EM03 - Dielectric Models

Material Dispersion

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

A better model of χ_e is to say the response is that of a linear system.

$$\epsilon_0 \vec{E} \longrightarrow \boxed{\chi_e} \longrightarrow \vec{P}$$

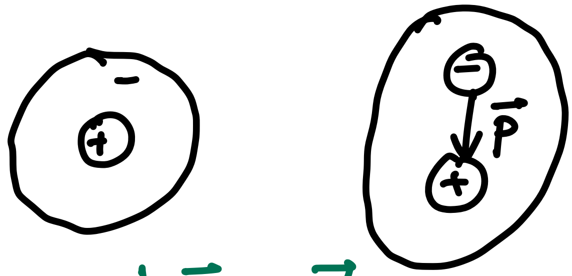
$$\vec{P} = \chi_e * \epsilon_0 \vec{E}$$

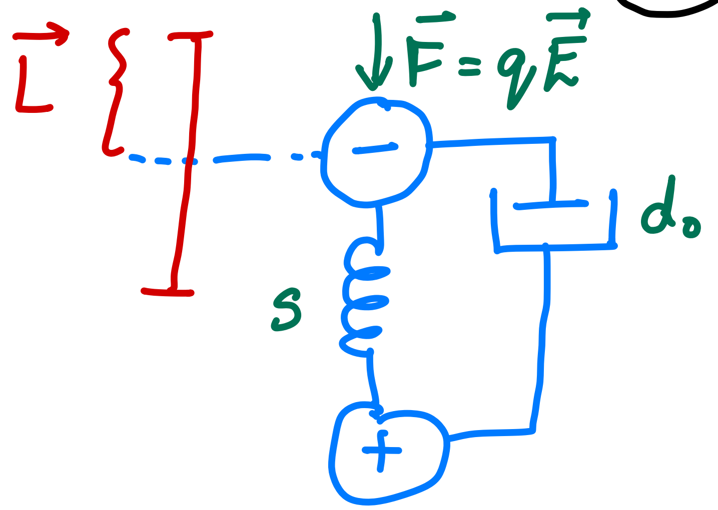
$$\vec{P}(t) = \epsilon_0 \int_{-\infty}^{t} \chi_e(t-\tau)\, \vec{E}(\tau)\, d\tau$$

$$\boxed{\vec{P}(\omega) = \epsilon_0 \underbrace{\chi_e(\omega)}_{?} \vec{E}(\omega)}$$

$$\boxed{\vec{M}(\omega) = \mu_0 \chi_m(\omega) \vec{H}(\omega)}$$

Lorentz Model





m^*

$$m\frac{d^2\vec{L}}{dt^2} + d_o\frac{d\vec{L}}{dt} + s\vec{L} = q\vec{E}$$

$$\vec{E}(t) = \mathrm{Re}\left\{\vec{\tilde{E}}\, e^{-i\omega t}\right\}$$

$$\vec{L}(t) = \mathrm{Re}\left\{\vec{\tilde{L}}\, e^{-i\omega t}\right\}$$

$$\vec{\tilde{L}} = \frac{\left(\frac{q}{m}\right)\vec{\tilde{E}}}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

resonant freq

$$\omega_0^2 = \frac{s}{m}$$

damping factor

$$\gamma = \frac{d_o}{m}$$

$$\vec{\tilde{P}} = Nq\vec{\tilde{L}} = \underbrace{\frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}}_{\chi_e}\epsilon_0\vec{\tilde{E}}$$

plasma freq.

