EM 07 - Time Harmonic EM Waves

> We will use the $e^{-i\omega t}$ time convention.

$$\tilde{\vec{E}}(\vec{r},\omega) = \int_{-\infty}^{\infty} \vec{E}(\vec{r},t)\, e^{i\omega t}\, dt$$

$$\vec{E}(\vec{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\vec{E}}(\vec{r},\omega)\, e^{-i\omega t}\, d\omega$$

If $\vec{E}$ is real, RHS

$$= \frac{1}{\pi}\int_{0}^{\infty} \mathrm{Re}\Big(\underbrace{\tilde{\vec{E}}(\vec{r},\omega)\, e^{-i\omega t}}_{\text{Phasor form.}}\Big)\, d\omega.$$

$$\frac{d}{dt} \rightarrow -i\omega$$

For simplicity we drop $\sim$ & $e^{-i\omega t}$!

Maxwell's Equations

$$\nabla \times \vec{E} = i\omega\mu \vec{H} - \vec{J}_m \qquad \nabla \cdot \vec{B} = \rho_m$$

$$\nabla \times \vec{H} = -i\omega\epsilon \vec{E} + \vec{J} \qquad \nabla \cdot \vec{D} = \rho$$

$$\nabla \times \vec{H} = -i\omega\epsilon'\vec{E} + \sigma\vec{E} + \vec{J}$$

$$= -i\omega\underbrace{\left(\epsilon' + i\frac{\sigma}{\omega}\right)}\vec{E} + \vec{J}$$

$$\epsilon \text{ complex} = \epsilon' - i\underbrace{\epsilon''}_{\hookrightarrow -\frac{\sigma}{\omega}}$$

Continuity

$$\nabla \cdot \vec{J} = i\omega\rho$$

$$\nabla \cdot \vec{J_m} = i\omega\rho_m$$

Polarization & Magnetization Currents

$$\vec{J_{ep}} = -i\omega\epsilon_0(\epsilon_r - 1)\vec{E}$$

$$\vec{J_{em}} = -i\omega\mu_0(\mu_r - 1)\vec{H}$$

where $\epsilon_r = \epsilon_r' - i\epsilon_r''$

Potentials

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\nabla\phi + i\omega\vec{A}$$

Gauge: $\nabla \cdot \vec{A} = i\omega\mu\epsilon\phi$

$$\Rightarrow \phi = \frac{-i\omega}{K^2} \nabla \cdot \vec{A}$$

, $K = \omega\sqrt{\mu\epsilon}$ is the wave number

$$\Rightarrow \vec{E} = \frac{i\omega}{K^2} \nabla\nabla \cdot \vec{A} + i\omega\vec{A}$$

$$\vec{H} = \frac{1}{\mu} \nabla \times \vec{A}$$

Wave equations $\rightarrow$ Helmholtz Eqns.

$$\nabla^2\vec{A} + K^2\vec{A} = -\mu\vec{J}$$

$$\nabla^2\phi + K^2\phi = \frac{-\rho}{\epsilon}$$

$$\nabla^2\vec{A_m} + K^2\vec{A_m} = -\epsilon\vec{J_m}$$

$$\nabla^2\phi_m + K^2\phi_m = \frac{-\rho_m}{\mu}$$

$$\boxed{\begin{aligned}\vec{E} &= \frac{i\omega}{k^2}\nabla\nabla\cdot\vec{A} + i\omega\vec{A} - \frac{1}{\epsilon}\nabla\times\vec{A}_m \\ \vec{H} &= \frac{1}{\mu}\nabla\times\vec{A} + \frac{i\omega}{k^2}\nabla\nabla\cdot\vec{A}_m + i\omega\vec{A}_m\end{aligned}}$$

Poynting Vector

$$\underbrace{\vec{S}(\vec{r},t)}_{\text{not a phasor}} = \mathrm{Re}\left[\vec{E}(\vec{r})e^{-i\omega t}\right] \times \mathrm{Re}\left[\vec{H}(\vec{r})e^{-i\omega t}\right]$$

$$= \left[\vec{E}_r(\vec{r})\cos(\omega t) + \vec{E}_i(\vec{r})\sin\omega t\right] \times \left[\vec{H}_r\cos(\omega t) + \vec{H}_i\sin\omega t\right]$$

$$\vec{S}(\vec{r},t) = \left(\vec{E}_r\times\vec{H}_r\right)\cos^2\omega t + \left(\vec{E}_i\times\vec{H}_i\right)\sin^2\omega t + \left(\vec{E}_r\times\vec{H}_i + \vec{E}_i\times\vec{H}_r\right)\cos\omega t\sin\omega t.$$

