Maxwell's Equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ Faraday's Law

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$ Modified Ampere's Law

$$\left.\begin{aligned} \nabla \cdot \vec{D} &= \rho \\ \nabla \cdot \vec{B} &= 0 \end{aligned}\right\}$$ Gauss' Laws of Electricity & Magnetism.

$\vec{E}$ = Electric Field Intensity $\left(\frac{V}{m}\right)$

$\vec{H}$ = Magnetic " " $\left(\frac{A}{m}\right)$

$\vec{D}$ = Electric Flux Density (C/m^2)

$\vec{B}$ = Magnetic Flux Density (Wb/m^2) - T

$\vec{J}$ = (~~Surface~~ Vol.) Current Density (A/m^2)

ρ = Volumetric charge Density (C/m^3)

$$\boxed{\underbrace{\nabla \cdot \nabla \times \vec{H}}_{0} = \nabla \cdot \left(\frac{\partial \vec{D}}{\partial t} + \vec{J}\right)}$$

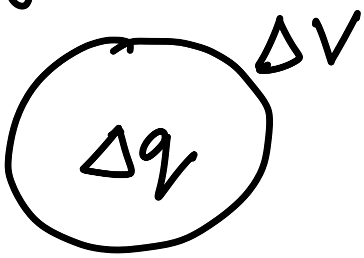
$$\Rightarrow \frac{\partial}{\partial t}(\nabla \cdot \vec{D}) = -\nabla \cdot \vec{J}$$

$$\Rightarrow \boxed{\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t}}$$ Continuity Equation "Conservation of charge".

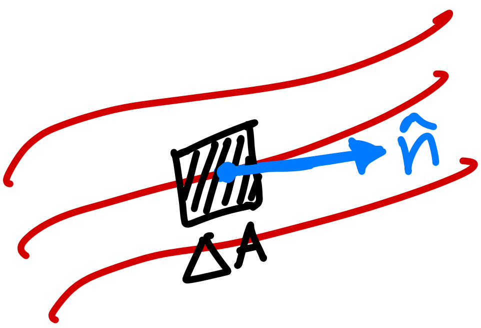
Define $\vec{J}$ & ρ

net charge.

$$\rho = \lim_{\Delta V\to 0}\frac{\Delta q}{\Delta V} = \frac{\partial q}{\partial V}$$



$$\vec{J} = \lim_{\Delta A\to 0}\frac{\Delta I}{\Delta A}\hat{n}$$

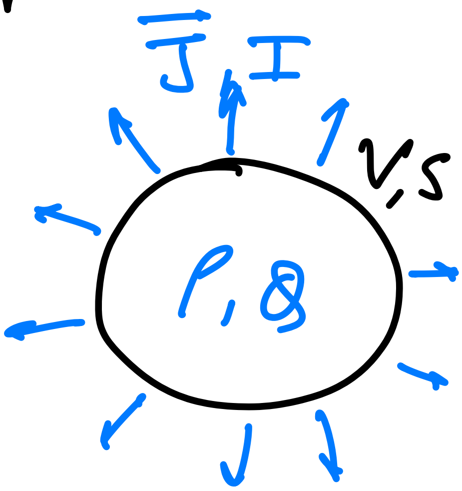


$$\vec{J} = \frac{\partial I}{\partial A}\hat{n}$$

$$I = \oiint_S \vec{J}\cdot\underbrace{\vec{ds}}_{dA\,\hat{n}} \qquad Q = \iiint_V \rho\, dV$$

$$I = -\frac{dQ}{dt} = -\iiint_V \frac{\partial\rho}{\partial t}\, dV$$

$$I = \iiint_V \nabla\cdot\vec{J}\, dV$$



$$\Rightarrow \boxed{\nabla\cdot\vec{J} = -\frac{\partial \rho}{\partial t}} \rightarrow$$ Is this a valid proof? NO!

