

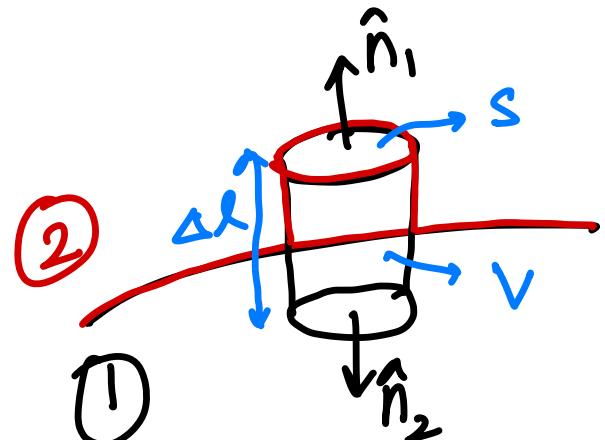


EM02 - Boundary Conditions & Constitutive Relations

Boundary Conditions

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

$$\iiint_V dV \text{ on both sides.}$$



$$\iiint_V \nabla \times \vec{E} dV = - \oint_S \vec{E} \times d\vec{s} = \iiint_V - \frac{\partial \vec{B}}{\partial t} dV$$

Take the limit as $\Delta l \rightarrow 0 \Rightarrow V \rightarrow 0$

$$\Rightarrow \text{RHS} = 0$$

$$\Rightarrow \oint_S \vec{E} \times d\vec{s} = 0$$

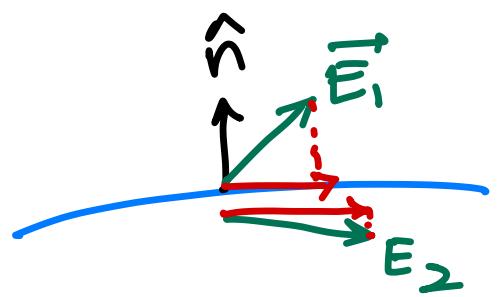
$$\Rightarrow \iint_S (\vec{E}_1 \times \hat{n}_1 + \vec{E}_2 \times \hat{n}_2) dS = 0$$

$$\Rightarrow \vec{E}_1 \times \hat{n}_1 + \vec{E}_2 \times \hat{n}_2 = 0$$

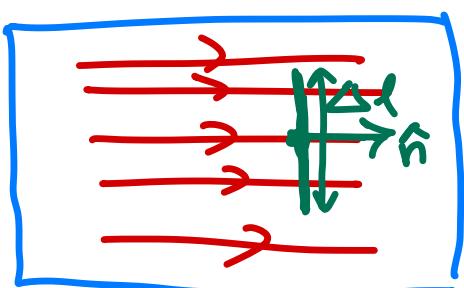
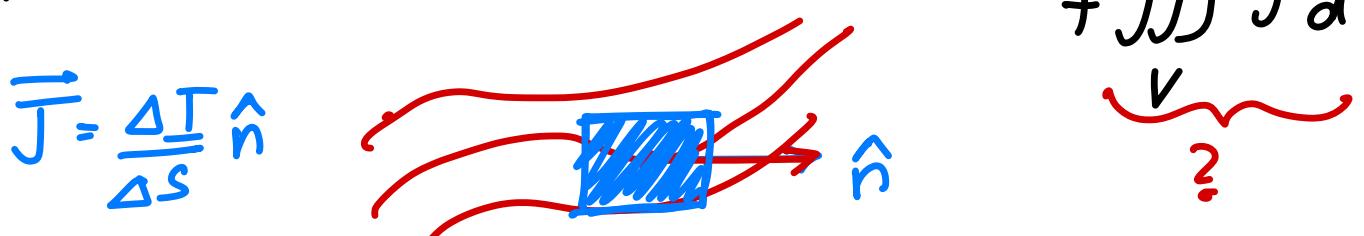
$$\Rightarrow \hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0$$

$$\hat{n}_1 = -\hat{n}_2 = \hat{n}$$

"Tangential \vec{E} must be continuous."



$$\iiint_V \nabla \times \vec{H} dV = - \oint_S \vec{H} \times d\vec{s} = \iiint_V \frac{\partial \vec{D}}{\partial t} dV + \iiint_V \vec{J} dV$$



$$\vec{J}_s = \frac{\Delta I}{\Delta l} \hat{n}$$

$$\vec{J}(x, y, z) = \vec{J}_s(x, y) \delta(z)$$

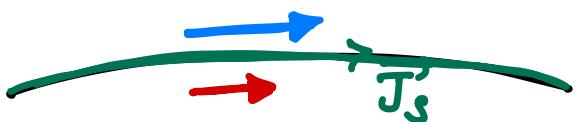
$$\iiint_{xyz} \vec{J} dx dy dz = \iint_{xy} \vec{J}_s(x, y) \underbrace{\left(\int_z \delta(z) dz \right)}_{1/m} dx dy$$

$$= \iint_{xy} \vec{J}_s dx dy$$

$$= \iint_S \vec{J}_s ds_{\parallel}$$

$$\iint_S (\hat{n}_1 \times \vec{H}_1 + \hat{n}_2 \times \vec{H}_2) dS = \iint_S \vec{J}_S dS$$

$$\Rightarrow \hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_S$$



$$\iiint_V \nabla \cdot \vec{B} = 0$$

✓

$$\Rightarrow \oint_S \vec{B} \cdot d\vec{s} = 0 \Rightarrow \hat{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0$$



$$\iiint_V \nabla \cdot \vec{D} = \iiint_V \rho dV$$

$$\Rightarrow \hat{n} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s \quad \text{net charge.}$$