Time averaged Poynting Vector.

$$\vec{S}_{avg}(\vec{r}) = \frac{1}{T}\int_t^{t+T}\vec{S}(\vec{r},t)\,dt$$

$$= \frac{1}{2}\left[\vec{E}_r \times \vec{H}_r + \vec{E}_i \times \vec{H}_i\right]$$

$$\vec{S}_{avg} = \frac{1}{2} \operatorname{Re}\left\{\vec{E} \times \vec{H}^*\right\}$$

The net power density flowing through a inf. surface.

$$\boxed{\vec{S}(\vec{r}) = \frac{1}{2} \vec{E}(\vec{r}) \times \vec{H}^*(\vec{r})}$$

Complex Poynting Vector.

$\operatorname{Im}(\vec{S}) \rightarrow$ Reactive Power Density

"Power oscillating at the boundary s".

V, S

Complex Poynting Theorem

$$\frac{1}{2} \oint_S (\vec{E} \times \vec{H}^*) \cdot \vec{ds} = -\frac{1}{2} \iiint_V (\vec{E} \cdot \vec{J}^* + \vec{J}_m \cdot \vec{H}^*)\, dv$$

$$+ i\omega \underbrace{\int_V \frac{1}{2}\mu |\vec{H}|^2 - \frac{1}{2} \epsilon^* |E|^2 \, dv}$$

$$W_e = \frac{1}{2}\int_V \frac{\epsilon' |\vec{E}|^2}{2}\, dv$$

$$i[(\mu' + i\mu'') - (\epsilon' + i\epsilon'')]$$

$$i(\mu' - \epsilon')$$

$$-(\mu'' + \epsilon'')$$

$$W_m = \frac{1}{2}\int_V \frac{\mu' |\vec{H}|^2}{2}\, dv$$

$$P_{loss} = \frac{1}{2}\int_V \left(\omega\mu'' |\vec{H}|^2 + \omega\epsilon'' |E|^2\right) dv$$

$$\boxed{P_{in}^c = P_{rad}^c + P_{loss} + i2\omega(W_e - W_m)}$$

$$\mathrm{Re}\{P_{in}^c\} = \mathrm{Re}\{P_{rad}^c\} + P_{loss}$$

$$\mathrm{Im}\{P_{in}^c\} = \mathrm{Im}\{P_{rad}^c\} + 2\omega(W_e - W_m).$$

Quality Factor

$$Q = \omega \cdot \frac{\text{max. stored energy}}{\text{avg. dissipated power}} = \omega \cdot \frac{\frac{1}{2}\epsilon' |\vec{E}|^2}{\frac{1}{2}\omega\epsilon'' |\vec{E}|^2}$$

$$\Rightarrow \boxed{Q = \frac{\epsilon'}{\epsilon''}}$$

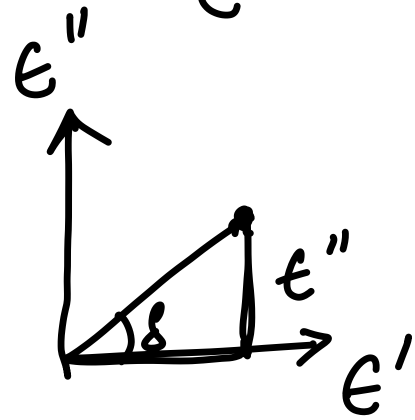
Loss tangent

$$\tan\delta = \frac{\text{dissipation current}}{\text{displacement current}}$$

$$\left[\ \vec{J}_p = -i\omega\epsilon_0(\epsilon_r - 1)\vec{E}(\vec{r})\ \right]$$

$$\tan\delta = \frac{\omega\epsilon_0\epsilon_r''\vec{E}}{\omega\epsilon_0\epsilon_r'\vec{E}} = \frac{\epsilon''}{\epsilon'}$$

$$\delta^{-1} = \tan\delta$$



Time Harmonic Retarded Potential

Radiation

$$\vec{A}(\vec{r},\vec{r}',t) = \frac{\mu}{4\pi}\iiint_V \frac{\vec{J}\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{u_p}\right)}{|\vec{r}-\vec{r}'|}\, d\vec{r}'$$

$$\vec{J}(\vec{r}',t) = I_0\, \vec{dl}\, \delta(\vec{r}')\, f(t)$$

$$f(t) = e^{-i\omega t}$$

$$\omega\sqrt{\mu\epsilon} = \frac{\omega}{u_p}$$

$$f\left(t-\frac{r}{u_p}\right) = e^{-i\omega\left(t-\frac{r}{u_p}\right)} = e^{-i\omega t}\, e^{ikr}$$

$$\vec{A}(\vec{r},\vec{r}',t) = \frac{\mu I_0}{4\pi r}\, \vec{dl}\, f\left(t-\frac{r}{u_p}\right)$$