Real Proof: Requires $\underbrace{\text{Gauge invariance!}}_{\text{(Noether's Theorem)}}$

Maxwell's Equations (Interdependant).

$$\nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{J}$$

$\nabla\cdot$ on both sides

$$0 = \frac{\partial}{\partial t}\nabla\cdot\vec{D} + \underbrace{\nabla\cdot\vec{J}}_{-\frac{\partial\rho}{\partial t}}$$

$$\Rightarrow \frac{\partial}{\partial t}\nabla\cdot\vec{D} = \frac{\partial}{\partial t}\rho$$

$$\Rightarrow \nabla\cdot\vec{D} = \underbrace{\rho}_{0} + \cancelto{0}{c}$$

$$\Rightarrow \boxed{\nabla\cdot\vec{D} = \rho}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\frac{\partial\,\nabla\cdot\vec{B}}{\partial t} = 0$$

$$\Rightarrow \nabla\cdot\vec{B} = c$$

$$\Rightarrow \boxed{\nabla\cdot\vec{B} = 0}$$

Maxwell's Equations in Integral Form

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\Rightarrow \iint_S (\nabla \times \vec{H}) \cdot \vec{ds} = \iint_S \left(\frac{\partial \vec{D}}{\partial t} + \vec{J} \right) \cdot \vec{ds}$$

S is stationary!

$$\Rightarrow \boxed{\oint_C \vec{H} \cdot \vec{dl} = I + \frac{d}{dt} \iint_S \vec{D} \cdot \vec{ds}}$$

Modified Ampere's Law.

$$\boxed{\oint_C \vec{E} \cdot \vec{dl} = -\frac{d}{dt} \iint \vec{B} \cdot \vec{ds}}$$

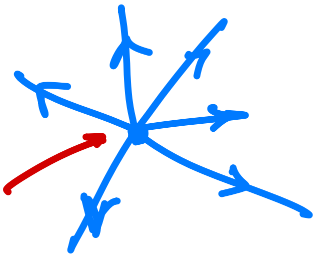
Faraday's Law.

$$\int_V \nabla \cdot \vec{D} = \int_V \rho \Rightarrow \boxed{\begin{array}{l} \oiint_S \vec{D} \cdot \vec{ds} = Q \\ \oiint_S \vec{B} \cdot \vec{ds} = 0 \end{array}}$$

Gauss' Laws.

$$\boxed{\nabla\cdot\vec{J} = -\frac{\partial \rho}{\partial t}}$$

0



KCL

$$\phi_B = Li \Rightarrow \frac{d\phi}{dt} = V$$
$$= L\frac{di}{dt}$$

$$\oint_C \vec{E}\cdot\vec{dl} = \underbrace{-\frac{d}{dt}\overbrace{\iint_S \vec{B}\cdot\vec{ds}}^{\phi_B}}_{V}$$

Coulomb's Law

$$\oiint_S \vec{D}\cdot\vec{ds} = Q$$

If Q is spherically symmetric

$\Rightarrow \vec{D}$ must be spherical symmetric

$$\Rightarrow \vec{D} = D_r\hat{r}$$
$$\vec{ds} = ds\,\hat{r}$$
$$= \underbrace{r^2\sin\theta\, d\theta\, d\phi}\,\hat{r}$$

