EM06 - EM Potentials

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$\rightarrow$ HME

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \quad \Rightarrow \quad \vec{B} = \nabla \times \vec{A}$$

Magnetic Vector Potential.

$$\nabla \times \vec{E} + \frac{\partial}{\partial t}(\nabla \times \vec{A}) = 0$$

$$\Rightarrow \nabla \times \Big(\underbrace{\vec{E} + \frac{\partial \vec{A}}{\partial t}}_{-\nabla \phi} \Big) = 0 \qquad \nabla \times \nabla \phi = 0$$

$-\nabla \phi \rightarrow$ electric scalar potential.

$$\boxed{\begin{aligned} \vec{E} &= -\nabla \phi - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \nabla \times \vec{A} \\ \vec{H} &= \frac{1}{\mu}(\nabla \times \vec{A}) \\ \vec{D} &= \epsilon \left(-\nabla \phi - \frac{\partial \vec{A}}{\partial t}\right) \end{aligned}}$$

What equations do the potentials satisfy?

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\frac{\nabla \times \nabla \times \vec{A}}{\mu} = \epsilon \frac{\partial}{\partial t}\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t}\right) + \vec{J}$$

$$\nabla \nabla . \vec{A} - \nabla^2 \vec{A} + \mu\epsilon \nabla \frac{\partial \phi}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t} = \mu \vec{J}$$

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 A}{\partial t^2} - \nabla\left(\nabla . \vec{A} + \mu\epsilon \frac{\partial \phi}{\partial t}\right) = -\mu \vec{J}$$

Have we uniquely defined $\vec{A}$ & ϕ?

$\vec{B} = \nabla \times \vec{A}$; $\vec{A}' = \vec{A} + \nabla \psi$

$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$; $\phi' = \phi - \frac{\partial \psi}{\partial t}$

$$\vec{B} = \nabla \times \vec{A}' = \nabla \times \vec{A} + \underbrace{\cancel{\nabla \times \nabla \psi}}_{0}$$

$$\vec{E} = -\nabla\left(\phi - \frac{\partial \psi}{\partial t}\right) - \frac{\partial}{\partial t}\left(\vec{A} + \nabla \psi\right)$$

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} + \cancel{\frac{\partial}{\partial t}\nabla\psi} - \cancel{\frac{\partial}{\partial t}\nabla\psi}$$

$$\boxed{\nabla\cdot\vec{A} = -\mu\epsilon\frac{\partial\phi}{\partial t}}$$ Lorentz Gauge!

$$\boxed{\nabla^2\vec{A} - \mu\epsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J}}$$

Vector wave Eqn.

In Cartesian it seperates in 3 scalar wave eqns.

$$\nabla\cdot\vec{D} = \rho$$

$$\nabla\cdot\left(-\epsilon\left(\nabla\phi + \frac{\partial\vec{A}}{\partial t}\right)\right) = \rho$$

$$\Rightarrow \nabla^2\phi + \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = \frac{-\rho}{\epsilon}$$

$$\Rightarrow \boxed{\nabla^2\phi - \mu\epsilon\frac{\partial^2\phi}{\partial t^2} = \frac{-\rho}{\epsilon}}$$

Scalar wave Eqn.

Strategy $\vec{J}, \rho \rightarrow \vec{A}, \phi \rightarrow \vec{E}, \vec{H}, \vec{D}, \vec{B}$

$$\left.\begin{aligned} \vec{A}' &\longrightarrow \vec{A} + \nabla\psi \\ \phi' &\longrightarrow \phi - \frac{\partial\psi}{\partial t} \end{aligned}\right\}$$ Gauge Invariance!

+

Noether's Theorem

= Conservation of charge!!

Potential for Magnetic Sources

$$\left.\begin{aligned}
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} - \vec{J}_m \\
\nabla \times \vec{H} &= \frac{\partial \vec{D}}{\partial t} \\
\nabla \cdot \vec{D} &= 0 \\
\nabla \cdot \vec{B} &= \rho_m
\end{aligned}\right\}
\qquad
\begin{aligned}
\vec{D} &= \nabla \times \vec{A}_m \\
\vec{H} &= -\nabla \phi_m - \frac{\partial \vec{A}_m}{\partial t}
\end{aligned}$$

electric vector potential

$$\nabla^2 \vec{A}_m - \mu \epsilon \frac{\partial^2 \vec{A}_m}{\partial t^2} = -\epsilon \vec{J}_m$$

mag. scalar potential.

