Topology Notes (review)

Def: Topology

ϕ, X ; Closed under arb unions & finite intersections.

Def: Basis

$\mathbb{B}$ covers X, closed under pairwise intersections.
($\exists B$ s.t $x \in B$ $\forall x$) ($x \in B_1 \cap B_2 \Rightarrow \exists B_3$ s.t $x \in B_3 \subseteq B_1 \cap B_2$)

Lemma: Finer ($\mathcal{T}' \supseteq \mathcal{T}$)

$\forall x \in X$ & $\forall \underset{\mathbb{B}}{\overset{\cap}{B}} \ni x$, $\exists B' \in \mathbb{B}'$ s.t $x \in B' \subset B$.

Order Topology

Basis: (a,b), $[\underset{\min}{a_0}, b)$, $(a, \underset{\max}{b_0}]$.

Product Topology

Basis: $\mathbb{B} = \{U \times V \mid U \text{ open in } X,\ V \text{ open in } Y\}$.

Subspace Topology

$Y \subseteq X$, $\mathcal{T}_y = \{Y \cap U \mid U \in \mathcal{T}\}$ forms the subspace topology.

Basis: $\mathbb{B}_y = \{B \cap Y \mid B \in \mathbb{B}\}$.

Thm: $A \subseteq X$, $B \subseteq Y$; then product topology on $A \times B$ is equal to subspace topology $A \times B \subseteq \underbrace{X \times Y}_{\text{Product topology}}$.

Def **Closed**.

$A \subseteq X$ is closed if $X \setminus A$ is open.

Def **Interior**

$\underline{Int(A)}$: Union of all open sets contained in A.

Def **Closure**.

$\underline{Cl(A)}$: Intersection of all closed sets containing A.

Thm: $Y \subseteq X$ subspace; $A \subseteq Y$; If $\overline{A}$ = closure of A in X, then

closure of A in Y = $\overline{A} \cap Y$.

Def **Neighbourhood**.

An open set containing a pt. is its' nbd.

* **Thm**: $x \in \overline{A}$ iff every nbd of x intersects A.

Def **Limit point**

Every neighbourhood of x intersects A in some pt. other than x.

$\Leftrightarrow \quad x \in \overline{A - \{x\}}$.

Thm: $\overline{A} = A \cup \underbrace{A'}_{\text{set of all limit pts. of } A}$.

Cor: A Closed $\Leftrightarrow$ A contains all its limit pts.

Def : Convergence.

$\{x_n\} \to x$ if $\forall$ nbd U of x $\exists N$ s.t $x_n \in U$ $\forall n \geq N$.

("A tail in every nbd").

Def : Hausdorff

Given 2 distinct points, there exist disjoint nbds.

$x_1, x_2 \in X$; $x_1 \neq x_2 \Rightarrow \exists U_1, U_2$ s.t $x_1 \in U_1$, $x_2 \in U_2$ & $U_1 \cap U_2 = \phi$.

Thm: In a Hausdorff space, every finite set is closed.

Thm: In a Hausdorff space convergence is unique.

Thm: Subspace of a HS is a HS. Product of HS are HS.
Every simply ordered set with order top. is HS.

Def: T1 axiom

"Every finite set is closed".

Thm: If X is T1, $(A \subseteq X)$ $x \in X$ is a limit pt of $A \Leftrightarrow$ every nbd of x contains infinitely many pts. in A.

Def: Continuous function

$f: X \to Y$ continuous if $\forall$ open $A \subseteq Y$; $f^{-1}(A)$ open $\subseteq X$.

(enough to show inverse image of basis elt is open)

Thm: TFAE

1) f continuous 2) $A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)}$ 3) f^{-1} of closed subset is closed.

4) $\forall x \in X$, and each nbd V of $f(x)$, $\exists$ nbd U of x s.t. $f(U) \subseteq V$.

Def: Homeomorphism

> $f: X \to Y$ bijection with inverse $f^{-1}: Y \to X$ s.t. both f, f^{-1} are continuous.

> $f: X \to Y$ bijection & $f(U)$ open $\Leftrightarrow$ U open ($\forall$ U open).

Thm: Constant fns., restrictions, inclusions and compositions are continuous.

Product Topology. $\prod_{\alpha \in J} X_\alpha$

Def: Box topology

Basis: $\prod_{\alpha \in J} U_\alpha$ where each U_α is an open set in X_α $\forall \alpha$.

Def: Product topology.

Basis: $\prod_{\alpha \in J} U_\alpha$ where each U_α & $U_\alpha = X_\alpha$ for all but finitely many α.

(2 Thms on Pg 30)

Metric Spaces.

: $d : X \times X \to \mathbb{R}$ is a metric on set X if

1) $d(x,y) \geq 0 \ \forall x,y \in X$ & $d(x,y) = 0$ iff $x=y$ semidefinite.

2) $d(x,y) = d(y,x)$ Symmetric

3) $d(x,y) + d(y,z) \geq d(x,z) \ \forall x,y,z \in X$.

Euclidean metric : $d(\vec{x},\vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$

Square metric : $\rho(\vec{x},\vec{y}) = \max \{|x_i - y_i|\}$

<u>Def</u>: <u>ϵ-Ball $B_d(x,\epsilon)$</u>.

$B_d(x,\epsilon) = \{y \in X \mid d(x,y) < \epsilon\}$.

<u>Def</u>: <u>Bounded</u>.
(X,d) Subset A is bounded if $\exists M$ s.t $d(a_1,a_2) \leq M \ \forall$ pairs $a_1, a_2 \in A$.

<u>Discrete metric</u>

$$d(x,y) = \begin{cases} 0 & x=y \\ 1 & \text{otherwise} \end{cases}$$

<u>Def</u>: <u>Metric Topology</u>.

<u>Basis</u>: Collection of all ϵ-balls $B_d(x,\epsilon)$.

<u>Def</u>: <u>Metrizable</u>.
A top. sp. X is metrizable if $\exists$ a metric d on X that induces X.

<u>Thm</u>: Topologies on $\mathbb{R}^n$ induced by d & ρ are same as the product top.

<u>Lemma</u>: d, d' metrics on X induce T, T'. T' is finer than T iff $\forall x \in X, \epsilon > 0 \ \exists \delta > 0$ s.t $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$.

Def: <u>Uniform metric</u> (a metric on $\mathbb{R}^\omega$) ($\to$ generates uniform Topology)

$\bar{\rho}(x,y) = \sup\{\min\{|x_i - y_i|, 1\}\}$

Thm: On $\mathbb{R}^\omega$ Box $\supsetneq$ Uniform $\supsetneq$ Product topologies.

Thm: Metrization of product topology.

Let $\bar{d}(a,b) = \min\{|a-b|, 1\}$ metric on $\mathbb{R}$. If $\vec{x}, \vec{y} \in \mathbb{R}^\omega$,

define $D(\vec{x}, \vec{y}) = \sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\} \to$ induces product topology.

<u>Continuous functions and metric spaces</u>

Thm: (X, d_X) and (Y, d_Y) metric spaces. Let $f: X \to Y$.

f is continuous iff $\forall x,y \in X$ & $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $d_X(x,y) < \delta$

then $d_Y(f(x), f(y)) < \epsilon$.

Lemma: <u>Sequence Lemma</u>.

If $\exists$ a sequence of pts in $A \subseteq X$ converging to $x \Rightarrow x \in \bar{A}$.

Converse is true if X is metrizable.

Thm: Continuity & convergence.

If $f: X \to Y$ continuous, then if $x_n \to x$ in X, then $f(x_n) \to f(x)$ in Y.

Converse is true if X is metrizable.

Def: <u>Uniform convergence</u>.

Let $f_n : X \to Y$ seq. of fns. Y metric space with metric d. (f_n) converges uniformly to $f: X \to Y$ if given $\epsilon > 0$, $\exists$ some $N \in \mathbb{N}$ s.t. $d(f_n(x), f(x)) < \epsilon$ $\forall n >$

Thm (Uniform Limit Theorem)

Let $f_n : X \to Y$ sequence of continuous functions. X top sp. & Y metric sp. (Y, d). If f_n converges uniformly to $f : X \to Y$, then f is continuous.

Quotient Topology.

Def: Quotient map

$p : X \to Y$ surjective and $p^{-1}(U)$ is open in $X \Longleftrightarrow U$ is open in Y.

"Different from open map since only open preimages mapped to open sets."

"Stronger than continuity".

Def: Quotient topology.

Given surjective $p : \overset{top}{X} \to \overset{set}{A}$, there is a unique topology on A s.t. p is a quotient map. This is the quotient topology.

$$\mathcal{T} = \{ U \subseteq A \mid p^{-1}(U) \text{ is open in } X \}.$$

Def: Quotient space.

X top. space. Let X^* be a partition of X. Let $p : X \to X^*$ be a surjective map which maps $x \in X$ to the elt of X^* which contains that pt. With the quotient topology, X^* is called a quotient space of X.

$\mathcal{T}_{X^*} = \{ U \subseteq X^* \mid p^{-1}(U) \text{ open in } X \} \to$ construction of top on X^* from top on X

Thm: $p : X \to Y$ quotient map. Let Z be a space and $g : X \to Z$ be a map that's constant on each set $p^{-1}(\{y\})$ $\forall y \in Y$.

Then g induces a map $f : Y \to Z$ s.t. $f \circ p = g$

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

① f cont. $\Leftrightarrow$ g cont.

② f quotient map $\Leftrightarrow$ g quotient map.

Corollary: $p : X \to X^*$ quotient map, s.t $X^* = \{ g^{-1}(\{z\}) \mid z \in Z \}$

where $g : X \to Z$ surjective continuous map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ X^* \arrow[r, "f"'] & Z \end{tikzcd}$$

Then g induces a bijective continuous map

$f : X^* \to Z$ s.t

① f is homeo $\Leftrightarrow$ g is quotient map

② If Z is Hausdorff, then so is X^*.

Connected Spaces.

Def: Seperation

A seperation of X is a pair U, V of dijoint, nonempty, open subsets of X whose union is X.

Def: Connected space.

- X is connected if there does not exist a seperation of X.
- X is connected iff the only subsets of X which are both open & closed are ϕ & X.

Lemma: Subspace connectedness

$Y \subseteq X$ subspace; A seperation of Y is a pair of disjoint, nonempty sets A & B whose union is Y, neither of which contains a limit pt. of the other. Y is connected if there exists no seperation of Y.

Thm: A union of connected subspaces of X that have a pt in common is connected.

Thm: Let A be a connected subspace of X, if $A \subset B \subset \bar{A}$; Then B is connected.

Thm: A finite Cartesian product of connected spaces is connected.

Thm: The image of a connected space under a continuous map is connected.

Connected Subspaces of $\mathbb{R}$.

Def: Least Upper Bound property (LUB or Sup).

An ordered set A has the LUB if every non empty subset $A_0 \subseteq A$ that is bounded above (by some element of A) has a least upper bound in A.

"Bounded above by $A \Rightarrow \text{Sup} \in A$".

Def: Linear Continuum

A simply ordered set L having more than one elt, is called a linear continuum if

1) L has LUB
2) If $x < y$, $\exists\, z$ s.t $x < z < y$.

Thm: If L is a linear continuum with order topology, then L is connected and so are intervals and rays in L.

Thm: (Intermediate Value theorem)

$f: X \to Y$ continuous map. X connected, Y is ordered set with order topology; if $a, b \in X$ and $r \in Y$ s.t $f(a) < r < f(b)$ then $\exists\, c \in X$ s.t $f(c) = r$.

Def: Path

Given $x, y \in X$, a path from x to y is a continuous map $f : [a,b] \longrightarrow X$ s.t $f(a) = x$, $f(b) = y$.

$[a,b] \subseteq \mathbb{R}$

Def: Path Connected (stronger than connectedness)

A space is path connected if every pair of pts. can be joined by a path in X.

Def: Components (or connected components)

Given X space, define equivalence relation on X by: $x \sim y$ if there exists a connected subspace of X containing both x and y. The equivalence classes are then the connected components or components of X.

Def: Path components (analogous to above def).

thm: The (path) components of X are (path) connected disjoint subspaces whose union is X s.t any non empty (path) connected subspace of X will intersect only one of them.

Remarks

1. Each component of X is closed in X.
2. If X is a finite union of its components, then each component is also open in X.
3. In general a component of X need not be open.
4. Path components need not be open or closed.

Def: Local Connectedness (or Local path connectedness)

X is locally (path) connected at x if $\forall$ nbd U of x, $\exists$ a (path) connected nbd V of x s.t $x \in V \subseteq U$.

Def: Locally connected (or Locally path connected)

X is locally (path) connected if every $x \in X$ is locally (path) connected.

Compact Spaces

Def: Covering (Open covering)

A collection $\mathcal{A}$ of subsets of a space X is a covering of X or covers X if $X = \bigcup_{A \in \mathcal{A}} A$. Open covering if each A is open in X.

Def: Compact

X is compact if every open covering of X contains a finite subcollection which also covers X. "Every open covering has a finite subcovering".

Def: Subspace Covering.

If $Y \subseteq X$ subspace, a collection $\mathcal{A}$ of subsets of X is said to cover Y if the union of its elements contains Y.

$$\left[\bigcup_{A \in \mathcal{A}} A \supseteq Y \right]$$

Lemma: Compact subspace

$Y \subseteq X$ subspace is compact iff every covering of Y by open sets in X contains a finite subcollection which also covers Y.

Thm: Every closed subspace of a compact space is compact.

Thm: The image of a compact space under a continuous functio is compact.

Thm: Every compact subspace of a Hausdorff space is closed.

Thm: Let $f: X \to Y$ be a bijection & continuous fn. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Thm: A finite product of compact spaces is also compact.

Thm: Extreme Value Theorem (EVT)

Let $f: X \to Y$ continuous, Y order topology. If X is compact, $\exists\, c, d \in X$ s.t $f(c) \leq f(x) \leq f(d) \quad \forall x \in X$.

Thm: X = simply ordered set with LUB. In the order top on X, each closed interval in X is compact.

Thm: $A \subseteq \mathbb{R}^n$ is compact iff A is closed and bounded.

Def: Uniformly Continuous

(X, d_X), (Y, d_Y) metric spaces. $f: X \to Y$ is uniformly continuous if $\forall \epsilon > 0 \; \exists \delta > 0$ s.t $d_X(x_0, x_1) < \delta \Rightarrow d_Y(f(x_0), f(x_1)) < \epsilon$ $\forall x_0, x_1 \in X$.

Thm: (Uniform Continuity Theorem)

$f: X \to Y$ continuous, (X, d_X) compact metric space, (Y, d_Y) metric sp. Then f is uniformly continuous.