$$= D_r \oiint_S ds = Q$$

$$= D_r r^2 \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi$$

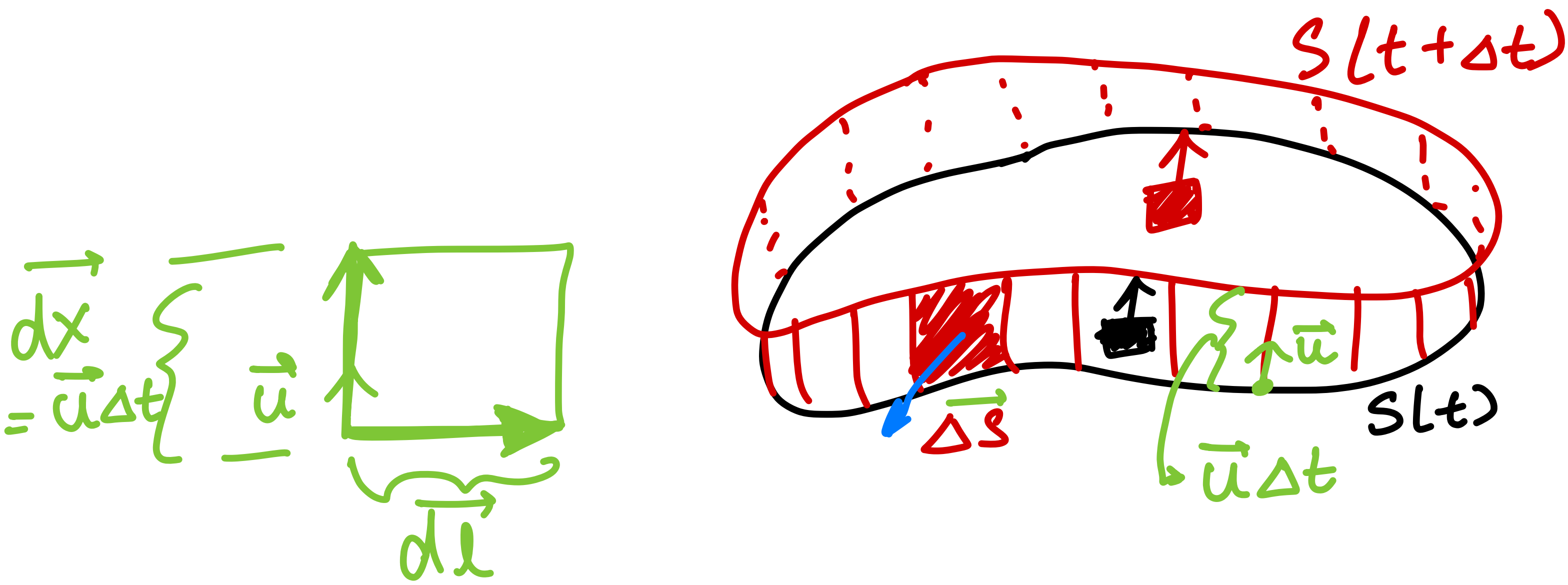
$$= D_r r^2 4\pi$$

$$\Rightarrow D_r = \frac{Q}{4\pi r^2}$$

Coulomb's Law.

$$\Rightarrow \boxed{\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}}$$

Maxwell's Equations under moving surface conditions



$$\left(\vec{\Delta s} = \vec{dl} \times \vec{dx}\right.$$

$$\left.\rightarrow \vec{\Delta s} = \vec{dl} \times \vec{u}\,\Delta t\right)$$

$$\oint_C \vec{E}\cdot\vec{dl} = -\frac{d}{dt}\iint_S \vec{B}\cdot\vec{ds} = -\tilde{V}_i$$

$$\Rightarrow \tilde{V}_i = \frac{d}{dt}\iint_{S(t)} \vec{B}(t)\cdot\vec{ds}$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\iint_{S(t+\Delta t)} \vec{B}(t+\Delta t)\cdot\vec{ds} - \iint_{S(t)} \vec{B}(t)\cdot\vec{ds}\right\}$$

Taylor series expansion for small Δt.

$$\vec{B}(t+\Delta t) = \vec{B}(t) + \frac{\partial \vec{B}(t)}{\partial t}\Delta t + \cancelto{0}{HOT}$$