> Imposing $\vec{E} \leftarrow \vec{H}$ B.Cs automatically ensures \vec{D} & \vec{B} .

> In a Perfect Electric Conductor (PEC), the "time varying" field quantities are all 0.

$$\hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0$$

$$\Rightarrow \hat{n} \times \vec{E} = 0$$

$$\hat{n} \times \vec{H} = \vec{J}_s$$

$$\begin{aligned} \hat{n} \cdot \vec{B} &= 0 \\ \hat{n} \cdot \vec{D} &= \rho_s \end{aligned}$$

Wave Matter Interactions & Constitutive Relations

In general,

$$\vec{D} = f_D(\vec{E}, \vec{H})$$

$$\vec{B} = f_B(\vec{E}, \vec{H})$$

Assume f_D, f_B are linear functions.

$$f(x) = x, \quad f(x) = x^2, \quad f(x) = x + 1?$$

$$f(ax+by) = af(x) + bf(y).$$

Case 1 Free space

$$\vec{D} = \epsilon_0 \vec{E}$$

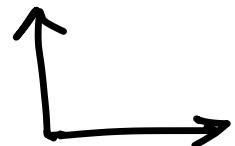
$$\epsilon_0 = 8.85 \times 10^{-12} \text{ F/m}$$

$$\vec{B} = \mu_0 \vec{H}$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m.}$$

Case 2 Isotropic & Homogeneous

Isotropic: Independant of vector direction
of $\frac{\vec{D}}{\vec{E}} \& \vec{H}$.



Homogeneous: Independant of position.

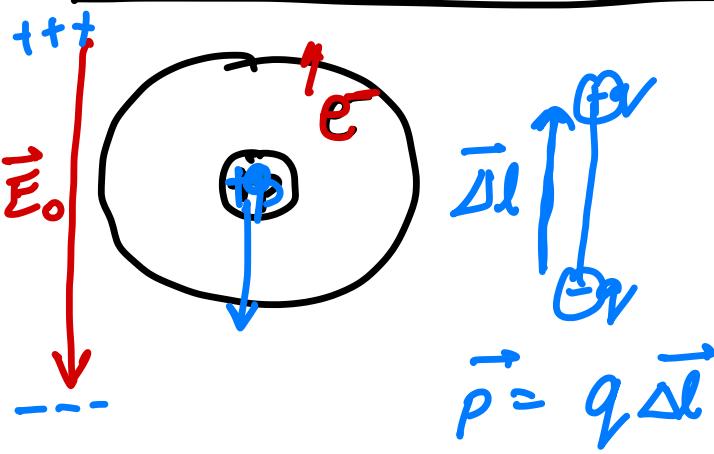
$$\vec{D} = \epsilon \vec{E} = \epsilon_r \epsilon_0 \vec{E}$$

↗ relative permittivity
(dielectric constant).

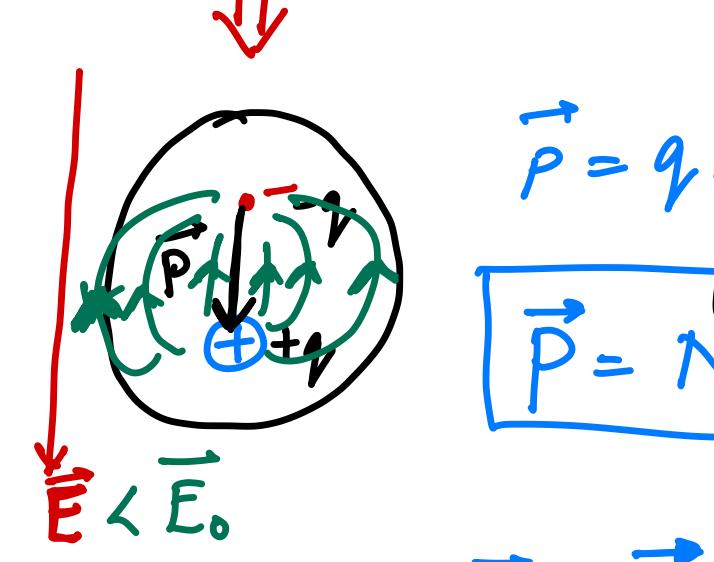
$$\vec{B} = \mu \vec{H} = \mu_r \mu_0 \vec{H}$$

↗ relative permeability.