Def: distance from x to A

(X, dx) metric space and $A \subseteq X$ nonempty subset. For any $x \in X$, $d(x, A) = \underset{(glb)}{\inf} \{ d(x, a) \mid a \in A \}$.

For a fixed A; $d(x, A)$ is continuous w.r.t x.

Thm: Lebesgue Number Theorem

Let $\mathcal{A}$ be an open covering of a metric space (X, d). If X is compact, there is a $\delta > 0$ s.t for each subset of X of diam $< \delta$, $\exists$ an element of $\mathcal{A}$ that contains it. (δ is called the Lebesgue no. of $\mathcal{A}$).

Post midterm

<u>Thm</u> X is locally (path) connected iff for every open set U of X, each (path) component of U is open in X.

<u>Def</u>: <u>Limit point compactness</u> (LPC)
X is LPC if every infinite subset of X has a limit point.

<u>Thm</u>: Compactness $\Rightarrow$ LPC.

<u>Def</u>: <u>Subsequence</u>
(x_n) is a sequence in X. If $n_1 < n_2 < \ldots < n_i < \ldots$ is an increasing sequence of pts. in $\mathbb{Z}_+$, then $y_i = x_{n_i}$ is a subsequence of (x_n).

<u>Def</u>: <u>Sequentially compact</u> (SC)
X is SC if every sequence of pts. of X has a convergent subsequence.

<u>Thm</u>: If X is metrizable, then C $\Leftrightarrow$ SC $\Leftrightarrow$ LPC

<u>Def</u>: <u>Local Compactness</u> (Later a more satisfying def is given which is equivalent).
A space X is locally compact at x if $\exists$ compact subspace C of X that contains a nbd of x.

<u>Thm</u> (One point compactification)
X locally compact Hausdorff iff $\exists$ Y s.t
1) X subspace of Y
2) Y−X consists of a single point
3) Y is compact Hausdorff.

Y is unique upto a homeomorphism & is the <u>one pt. compactification</u> of

Countability and Seperation Axioms

Def: First countable / Countable basis → "Local property"

→ X has a countable (local) basis at x if $\exists$ a countable collection $\mathcal{B}$ of nbds of x s.t every nbd of x contains a $B \in \mathcal{B}$.

↳ Given nbd U, $\exists B$ s.t $x \in B \subseteq U$

→ First countable if true $\forall x \in X$.

Def: Second countable → "Global property"

If X has a countable basis it is second countable.

Thm: Subspaces & products of $1^{st}/2^{nd}$ countable spaces are also $1^{st}/2^{nd}$ countable

Def: Dense subsets

A subset $A \subseteq X$ is dense in X if $\overline{A} = X$.

Def: Lindelöf space

Every open covering has a countable subcovering.

Def: Seperable space

$\exists$ a countable subset of X that is dense in X.

Thm: Second countable $\Rightarrow$ Lindelöf & seperable.

Seperation axioms

Def: Regular and Normal

Suppose T1 is satisfied (i.e one pt. sets are closed)

Regular: For each pair consisting of a point $x \in X$ & a closed set B disjoint from x, $\exists$ open disjoint set $U \ni x$ and $V \supset B$.

Normal: For each pair of disjoint closed sets A, B; $\exists$ open disjoint sets $U \supset A$ & $V \supset B$.

Rmk: Normal $\Rightarrow$ Regular $\Rightarrow$ Hf $\Rightarrow$ T1

Lemma: Alternative definition

Suppose T1 is satisfied,

a) Regular: Iff given $x \in X$ and nbd U of x, $\exists$ nbd V of x s.t $\overline{V} \subset U$

b) Normal: Iff given closed set A & open set $U \supset A$ $\exists$ open $V \supset A$ s.t $\overline{V} \subset U$.

Thm Subspace & product of regular/Hf spaces are regular/Hf.

Thm: Every metrizable space is Normal.

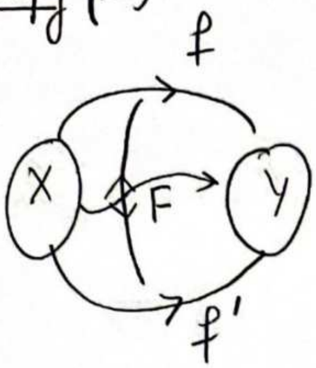
Thm: Every regular space with a countable basis is normal.

Thm: Every compact Hf space is normal.

Thm: Every well ordered set is normal in the order topology

ALGEBRAIC TOPOLOGY

Def: Homotopy ($\simeq$)



$F : X \times \underset{[0,1]}{I} \to Y$ continuous.

$F(x,0) = f(x)$

$F(x,1) = f'(x)$

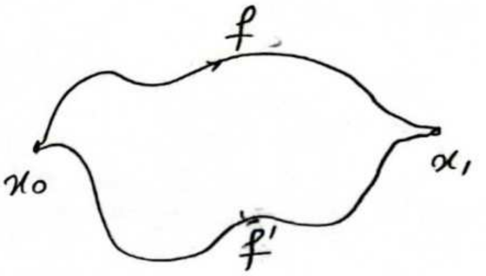
$f \simeq f' \to$ homotopic ; F is the homotopy

If f' is a constant map, the f is nul homotopic.

Def : Path

$f : \underbrace{[0,1]}_{I} \to X$ continuous S.T $f(0) = x_0$, $f(1) = x_1$

Def : Path Homotopy ($\simeq_p$)



f, f' two paths if they have same end points & $\exists$ continuous $F : I \times I \to X$

S.T

$F(s,0) = f(s)$ $\quad F(s,1) = f'(s)\}$ Homotopy

$F(0,t) = x_0$ $\quad F(1,t) = x_1\}$ Fixed end points.

$F(\underset{\sim}{s}, \underset{\sim}{t})$ — location, time ensemble

Lemma 51.1 $\simeq$ & $\simeq_p$ are equivalence relations. [f]

- Straight line Homotopy $F(x,t) = (1-t) f(x) + t\, g(x)$.

Def: Product of paths $h(s) = \begin{cases} f(2s) & \text{for } s \in [0, \frac{1}{2}] \\ g(2s-1) & \text{for } s \in [\frac{1}{2}, 1] \end{cases}$

$[f] * [g] = [f * g]$

$*$ on path homotopy classes behaves like a group.
Only difference being the constraint $f(1) = g(0)$.

<u>Associative:</u> $[f] * ([g] * [h]) = ([f] * [g]) * [h]$

<u>Identity:</u> Let $e_x : I \to X$ be constant taking $I \to x \in X$.

$[f] * [e_{x_1}] = [e_{x_0}] * [f] = [f]$.

<u>Inverse:</u> $\bar{f}(s) = f(1-s) \Rightarrow [f] * [\bar{f}] = [e_{x_0}]$
$[\bar{f}] * [f] = [e_{x_1}]$

Fundamental Group

Path homotopy classes $[f]$ behave <u>exactly</u> like elements of a group under $*$ when we restrict the initial & end pt to be equal.

Groups Review

<u>Homomorphism:</u> $f : G \to G' \Rightarrow f(x \cdot y) = f(x) f(y)$ &
automatically satisfies $f(x^{-1}) = f(x)^{-1}$ & $f(e) = e'$
inverse
identities in G, G'.

<u>Kernel:</u> $f^{-1}(e')$ is a subgroup of G.

Monomorphism: Homomorphism that is injective i.e $f^{-1}(e') = e$.

Epimorphism: Surjective homomorphism

Isomorphism: Bijective homomorphism.

$G = (\mathbb{Z}/8\mathbb{Z}, +)$
$H = \{0,4\}$, the cosets are $\{0,4\}$; $\{1,5\}$; $\{2,6\}$ and $\{3,7\}$.
i.e. H, $H+1$, $H+2$, $H+3$
$\Rightarrow$ index $[G:H] = 4$

Left coset: H subgroup of G. $xH = \{x.h \mid h \in H\}$ for each $x \in G$

The collection of these left cosets partition G. Similarly for right cosets. Each subgroup has a left & right coset.

Normal Subgroup: If $x.h.x^{-1} \in H$ $\forall x \in G$ & $h \in H$.

Then $xH = Hx$ $\forall x \in G$ and the 2 partitions are equal.

This partition is then denoted G/H and $(xH).(yH) = (x.y)H$ is well defined forming a new group called the quotient of G by H. {The left & right partitions being equal form the quotient group & obviously $H \sim 0$ is the kernel}

$f: G \rightarrow G/H$ is an epimorphism with kernel H.

Conversely, if $f: G \rightarrow G'$ is an epimorphism, it's kernel forms a normal subgroup N and f induces an isomorphism from $G/N \rightarrow G'$ that carries xN to $f(x)$ $\forall x \in G$.

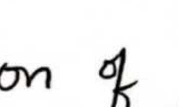
Interesting

If H is not a normal subgroup, G/H denotes the collection of right cosets of H in G.

Def: <u>Fundamental Group</u> (aka First Homotopy group of X)

$\pi_1(X, x_0)$ is the collection of path homotopy classes of loops at x_0 with $*$.

$\pi_1(\mathbb{R}^n, x_0)$ is the trivial group (identity is the only element).
i.e $\pi_1(\mathbb{R}^n, x_0) = 0$



$\bar{\alpha} = \alpha^{-1}$ Def: $\hat{\alpha}: \pi_1(X, x_0) \to \pi_1(X, x_1)$ by
$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$

Def: <u>Simply connected</u>

Path connected and $\pi_1(X, x_0) = 0$ for some $x_0 \in X$.

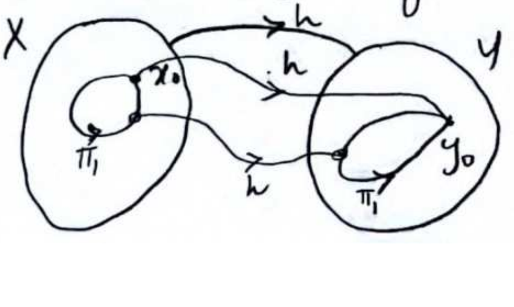
Lemma: In a simply connected space any two paths having the same end points are path homotopic.

Def: <u>Homomorphism induced by map h</u>

Let $h: (X, x_0) \to (Y, y_0)$ continuous & $h(x_0) = y_0$

$h_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined as $h_*([f]) = [h \circ f]$

h_* ↓ homomorphism coming from map h.



<u>Homomorphism</u>
$(h \circ f) * (h \circ g) = h \circ (f * g)$

Thm Functorial Properties of h_*

1) If $h:(X,x_0)\to(Y,y_0)$ and $k:(Y,y_0)\to(Z,z_0)$ are continuous, then $(k\circ h)_* = k_*\circ h_*$

2) If $i:(X,x_0)\to(X,x_0)$ is the identity map, then i_* is the identity homomorphism.

Cor: Topological invariance of π_1

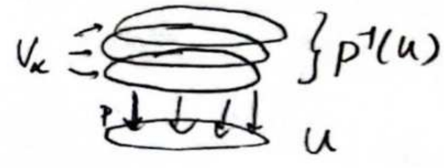
h is homeomorphism $\Rightarrow$ h_* is isomorphism.

Def: Evenly covered & Slices

Let $p: E\to B$ be a continuous surjective map. The open set U of B is evenly covered by p if

$p^{-1}(U) = \bigcup V_\alpha$ where V_α are disjoint open sets in E

$p\big|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism for each α.



$\{V_\alpha\}$ partitions $p^{-1}(U)$ into slices

Def: Covering map

Let $p: E\to B$ cont. surj. If every pt. $b\in B$ has nbd U that is evenly covered by p, then p is called a covering map.

Thm $p:\mathbb{R}\to S^1$ given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is a covering map.

Def <u>Local homeomorphism</u> (Covering map ⇒ local homeo)

$p : E \to B$ cont. For each $e \in E$ $\exists$ nbd that is mapped homeomorphically onto an open subset of B.

<u>Thm</u>: $p : E \to B$ covering map. If $B_0 \subseteq B$ subspace and $E_0 = p^{-1}(B_0)$ then $p_0 : E_0 \to B_0$ where $p_0 = p|_{E_0}$ is a covering map.

<u>Thm</u>: If $p : E \to B$ and $p' : E' \to B'$ are covering maps, then $p \times p' : E \times E' \longrightarrow B \times B'$ is a covering map.

<u>Def</u>: <u>Lifting</u>

Let $p : E \to B$ be a map. If $f : X \to B$ continuous, a <u>lift</u> of f is a map $\tilde{f} : X \to E$ S.T $p \circ \tilde{f} = f$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

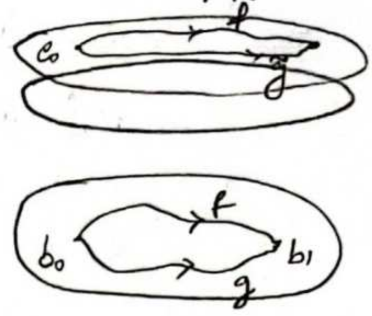
<u>Key Ideas</u>

* If p is a covering map, paths in B can be lifted to paths in E.

<u>Lemma</u>: Let $p : E \to B$ covering map. and $p(e_0) = b_0$. Any path $f : [0, 1] \to B$ beginning at b_0 has a unique lift to a path $\tilde{f}$ in E beginning at e_0

<u>Lemma</u>: Let $p : E \to B$ be a covering map; Let $p(e_0) = b_0$. Let the map $F : I \times I \to B$ be continuous, with $F(0,0) = b_0$. There is a unique lifting of F to a continuous map $\tilde{F} : I \times I \to E$ S.T $\tilde{F}(0,0) = e_0$ (F is path homot[illegible]
⇒ $\tilde{F}$ is path homot[illegible]

Thm : $p: E \to B$ covering map; $p(e_0) = b_0$. Let f and g be two paths in B from b_0 to b_1, let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths in E beginning at e_0. If f and g are path homotopic, then $\tilde{f}$ and $\tilde{g}$ end at the same pt. of E and are path homotopic.



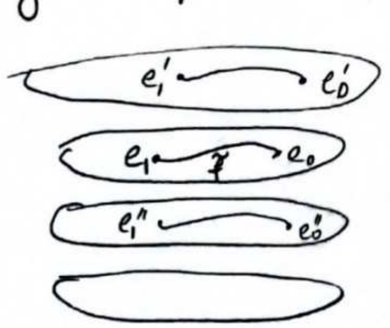
Def: Lifting correspondence

$p: E \to B$ covering map, $b_0 \in B$. Choose e_0 s.t. $p(e_0) = b_0$.
Given $[f] \in \pi_1(B, b_0)$, let $\tilde{f}$ be the lifting of f to a path in E beginning at e_0. Let $\phi([f])$ denote the end pt. of $\tilde{f} = \tilde{f}(1)$.