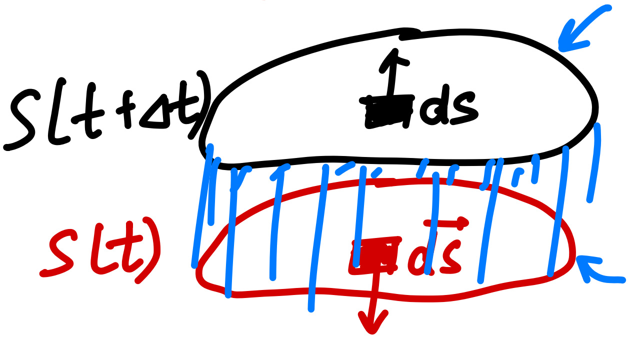
$$\tilde{V}_i = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{ \iint_{S(t+\Delta t)} \left(\vec{B}(t) + \frac{\partial \vec{B}(t)}{\partial t}\Delta t\right)\cdot \vec{ds} - \iint_{S(t)} \vec{B}(t)\cdot\vec{ds} \right\}$$

$$\tilde{V}_{it} \triangleq \lim_{\Delta t \to 0} \frac{1}{\Delta t} \iint_{S(t+\Delta t)} \left(\frac{\partial \vec{B}(t)}{\partial t}\Delta t\right)\cdot\vec{ds}$$

$$\tilde{V}_{it} = \iint_{S(t)} \frac{\partial \vec{B}}{\partial t}\cdot\vec{ds} \quad \checkmark$$

$$\tilde{V}_{im} = \lim_{\Delta t \to 0}\frac{1}{\Delta t}\left\{ \overbrace{\iint_{S(t+\Delta t)} \vec{B}(t)\cdot\vec{ds} + \iint_{\underbrace{S(t)}} \vec{B}(t)\cdot\vec{ds}} \right\}$$

$$S_0 = \underbrace{S_{top} + S_{bot}} + S_{wall}$$



$$\tilde{V}_{im} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \cancelto{0}{\oiint_{S_0} \vec{B}(t) \cdot \vec{ds}} - \iint_{S_{wall}} \vec{B}(t) \cdot \vec{ds} \right\}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \iint_{S_{wall}} \vec{B}(t) \cdot (\vec{dl} \times \vec{u}) \Delta t \right.$$

$$= \oint_C \vec{B} \cdot (\vec{dl} \times u)$$

$$F = q \underbrace{\vec{u} \times \vec{B}}_{E}$$

$$\tilde{V}_{im} = \oint_C (\vec{u} \times \vec{B}) \cdot \vec{dl}$$

$$\vec{F} = q\vec{E}$$

$$\boxed{\oint_C \vec{E} \cdot \vec{dl} = \underbrace{\iint_S -\frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}}_{\text{Transformer induction voltage}} + \underbrace{\oint_C (\vec{u} \times \vec{B}) \cdot \vec{dl}}_{\text{Motional induction voltage.}}}$$

$$\oint_C \vec{H} \cdot \vec{dl} = \underbrace{I}_{\text{Conduction}} + \underbrace{\iint_S \frac{\partial \vec{D}}{\partial t} \cdot \vec{ds}}_{\text{Displacement}} + \underbrace{\iint_S (\underbrace{\nabla \cdot \vec{D}}_{\rho}) \vec{u} \cdot \vec{ds}}_{\text{Drift}} - \underbrace{\oint_C (\vec{u} \times \vec{D}) \cdot \vec{dl}}_{\text{motional.}}$$

$$\oint_C \vec{E} \cdot \vec{dl} = \iint_S -\frac{\partial \vec{B}}{\partial t} \cdot \vec{ds} + \oint_C (\vec{u} \times \vec{B}) \cdot \vec{dl}$$

$$\oint_C \left(\vec{E} - \vec{u} \times \vec{B}\right) \cdot \vec{dl} = \iint_S -\frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$$

$$\iint_S \nabla \times \left(\vec{E} - \vec{u} \times \vec{B}\right) \cdot \vec{ds} = \iint_S -\frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$$

$$\boxed{\nabla \times \vec{E} - \nabla \times (\vec{u} \times \vec{B}) = -\frac{\partial \vec{B}}{\partial t}}$$

$$\boxed{\nabla \times \vec{H} + \nabla \times (\vec{u} \times \vec{D}) = \frac{\partial \vec{D}}{\partial t} + \rho \vec{u} + \vec{J}}$$