$$\nabla^2 \phi_m - \mu \epsilon \frac{\partial^2 \phi_m}{\partial t} = \frac{-\rho_m}{\mu}$$

$$\boxed{\begin{aligned}
\vec{E} &= -\nabla \phi - \frac{\partial \vec{A}}{\partial t} - \frac{1}{\epsilon} \nabla \times \vec{A}_m \\
\vec{H} &= \frac{1}{\mu} \nabla \times \vec{A} - \frac{\partial \vec{A}_m}{\partial t} - \nabla \phi_m
\end{aligned}}$$

Hertz Vector Potential

$$\vec{A} = \mu\epsilon \frac{\partial \vec{\pi}}{\partial t} \qquad \phi = -\nabla . \vec{\pi}$$

Hertz Electric Vec. Pot.

$$\vec{E} = \nabla\nabla . \vec{\pi} - \mu\epsilon \frac{\partial^2 \vec{\pi}}{\partial t^2}$$

$$\vec{H} = \epsilon \nabla \times \frac{\partial \vec{\pi}}{\partial t}$$

$$\nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} + \vec{J}$$

$$\epsilon \frac{\partial}{\partial t}\left(\nabla \times \nabla \times \vec{\pi}\right) = \epsilon \frac{\partial}{\partial t}\left(\nabla\nabla . \vec{\pi} - \mu\epsilon \frac{\partial^2 \vec{\pi}}{\partial t^2}\right) + \vec{J}$$

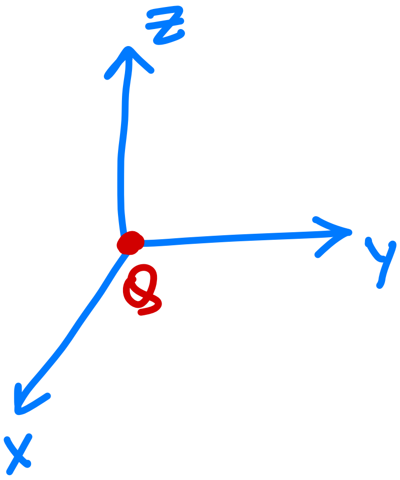
$$\Rightarrow \frac{\partial}{\partial t}\left(\underbrace{\nabla \times \nabla \times \vec{\pi} - \nabla\nabla . \vec{\pi}}_{-\nabla^2 \vec{\pi}} + \mu\epsilon \frac{\partial^2 \vec{\pi}}{\partial t^2}\right) = \frac{\vec{J}}{\epsilon}$$

$$\Rightarrow \boxed{\nabla^2 \vec{\pi} - \mu\epsilon \frac{\partial^2 \vec{\pi}}{\partial t^2} = -\frac{1}{\epsilon} \int \vec{J}\, dt}$$

$$\nabla^2 \vec{\pi}_m - \mu\epsilon \frac{\partial^2 \vec{\pi}_m}{\partial t^2} = -\frac{1}{\mu} \int \vec{J}_m\, dt$$

$$\boxed{\begin{aligned}\vec{E} &= \nabla\nabla\cdot\vec{\Pi} - \mu\epsilon\frac{\partial^2\vec{\Pi}}{\partial t^2} - \mu\nabla\times\frac{\partial\vec{\Pi}_m}{\partial t}\\ \vec{H} &= \nabla\nabla\cdot\vec{\Pi}_m - \mu\epsilon\frac{\partial^2\vec{\Pi}_m}{\partial t^2} + \epsilon\nabla\times\frac{\partial\vec{\Pi}}{\partial t}\end{aligned}}$$

Solution of the Wave equation (Heuristic)



$$\nabla^2\Phi - \mu\epsilon\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\epsilon}$$

$$\rho(\vec{r}) = Q\,\delta(\vec{r})$$

$$\hookrightarrow \delta(x)\delta(y)\delta(z)$$

$$\frac{1}{\rho}\delta(\rho)\delta(\phi)\delta(z)$$

$$\frac{1}{r^2\sin\theta}\delta(r)\delta(\phi)\delta(\theta)$$

$$\Phi(\vec{r}) = \Phi(r) \quad ; \quad \frac{\partial\Phi}{\partial\theta} = \frac{\partial\Phi}{\partial\phi} = 0$$