$$\boxed{\omega_p^2 = \frac{Nq^2}{m\epsilon_0}}$$

$$\boxed{\chi_e = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}}$$

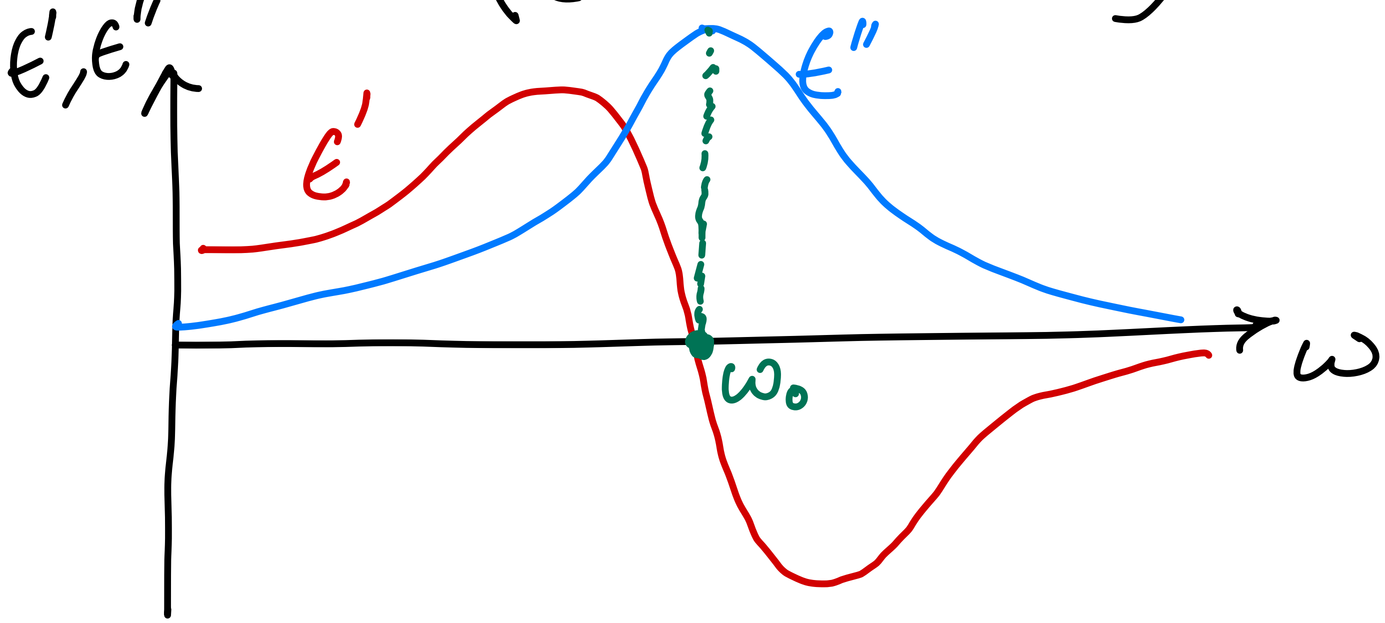
$$\epsilon = \epsilon_0 (1 + \chi_e)$$

$$= \epsilon_0 \left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega} \right)$$

$$\epsilon = \epsilon' + i\epsilon''$$

$$\epsilon' = \epsilon_0 \left(1 + \frac{\omega_p^2 (\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \right)$$

$$\epsilon'' = \epsilon_0 \left(\frac{\omega_p^2 \gamma \omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} \right)$$



Approximations

$\omega \ll \omega_0$

$$\epsilon' \approx \epsilon_0 \left(1 + \frac{\omega_p^2}{\omega_0^2} \right)$$ → independent of freq.

$$\epsilon'' \approx \epsilon_0 \left(\frac{\gamma \omega_p^2 \omega}{\omega_0^4} \right)$$ → linear with freq.

Loss tangent: $\frac{\epsilon''}{\epsilon'}$

$\omega \gg \omega_0$

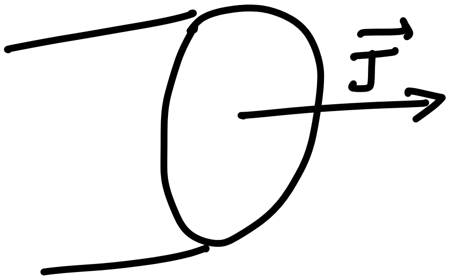
$$\epsilon' \approx \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2} \right) \qquad \epsilon'' \approx \epsilon_0 \left(\frac{\gamma \omega_p^2}{\omega^3} \right)$$

As $\omega \to \infty$, $\epsilon' = \epsilon_0$; $\epsilon'' = 0$

$\Rightarrow$ Material disappears!

"Ultraviolet transparency."

Conducting Media (Drude Model)



$$\vec J = Nq\,\frac{d\vec L}{dt}$$

$$= Nq\,\frac{d}{dt}\left(\frac{q\vec E}{\omega_0^2-\omega^2-i\omega\gamma}\cdot\frac{1}{m}\right)$$

$s = 0$

$\Rightarrow \underbrace{\omega_0 = 0}_{\text{Drude model.}}$

$$\tilde{\vec J} = \frac{-i\omega q^2 N/m}{\cancel{\omega_0^2}^{\,0}-\omega^2-i\omega\gamma}\,\tilde{\vec E}$$

$$\tilde{\vec J} = \frac{Nq^2}{\underbrace{m(\gamma-i\omega)}_{\sigma}}\,\tilde{\vec E}$$

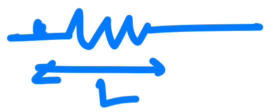
At $\omega \ll \gamma$

Conductivity

$$\tilde{\vec J} \simeq \frac{Nq^2}{\underbrace{m\gamma}_{\sigma}}\,\vec E \;\Rightarrow\; \boxed{\vec J = \sigma \vec E}$$

Ohm's Law.