Then ϕ is a well-defined set map.

$$\phi: \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$$

↳ Lifting correspondence.



ϕ maps loop classes to their corresponding lifted end pts — end pts. $e_1, e_1', e_1'' \ldots$ all $\in p^{-1}(b_0)$. $p^{-1}(b_0)$ also contains $e_0, e_0', e_0'' \ldots$, These pts. are also end pts. of the trivial loop which $\in [f]$. So p maps $[f]$ to all the (start & end pts of loops $[f]$.
↓ also end pts.

$p^{-1}(b_0)$ is a set with the discrete topology. Its elts are "fibers" in each slice.

Thm: E is path connected $\Rightarrow$ ϕ is surjective

E is simply connected $\Rightarrow$ ϕ is bijective.

Thm: $\pi_1(S^1, b_0) \cong (\mathbb{Z}, +)$

Def: Generator of a group (Cyclic Group)

Let G be a group, let $x \in G$. If $x^m = G$ for $m \in \mathbb{Z}$,

G is a cyclic group and x is a generator of G.

Order of a group = cardinality (i.e. no. of elements).

Def: Retraction / Retract

If $A \subseteq X$, a retraction of X onto A is a continuous map $r : X \to A$ s.t. $r|_A$ is the identity map of A. If r exists, A is a retract of X.

Lemma: If A is a retract of X, then the inclusion map $j : A \to X$ induces an injective homomorphism

$$j_* : \pi_1(A, a) \to \pi_1(X, a)$$

Thm: There is no retraction of B^2 onto S^1.

2D ball "disk"

<u>Thm</u>: <u>(Brouwer Fixed Point Thm for B^2)</u>

- If $f: B^2 \to B^2$ continuous, $\exists$ some $x \in B^2$ s.t $f(x) = x$.

<u>Lemma</u>: Let $h: S^1 \to X$ continuous map. TFAE.

1) h is nulhomotopic
2) h extends to a continuous map $K: B^2 \to X$ $\left(K|_{S^1} = h \right)$
3) h_* is the trivial homomorphism on the fundamental group.

<u>Cor</u>: The inclusion map $j: S^1 \to \mathbb{R}^2 - \{0\}$ is not nulhomotopic.
The identity map $id: S^1 \to S^1$ is not nulhomotopic.

- <u>Thm</u>* : <u>Fundamental Thm of Algebra</u>.

A polynomial equation (with $\mathbb{R}$ or $\mathbb{C}$ coefficients)
$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$, $n > 0$ has atleast one (real or complex) root.

<u>Lemma</u>: $h, k: (X, x_0) \to (Y, y_0)$ continuous maps. If h and k are homotopic, and the image of the base point x_0 of X remains fixed at y_0, during the homotopy, then the homomorphisms h_* and k_* are equal.

- <u>Thm</u>: The inclusion map $j: S^n \to \mathbb{R}^{n+1} - \{0\}$ induces an isomorphism of fundamental groups.

Def: **Deformation retract**

$A \subseteq X$, A is a deformation retract of X if $id_X : X \to X$ is homotopic to a map that carries all of X into A s.t each point of A remains fixed during the homotopy.

i.e $\exists$ continuous map $H : X \times I \to X$ s.t

$$H(x,0) = x \quad \forall x \in X$$

$$H(x,1) \in A \quad \forall x \in X$$

$$H(a,t) = a \quad \forall a \in A.$$

H is called a deformation retraction of X onto A.

Note: The map $r : X \to A$ defined by $r(x) = H(x,1)$ is a retraction of X onto A, and H is a homotopy b/w the identity map of X and the map $j \circ r$, where $j : A \to X$ is the inclusion map

inclusion retraction

$(j \circ r : X \to X$ and $id_X : X \to X)$

Thm: Let A be a deformation retract of X, let $x_0 \in A$. Then the inclusion $j : (A, x_0) \to (X, x_0)$ induces an isomorphism of fundamental groups (of X and A).

Def : Homotopy equivalences, inverses and types

- Let $f : X \to Y$ and $g : Y \to X$ be continuous maps. Suppose that the map $g \circ f : X \to X$ is homotopic to the identity map of X, and the map $f \circ g : Y \to Y$ is homotopic to the identity map of Y. Then the maps f and g are called <u>homotopy equivalences</u> and each is said to be a <u>homotopy inverse</u> of the other.

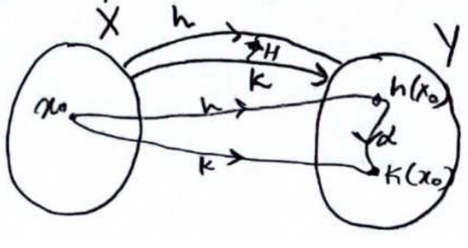
<u>Thm</u> : The relation of homotopy equivalence is an equivalence relation.

- Two spaces that are homotopy equivalent are said to have the same <u>homotopy type</u>.

<u>Thm</u> If $f : X \to Y$ is a homotopy equivalence, then $\forall x_0 \in X$,

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0)) \text{ is an isomorphism.}$$

<u>Lemma</u> Let $h, k : X \to Y$ continuous and $h \simeq k$ via homotopy $H : X \times I \to Y$. Let $x_0 \in X$. Then $\exists$ a path α in Y from $h(x_0)$ to $k(x_0)$ s.t. $k_* = \hat{\alpha} \circ h_*$ (h_*, k_* differ by base pt. change map). Indeed α is the path $\alpha(t) = H(x_0, t)$.



Thm: Let $f: X \to Y$ continuous. Let $f(x_0) = y_0$. If f is a homotopy equivalence, then

$$f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0) \text{ is an isomorphism.}$$

Thm: Let $X = U \cup V$, where U and V open in X. Suppose $U \cap V$ path connected, $x_0 \in U \cap V$. Let $i_U: U \to X$, $i_V: V \to X$ inclusion maps. Then the images of

$$i_{U*}: \pi_1(U, x_0) \longrightarrow \pi_1(X, x_0)$$
$$i_{V*}: \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

generate $\pi_1(X, x_0)$

Cor: Suppose $X = U \cup V$, where U, V open in X, $U \cap V \neq \phi$, path connected. If U and V simply connected, then X is simply connected.

Thm: If $n \geq 2$, the n-sphere S^n is simply connected.

Thm: $\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$

Recall group structure on $A \times B$ is $(a \times b) \cdot (a' \times b') = (a \cdot a') \times (b \cdot b')$

Cor: $\pi_1(\underbrace{S^1 \times S^1}_{\text{Torus}}, b_0) \cong \mathbb{Z} \times \mathbb{Z}$

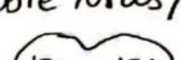
Def: Projective plane

The **projective plane** P^2 is the quotient space obtained from S^2 by identifying each pt. x of S^2 with its antipodal point $-x$.

Thm: $\pi_1(P^2, y)$ is a group of order 2 $(\cong \mathbb{Z}/2\mathbb{Z})$

Lemma: π_1 (Fig. 8, b_0) is not abelian.

Thm: $\pi_1(\Sigma_2, b_0)$ is not abelian. $\Sigma_2 \rightarrow$ double torus / Genus 2



MATH 590 - TOPOLOGY

Introduction

Topology is the study of topological spaces, continuous maps between them and properties of spaces preserved by continuous maps.

Goal: Define topological invariants. Understand what properties are preserved under continuous maps.

How to define a topological space?

The def. should be broad enough to include:

- Euclidean space $\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$
- ∞-dim Euclidean space $\mathbb{R}^\infty$
- function spaces, eg: $f: \mathbb{R} \to \mathbb{R}$
- Circle S^1, sphere S^2
- Products $S^1 \times S^1$, $S^1 \times S^1 \times S^1$, $S^1 \times \mathbb{R}$
- metric spaces

- open intervals $\neq$ open sets. ( ) $\to$ open interval.

$(0,1) \cup (2,4)$ is not an open interval but it is open in τ_{std} of $\mathbb{R}$

Ch 2.12

Def. A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1) ϕ & X are in $\mathcal{T}$.
2) The arbitrary unions of elements of $\mathcal{T}$ are also in $\mathcal{T}$.
3) Finite intersections of elements of $\mathcal{T}$ are also in $\mathcal{T}$.

$(X, \mathcal{T})$ or in short (X) is called a topological space.

In the def:

index set could be ∞

(2) $\iff$ If $A_\alpha \in \mathcal{T}$ for $\alpha \in I$, then $\bigcup_{\alpha \in I} A_\alpha \in \mathcal{T}$.

(3) $\iff$ If $A_1, \ldots, A_m \in \mathcal{T}$, then $A_1 \cap \ldots \cap A_m \in \mathcal{T}$.

Why not infinite intersections?

Eg: $\bigcap_{n \to \infty} \left(\frac{1}{n}, \frac{1}{n} \right) = \{0\}$ in $\mathbb{R}$.

open sets — closed

Def: A subset U of X is an open set of X if $U \in \mathcal{T}$.

'Elements of a topology are open'.

Ex: The standard topology on $\mathbb{R}$.

T_{std} = { open subsets of $\mathbb{R}$ }

includes the following elements : ϕ, $\mathbb{R}$

- $(a,b) \in T_{std}$
- (a, ∞)
- $(-\infty, b)$
- $(a,b) \cup (c,d)$
- $(a,b) \cap (c,d) = \begin{cases} \phi & \text{if } a<b\le c<d \\ (c,b) & \text{if } c<b \end{cases}$

Ex: X set ; $\mathcal{T} = \{\phi, X\}$ is the trivial topology.

Ex: X set; $\mathcal{T}$ = collection of all subsets of $X = P(X)$ powerset

$(X, \mathcal{T})$ is the discrete topology.

A set X may have different topologies on it.

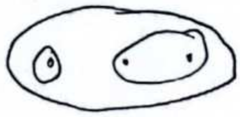
Ex: Let $X = \{a, b, c\}$, which of the following represent a topology on X?

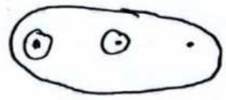
① 

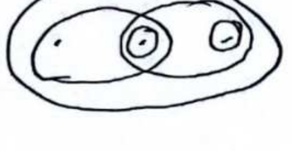
$T = \{\phi, X\}$ ✓

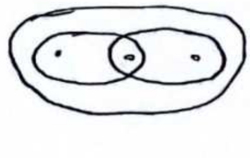
② 

$T = \{\phi, \{a\}, \{a,b\}, X\}$ ✓

③  ✓

④  X since $a \cup b \notin T$.

⑤  ✓

⑥  X $\{a,b\} \cap \{b,c\} = \{b\} \notin T$

 Basis for a Topology.

Recall: $(X, \mathcal{T})$; $\mathcal{T}$ includes ϕ, X & closed under arb $\cup$ & finite $\cap$.

Goal: Instead of defining the topology $\mathcal{T}$ by explicitly defining the entire collection of open sets, use a smaller collection of subsets of X to "generate" $\mathcal{T}$.

Def If X is a set, a basis for a topology on X is a collection of subsets of X (called basis elements) such that

(1) For each $x \in X$, there is atleast one basis element $B \in \mathbb{B}$ containing x. ($\mathbb{B}$ covers X)

(2) If $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathbb{B}$, then there is a basis element $B_3 \in \mathbb{B}$ containing X s.t $B_3 \subset B_1 \cap B_2$. (Closed under pairwise intersections)

Given a basis $\mathbb{B}$, the topology $\mathcal{T}$ generated by $\mathbb{B}$ is defined by:

- $U \subseteq X$ is open (that is, $U \in \mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathbb{B}$ s.t $x \in B$ & $B \subseteq U$

$\boxed{x \in B \subseteq U \ \forall x \in U}$

Proof in text that $\mathcal{T}$ is a topology. (Open sets are sets generated by unions of $\mathbb{B}$. ϕ is generated by union of an "empty" collection.)

Ex: If X is any set & $\mathbb{B}$ is the collection of all one-point subsets of X i.e. $\mathbb{B} = \{ \{a\} \mid a \in X \}$ then $\mathbb{B}$ is a basis for the discrete topology X.

Discrete basis $\longrightarrow$ Discrete topology

Trivial basis (ie. X) $\rightarrow$ Trivial topology

Proof: $\mathcal{T}_{\text{discrete}} = \{U \subset X\} \supset \mathcal{T}_{\text{gen by } \mathbb{B}}$ (trivial)

($\text{Let } U \in \mathcal{T}_{\text{gen by } \mathbb{B}} \Rightarrow U \subseteq X \Rightarrow U \in \mathcal{T}_{\text{discrete}}$)

To show $\mathcal{T}_{\text{discrete}} \subset \mathcal{T}_{\text{gen by } \mathbb{B}}$:

Let $U \subseteq X$. Show $U \in \mathcal{T}_{\text{gen } \mathbb{B}}$. For each $x \in U$, the basis element $\{x\} \in \mathbb{B}$ satisfies $x \in \{x\} \subseteq U$. ∎

Lemma: Let X be a set. Let $\mathbb{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathbb{B}$.

Proof: Note $B \in \mathbb{B} \Rightarrow B \in \mathcal{T}$.

($\supseteq$) - Because $\mathcal{T}$ is a topology, a union of elements in $\mathcal{T}$ must also be in $\mathcal{T}$.

($\subseteq$) - Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$, a basis element $B_x \in \mathbb{B}$ s.t. $x \in B_x \subset U$, then $U = \bigcup_{x \in U} B_x$, so U is a union of elements of $\mathbb{B}$. ∎

Lemma $\Rightarrow$ an open set is a union of basis elements.

Ex: The <u>standard topology</u> on $\mathbb{R}$ is given by basis

$$\mathbb{B} = \{ (a,b) \mid a,b \in \mathbb{R} \}.$$

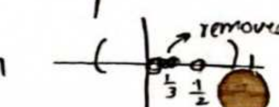
Ex: $\mathbb{B}_\ell = \{ [a,b) \mid a,b \in \mathbb{R} \}$ is a basis for the <u>lower limit topology</u> on $\mathbb{R}$.

$(\mathbb{R}, \mathcal{T})$ given by $\mathbb{B}_\ell$ is denoted $\mathbb{R}_\ell$.

Ex: $\mathbb{B}_K = \{(a,b)\} \cup \{(a,b) \setminus K\}$ where $K = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$

$\mathbb{B}_K$ is basis for the K- topology on $\mathbb{R}$.

$\mathbb{R}_K$ is $\mathbb{R}$ with the topology generated by $\mathbb{B}_K$. (It is finer than the standard topology since it contains subsets of the form



Def: If $\mathcal{T}$ & $\mathcal{T}'$ are two topologies on a set X:

1) If $\mathcal{T}' \supset \mathcal{T}$ then $\mathcal{T}'$ is finer than $\mathcal{T}$. (since it contains more subsets or singletons)

2) If $\mathcal{T}' \supsetneq \mathcal{T}$ then strictly finer

otherwise coarser or strictly coarser. . $\mathcal{T}$ & $\mathcal{T}'$ are comparable if $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}'$.