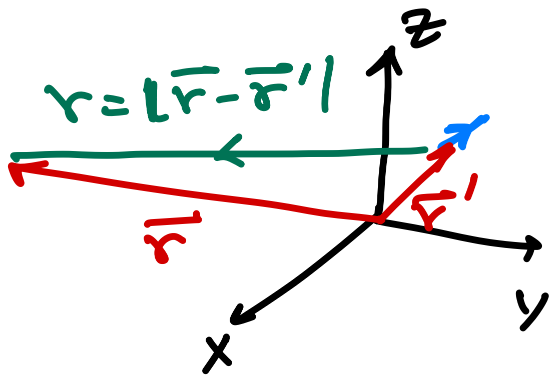
$r \rightarrow |\vec{r}-\vec{r}'|$

FT on both sides.

$$\vec{A}(r) = \mu I_0 \frac{e^{ikr}}{4\pi r}\, \vec{dl}$$

$$\boxed{\vec{A}(\vec{r},\vec{r}') = \mu I_0 \frac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi |\vec{r}-\vec{r}'|}\, \vec{dl}}$$

$$\text{phase} = k|\vec{r}-\vec{r}'|$$
$$= kr$$
$$= \frac{2\pi}{\lambda}\cdot r = 2\pi\,(\text{no. of wavelengths})$$



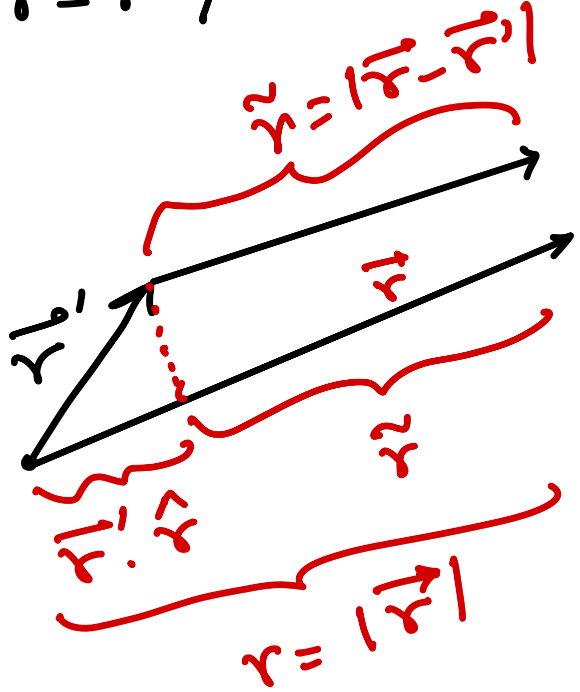
$$\vec{H}(\vec{r}) = \frac{I_0 k^2}{4\pi}\left(-i + \frac{1}{k\tilde{r}}\right)\frac{e^{ik\tilde{r}}}{k\tilde{r}}\,\vec{dl} \times \frac{\vec{r}-\vec{r}\,'}{\tilde{r}}$$

$$\vec{E}(\vec{r}) = \frac{I_0 k^2}{4\pi}\,\eta\,\frac{e^{ikr}}{k\tilde{r}}\left\{\left[\frac{2}{k\tilde{r}} + \frac{i2}{k^2\tilde{r}^2}\right]\frac{(\vec{r}-\vec{r}\,')\cdot\vec{dl}}{\tilde{r}^2}(\vec{r}-\vec{r}\,') - \left(\frac{1}{k\tilde{r}} + \frac{i}{k^2\tilde{r}^2} - i\right)\frac{\vec{r}-\vec{r}\,'}{\tilde{r}} \times \left(\vec{dl} \times \frac{(\vec{r}-\vec{r}\,')}{\tilde{r}}\right)\right\}$$

$$\tilde{r} = |\vec{r}-\vec{r}\,'| \;, \quad \eta = \sqrt{\frac{\mu}{\epsilon}}$$

