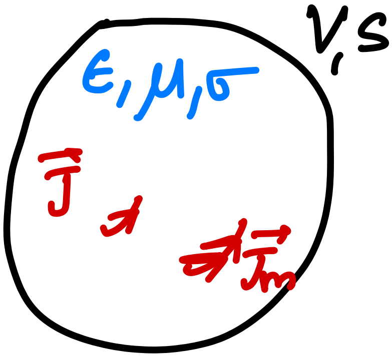
EM05 - EM Theorems

Poynting Theorem (Power Flow)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{J}_m \quad \text{FL}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \sigma \vec{E} + \vec{J} \quad \text{MAL}$$



$\vec{E}.(\text{AL}) - \vec{H}.(\text{FL})$

$$\Rightarrow \underbrace{\vec{E}.\nabla \times \vec{H} - \vec{H}.\nabla \times \vec{E}}_{-\nabla.(\vec{E}\times\vec{H})} = \vec{E}.\frac{\partial \vec{D}}{\partial t} + \sigma \vec{E}.\vec{E} + \vec{E}.\vec{J} + \vec{H}.\frac{\partial \vec{B}}{\partial t} + \vec{H}.\vec{J}_m$$

$$\vec{E}.\frac{\partial \vec{D}}{\partial t} = \epsilon \vec{E}.\frac{\partial \vec{E}}{\partial t} = \frac{\epsilon}{2}\frac{\partial}{\partial t}(\vec{E}.\vec{E})$$

$$\Rightarrow \nabla.(\vec{E}\times\vec{H}) + \frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon|\vec{E}|^2 + \mu|\vec{H}|^2\right) + \sigma|\vec{E}|^2$$

$$= -\vec{J}.\vec{E} - \vec{J}_m.\vec{H}$$

$\iiint_V$ on both sides & apply Div Thm

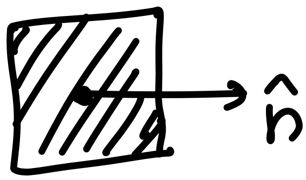
$$\oint\!\!\!\oint_S (\vec{E}\times\vec{H})\cdot d\vec{S} + \frac{\partial}{\partial t}\iiint_V \Big(\overbrace{\frac{\epsilon}{2}|E|^2}^{\frac{1}{2}CV^2} + \overbrace{\frac{\mu}{2}|H|^2}^{\frac{1}{2}LI^2}\Big)dV$$

$$+ \underbrace{\iiint_V \sigma |E|^2 dV}_{\frac{V^2}{R}} = \underbrace{-\iiint_V \vec{J}\cdot\vec{E}\, dV}_{\langle V, I\rangle} - \underbrace{\iiint_V \vec{J}_m\cdot\vec{H}\, dV}_{\langle V, I\rangle}$$

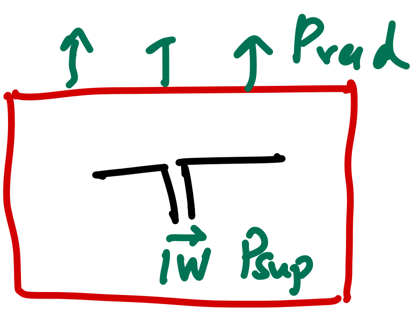
$$\boxed{\vec{S} = \vec{E}\times\vec{H}}$$

Power flow per unit area along $\hat{n}$.

Poynting Vector!

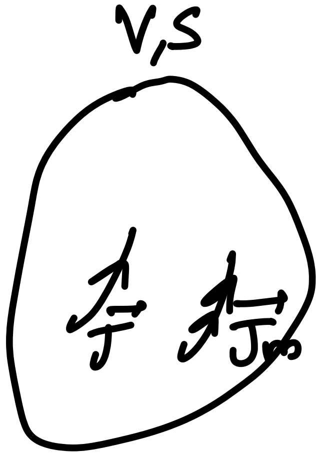


$$P_{rad} = \oint\!\!\!\oint_S (\vec{E}\times\vec{H})\cdot d\vec{S}$$



$$\eta_{ant} = \frac{P_{rad}}{P_{sup}}$$

Uniqueness Theorem



Given a region $\{V, S\}$ with sources $\vec{J}, \vec{J}_m$, the solution to ME is unique in V under the following conditions.

IC $\{$ i) $\vec{E}$ & $\vec{H}$ are unique known everywhere in V at some $t=0$.

BC $\{$ ii) $\vec{E}_{tan}$ or $\vec{H}_{tan}$ are unique known on the surface S at any time t.