Ex: Trivial topology is 'coarsest' & discrete topology is 'finest'.

<u>Lemma:</u> $\mathbb{B}$, $\mathbb{B}'$ bases for $\mathcal{T}$, $\mathcal{T}'$ topology on X. The following are equivalent.

1) $\mathcal{T}'$ is finer than $\mathcal{T}$.

2) For each $x \in X$ & each basis element $B \in \mathbb{B}$ containing x, there is a basis element $B' \in \mathbb{B}'$ s.t $x \in B' \subset B$.

<u>Proof</u> (2) $\Rightarrow$ (1) Let $U \in \mathcal{T}$, show $U \in \mathcal{T}'$.

$U = \bigcup_{x \in U} B_x$, basis elts. $B_x \in \mathbb{B}$ with $x \in B_x$ & $B_x \subset U$.

By (2), $\exists$ B_x' s.t $x \in B_x' \subset B_x \subset U$.

$\Rightarrow U = \bigcup_{x \in U} B_x'$, basis elts $B_x' \in \mathbb{B}'$.

$\Rightarrow U \in \mathcal{T}'$.

(1) $\Rightarrow$ (2) $\mathcal{T}' \supset \mathcal{T}$.

Let $x \in B \in \mathbb{B}$. Then $B \in \mathcal{T} \subset \mathcal{T}'$.

Since B is open in $\mathcal{T}'$, there is a basis elt $B' \in \mathbb{B}'$ s.t $x \in B' \subset B$.

■

Lec 3

Ch1-3 & Ch 2.14

Def: A relation $<$ on a set A is called an order relation (or linear order or simple order) if:

① If $x \neq y$ then either $x<y$ or $y<x$.

② For no $x \in A$ does $x<x$ hold $(x \nless x \ \forall x)$

③ If $x<y$ & $y<z$, then $x<z$.

ex: In $\mathbb{R}$, usual order relation $<$.

ex: $\mathbb{R}$, $<_{sq}$ defined below is an order relation.

Define $x <_{sq} y$ if $x^2<y^2$ or if $x^2=y^2$ & $x<y$.

Def: If X set, $<$ order relation, then

open $(a,b) = \{x \mid a<x<b\}$

half open $(a,b] = \{x \mid a<x\leq b\}$

half open $[a,b) = \{x \mid a\leq x<b\}$

closed $[a,b] = \{x \mid a\leq x\leq b\}$

Intervals

$(a,\infty) = \{x \mid x>a\}$

$(-\infty, a)$

$(-\infty, a]$ & $[a,\infty)$

Rays.

Def. The <u>dictionary order</u> (or lexicographic order) $<$ on $A \times B = \{a \times b \mid a \in A, b \in B\}$.

If $(A, <_A)$ and $(B, <_B)$ are simply ordered, define an order relation $<$ on $A \times B$ $a_1 \times b_1 < a_2 \times b_2$ if:

- $a_1 <_A a_2$
- OR if $a_1 = a_2$ & $b_1 < b_2$.

<u>Remark:</u> Easily extended to countable products (of simply ordered sets)

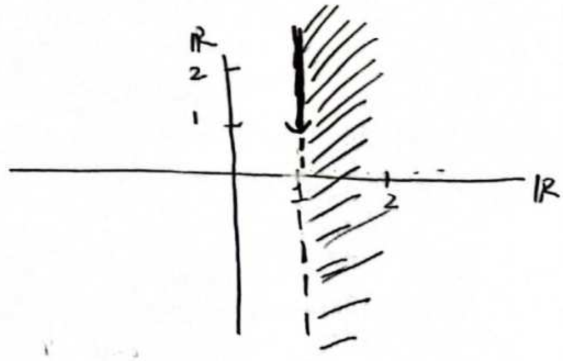
<u>Ex:</u> A = English alphabet, $<$ $(a < b < \dots)$

In $A \times A \times A$, "car" $<$ "cat" $<$ "dog".

<u>Ex:</u> The dictionary order on $\mathbb{R} \times \mathbb{R}$

Let $a = 1 \times 2 \in \mathbb{R} \times \mathbb{R}$

$(a, \infty) = \{ x \times y \mid \begin{smallmatrix} x = 1, \ y > 2 \\ \text{or } x > 1 \end{smallmatrix} \}$



If X is a set with a simple order, there is a standard topology for X, the <u>order topology</u> induced by the order relation.

<u>Def</u> Let X be a set with a simple order relation. Assume X has more than one element. Let $\mathcal{B}$ be the basis for the topology on X, where $\mathcal{B}$ consists of:

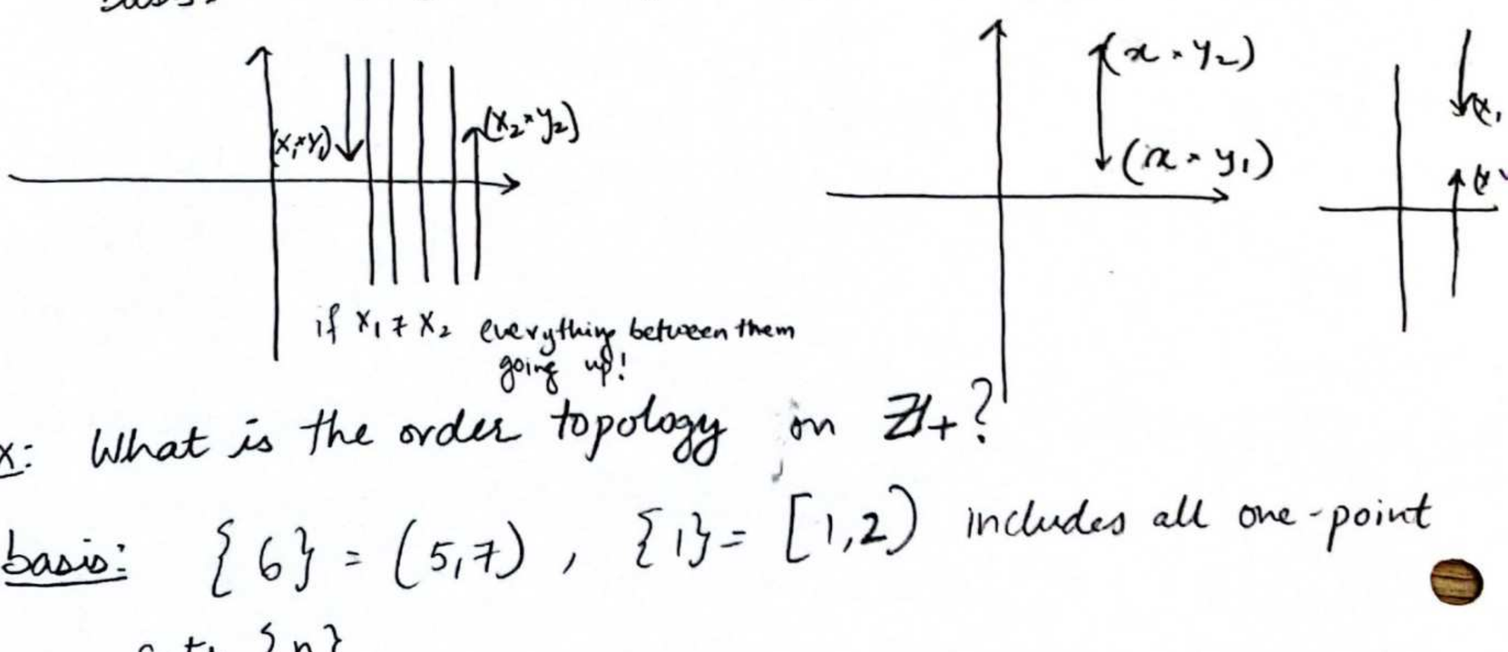
- open intervals (a,b), $a,b \in X$.
- intervals $[a_0, b)$, where a_0 is the smallest element (if any) of X.
- intervals $(a, b_0]$ where b_0 is the largest element (if any) of X.

The topology generated by the basis $\mathcal{B}$ is the <u>order topology</u>.

<u>EX:</u> The standard topology on $\mathbb{R}$ is the ordered topology on $\mathbb{R}$

<u>Ex:</u> Order topology on $\mathbb{R}\times\mathbb{R} = \{x \times y \mid x, y \in \mathbb{R}\}$

basis: $(x_1 \times y_1, x_2 \times y_2) = \{a \times b \mid x_1 \times y_1 < a \times b < x_2 \times y_2\}$



<u>EX:</u> What is the order topology on $\mathbb{Z}_+$?

<u>basis:</u> $\{6\} = (5,7)$, $\{1\} = [1,2)$ includes all one-point sets $\{n\}$

This is the same as the discrete topology.

Ex: The set $X = \{1,2\} \times \mathbb{Z}_+$ has a smallest element : 1×1.

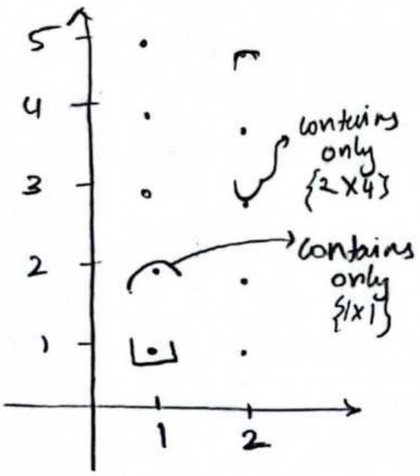
- no largest element
- basis $\{[1\times 1, \underset{\{1,2\}}{\underset{\cap}{a}} \times \underset{\mathbb{Z}_+}{\underset{\cap}{b}})\} \cup \{(a_1 \times b_1, a_2 \times b_2)\}$ $\quad a_i \in \{1,2\}\ b_i \in \mathbb{Z}_+$

Is this the same as the discrete topology? NO

There exists a one point set $\{2\times 1\}$ which is not open.
Any open set containing 2×1 must contain a basis element

$2\times 1 \in (a_1 \times b_1, a_2 \times b_2)$ ← basis elt.

$\Leftrightarrow \quad 2\times 1 \in (1\times b_1, 2\times b_2) \Rightarrow$ A basis elt containing 2×1 must contain more than one point.



order topology basis by defn.

$\underbrace{[1\times 1, a\times b)}_{B1} \cup \underbrace{(a_1 \times b_1, a_2 \times b_2)}_{B2}$

> 1×1 is contained in B1

> except 2×1 all other singletons are contained in some $(a_1\times b_1, a_2 \times b_2)$

> So order topology does not contain $\{2\times 1\}$ but discrete topology does.

Lec 4 Ch 2 §15 Product Topology $X \times Y$

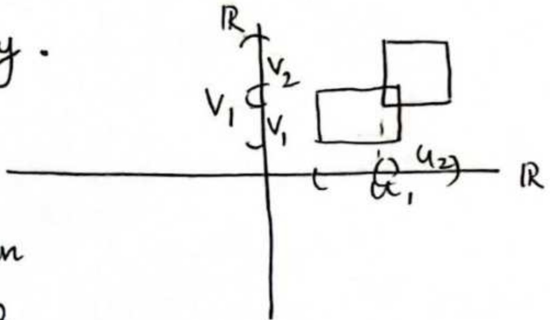
Def: Let X & Y be topological spaces. Then product topology on $X \times Y$ has basis $\mathbb{B} = \{ U \times V \mid U \text{ open in } X,\ V \text{ open in } Y \}$.

Check $\mathbb{B}$ is a basis.

① Since $x \times y \in X \times Y$, and X trivially open in X, Y open in Y;
$x \times y$ belongs to a basis element $X \times Y \in \mathbb{B}$

② If $x \times y \in (U_1 \times V_1) \cap (U_2 \times V_2)$ then $x \times y \in (U_1 \cap U_2) \times (V_1 \cap V_2)$
$\subseteq (U_1 \times V_1) \cap (U_2 \times V_2)$.

Note $\mathbb{B}$ itself is not a topology.



$\Rightarrow$ $(U_1 \times V_1) \cup (U_2 \times V_2)$ is open but not an element of $\mathbb{B}$.

Thm: If $\mathbb{B}$ is a basis for X, $\mathcal{C}$ is a basis for Y, then
$\mathcal{D} = \{ B \times C \mid B \in \mathbb{B},\ C \in \mathcal{C} \}$ is a basis for the product topology of $X \times Y$.

Ex: The standard topology on $\mathbb{R}^2$ is the product topology on $\mathbb{R} \times \mathbb{R}$.

Basis $\mathbb{B}$ = product of all open sets in $\mathbb{R}$ OR Basis $\mathbb{B} = \{ (a,b) \times (c,d) \mid a,b,c,d \in \mathbb{R} \}$

Ex: If X & Y have the discrete topology, what is the product topology on $X \times Y$?

$\{x\}$ is open in X, $\{y\}$ is open in Y.

$\Rightarrow$ $\{x\} \times \{y\} = \{x \times y\}$ open in $X \times Y$

$\Rightarrow$ discrete topology on $X \times Y$.

Ex: (HW2) Order topology on $\mathbb{R} \times \mathbb{R}$ is same as the product topology on $\mathbb{R}_{\text{discrete}} \times \mathbb{R}$.

§16 Subspace Topology

Def: Let $(X, \mathcal{T})$ be a topol. sp. If $Y \subseteq X$ is a subset of X, the collection $\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$ is a topology on Y called the subspace topology.

With this topology on Y, Y is called a <u>subspace</u> of X.

Open sets in Y are intersections of open sets in X with Y.

Ex: $[0,1] \subseteq \mathbb{R}$

$[0, \frac{1}{2})$ not open in $\mathbb{R}$ (due to 0) but open in $[0,1]$

Since $[0,\frac{1}{2}) = [0,1] \cap (-\frac{1}{2}, \frac{1}{2})$ open in $[0,1]$

$\Rightarrow$ $Y \subseteq X$: open in Y $\neq$ open in X.

<u>Lemma</u> : If $Y \subseteq X$ is open in X, and U is open in Y, then U is open in X.

<u>Proof</u> : U open in $Y \Rightarrow U = V \cap Y$, V open in X.

$\Rightarrow$ Since finite intersections of open sets are open, U open in X ∎.

<u>Lemma</u> : If $\mathbb{B}$ is a basis for topology of X then $\mathbb{B}_Y = \{ B \cap Y \mid B \in \mathbb{B} \}$ is a basis for subspace topology of Y.