$\rho = \frac{1}{\sigma}$ → resistivity



$$\int \vec E\cdot d\vec l = \int \rho\,\vec J\cdot d\vec l$$

$$V = \rho\int \frac{I}{A}\,\hat n\cdot dl\,\hat n = \frac{\rho I}{A}L = \underline{IR}$$

Relationship b/w σ & ϵ

$$\epsilon = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2 + i\gamma\omega}\right)$$ when $\omega_0 \to 0$

$$\Rightarrow \quad \epsilon_r' = 1 - \frac{\omega_p^2}{\omega^2+\gamma^2} \qquad \epsilon_r'' = \frac{\omega_p^2 \gamma}{\omega(\omega^2+\gamma^2)}$$

When $\omega \ll \gamma$

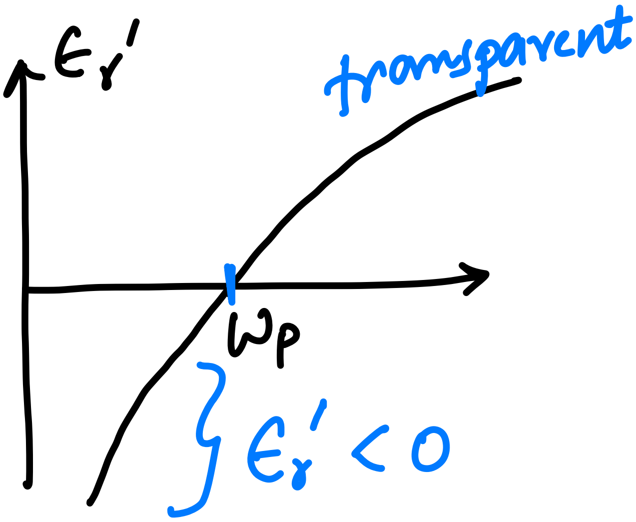
$$\epsilon_r'' = \frac{Nq^2}{m\epsilon_0 \omega \gamma} = \frac{\sigma}{\omega \epsilon_0}$$

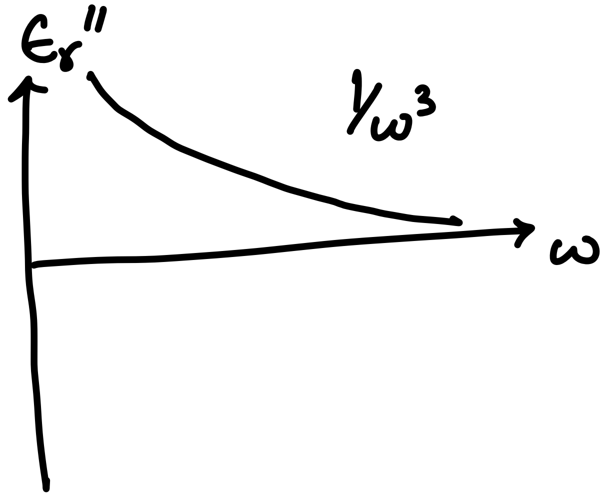
$$\boxed{\epsilon_r'' = \frac{\sigma}{\omega \epsilon_0}}$$

$$\boxed{\epsilon'' = \frac{\sigma}{\omega}}$$

$\omega \gg \gamma$

$$\epsilon_r' = 1 - \frac{\omega_p^2}{\omega^2} \qquad \epsilon_r'' = \frac{\omega_p^2 \gamma}{\omega^3}$$





$\Rightarrow$ reflection.

> ω_p for most metals $\simeq 10^{15} - 10^{16}$ Hz.

$\Rightarrow$ high UV

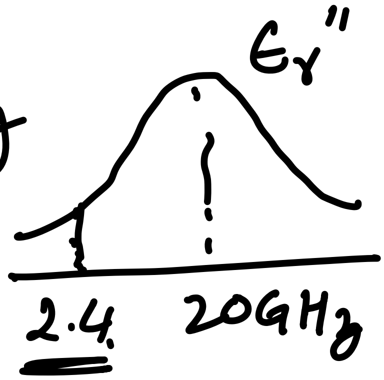
$\Rightarrow$ Al & Silver are reflective.

Cu & Gold? $\rightarrow$ They have resonances around blue-green region of the spectrum.

$\Rightarrow$ yellow-red are reflective.

> Water has a resonance at $\simeq$ 20 GHz.

Microwave oven $\simeq$ 2.4 GHz



> $O_2 \rightarrow \underline{60\,GHz}$. {Secure comm.}

> Radiative cooling.