Proof: Consider two sets of solutions to Maxwell's Eqns $\{\vec{E}_1, \vec{H}_1\}$ & $\{\vec{E}_2, \vec{H}_2\}$

$$\tilde{E} = \vec{E}_1 - \vec{E}_2 \; ; \quad \tilde{H} = \vec{H}_1 - \vec{H}_2$$

$$\Rightarrow \quad \nabla \times \tilde{E} = -\mu \frac{\partial \tilde{H}}{\partial t}$$

$$\nabla \times \tilde{H} = \epsilon \frac{\partial \tilde{E}}{\partial t} + \sigma \tilde{E}$$

Apply Poynting Thm,

$$\frac{\partial}{\partial t}\int_v \overbrace{\left(\frac{1}{2}\epsilon|\tilde{E}|^2+\frac{1}{2}\mu|\tilde{H}|^2\right)}^{\text{Arg}} dv$$

$$= -\int_v \sigma|\tilde{E}|^2 dv - \underbrace{\oint_S (\tilde{E}\times\tilde{H})\cdot\vec{ds}}_{0}$$

E_{tan} or H_{tan} are known or unique.

$\Rightarrow$ $\tilde{E}_{tan}=0$ or $\tilde{H}_{tan}=0$

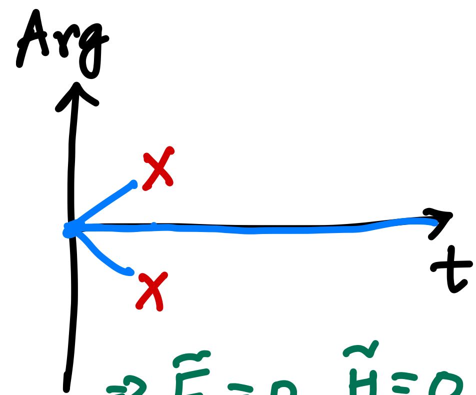
RHS ≤ 0

At $t=0$, $\tilde{E}=0$ & $\tilde{H}=0$

Arg $=0$ at $t=0$

Arg ≥ 0 at $t\geq 0$

$\frac{\partial}{\partial t}(\text{Arg}) \leq 0$ at $t \geq 0$

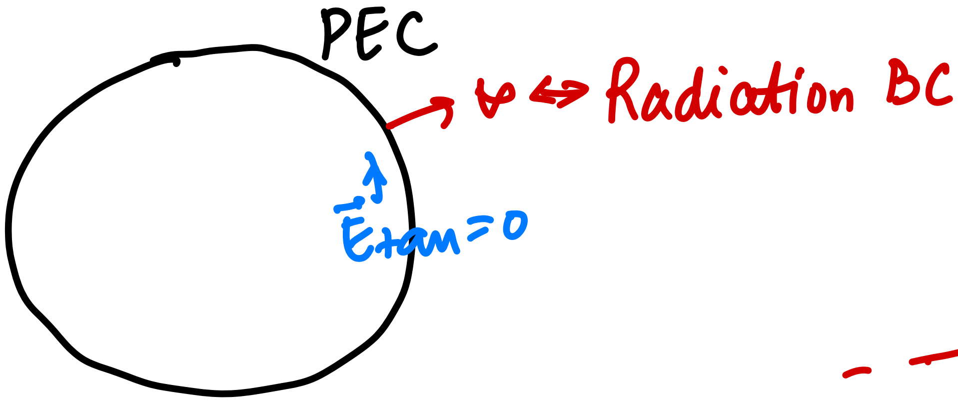


$\Rightarrow \tilde{E}=0, \tilde{H}=0$ $\forall t$

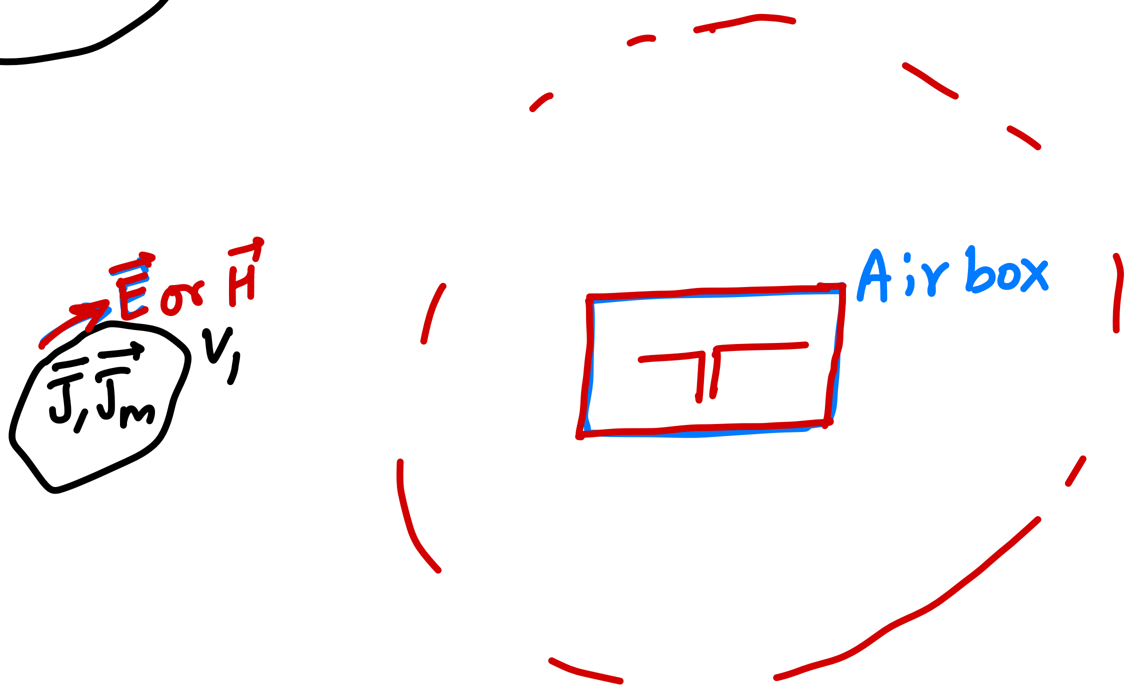
$\blacksquare$

Implications

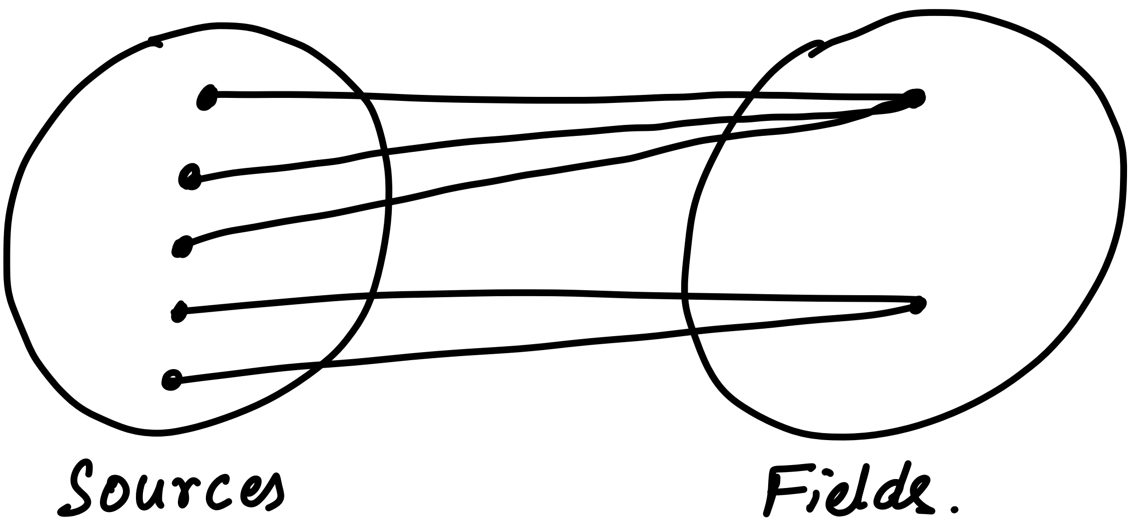
1)

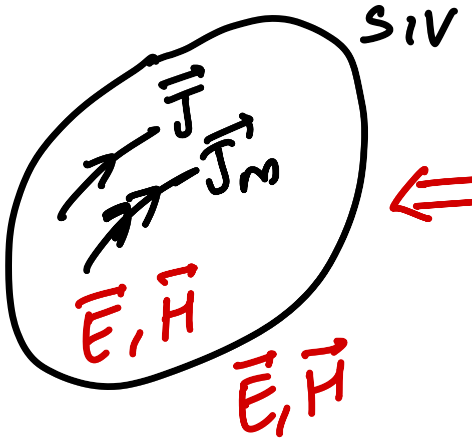


2)