<u>Ex</u>: $[0,1] \subseteq \mathbb{R}$. The subspace topology for $[0,1]$ has basis

$$\mathbb{B}_{[0,1]} = \{ (a,b) \cap [0,1] \mid a,b \in \mathbb{R} \}$$

$$(a,b) \cap [0,1] = \begin{cases} (a,b) & \text{if } a,b \in [0,1] \\ (a,1] & \text{if } a \in [0,1],\ b \notin [0,1] \\ [0,b) & \text{if } b \in [0,1],\ a \notin [0,1] \\ [0,1] & \text{if } a,b \notin [0,1] \end{cases}$$

Thus, for $[0,1] \subseteq \mathbb{R}$, the subspace topology is the same as the order topology.

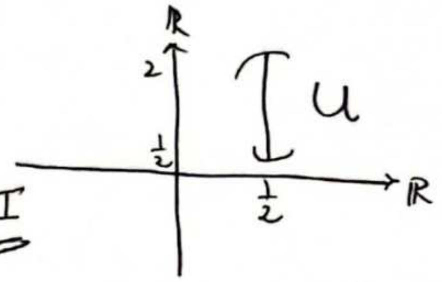
<u>EX</u>: (HW 2) Subspace Topol. $\neq$ order topology.

<u>EX</u>: $\underbrace{[0,1]}_{I} \times \underbrace{[0,1]}_{I} \subseteq \mathbb{R} \times \mathbb{R}$, Subspace topol on $I \times I$ is not the same as order topology on $I \times I$.

$\{\frac{1}{2}\} \times (\frac{1}{2}, 1]$

① $\{\frac{1}{2}\} \times (\frac{1}{2}, 1]$ is open in subspace topology.

The open interval $U = (\frac{1}{2} \times \frac{1}{2}, \frac{1}{2} \times 2)$ in $\mathbb{R} \times \mathbb{R}$ → Here we use the order topology



$\{\frac{1}{2}\} \times (\frac{1}{2}, 1] = U \cap (I \times I)$

open in subspace $\underline{\underline{I \times I}}$

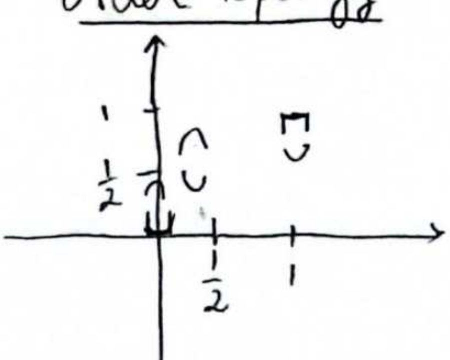
Why exactly is U open in $\mathbb{R} \times \mathbb{R}$? Because it is a basis in the order top. of $\mathbb{R}^2$.

② $\{\frac{1}{2}\} \times (\frac{1}{2}, 1]$ is <u>not</u> open in order top on $I \times I$.

$\frac{1}{2} \times 1$ is not a max in $I \times I$. If $\frac{1}{2} \times 1 \in B$ basis, then

B contains pts. $p \times q > \frac{1}{2} \times 1$. $B \not\subseteq \{\frac{1}{2}\} \times (\frac{1}{2}, 1]$ //

<u>Order topology</u>



Since $\frac{1}{2} \times 1$ is not a max we cannot close at $(\frac{1}{2}, 1]$

Lec (5) §16 continued

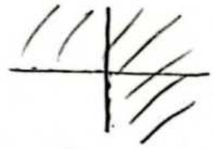
Thm: If $A \subseteq X$ and $B \subseteq Y$, then the product topology on $A \times B$ is the same as the subspace topology on $A \times B \subseteq \underbrace{X \times Y}_{\text{product topology}}$.

§17 Closed sets, interior and closure of a set

Def A subset $A \subseteq X$ is closed if $X \setminus A$ is open.

ex: $[a,b] \subseteq \mathbb{R}$ closed because $[a,b]^c = (-\infty, a) \cup (b, \infty)$

ex: $\{x \times y \mid x \leq 0,\ y \leq 0\} \subseteq \mathbb{R}^2$ closed



$\mathbb{R} \times (0, \infty) \cup (0, \infty) \times \mathbb{R}$ is the compliment which is open.

ex: In discrete topology of X, every set is closed since it's compliment is open.

ex: $Y = [0,1] \cup (2,3) \subseteq \mathbb{R}$ with subspace topology.

$[0,1]$ is open in Y since $[0,1] = (-1,2) \cap Y$

$\Rightarrow Y \setminus [0,1]$ is closed $\Rightarrow (2,3)$ is closed in Y.

$(2,3)$ is open in Y $\Rightarrow Y \setminus (2,3) = [0,1]$ is closed in Y.

Thm. Let X be a topological space. Then

① X and ϕ are closed.

② Arbitrary intersections of closed sets are closed.

③ Finite unions of closed sets are closed.

Remark Can specify a topology on a set by specifying the collection of closed sets instead of open sets.

Sketch of proof ① Trivial.

② $$\underbrace{\bigcap_\alpha Z_\alpha}_{\text{closed sets}} = \underbrace{\bigcap_\alpha U_\alpha^c}_{\text{opensets}} = \Big(\underbrace{\bigcup_\alpha U_\alpha}_{\text{open}}\Big)^c \Rightarrow \text{closed.}$$

③ Similar.

Thm: Let $Y \subseteq X$ subspace. Then $A \subseteq Y$ is closed in Y iff A equals the intersection of a closed set of X with Y.

Proof $(\Leftarrow)$ Suppose $A = C \cap Y$, C closed in X

$\Rightarrow$ $X \setminus C$ is open in X.

$$\underbrace{(X \setminus C)}_{\text{open}} \cap \underbrace{Y}_{\text{open}} = Y \setminus (Y \cap C) = Y \setminus A$$

$\Rightarrow$ A^c is open in Y $\Rightarrow$ A is closed in Y.

$(\Rightarrow)$ (Similar).

Thm: Let $Y \subseteq X$ subspace. If A closed in Y, and Y is closed in X, then A is closed in X.

Closure and Interior of a set

Def: Let A be a subset of a topological space X. The interior of A, denoted by $\operatorname{Int} A$ or $\mathring{A}$, is the union of all open sets contained in A.

Def The closure of A, denoted by $\operatorname{Cl} A$ or $\bar{A}$ is the intersection of all closed sets containing A.

Note: $\operatorname{Int} A \subseteq A \subseteq \bar{A}$.

$\operatorname{Int} A$ is open & $\bar{A}$ is closed.

If A is open, $\operatorname{Int} A = A$. If A is closed, $\bar{A} = A$.

Ex: $A = [0,1) \subseteq \mathbb{R}$. $\operatorname{Int} A = (0,1)$, $\bar{A} = [0,1]$

Notation If $A \subseteq Y \subseteq X$, the closure of A in Y might NOT be the same as the closure of A in X.

Thm: If $Y \subseteq X$ subspace and $A \subseteq Y$, and let $\bar{A}$ = closure of A in X then the closure of A in Y is equal to $\bar{A} \cap Y$.

ex: $Y = [0,1] \subseteq \mathbb{R}$. Let $A = (0, \frac{1}{2}) \subseteq Y$. The closure of A in $\mathbb{R}$ is $[0, \frac{1}{2}]$. The closure of A in Y is $(0, \frac{1}{2}]$.

Proof: Let B = closure of A in Y.

($\subseteq$) $\bar{A}$ is closed in $X \Rightarrow \bar{A} \cap Y$ is closed in Y.

Since $\bar{A} \cap Y \supseteq A$ and B is the intersection of all closed sets in Y containing A,

$\Rightarrow B \subseteq \bar{A} \cap Y$.

($\supseteq$) B is closed in $Y \Rightarrow B = C \cap Y$, C is closed in X.

and since $B \supseteq A$, $C \supseteq A$. Since $\bar{A}$ is the intersection of all closed sets in X containing A, $\bar{A} \subseteq C$.

Thus, $\bar{A} \cap Y \subseteq C \cap Y = B$ ■

Terminology

- A set A intersects B if $A \cap B \neq \phi$
- If $U \subseteq X$ is open set containing $p \in X$, then we say "U is a neighbourhood of p".

Ex: $(-\epsilon, \epsilon) \subseteq \mathbb{R}$ is a neighbourhood of 0.

Thm: $A \subseteq X$ top space.

$x \in \overline{A}$ iff every nbd of X intersects A.

Lecture.

§17 Limit Points, Hausdorff spaces.
(aka cluster points or points of accumulation)

Def: Let A subset of a top. sp. X and $x \in X$. We say that x is a limit point of A if every nbd of x intersects A in some point other than x.

i.e. x is a limit pt. of A if $x \in \overline{A - \{x\}}$

(it doesn't matter if $x \in A$ or not)

ex In $\mathbb{R}$, let $B = \{ \frac{n+1}{n} \mid n \in \mathbb{Z}_+ \}$. $1 \in \mathbb{R}$ is a limit pt. of B.

ex In $\mathbb{R}$, Let $A = (0,1]$. The limit pts. are all pts in $[0,1]$.

ex In $\mathbb{Q} \subseteq \mathbb{R}$. What are the limit pts. of $\mathbb{Q}$? All $\mathbb{R}$.

Another way to define the closure of a set.

Thm: $A \subseteq X$ top. sp. Let A' = set of all limit points of A.
Then $\overline{A} = A \cup A'$.

Proof: ($\supseteq$) $\overline{A} \supseteq A$. If $x \in A'$, then every nbhd of x intersects A $\Rightarrow x \in \overline{A}$.

($\subseteq$) Let $x \in \overline{A}$. If $x \notin A$, every nbd of x intersects A in some $x' \neq x \Rightarrow x$ is a limit pt $\Rightarrow x \in A'$. ∎

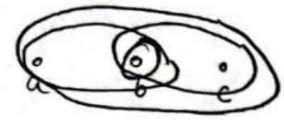
<u>Corollary</u> A subset A of a top. sp. X is closed iff it contains all of its limit points.

<u>Proof</u> : A is closed $\Leftrightarrow A = \bar{A} \underset{\text{by Thm}}{\Longleftrightarrow} A = A \cup \bar{A} \Longleftrightarrow A' \subseteq A$. ∎

<u>Hausdorff spaces</u>

In familiar top spaces, a one pt. $\{x\}$ is closed.

But this is not true in every top space.

<u>ex</u> 

$\{b\}$ is not closed. (since (a,c) is not open)

Other strange behaviour

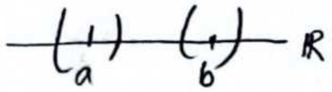
A seq. of pts converge to more than one pt.

<u>Def</u> $\{x_n\}$ <u>converges</u> to $x \in X$ if $\forall$ nbd U of x there is some N s.t $x_n \in U$ $\forall n \geq N$.

In the previous example the sequence $x_n = b$ converges to b, a & c.

To avoid such pathologies we use

<u>Def</u>: A top space X is called a <u>Hausdorff space</u> if for each pair of distinct pts. $x_1, x_2 \in X$, there exist nbds U_1 of x_1, U_2 of x_2 that are disjoint.

<u>eg</u> 

<u>Thm</u> In a Hausdorff space, every finite set is closed.

<u>Proof</u>: Suffices to show $\{x\}$ is closed.

i.e. $X \setminus \{x\}$ is open.

Let $y \in X \setminus \{x\}$. $\exists$ disjoint nbds U_y of x & V_y of y.

Then $X \setminus \{x\} = \bigcup_{\substack{y \neq x \\ y \in X}} V_y$ is open ■.

<u>Def</u> We say a top sp. X in which every finite set is closed satisfies the T_1 axiom.

The previous thm says that Hausdorff space $\underset{\text{(stronger than)}}{\Rightarrow}$ T_1 axiom.

Q: Does $T_1 \Rightarrow$ Hausdorff? No, the cofinite topology on $\mathbb{R}$.

<u>Thm</u>: If X is Hsdf, then the sequence of pts. in X converges to at most one point of X.

<u>eg</u>: $B = \left\{ \frac{n+1}{n} \mid n \in \mathbb{Z}_+ \right\} \subseteq \mathbb{R}$ converges to $1 \in \mathbb{R}$.

<u>ex</u>: $C = \{0, 1, 0, 1 \ldots\}$ does not converge.

Most spaces that topologists study are Hausdorff.

Thm A subspace of a HS is also a HS.

The product of two HS are also HS.

Every simply ordered set with order top. is HS.

Proof : Exercise.

Thm Let X be a space satisfying the T_1 axiom, let $A \subseteq X$.

Then $x \in X$ is a limit point of $A \Leftrightarrow$ every nbd of x contains infinitely many pts. of A.

Proof ($\Leftarrow$) Trivial

($\Rightarrow$) Suppose $x \in X$ is a limit pt. of A.

Suppose U nbd of x. Suppose $U \cap A$ is finite.

$\Rightarrow U \cap (A - \{x\})$ finite. By T_1 axiom, $U \cap (A - \{x\})$ is closed

Let $V = X \setminus (U \cap (A - \{x\}))$ open nbd of x.

But $U \cap V$ nbd of x, doesn't contain any pt. of $A - \{x\}$.

$\divideontimes$ contradiction to x being a limit point. ∎

Ch 2 §18 Continuous Functions

- Let X & Y be topological spaces.

<u>Def</u>: $f: X \to Y$ is called a continuous function if for every open subset A in Y, the preimage $f^{-1}(A)$ is an open subset of X.

<u>Note</u>: Continuity of f depends on f, and on the topologies on X and Y.

To prove continuity of f, it suffices to show the inverse image of every basis elt is open.

- (If $V = \bigcup_\alpha B_\alpha$ open in Y, then $f^{-1}(V) = \bigcup_\alpha f^{-1}(B_\alpha)$ is open).
  B_α ↑ basis elts

<u>ex</u> $f: \mathbb{R} \to \mathbb{R}$ In analysis, "$\varepsilon - \delta$" def of continuity agrees with this def of continuity.

<u>ex</u>: $\mathbb{R}_\ell$: lower limit topology, basis $\{[a,b) \mid a,b \in \mathbb{R}\}$

$f: \mathbb{R} \to \mathbb{R}_\ell$ given by $f(x) = x$ for every real x

Not continuous because $f^{-1}([a,b)) = [a,b)$ not open in $\mathbb{R}$.

But, $g: \mathbb{R}_\ell \to \mathbb{R}$ given by $g(y) = y$ for every real y.

- g is continuous, $\mathbb{R}_\ell$ is finer than $\mathbb{R}$, $g^{-1}((a,b)) = (a,b)$ open in $\mathbb{R}_\ell$.

<u>Thm</u>: Let $f: X \to Y$. The following are equivalent (TFAE).

1) f is continuous.

