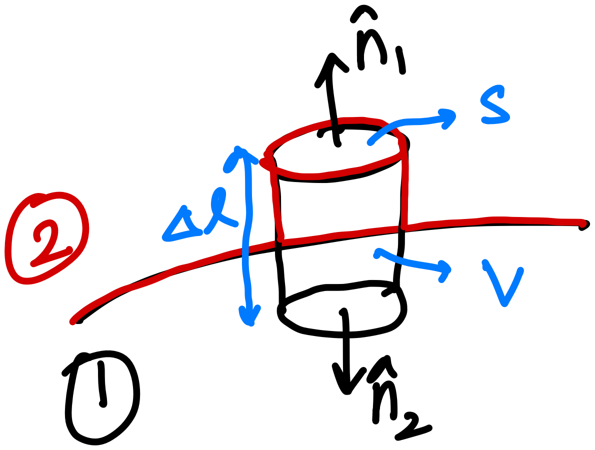
EM02 - Boundary Conditions & Constitutive Relations

Boundary Conditions



$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$\iiint_V dV$ on both sides.

$$\iiint_V \nabla \times \vec{E}\, dV = -\oint\!\!\oint_S \vec{E} \times d\vec{s} = \iiint_V -\frac{\partial \vec{B}}{\partial t} dV$$

Take the limit as $\Delta l \to 0 \Rightarrow V \to 0$

$\Rightarrow$ RHS $= 0$

$$\Rightarrow \oint\!\!\oint_S \vec{E} \times d\vec{s} = 0$$

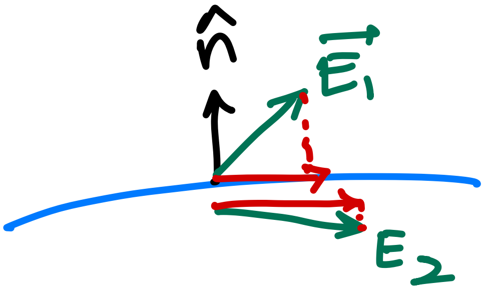
$$\Rightarrow \iint_S (\vec{E}_1 \times \hat{n}_1 + \vec{E}_2 \times \hat{n}_2)\, ds = 0$$

$$\Rightarrow \vec{E}_1 \times \hat{n}_1 + \vec{E}_2 \times \hat{n}_2 = 0$$

$$\Rightarrow \boxed{\hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0}$$

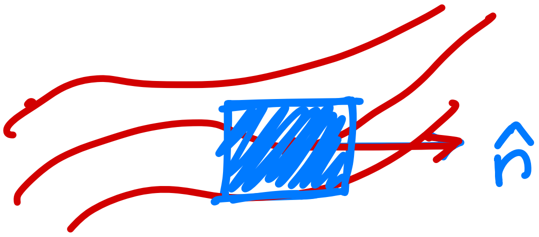
$\hat{n}_1 = -\hat{n}_2 = \hat{n}$

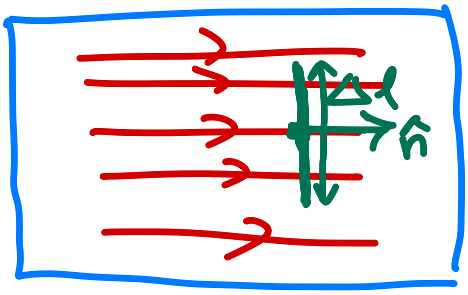
"Tangential $\vec{E}$ must be continuous."

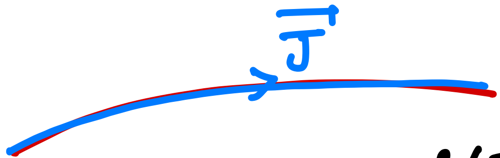


$$\iiint_V \nabla\times\vec{H}\,dV = -\oint_S \vec{H}\times d\vec{s} = \cancel{\iiint_V \frac{\partial \vec{D}}{\partial t}\,dV}^{\,0} + \underbrace{\iiint_V \vec{J}\,dV}_{?}$$

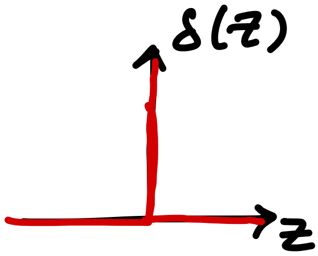
$$\vec{J} = \frac{\Delta I}{\Delta S}\,\hat{n}$$







$$\vec{J}_s = \frac{\Delta I}{\Delta l}\,\hat{n}$$



$$\underbrace{\int_{-\infty}^{\infty} \delta(z)\,dz}_{\frac{1}{m}} = 1$$

$$\vec{J}(x,y,z) = \vec{J}_s(x,y)\,\delta(z)$$

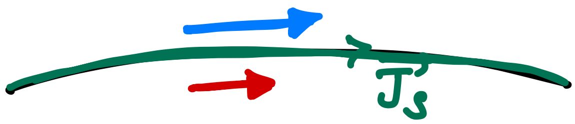
$$\iiint_{xyz} \vec{J}\,dx\,dy\,dz = \iint_{xy} \vec{J}_s(x,y)\underbrace{\left(\int_z \delta(z)\,dz\right)}_{1}dx\,dy$$