What is ϵ_r & μ_r ?


 A diagram showing a circular cross-section of a dielectric medium. On the left, a vertical red arrow labeled \vec{E}_0 points downwards. Inside the circle, there are positive charges ($++$) at the top and negative charges ($--$) at the bottom. A central nucleus with a positive charge $+e$ has a negative electron $-e$ orbiting it. A dipole moment vector \vec{p} is shown, pointing from the center to the electron, with a length labeled $q\Delta l$. To the right, a rectangular region contains several atoms with positive nuclei and negative electrons, separated by dashed lines representing lattice sites.

\downarrow


 A diagram showing the same circular cross-section after polarization. The external electric field \vec{E} is now shown, which is less than \vec{E}_0 ($\vec{E} < \vec{E}_0$). The internal dipole moment \vec{p} is now aligned with the external field. Green arrows indicate the movement of electrons towards the positive side of the field, creating a net negative charge on the left surface and a net positive charge on the right surface.

$\vec{P} = q\vec{\Delta l}$ no. of atoms per unit vol.

$\boxed{\vec{P} = Nq\vec{\Delta l}}$ Polarization vector.
 (Dipole moment per unit volume).

$\vec{P} \propto \vec{E}$

$\boxed{\vec{P} = \epsilon_0 \chi_e \vec{E}}$ electric susceptibility.

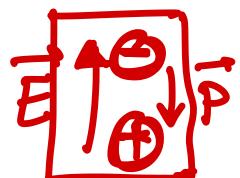
$\boxed{\vec{D} = \epsilon_0 \vec{E} + \vec{P}}$

$= \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon_0 (1 + \chi_e) \vec{E}$

$\Rightarrow \epsilon_r = 1 + \chi_e$

Typically $\chi_e > 0 \Rightarrow \epsilon_r > 1$

But, this assumes that the material response is instantaneous.



> In reality χ_e can be engineered or can look like it is negative

⇒ meta materials

Similarly,

$$\vec{B} = \mu \vec{H} = \mu_r \mu_0 \vec{H} = \mu_0 (1 + \chi_m) \vec{H}$$

↗ magnetic susceptibility.

$$= \mu_0 \vec{H} + \underbrace{\mu_0 \chi_m \vec{H}}_{\vec{M} \text{ magnetization vector.}}$$

i) $\mu_r > 1$: Paramagnetism.

Weak & not permanent

ii) $\mu_r < 1$: Diamagnetic (Levitating frog).
Very weak & opposite direction
repelled by permanent magnets.

iii) $M_r \gg 1$: Ferromagnetism

Strong; nonlinear & hysteresis.

Permanent magnets. "sub-domains"

$\downarrow \uparrow$
 $\downarrow \uparrow$
[Antiferro, Ferri, Super para, Ferroelec etc.]
 $E \propto M$

Case 3 Inhomogeneous

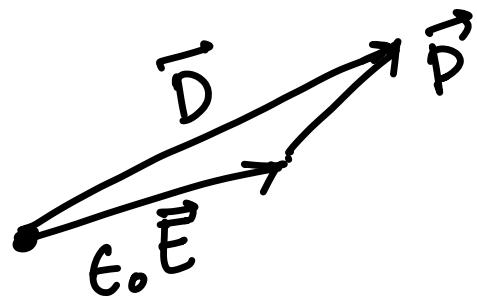
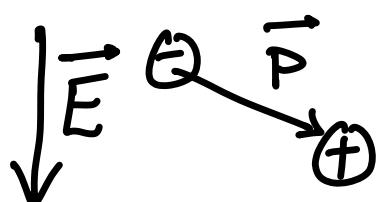
$$\vec{D}(\vec{r}) = \epsilon(\vec{r}) \vec{E}(\vec{r})$$

$$\vec{B}(\vec{r}) = \mu(\vec{r}) \vec{H}(\vec{r})$$

$$\begin{aligned}\nabla \cdot \vec{D} &= \rho \\ \nabla \cdot \epsilon \vec{E} &= \rho \\ \neq \nabla \cdot \vec{B} &= \frac{\rho}{\mu}\end{aligned}$$

Case 4 Anisotropic

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$



$\vec{D}, \vec{E} \rightarrow$ Can be related using a matrix?

$$\vec{D} = \bar{\epsilon} \vec{E}$$

↳ tensor or matrix.

$$\bar{\epsilon} = \epsilon_0 \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

$\bar{\epsilon}_r$

These could be dep. on \vec{r} .

$$\bar{\epsilon}_r = \begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} \rightarrow \text{Biaxial medium.}$$

$$= \begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_x & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} \rightarrow \text{Uniaxial medium.}$$

Case 5 Biamisotropic media

$$\vec{D} = \bar{\epsilon} \vec{E} + \bar{\zeta} \vec{H}$$

Zeta → magneto electric tensor

$$\vec{B} = \bar{\xi} \vec{E} + \bar{\mu} \vec{H}$$

↳ electro magnetic tensor.