2) for every subset A of X, $f(\bar{A}) \subseteq \overline{f(A)}$

3) for every closed subset B of Y, $f^{-1}(B)$ is closed in X

4) For each $x \in X$ & each nbd V of $f(x)$, there is a nbd U of x s.t $f(U) \subseteq V$.

ϵ-δ def of cont. for $f: \mathbb{R} \to$ at a pt

<u>Proof</u> $(1) \Rightarrow (4)$

Let $x \in X$, let V be a nbd of $f(x)$

then $\underset{\uparrow \text{nbd of } x}{U} = f^{-1}(V)$ is open and $f(U) \subseteq V$.

$(4) \Rightarrow (1)$ Let V be open set in Y. Let $x \in f^{-1}(V)$. Then $f(x) \in V$.

By (4), there is a nbd U_x of x s.t $f(U_x) \subseteq V \Leftrightarrow U_x \subseteq f^{-1}(V)$.

Then $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ is open in X.

$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$ [Munkres] ■

Def Let $f: X \to Y$ be a bijection with inverse $f^{-1}: Y \to X$. If both f & f^{-1} are continuous, then we call f a homeomorphism.

Equiv. def. A bijection $f: X \to Y$ is a homeomorphism if

$f(U)$ is open iff U is open. ($\forall$ U open).

uses $(f^{-1})^{-1}(U) = f(U)$.

A homeomorphism $f: X \longrightarrow Y$ gives a bijective correspondence b/w X & Y and also the open sets of X and of Y.

Homeomorphisms notion of equivalence for top. space like isomorphisms in algebra.

ex: $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = 3x+1$ is a homeo with inverse,

$g: \mathbb{R} \to \mathbb{R}$ $\quad g(y) = \frac{y-1}{3}$. Check $g(f(x)) = x$ & $f(g(y)) = y$ $\forall$ reals x, y.

ex: $f: (a,b) \longrightarrow (0,1)$, $\quad f(x) = \frac{(x-a)}{b-a}$ homeo with inverse

$g(y) = (b-a)y + a$.

Thm (Rules for constructing cts fns)

(1) Constant fns. are continuous.

(2) Inclusion maps $f: A \longrightarrow X$; $A \subseteq X$ are cts. (maps every element in A to itself)

(3) Compositions of cts. fns are cts.

(4) Restrictions $f\big|_A : A \longrightarrow Y$, $A \subseteq X$ of a cts. $f: X \rightarrow Y$ are cts.

(5) Local formulation of continuity.

If $X = \bigcup_\alpha U_\alpha$, U_α open in X, then $f: X \rightarrow Y$ is continuous if $f\big|_{U_\alpha}$ is cts. for each α.

Thm (Pasting Lemma)

Let $X = A \cup B$; A, B closed in X (important). Let $f: A \longrightarrow Y$, $g: B \rightarrow Y$ cts. fns. that agree on $A \cap B$. (ie. $f(x) = g(x)\ \forall x \in A \cap B$)

Then f, g glue together to make a cts. function $h: X \rightarrow Y$ defined by

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}.$$

Proof Check h is cts. Let C be closed in Y.

Now $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$, since $X = A \cup B$.
& by def. of h.

Since f cts. $f^{-1}(C)$ closed in A, and closed in X.

g cts. $g^{-1}(C)$ closed in X for same reason.

Thus, $h^{-1}(C)$ is closed in X. ∎

Ch2 §19 Product Topology on $\prod_{\alpha \in J} X_\alpha$ $\hookrightarrow$ $X_1 \times X_2 \ldots$

ex $X_1 \times X_2 \times \ldots \times X_n$ finite cartesian product

ex $X_1 \times X_2 \times \ldots$ infinite cartesian product

ex $\prod_{n \in \mathbb{Z}_+} X_n$, $X_n = \mathbb{R}$.

$\mathbb{R}^\omega = \prod_{n \in \mathbb{Z}^+} X_n = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ldots$

Def: The box topology on $\prod_{\alpha \in J} X_\alpha$ has basis all sets of the form $\prod_{\alpha \in J} U_\alpha$ where each U_α is an open set in X_α for each α.

Def: The product topology on $\prod_{\alpha \in J} X_\alpha$ has basis all sets of the form $\prod_{\alpha \in J} U_\alpha$ where each U_α is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many α.

Rmk: If $\prod_{\alpha \in J} X_\alpha$ is a finite Cartesian product, then the box and product topology are the same.

Rmk: Box topology is finer than the product topology.

* We assume that $\prod_{\alpha \in J} X_\alpha$ has the product topology on it, unless stated otherwise.

Thm (Properties that hold for both box and product top.)

① If X_α has basis B_α, then

$\{\prod B_\alpha\}$ is a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$.

$\{\prod B_\alpha\}$ is a basis for the product topology on $\prod_{\alpha \in J} X_\alpha$.
$B_\alpha = X_\alpha$ $\forall$ but finitely many αs.

② If A_α is a subspace of X_α, then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$, if both are given box top or product top.

③ If X_α is Hausdorff, then $\prod X_\alpha$ is Hausdorff (in either box or product)

Thm Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by $f(a) = (f_\alpha(a))_{\alpha \in J}$

Where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod_{\alpha \in J} X_\alpha$ be given the product topology.

Then f is continuous iff each f_α is continuous.

ex: $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \ldots$. Let $f : \mathbb{R} \to \mathbb{R}^\omega$ be given by

$f(t) = (t, t, t, \ldots)$. If $\mathbb{R}^\omega$ is given product topology, then f is continuous.

But $\mathbb{R}^\omega$ is given box top. then f not continuous.

The set $B = (-1,1) \times (-\frac{1}{2}, \frac{1}{2}) \times (\frac{-1}{3}, \frac{1}{3}) \times \cdots$ open in $\mathbb{R}^\omega$ since it is a basis elt.

The preimage $f^{-1}(B) = \{0\}$ ↳ not open in $\mathbb{R}$. since for any $\underset{\text{basis elt}}{(-\epsilon, \epsilon)} \not\subset \{0\}$

§ 20 Metric Spaces.

Def: A metric on a set X is a function $d: X \times X \to \mathbb{R}$ satisfying

① $d(x,y) \geq 0 \ \forall x,y \in X$ and $d(x,y) = 0$ iff $x = y$

② $d(x,y) = d(y,x)$

③ $d(x,y) + d(y,z) \geq d(x,z) \quad \forall x,y,z \in X$

ex: $\mathbb{R}^n$, Euclidean distance d

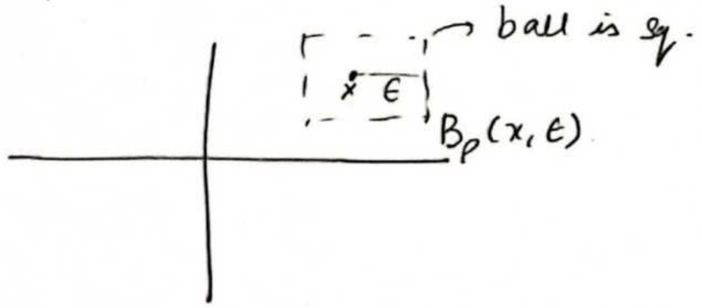
$$d(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2} \quad \text{where } \vec{x} = (x_1 \cdots x_n),\ \vec{y} = (y_1 \cdots y_n)$$

ex: Square metric ρ on $\mathbb{R}^n$ $\quad \rho(\vec{x}, \vec{y}) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$.

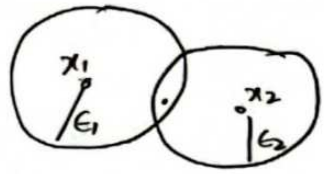
Given a metric d on a set X, $d(x,y) =$ dist. b/w x & y.

Given $\epsilon > 0$, the set $B_d(x, \epsilon) = \{y \in X \mid d(x,y) < \epsilon\} = B(x, \epsilon)$ is called the ϵ-ball centred at $x \in X$.

ex: $(\mathbb{R}^2, \rho)$ sq. metric



Def: If d is a metric on a set X, the collection of all ϵ-balls $B_d(x,\epsilon)$ for $x \in X$, $\epsilon > 0$, is a basis for a topology in X, called metric topology (induced by d). X is called a metric space



① Every $x \in B_d(x, \epsilon)$

② Let $y \in B(x_1, \epsilon_2) \cap B(x_2, \epsilon_2)$

Let $\delta_1 = \epsilon_1 - d(x_1, y)$ $\quad \delta_2 = \epsilon_2 - d(x_2, y)$

Let $\delta = \min(\delta_1, \delta_2)$

$\Rightarrow$ Claim: $y \in B(y, \delta) \subseteq B(x_1, \epsilon_1) \cap B(x_2, \epsilon_2)$.

Let $z \in B(y, \delta)$. Then $d(y, z) < \delta \leq \delta_1 = \epsilon_1 - d(x_1, y)$.

$$\Rightarrow \quad d(x_1, y) + d(y, z) < \epsilon_1$$

$$\Rightarrow \quad d(x_1, z) \leq \epsilon_1 \quad \Rightarrow \quad z \in B(x_1, \epsilon_1)$$

Similarly $z \in B(x_2, \epsilon_2)$ $\Rightarrow z \in B_1 \cap B_2$

A set U is open in the metric topology

$\Leftrightarrow$ For each $y \in U$, $\exists \delta > 0$ S.T $B(y, \delta) \subseteq U$

ex Set X, $d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \neq y \end{cases}$

Then d is a metric and the metric topology is the same as the discrete topology on X; the basis elt $B(x, \underbrace{\varepsilon}_{\varepsilon \leq 1}) = \{x\}$

ex $\mathbb{R}$ with $d(x,y) = |x-y|$. Metric top is same as order top.

Basis elt in metric top; $B(x, \epsilon) = (x-\epsilon, x+\epsilon)$ open in ord top.

and $(a,b) = B\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$

Lec §20-21 Metric Topology.

Def A topological space X is metrizable if there exists a metric d on X that induces the topology of X.

ex $\mathbb{R}^n$, $d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$

ex $\mathbb{R}^n$, $\rho(x,y) = \max\{|x_i - y_i|\}$

thm The topologies on $\mathbb{R}^n$ induced by d and ρ are same as the product topology on $\mathbb{R}^n$.

Lemma Let d, d' metrics on X that induce topols $\mathcal{T}$ & $\mathcal{T}'$ on X. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ iff $\forall x \in X$, $\epsilon > 0$, $\exists \delta > 0$ S.T

$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.

Proof of Lemma.

Suppose $\mathcal{T}'$ is finer than $\mathcal{T}$. Given a basis elt $B_d(x, \epsilon)$ for $\mathcal{T}$, $B_d(x, \epsilon)$ is open in $\mathcal{T}'$, can find basis elt B' of $\mathcal{T}'$ s.t $x \in B' \subseteq B_d(x, \epsilon)$.

Within B', find basis elt $x \in B_{d'}(x, \delta) \subseteq B' \subseteq B_d(x, \epsilon)$.

Conversely, supp. "ϵ-δ" condition holds. Given a basis elt B for $\mathcal{T}$ containing x, and find $B_d(x, \epsilon) \subseteq B$. Then $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$. Thus $\mathcal{T}'$ is finer than $\mathcal{T}$. ■

Proof of Thm

Let $\vec{x} = (x_1 \dots x_n)$; $\vec{y} = (y_1 \dots y_n)$. Can verify $\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$

$B_d(x, \epsilon) \subseteq B_\rho(x, \epsilon)$ (By 1st ineq). By 2nd ineq, $B_\rho(x, \frac{\epsilon}{\sqrt{n}}) \subseteq B_d(x, \epsilon)$.

By lemma, the two topologies induced by ρ & d are the same.

Next, show product topology on $\mathbb{R}^n$ = metric topology induced by ρ

Let $B = (a_1, b_1) \times \dots \times (a_n, b_n)$ basis elt in $\mathbb{R}^n$. Pick $x \in B$, $x = (x_1 \dots x_n)$, for each i, $\exists\ \epsilon_i > 0$ s.t

$$x_i \in (x_i - \epsilon_i, x_i + \epsilon_i) \subseteq (a_i, b_i).$$

Let $\epsilon = \min_{i=1\dots n} \{\epsilon_i\}$.

$$x \in \underbrace{(x_1 - \epsilon, x_1 + \epsilon) \times \dots \times (x_n - \epsilon, x_n + \epsilon)} \subseteq B.$$

$x \in B_\rho(x, \epsilon) \subseteq B \Rightarrow B$ is open in ρ-metric top.

Conversely,

let $B_\rho(x, \epsilon)$ be a basis elt in ρ-topology.

$\Rightarrow$ is also a basis elt in Π - topology. ■

$\to$ countable $\mathbb{R} \times \mathbb{R} \dots$

> Is $\mathbb{R}^\omega$ = metrizable?

<u>Attempts</u>: $d(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{\infty} (x_i - y_i)^2 \right)^{1/2}$ ← might not converge.

$\rho(\vec{x}, \vec{y}) = \sup \{ |x_i - y_i| \}$ might not be defined.

<u>Def</u>: $\bar{\rho}(x,y) = \sup \{ \min \{ |x_i - y_i|, 1 \} \}$

$\bar{\rho}$ is a metric on $\mathbb{R}^\omega$, called uniform metric on $\mathbb{R}^\omega$. $(\mathbb{R}^\omega, \bar{\rho})$ is called the uniform topology.

<u>Thm</u>: On $\mathbb{R}^\omega$, box topology strictly finer than uniform topology is strictly finer than product topology.

<u>Thm</u>: Let $\bar{d}(a,b) = \min \{ |a-b|, 1 \}$ metric on $\mathbb{R}$. If $\vec{x}, \vec{y} \in \mathbb{R}^\omega$ define $D(\vec{x}, \vec{y}) = \sup \{ \frac{\bar{d}(x_i, y_i)}{} \}$ is a metric

that induces the product topology on $\mathbb{R}^\omega$.

<u>Proof</u> $\forall i$, $\dfrac{\bar{d}(x_i, z_i)}{i} \leq \dfrac{\bar{d}(x_i, y_i)}{i} + \dfrac{\bar{d}(y_i, z_i)}{i}$

RHS $\leq D(\bar{x}, \bar{y}) + D(\bar{y}, \bar{z})$ because of sup.

$\Rightarrow \sup\left\{\dfrac{\bar{d}(x_i, z_i)}{i}\right\} \leq D(\vec{x}, \bar{y}) + D(\bar{y}, \vec{z})$.

Thus D is a metric.

$(\mathcal{T}_D \leq \mathcal{T}_\pi)$. Let $U \subseteq \mathcal{T}_D$, let $\vec{x}\, U$. We need to find an open set $V \in \mathcal{T}_\pi$ s.t $x \in V \subseteq U$. Find $\vec{x} \in B_D(x, \epsilon) \subseteq U$.