$$= \iint_{xy} \vec{J}_s\,dx\,dy$$

$$= \iint_S \vec{J}_s\,ds \;//$$

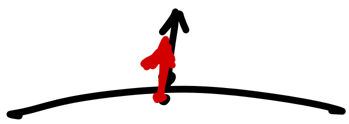
$$\iint_S \left(\hat{n}_1 \times \vec{H}_1 + \hat{n}_2 \times \vec{H}_2\right) ds = \iint_S \vec{J}_s \, ds$$

$$\Rightarrow \boxed{\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s}$$



$$\iiint_V \nabla \cdot \vec{B} = 0$$

$$\Rightarrow \oint\!\!\oint_S \vec{B} \cdot \vec{ds} = 0 \Rightarrow \boxed{\hat{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0}$$



$$\iiint_V \nabla \cdot \vec{D} = \iiint_V \rho \, dV$$

$$\Rightarrow \boxed{\hat{n} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s}$$

net charge.

> Imposing $\vec{E}$ & $\vec{H}$ B.Cs automatically ensures $\vec{D}$ & $\vec{B}$.

> In a Perfect Electric Conductor (PEC), the "time varying" field quantities are all 0.

$$\hat{n} \times (\vec{E}_1 - \cancelto{0}{\vec{E}_2}) = 0$$

$$\Rightarrow \boxed{\hat{n} \times \vec{E} = 0}$$

$$\boxed{\hat{n} \times \vec{H} = \vec{J}_s}$$

$$\boxed{\begin{array}{l} \hat{n} \cdot \vec{B} = 0 \\ \hat{n} \cdot \vec{D} = \rho_s \end{array}}$$

Wave Matter Interactions & Constitutive Relations

In general,

$$\vec{D} = f_D(\vec{E}, \vec{H})$$

$$\vec{B} = f_B(\vec{E}, \vec{H})$$

Assume f_D, f_B are <u>linear</u> functions.

$f(x) = x$, $f(x) = x^2$, $f(x) = x + 1$?

$$f(ax+by) = af(x) + bf(y).$$

Case 1 Free space

$$\vec{D} = \epsilon_0 \vec{E}$$

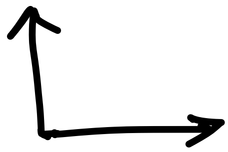
$$\vec{B} = \mu_0 \vec{H}$$

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ F/m}$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m}.$$

Case 2 Isotropic & Homogeneous

Isotropic: Independant of vector direction of $\vec{E}$ & $\vec{H}$.



Homogeneous: Independant of position.

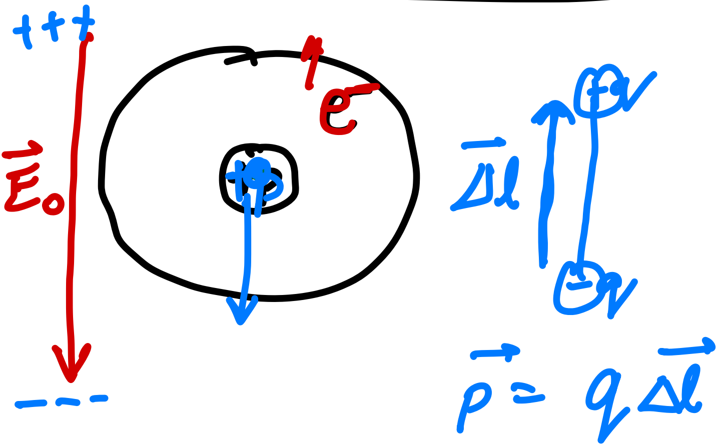
relative permittivity (dielectric constant).

$$\vec{D} = \epsilon \vec{E} = \epsilon_r \epsilon_0 \vec{E}$$

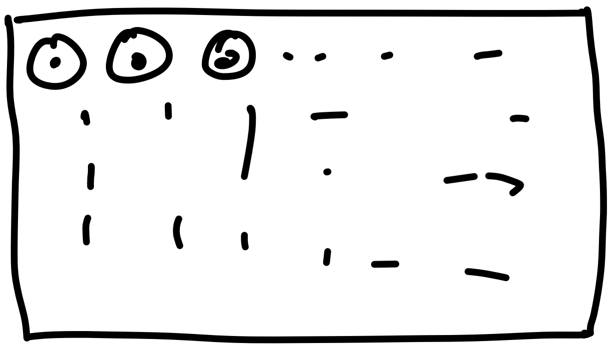
relative permeability.

$$\vec{B} = \mu \vec{H} = \mu_r \mu_0 \vec{H}$$

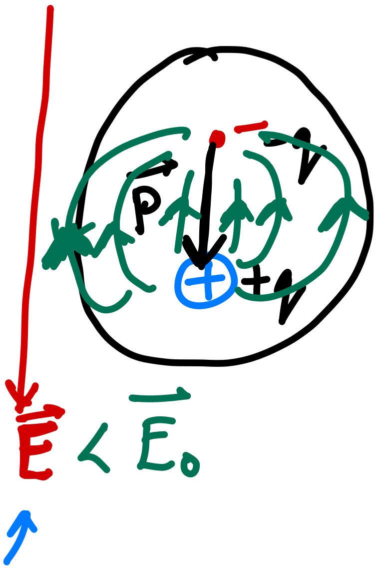
What is ϵ_r & μ_r?