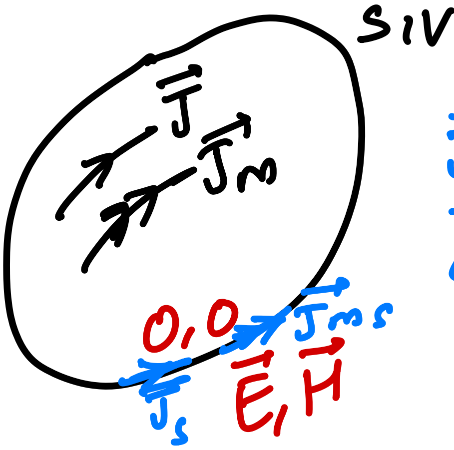


Corollary: Field Equivalence Principle





$\Longleftrightarrow$

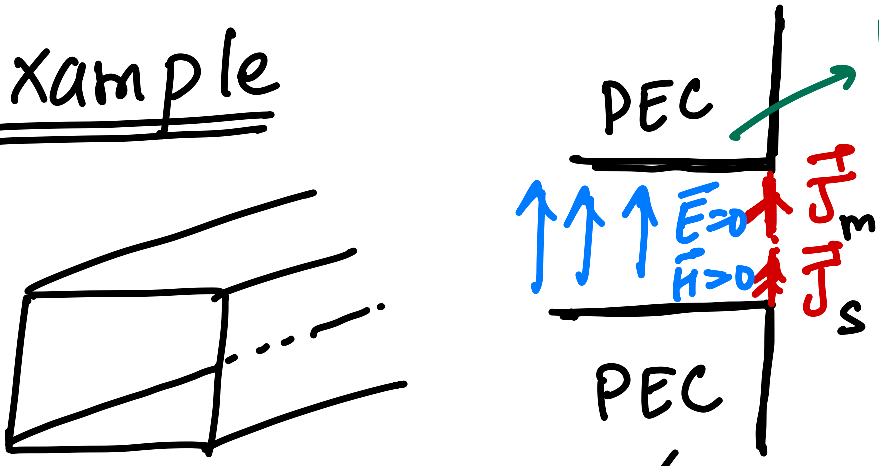


$$\vec{J}_{ms} = -\hat{n} \times \vec{E}$$
$$\vec{J}_s = \hat{n} \times \vec{H}$$

$\Longleftrightarrow$

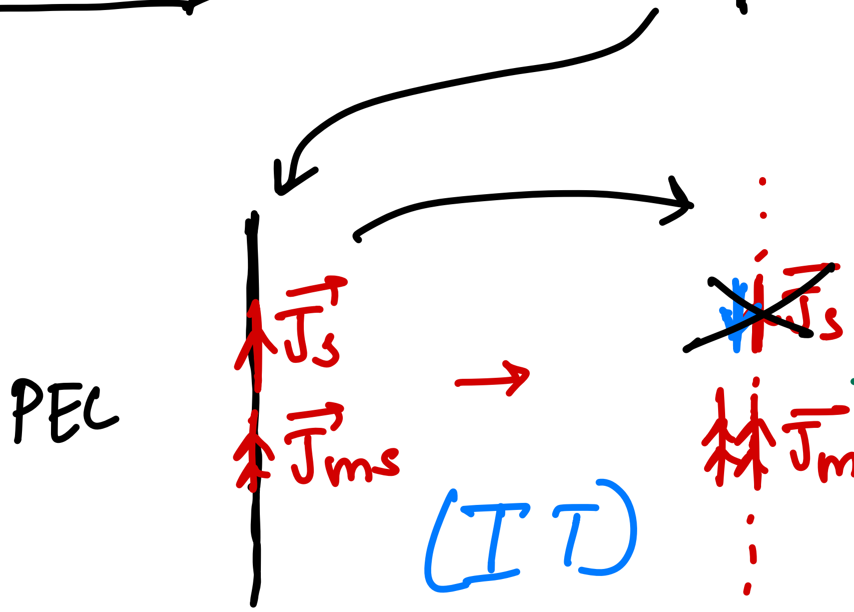
$$\vec{J}_{ms} = -\hat{n}(\vec{E} - \vec{E}')$$
$$\vec{J}_s = \hat{n}(\vec{H} - \vec{H}')$$

Example



(FEP)

$$\vec{J}_{ms} = -\hat{n} \times \vec{E}$$
$$\vec{J}_s = \hat{n} \times \vec{H}$$



$$\vec{J}_{ms} = -2(\hat{n} \times \vec{E})$$

Green's Fns.

(IT)