Choose N s.t $\frac{1}{N} < \epsilon$. Let V be the basis elt in product topology defined by $V = (x_1 - \epsilon, x_1 + \epsilon) \times \dots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \mathbb{R} \times \dots$

Given any $y \in \mathbb{R}^\omega$, $\dfrac{\bar{d}(x_i, y_i)}{i} \leq \dfrac{1}{N} \; \forall i \geq N$.

$\Rightarrow D(\vec{x}, \vec{y}) \leq \max\left\{\dfrac{\bar{d}(x_1, y_1)}{1}, \dots, \dfrac{\bar{d}(x_N, y_N)}{N}, \dfrac{1}{N}\right\}$

If $\vec{y} \in V$, $D(\vec{x}, \vec{y}) < \epsilon$, so $V \subseteq B_D(x, \epsilon)$.

$(\mathcal{T}_\pi \supseteq \mathcal{T}_D) \rightarrow$ in {Munkres} //

§21 Metric Topology

<u>EX</u> : Every metric space is Hausdorff. If $x, y \in (X, d)$, let $\epsilon = \frac{1}{2} d(x,y)$ Then $B_d(x, \epsilon) \cap B_d(y, \epsilon) = \phi$.

> Countable product of metrizable spaces are metrizable (Pf similar to $\mathbb{R}^\omega$)

> If A subspace of (X, d) then $d|_{A \times A}$ is a metric for topology of A.

> Next, Continuous functions and metric spaces.

<u>Thm</u> : (X, d_X) and (Y, d_Y) metric spaces. Let $f : X \to Y$. f is continuous iff $\forall x \in X$ and $\epsilon > 0$, $\exists$ $\delta > 0$ S.T $d_X(x,y) < \delta$ then $d_Y(f(x), f(x)) < \epsilon$.

"ϵ-δ condition for continuity".

<u>Proof</u> ($\Rightarrow$) Suppose f is continuous, Let $x \in X$, $\epsilon > 0$. Consider $f^{-1}(B(f(x), \epsilon))$. open in X & contains x. Choose $\delta > 0$ S.T $x \in B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon))$. Then if $d(x,y) < \delta$ $\Rightarrow d_Y(f(x), f(y)) < \epsilon$.

$\Leftarrow$) Suppose "ϵ-δ" holds. Let $V \subseteq Y$ open. Want to show $f^{-1}(V)$ open in X. Let $x \in f^{-1}(V)$. Since $f(x) \in V$ & V open in Y, can find ball $B(f(x), \epsilon)$ s.t.

$f(x) \in B(f(x), \epsilon) \subseteq V$. By "$\epsilon$-$\delta$", $\exists$ $\delta > 0$,

$B(x, \delta)$ s.t $f(B(x,\delta)) \subseteq B(f(x), \epsilon) \subseteq V$.

$\Rightarrow x \in B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon)) \subseteq f^{-1}(V)$ ■

$\left(\text{open} \because f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} B(x, \delta_x) \right)$

Recall,

<u>def</u>: $x_n \to x$ if for each nbd of U of x, $\exists N$ s.t $\forall n \geq N$, $x_n \in U$.

<u>Sequence Lemma.</u>

Let $A \subseteq X$ ($\Rightarrow$) If there exists a seq. of pts of A converging to x, then $x \in \overline{A}$. ($\Leftarrow$) Converse is true if X is metrizable.

<u>Proof</u> ($\Rightarrow$) $\underset{A}{\overset{a_n}{\cap}} \to x$ Every nbd of x contains a pt. in A. $\Rightarrow x \in \overline{A}$.

($\Leftarrow$) (X, d) is a metric space. Let $x \in \overline{A}$, $A \subseteq X$.

$a_n \in B_d(x, \frac{1}{n}) \cap A$, choose such a_n. Then $a_n \to x$. ■

<u>Thm</u> If $f : X \to Y$ continuous, then if $x_n \to x$ in X, then $f(x_n) \to f(x)$ in Y.

Converse is true if X is metrizable.

<u>Proof</u>

($\Rightarrow$) f is continuous, $x_n \to x$ in X. Let V be a nbd of $f(x)$

$\Rightarrow f^{-1}(V)$ is a nbd of x. $\exists N$ s.t $n \geq N \Rightarrow x_n \in f^{-1}(V)$.

$\Rightarrow f(x_n) \in V$. $\forall n \geq N$.

$\Rightarrow f(x_n) \to f(x)$.

($\Leftarrow$) (X, d) metric space. Suppose every $x_n \to x$ implies $f(x_n) \to f(x)$ in Y. WTS (want to show) f cont.

i.e. $\forall A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$ → eqv condition for cont.

By previous lemma, $x \in \overline{A}$, then $\exists$ seq. $\underset{A}{\underset{\cap}{a_n}} \to x$.

$\Rightarrow \underset{f(A)}{\underset{\cap}{f(a_n)}} \to f(x)$ by assumption $\underset{\text{by prev Lemma}}{\Rightarrow}$ $f(x) \in \overline{f(A)}$. ■

<u>Def</u>: Let $f_n : X \to Y$ seq. of functions. Y, metric space with metric d. We say (f_n) <u>converges uniformly</u> to $f : X \to Y$ if given $\epsilon > 0$, there exists some integer N s.t $d(f_n(x), f(x)) < \epsilon$ $\forall n > N$ & $\forall x \in X$

Thm (Uniform Limit Theorem)

Let $f_n : X \to Y$ seq. of continuous functions (where X is top. space, (Y, d) metric space). If (f_n) converges uniformly to $f : X \to Y$, then f is continuous.

Proof Let V open in Y. WTS $f^{-1}(V)$ is open in X.

Let $x_0 \in f^{-1}(V)$. WTS: $\exists$ nbds U s.t $x_0 \in U \subseteq f^{-1}(V)$
$\Leftrightarrow f(U) \subseteq V$. Choose ϵ-ball $B(f(x_0), \epsilon) \subseteq V$.

By uniform convergence choose N s.t $\forall\ n \geq N$ & all $x \in X$, $d(f_n(x), f(x)) < \frac{\epsilon}{3}$. By continuity of f_N, and "ϵ-δ", can choose nbd $U \ni x_0$ s.t $f_N(U) \subseteq B(f_N(x_0), \frac{\epsilon}{3})$

Claim $f(U) \subseteq B(f(x_0), \epsilon)$.

If $x \in U$, then $d(f(x), f_N(x)) < \frac{\epsilon}{3}$

$d(f_N(x), f_N(x_0)) < \frac{\epsilon}{3}$

$d(f_N(x_0), f(x_0)) < \frac{\epsilon}{3}$

By triangle inequality $d(f(x), f(x_0)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$

$\Rightarrow f(U) \subseteq V$

Lecture. §22 Quotient Topology

<u>Def</u> $\sim$ is an equivalence relation on a set A if

(i) $x \sim x \quad \forall x \in A$

(2) If $x \sim y$, then $y \sim x$

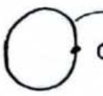
(3) If $x \sim y$, $y \sim z$; then $x \sim z$.

<u>Lemma.</u> Two equivalence classes are either disjoint or equal.

If $\sim$ is an equivalence relation on a set A, and ξ = collection of all eq. classes on A by $\sim$ then

$A = \bigcup_{E \in \xi} E$. ξ is a <u>partition</u> of A, or a collection of disjoint nonempty subsets of A whose union is all of A.

<u>First examples of Quotient spaces.</u>

(1) $\underbrace{[0,1] / 0 \sim 1}$ $\underset{\text{Homeo}}{\cong}$  $0\sim1$ → This is called S^1 → 1D sphere → subspace top on $\mathbb{R}^2$.

→ all pts. except 0,1 are eq. to only themselves & $0 \sim 1$.

(2)

$/ \; p \text{ bdry} \sim q \quad \cong$  S^2

(3) 

$\cong$ 

Def: X, Y top spaces. Let $p: X \to Y$ surjective map. The map p is called a quotient map if $p^{-1}(U)$ is open in $X \iff U$ is open in Y.
$U \subseteq Y$.

("like a strong continuity condition".)

(Need not be open since only open preimages are mapped to open images.)

Def: Given surjective $p: \underset{\substack{\uparrow \\ \text{top sp.}}}{X} \longrightarrow \underset{\substack{\uparrow \\ \text{a set}}}{A}$, there is a unique topology on A s-t p is a quotient map. This topology $\mathcal{T}$ in A is called the quotient topology.

$$\mathcal{T} = \{ U \subseteq A \mid p^{-1}(U) \text{ is open in } X \}.$$

Def X top. space. Let X^* be a partition of X. Let $p: X \to X^*$ be surjective map which maps $x \in X$ to the elt of X^* which contains that point.

With quotient topology, X^* is called a quotient space of X.

ex $X =$  unit disk w/ bdry

$$X^* = \{ \{x\} \mid x \in \text{Int} X \} \cup \{ \text{Bdry of } X \}$$

Can show X^* is homeo to S^2: unit sphere in $\mathbb{R}^3$.

ex: Composition $q \circ p$ of two quotient maps q, p are quotient map.

This follows from $(q \circ p)^{-1}(U) = p^{-1}(q^{-1}(U))$.

Thm: Let $p : X \to Y$ quotient map. Let Z be a space and $g : X \to Z$ be a map that's constant on each set pre-image $p^{-1}(\{y\})$, $\forall y \in Y$.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, dashed, "f"'] & Z \end{tikzcd}$$

Then g induces a map $f : Y \to Z$ s.t. $f \circ p = g$.

① f is cont $\Leftrightarrow$ g is cont.

② f is quotient map $\Leftrightarrow$ g is quotient map.

Proof: Define $f : Y \to Z$. For $y \in Y$, $g(p^{-1}(\{y\})) = \{z_y\}$ one pt. Set in Z

Let $f(y) = z_y$. Then $f(p(x)) = g(x)$

① Suppose f is continuous. Then g is composition of 2 cont. fns $\Rightarrow g$ is continuous.

Conversely suppose g is cont. Let $V \subseteq Z$ open set. WTS $f^{-1}(V)$ open in Y. Since g cont, $g^{-1}(V)$ open in X.

$$g^{-1}(V) = p^{-1}(f^{-1}(V)) \text{ open in } X \underset{\substack{p \text{ is} \\ q\text{-map}}}{\Leftrightarrow} f^{-1}(V) \text{ open in } Y.$$

② Suppose f is a quotient map. $\Rightarrow g = f \circ p$ is quotient map.

Conversely suppose g is a q-map $\Rightarrow g$ surjective $\Rightarrow f$ surjective.

Let $V \subseteq Z$.

V open in Z $\underset{\substack{g \text{ is} \\ \text{q-map}}}{\Longleftrightarrow}$ $g^{-1}(V)$ is open in X

$\qquad\qquad\qquad\qquad\qquad \parallel$

$\qquad\qquad p^{-1}(f^{-1}(V))$ is open in X $\underset{\substack{p \text{ is a} \\ \text{q-map}}}{\Longleftrightarrow}$ $f^{-1}(V)$ is open in Y.

$\blacksquare$

<u>Corollary</u>

Let $p : X \to X^*$ quotient map, s.t. $X^* = \{ g^{-1}(\{z\}) \mid z \in Z \}$.

where $g : X \to Z$ surj. continuous map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ X^* \arrow[r, dashed, "f"'] & Z \end{tikzcd}$$

Then the map g induces a bijective cont. map $f : X^* \to Z$ s.t.

① f is homeo $\Leftrightarrow$ g is quotient map.

② If Z is Hausdorff, then so is X^*.

<u>Proof</u> By prev. thm, g is continuous and g induces cont. $f : X^* \to Z$.

Clear that f is bijective.

① ($\Rightarrow$) If f is homeo, then f is a q-map. Then g = composition of quotient maps is a quotient map.

($\Leftarrow$) Conversely supp. g is a q-map, by previous thm f is a q-map.

$\Rightarrow$ Since f is a bijection, f^{-1} is continuous $\Rightarrow$ f homeo.

② ... [Munkres]

$\blacksquare$

Ch 3 - § 23 Connected Spaces

Connectedness.

Def: A seperation of X is a pair U, V of disjoint nonempty open subsets of X whose union is X.

A space X is connected if there does not exist a seperation of X.

* Equivantly: A space X is connected iff the only subsets of X which are both open and closed are ϕ & X.

Proof: $\Rightarrow$ $\phi \subsetneq A \subsetneq X$, A open & closed.

A closed $\Rightarrow$ $X \setminus A$ open

$\Rightarrow$ $\{A, X \setminus A\}$ form a seperation.

$\Leftarrow$ Suppose X not connected. U, V seperation of X ; $U = X \setminus V$ closed + open. ∎

Connectedness of a subspace. (via limit points).

Lemma. $Y \subseteq X$ (subspace)

A seperation of Y is a pair of disjoint, nonempty sets A and B whose union is Y, neither of which contains a limit pt. of the other.

ex: $X = \mathbb{R}$; $Y = \underset{A}{[0,1)} \cup \underset{B}{\{2\}}$

Proof : (old def $\rightarrow$ new def)

Suppose A, B form seperation of Y. Then A is both open + closed in Y.

The closure of A in Y is $\underbrace{\bar{A}}_{\text{in } X} \cap Y$. Since A is closed in Y, $A = \bar{A} \cap Y$.

$\Rightarrow B \cap \bar{A} = \phi$.

$\bar{A} = A \cup \{\text{limit pts. in } X \text{ of } A\}$, B contains no limit pts. of A.

Similarly, A contains not limit pts. of B.

(new $\rightarrow$ old)

Suppose A, B pair of disjoint nonempty sets A & B whose union is Y, neither of which contains a limit pt. of the other.

$A \cap \bar{B} = \phi$ and $B \cap \bar{A} = \phi$.

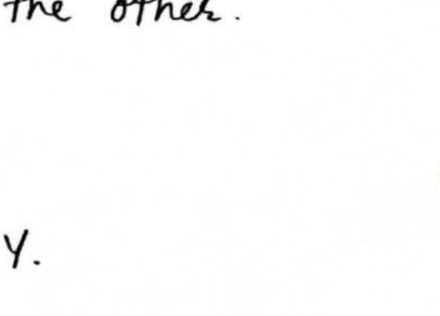
$\Rightarrow Y \cap \bar{B} = B$ and $Y \cap \bar{A} = A \Rightarrow A, B$ closed in Y.

$$\left.\begin{array}{l} A = Y \setminus B \text{ open in } Y \\ B = Y \setminus A \text{ open in } Y \end{array}\right\} \Rightarrow A, B \text{ Seperation of } Y.$$

ex: In $\mathbb{R}^2$, consider the subset

$$X = \{x \times y \mid y = 0\} \cup \{x \times y \mid x > 0 \text{ \& } y = \tfrac{1}{x}\}, \quad X \subseteq \mathbb{R}^2$$

Q Is X connected.



