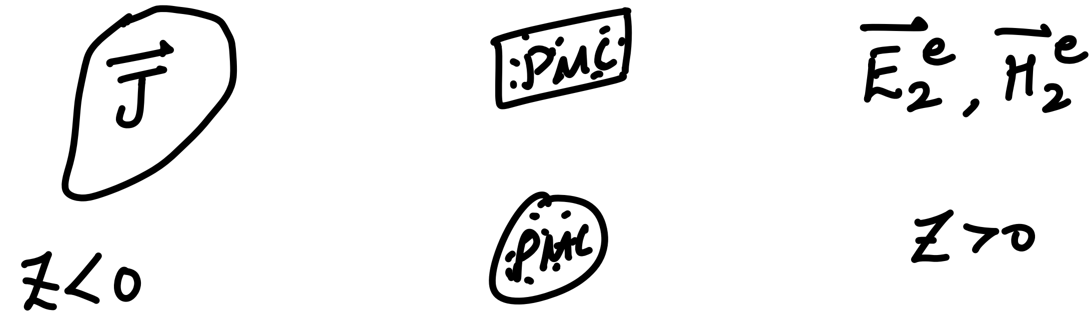


$$\vec{E}_2^m = \vec{E}_i^m + \vec{E}_s^m$$
$$\vec{H}_2^m = \vec{H}_i^m + \vec{H}_s^m$$

$$\vec{E}_1^e = -\sqrt{\frac{\mu}{\epsilon}}\, \vec{H}_s^m$$

$$\vec{H}_1^e = \sqrt{\frac{\epsilon}{\mu}}\, \vec{E}_s^m$$

<u>Proof:</u> i) Apply complementary theorem to a).



$$\vec{E}_1^e + \vec{E}_2^e = \vec{E}_i^e$$

$$\vec{H}_1^e + \vec{H}_2^e = \vec{H}_i^e$$

$$\vec{E}_2^e \longrightarrow \sqrt{\frac{\mu}{\epsilon}}\, \vec{H}_2^m = \sqrt{\frac{\mu}{\epsilon}}\left(\vec{H}_i^m + \vec{H}_s^m\right)$$

$$\vec{H}_2^e \longrightarrow -\sqrt{\frac{\epsilon}{\mu}}\, \vec{E}_2^m = -\sqrt{\frac{\epsilon}{\mu}}\left(\vec{E}_i^m + \vec{E}_s^m\right)$$

$$\vec{E}_i^e \longrightarrow \sqrt{\frac{\mu}{\epsilon}}\, \vec{H}_i^m$$

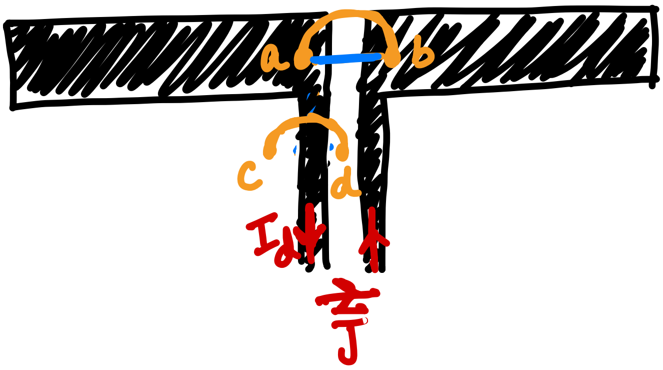
$$\vec{H}_i^e \longrightarrow -\sqrt{\frac{\epsilon}{\mu}}\, \vec{E}_i^m$$

$$\boxed{\begin{aligned} \vec{E}_1^e &= -\sqrt{\frac{\mu}{\epsilon}}\, \vec{H}_s^m \\ \vec{H}_1^e &= \sqrt{\frac{\epsilon}{\mu}}\, \vec{E}_s^m \end{aligned}}$$

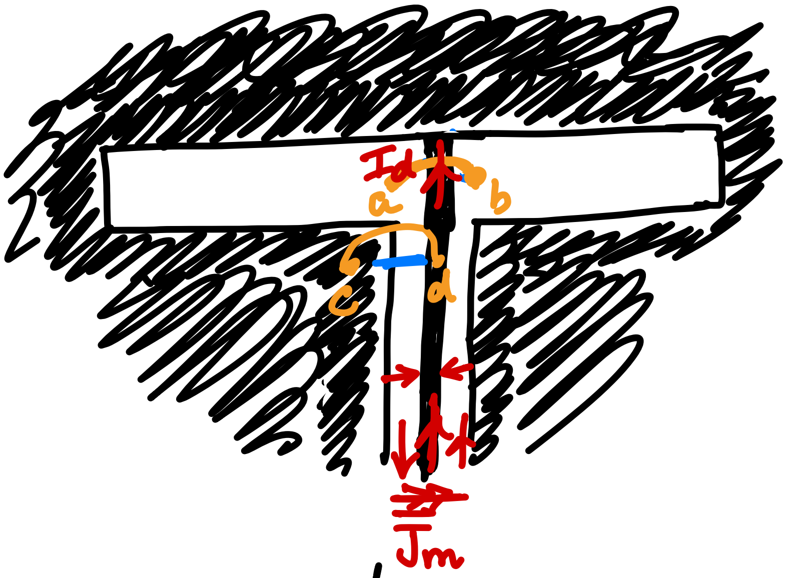
[redacted]

Complimentary Antennas

Dipole Antenna



Slot Antenna



$$V_d = \int_a^b \vec{E}_1^e \cdot \vec{dl}$$

$$I_d = -2\int_c^d \vec{H}_1^e \cdot \vec{dl}$$

$$V_s = \int_c^d \vec{E}_2^m \cdot \vec{dl}$$

$$I_s = +2\int_a^b \vec{H}_2^m \cdot \vec{dl}$$

$$V_d = -\eta \int_a^b \vec{H}_s^m \cdot \vec{dl}$$

$$I_d = -\frac{2}{\eta}\int_c^d \vec{E}_s^m \cdot \vec{dl}$$

$$Z_d = \frac{V_d}{I_d} = \frac{\eta^2}{2} \frac{\int_a^b \vec{H}_s^m \cdot \vec{dl}}{\int_c^d \vec{E}_s^m \cdot \vec{dl}} \simeq \frac{\eta^2}{2}\frac{1}{2Z_s}$$

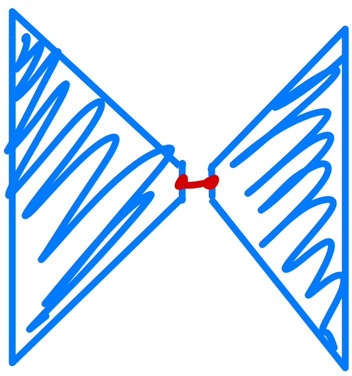
$$\Rightarrow Z_d Z_s = \frac{\eta^2}{4} \,//$$

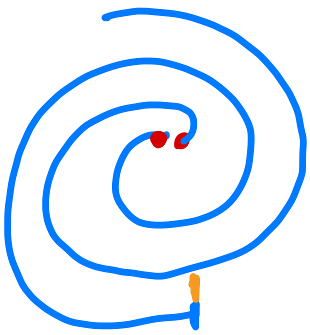
$$\Rightarrow \boxed{Z_A Z_C = \frac{\eta^2}{4}}$$

η is a constant!

Frequency Independent Antenna

$$\boxed{Z_{SCA} = \frac{\eta}{2}}$$





"Sinuous"