$$\Rightarrow \nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)$$

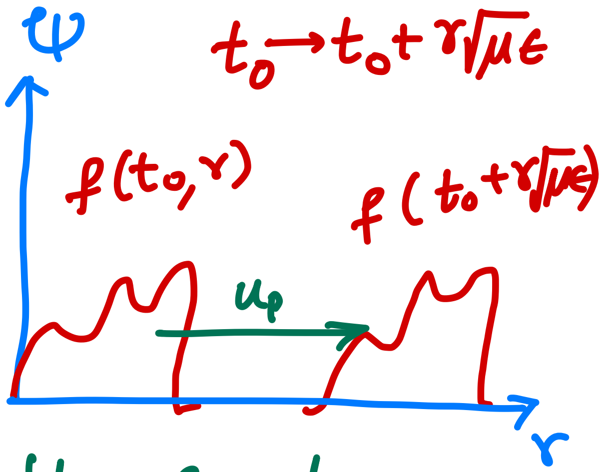
$$\psi(r) = r\,\phi(r)$$

$$\nabla^2 \phi = \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2}$$

$$\frac{\partial^2 \psi}{\partial r^2} - \mu\epsilon \frac{\partial^2 \psi}{\partial t^2} = -\frac{Q}{\epsilon} r\, \delta(\vec{r})$$

At $r \neq 0$, $RHS = 0$

$$\Rightarrow \boxed{\frac{\partial^2 \psi}{\partial r^2} - \mu\epsilon \frac{\partial^2 \psi}{\partial t^2} = 0}$$



$$\psi = f(t - r\sqrt{\mu\epsilon})$$

$$\Rightarrow \phi = \frac{f(t - r\sqrt{\mu\epsilon})}{r}$$

U_p = speed wave.

$$U_p = \frac{1}{\sqrt{\mu\epsilon}}$$

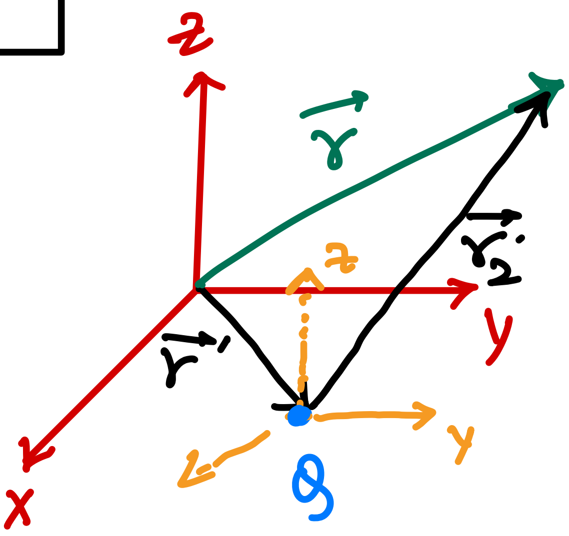
What is f?

We rely on Coulomb's Law in quasistatics.

$$\phi = \frac{Q(t)}{4\pi\epsilon r}$$

$$f(t) = \frac{Q(t)}{4\pi\epsilon}$$

$$\boxed{\phi(r,t) = \frac{Q\left(t - \frac{r}{u_p}\right)}{4\pi\epsilon r}}$$



$$\vec{r}_2 = \vec{r} - \vec{r}\,'$$

$$r = |\vec{r} - \vec{r}\,'|$$

$$\Phi(\vec{r}, \vec{r}\,', t) = \frac{Q\left(t - \frac{|\vec{r} - \vec{r}\,'|}{u_p}\right)}{4\pi\epsilon\, |\vec{r} - \vec{r}\,'|}$$

$$\boxed{\Phi(\vec{r}, t) = \frac{1}{4\pi\epsilon} \iiint_V \frac{\rho_v\left(\vec{r}\,', t - \frac{|\vec{r} - \vec{r}\,'|}{u_p}\right)}{|\vec{r} - \vec{r}\,'|}\, d\vec{r}\,'}$$

$$\boxed{\vec{A}(\vec{r}, t) = \frac{\mu}{4\pi} \iiint_V \frac{\vec{J}\left(\vec{r}\,', t - \frac{|\vec{r} - \vec{r}\,'|}{u_p}\right)}{|\vec{r} - \vec{r}\,'|}\, d\vec{r}\,'}$$