<u>Far-field approx.</u> $k\,|\vec{r}-\vec{r}\,'| \gg 1$

$\tilde{r} = r - \vec{r}\,'\cdot\hat{r}$



<u>Phase</u>

$|\vec{r}-\vec{r}\,'| = r - \vec{r}\,'\cdot\hat{r}$

<u>Amp</u>

$|\vec{r}-\vec{r}\,'| \simeq r$

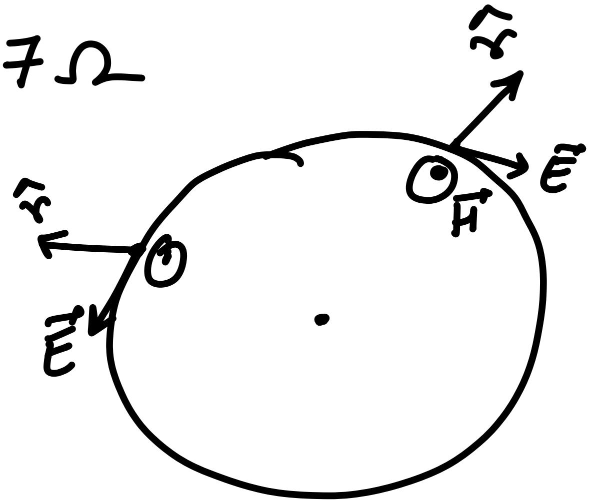
$\frac{1}{r}\,, \quad k\tilde{r}$

$$\vec{E} = -ik\eta \frac{e^{ikr}}{4\pi r} e^{-ik\vec{r}'\cdot\hat{r}} (\vec{dl} \times \hat{r}) \times \hat{r}$$

$$\vec{H} = -ik \frac{e^{ikr}}{4\pi r} e^{-ik\vec{r}'\cdot\hat{r}} (\vec{dl} \times \hat{r})$$

$$\eta = \frac{|\vec{E}|}{|\vec{H}|} = \sqrt{\frac{\mu}{\epsilon}}$$ → Impedance of the medium.

In free space $\eta_0 \simeq 377\,\Omega$

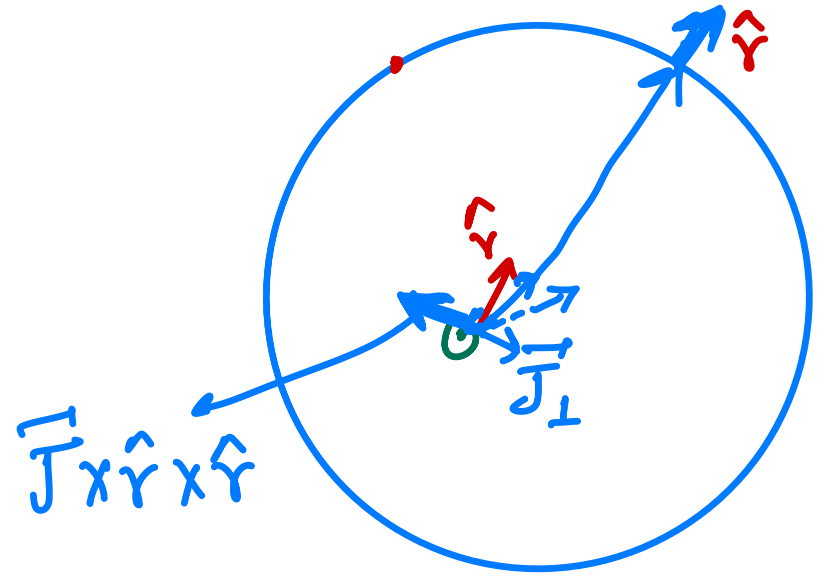


$$\underbrace{\vec{E} \perp \vec{H} \perp \hat{r}}_{\text{TEM wave.}}$$

Generalizing to an arbitrary current distribution (Far-Field)

$$\vec{E}(\vec{r}) \simeq -ik\eta \frac{e^{ikr}}{4\pi r} \int_V \left[\left(\vec{J}(\vec{r}') \times \hat{r}\right) \times \hat{r}\right] e^{-ik\vec{r}'\cdot\hat{r}}\, dv'$$

$$\vec{H}(\vec{r}) \simeq -ik \frac{e^{ikr}}{4\pi r} \int_V \left[\vec{J}(\vec{r}') \times \hat{r}\right] e^{-ik\vec{r}'\cdot\hat{r}}\, dv'$$



$$\iiint_{xyz} f(x,y,z)\, e^{ik_x x} e^{ik_y y} e^{ik_z z}\, dx\,dy\,dz$$

$$= \int_V f(\vec{r})\, e^{i\vec{K}\cdot\vec{r}}\, dv$$

$$\vec{K} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$$

$$\underbrace{\int_V (-\vec{J}_\perp)\, e^{i\vec{k}\cdot\vec{r}'}}_{\text{3D F.T of the source currents!}}$$

$$\vec{K} = K\hat{r}$$