$\vec{J} = \sigma \vec{E}$ PEC $\sigma = \infty \Rightarrow \rho = 0$

$$\Rightarrow \vec{E} = \vec{J}\rho = 0.$$

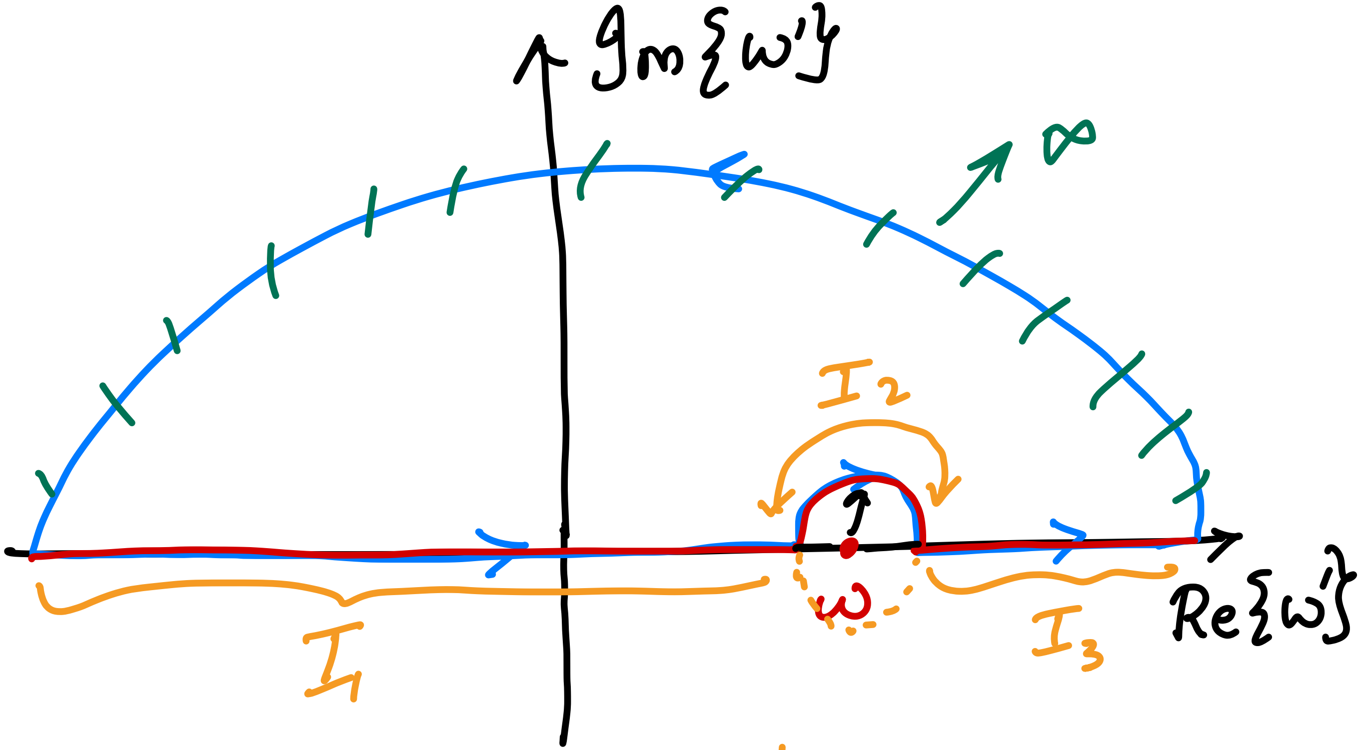
$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = \nabla \cdot \sigma \vec{E} = \sigma \, \nabla \cdot \vec{E} = \frac{\sigma \rho}{\epsilon}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\sigma \rho}{\epsilon} = 0 \Rightarrow \boxed{\rho(t) = \rho_0 e^{-\frac{\sigma}{\epsilon} t}.}$$

Kramers Kronig Relations

> Causality, Stability, linearity.

"$\chi_e(\omega)$ is "analytic" in the upper half complex ω plane"



$$\oint_C \frac{\chi_e(\omega')}{\omega'-\omega}\,d\omega' = 0$$

$$\int_C \frac{\chi_e(\omega')}{\omega'-\omega}\,d\omega' = 0$$

$$I_1 + I_2 + I_3 = 0$$

$$\Rightarrow \underbrace{I_1 + I_3}_{\int_{-\infty}^{\infty} \frac{\chi_e(\omega')}{\omega'-\omega}d\omega'} = -I_2$$

$$I_2 = \frac{1}{2}\{\text{Residue}\}$$

$$\int_{-\infty}^{\infty} \frac{\chi_e(\omega')}{\omega'-\omega}\,d\omega' = \underbrace{i\pi\chi_e(\omega)}_{\text{Residue}/2}$$

$$\chi_e'(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\chi_e''(\omega)}{\omega' - \omega}\, d\omega'$$

$$\chi_e''(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\chi_e'(\omega)}{\omega' - \omega}\, d\omega'$$

Hilbert.

- $\epsilon' \rightarrow$ propagation of phase
  $\epsilon'' \rightarrow$ loss or absorption.

> Phase retrieval!

> KK retrieval!