$\vec{p} = q\vec{\Delta l}$



$\Downarrow$



$\vec{p} = q\vec{\Delta l}$

$$\boxed{\vec{P} = Nq\vec{\Delta l}}$$

N → no. of atoms per unit vol.

Polarization vector.

(Dipole moment per unit volume).

$\vec{P} \propto \vec{E}$

$$\boxed{\vec{P} = \epsilon_0 \chi_e \vec{E}}$$

χ_e → electric susceptibility.

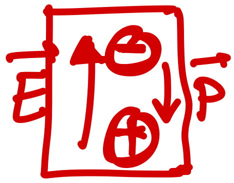
$$\boxed{\vec{D} = \epsilon_0 \vec{E} + \vec{P}}$$

$$= \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon_0 (1+\chi_e)\vec{E}$$

$$\Rightarrow \boxed{\epsilon_r = 1 + \chi_e}$$

Typically $\chi_e > 0 \Rightarrow \epsilon_r > 1$

But, this assumes that the material response is <u>instantaneous</u>.



> In reality χ_e can be engineered or can look like it is negative

$\Rightarrow$ <u>meta materials</u>

Similarly,

magnetic susceptibility

$$\vec{B} = \mu\vec{H} = \mu_r\mu_0\vec{H} = \mu_0(1+\chi_m)\vec{H}$$

$$= \mu_0\vec{H} + \underbrace{\mu_0\chi_m\vec{H}}_{\vec{M}}$$

$\vec{M}$ magnetization vector.

i) $\underline{\mu_r > 1}$: <u>Paramagnetism</u>.

Weak & not permanent

ii) $\underline{\mu_r < 1}$: <u>Diamagnetic</u> (Levitating frog).

Very weak & opposite direction

repelled by permanent magnets.

iii) $\mu_r \gg 1$: **Ferromagnetism**

Strong; nonlinear & hysteris.

Permanent magnets. "sub-domains"

[Anti ferro ($\downarrow\uparrow$), Ferri ($\downarrow\uparrow$), Super para, Ferroele etc.]

E & M

Case 3 Inhomogeneous

$$\vec{D}(\vec{r}) = \epsilon(\vec{r})\,\vec{E}(\vec{r})$$

$$\vec{B}(\vec{r}) = \mu(\vec{r})\,\vec{H}(\vec{r})$$

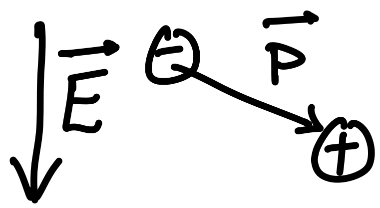
$$\nabla\cdot\vec{D} = \rho$$

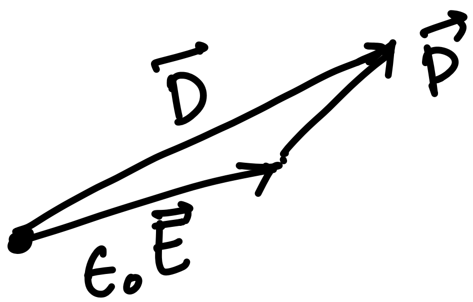
$$\nabla\cdot\epsilon\vec{E} = \rho$$

$$\neq \quad \nabla\cdot\vec{E} = \frac{\rho}{\epsilon}$$

Case 4 Anisotropic

$$\vec{D} = \epsilon_0\vec{E} + \vec{P}$$





$\vec{D}, \vec{E} \rightarrow$ can be related using a matrix?

$$\vec{D} = \bar{\bar{\epsilon}}\,\vec{E}$$

$\bar{\bar{\epsilon}}$ ↳ tensor or matrix.

$$\bar{\bar{\epsilon}} = \epsilon_0 \underbrace{\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}}_{\bar{\bar{\epsilon}}_r}$$

→ These could be dep. on $\vec{r}$.

$$\bar{\bar{\epsilon}}_r = \begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} \rightarrow \text{Biaxial medium.}$$

$$= \begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_x & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} \rightarrow \text{Uniaxial medium.}$$

Case 5 Bianisotropic media

$$\vec{D} = \bar{\bar{\epsilon}}\,\vec{E} + \bar{\bar{\zeta}}\,\vec{H}$$

$\bar{\bar{\zeta}}$ → Zeta → magneto electric tensor

$$\vec{B} = \bar{\bar{\xi}}\,\vec{E} + \bar{\bar{\mu}}\,\vec{H}$$

$\bar{\bar{\xi}}$ ↳ electromagnetic tensor.