Use lemma, A B disjoint nonempty, $A \cup B = X$

Each A, B does not contain limit pt. of the other set.

Any pt. has a small enough nbd that does not intersect the other.

ex: ($\mathbb{R}^\omega$, box topology) is not connected.

$\mathbb{R}^\omega = A \cup B$

where $A = \{$ unbounded sequences $\}$
$B = \{$ bounded sequences $\}$ are disjoint & open.

$b = (b_i)$ $\quad b_i \le M$ integer $\forall i$.

If $a = (a_i) \in A$, then $a \in U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \dots$
$U \subseteq A$

If $a = (a_i) \in B$, then $a \in U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \dots$
$U \subseteq B$

A, B seperation.

ex: ($\mathbb{R}^\omega$, product topology) is connected. (Munkres)

ex: $\mathbb{R}$ connected (next time).

Thm: A union of connected subspaces of X that have a point in common is connected.

Thm: Let A be a connected subspace of X, if $A \subset B \subset \bar{A}$, then B is connected.

Thm: A finite Cartesian product of connected spaces is connected

Thm: The image of a connected space under a continuous map is connected.

Proof: Let $f: X \to Y$ continuous, X connected. WTS $f(X)$ is connected.
The map $g: X \to f(X)$, $g(a) = f(a)\ \forall a \in X$, is also continuous. ($g$ is surjective).
(Contradiction). Suppose $f(X)$ has a seperation A, B. Then
$g^{-1}(A) \cup g^{-1}(B) = X$ and $g^{-1}(A)$ and $g^{-1}(B)$ are disjoint (g is well defined)
& from continuity of g, $g^{-1}(A)$ & $g^{-1}(B)$ open in X.
Since g surj, $g^{-1}(A)$, $g^{-1}(B)$ are nonempty.
Thus X has a seperation $g^{-1}(A), g^{-1}(B)$ $\divideontimes$ ∎

Lecture §24
Connected Subspaces of $\mathbb{R}$

Def An ordered set A has the least upper bound property (LUB) if every nonempty subset $A_0 \subseteq A$ that is bounded above (by some elt of A) has a least upper bound (called the supremum (sup)) in A.

ex: $\mathbb{R}$ has LUB ; $(0,1)$ has LUB

ex: $\mathbb{Q}$ does not have LUB. $A_0 = \{x \in \mathbb{Q} \mid x < \sqrt{2}\}$

ex: $A = (0,1) \cup (1,2)$, $A_0 = (0,1)$;
↳ NOT LUB

Def A simply ordered set L having more than one elt, is called a linear continuum if

(1) L has LUB

(2) If $x<y$, $\exists\, z$ s.t $x<z<y$.

ex: $\mathbb{R}$, $(0,1)$

nonex: $\mathbb{Z}$ does not satisfy (2).

Thm: If L is a linear continuum, with order topology, then L is connected, and so are the intervals & rays in L.

Pf: If Y is a convex subset of L, we will prove Y is connected.

((Convex: $\forall a,b \in L$; $[a,b] \subseteq L$) WLOG $a<b$.

Suppose $Y = A \cup B$; A, B disjoint non-empty subsets of Y.

$\Rightarrow$ Since Y convex, if $a \in A$, $b \in B$; $[a,b] \subseteq Y = A \cup B$.

$\Rightarrow [a,b] = A_0 \cup B_0$ where $A_0 = [a,b] \cap A$, $B_0 = [a,b] \cap B$; A_0, B_0 disjoint ($a \in A_0$, $b \in B_0$) $\Rightarrow$ nonempty; open subsets of $[a,b]$ from subspace topology = order topology).

$\Rightarrow [a,b]$ seperation A_0, B_0.

A_0 has an upper bound $\Rightarrow$ let $c = \sup A_0$ (exists since L is a lin. continuum).

We will show $c \notin A_0$ and $c \notin B_0$, contradicts $c \in [a,b]$.

Case 1 Suppose $c \in B_0$. Then $c \neq a$, so $c = b$ or $a < c < b$.

$c \in B_0 \Rightarrow (d, c] \subseteq B_0$.

↑ open in $[a,b]$

- If $c = b$, then d is a smaller upper bound for A_0 than c, contradicting $c = \sup A$.
- If $a < c < b$, then $[c, b] \cap A_0 = \phi \Rightarrow (d, b] \cap A_0 = \phi$

  since $(d, c] \cap A_0 = \phi$ & $[c, b] \cap A_0 = \phi$.

  $\Rightarrow d$ is an upper bound for A_0, contradicts $c = \sup A_0$.

Case 2 Suppose $c \in A_0$. Then $c = a$ or $a < c < b$.

$c \in A_0$ ← open subset in $[a,b]$ $\Rightarrow [c, e) \subseteq A_0$.

By (2) of lin. cont., $\exists \underset{A_0}{\overset{}{z}}$ s.t $c < z < e$ Contradicting $c = \sup A_0$ ∎

Corollary $\mathbb{R}$ is connected, and so are intervals & rays in $\mathbb{R}$.

Thm (Intermediate Value Thm).

Let $f : X \to Y$ continuous map, X connected, Y is ordered set with order topology; if $a, b \in X$ and $r \in Y$ s.t $f(a) < r < f(b)$ then $\exists\, c \in X$ s.t $f(c) = r$.

ex: $f : [a,b] \to \mathbb{R}$ cont. → IVT from calc.

Proof If $\nexists\, c \in X$ s.t $f(c) = r$.

$$f(X) = A \cup B \;, \quad \begin{array}{l} A = (-\infty, r) \cap f(X) \\ B = (r, \infty) \cap f(X) \end{array}$$

A, B nonempty since $f(a) \in A$; $f(b) \in B$, disjoint, open in $f(X)$

$\Rightarrow f(X)$ has a seperation.

But image of connected space under cont. map is connected. $\divideontimes$ ∎

ex Linear continuum : $I \times I$ dictionary order, $I = [0,1]$

Path Connectedness $\Rightarrow$ connectedness.

Def: Given $x, y \in X$; a path from x to y is a cont. map

$$f : \underset{\mathbb{R}}{\underset{\cap}{[a,b]}} \longrightarrow X \qquad \text{s.t } f(a) = x, \; f(b) = y.$$

A space is path connected if every pair of pts. in X can be joined by a path in X.

ex: $\mathbb{R}$, Sphere S^2, ...

ex: Ball B^n unit ball $\subseteq \mathbb{R}^n$; $B^n = \{\vec{x} \in \mathbb{R}^n \mid d(\vec{x}, \vec{0}) \leq 1\}$

ex: $\mathbb{R}^n \setminus \{\vec{0}\}$.

Lec (§24 contd.)

Path-connected $\Rightarrow$ connectedness

X $\rightarrow$ path conn, let $f : [a,b] \to X$; $[a,b]$ conn $\Rightarrow f([a,b])$ conn.

Suppose X is not connected. Find $X = A \cup B$ sep of X.

$\Rightarrow f([a,b])$ lies entirely in A or B (otherwise $A \cap \operatorname{Im} f$ & $B \cap \operatorname{Im} f$ seperate $f([a,b])$). There is no path connecting $a \in A$ to $b \in B$. Contradiction to X being path-conn.

ex: $I \times I$, dictionary order. (not path conn but conn).

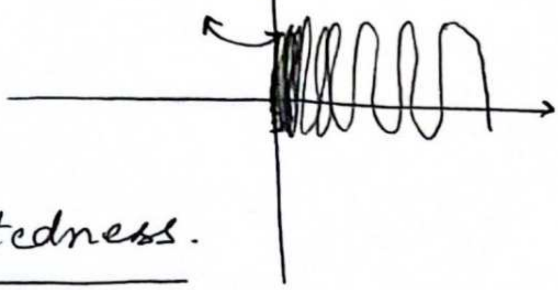
ex: Topologist's sine curve $\bar{S} \subseteq \mathbb{R}^2$.

$$S = \{ x \times \sin\left(\tfrac{1}{x}\right) \mid 0 < x \leq 1 \}$$

S is image of conth. fn $f(x) = \sin\left(\frac{1}{x}\right)$, $f((0,1])$ is conn.

$\Rightarrow S$ conn $\Rightarrow \bar{S}$ conn.

$\bar{S}$ includes $\{0\} \times [-1,1] \cup S$.



$\bar{S}$ is not path connected.

§25 Components and Local Connectedness.

Def Given X space, define an equivalence relation on X by:
$x \sim y$ if there exists a connected subspace of X containing both x & y.

The equivalence classes are called the <u>conn. components</u> or just components of X.

ex: $\mathbb{R} \supseteq Y = (0,1] \cup \{2\}$ ← two components.

ex. $\mathbb{Q} \subseteq \mathbb{R}$ components of $\mathbb{Q}$? Singletons of $\mathbb{Q}$.

<u>Thm:</u> The components (or path components) of X are connected (or path conn.) disjoint subspaces whose union is X, S.T. any nonempty connected (or pathconn.) subspace of X will intersect only one of them.

<u>Proof:</u> • Components C_i of X are equivalence classes
$\Rightarrow$ disjoint, union is X

• Let A be nonempty conn. subspace. Suppose $a \in A \cap C_1$ and $b \in A \cap C_2$. Then $a \sim b \Rightarrow C_1 = C_2$.

• Show C_i is connected. Pick $x_0 \in C_i$ $\forall x \in C_i$, $x \sim x_0$. $\exists$ conn. subspace A_x containing x & x_0.
$\Rightarrow A_x \subseteq C_i$ by above. $C_i = \bigcup_{x \in C_i} A_x$ ← $x_0 \in A_x$ is connected ■

<u>Path Components.</u>

Def X space, define eq. rel. $x \sim y$ if there is a path in X from x to y. The equivalence classes are called <u>path components</u> of X. [Above Thm is also true for path connectedness (analogous proof)]

Rmks: ① Each component of X is closed in X.

(Why? Recall thm: A conn. $\subseteq B \subseteq \bar{A} \Rightarrow B$ conn.

B: component of $X \Rightarrow B$ conn. $\Rightarrow B \subseteq \bar{B} \subseteq B \Rightarrow B = \bar{B}$)

② If X is a finite union of its components, then each component is also open in X.

③ In general, a component of X need not be open.

ex: $\mathbb{Q} \subseteq \mathbb{R}$.

④ Path components need not be open or closed.

ex: topol. sine curve in $\mathbb{R}^2$.

Local Connectedness

Def X is locally conn. at x if $\forall$ nbd U of x, there is a conn. nbd V of x s.t. $x \in V \subseteq U$.

X is locally path conn. at x " " " . . path conn. nbd V of x s.t. $x \in V \subseteq U$.

If X locally conn. at every $x \in X$, then say X is locally conn.

If X locally path conn. " " " , ... path conn.

ex:

1. Intervals & rays in $\mathbb{R}$ (C, PC, LPC, LC)
2. $[-1,0) \cup (0,1) \subseteq \mathbb{R}$ (LPC, LC)
3. $\mathbb{Q} \subseteq \mathbb{R}$ (neither)
4. Topologists' sine curve $\subseteq \mathbb{R}^2$ (C)

Lec: §26 Compact spaces.

Def: A collection $\mathcal{A}$ of subsets of a space X is a covering of X or covers X if $\bigcup_{A \in \mathcal{A}} A = X$. It is an open covering if the elts in A are open in X.

Def: A space X is compact if every open covering of X contains a finite subcollection which also covers X

ex: $\mathbb{R}$ not compact
$\mathcal{A} = \{(n, n+3) \mid n \in \mathbb{Z}\}$ is open covering of $\mathbb{R}$, but no finite subcovering

ex: Any space X containing only a finite # of points $X = \{x_1 \dots x_n\}$
X is compact.

ex: The interval $(0,1] \subseteq \mathbb{R}$ not compact
$\mathcal{A} = \{(\frac{1}{n}, 1] \mid n \in \mathbb{Z}_+\}$ open covering, no finite subcover.

ex: $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\} \subseteq \mathbb{R}$. is compact.
Let $\mathcal{A}$ open cover of X. Then $\exists\, U \in \mathcal{A}$ s.t. $0 \in U \subseteq X$.

U also contains all but finitely many of the points $\frac{1}{n}$, $n \in \mathbb{Z}_+$.

For each $y \notin U$, choose $U_y \in \mathcal{A}$. U & $\bigcup U_y$ is a finite covering.

ex

Next time: $[a,b] \subseteq \mathbb{R}$ compact.

<u>Def</u>: If $Y \subseteq X$ subspace, a collection $\mathcal{A}$ of subsets of X is said to <u>cover Y</u> if the union of its elts. contains Y.

$$\Big(\bigcup_{A \in \mathcal{A}} A \supseteq Y \Big).$$

<u>Lemma</u>: $\underset{\text{Subspace}}{Y \subseteq X}$ is compact iff every covering of Y by open sets in X contains a finite subcollection which also covers Y.

<u>Proof</u>: ($\Rightarrow$) Let $\mathcal{A}$ covering of Y by open sets in X, & Y compact.

$\mathcal{A} = \{A_\alpha\}_{\alpha \in J} \Rightarrow \{A_\alpha \cap Y\}_{\alpha \in J}$ open sets in Y covering Y.

$\Rightarrow$ finite subcover. $\{A_{\alpha_1} \cap Y, \dots, A_{\alpha_n} \cap Y\}$ cover Y

$\Rightarrow$ $\underbrace{\{A_{\alpha_1}, \dots, A_{\alpha_n}\}}_{\subseteq \mathcal{A}}$ cover Y.

($\Leftarrow$) Let $\mathcal{A}' = \{A'_\alpha\}$ be open sets in Y covering Y.

Each $A'_\alpha = U_\alpha \cap Y$, U_α open in X.

Then $\mathcal{A} = \{U_\alpha\}$ open sets in X which cover Y.

$\exists\ U_{\alpha_1}, \ldots, U_{\alpha_n}$ which also cover Y

$\Rightarrow A'_{\alpha_1}, \ldots, A'_{\alpha_n}$ is a finite subcoll of $\mathcal{A}'$ which covers Y

$\Rightarrow Y$ compact. ■

<u>Theorem</u> Every closed subspace of a compact space is compact.

<u>Proof</u>: Let $\underset{\text{closed}}{Y} \subseteq X$, X compact WTS Y compact.

Let $\mathcal{A}$ be a cover of Y by open sets in X.

X has an open cover by $\mathcal{A} \cup (X \setminus Y)$

$\underset{X \text{ compact}}{\exists}$ finite subcollection of $\mathcal{A} \cup (X \setminus Y)$ which covers X.

$\Rightarrow$ " " " $\mathcal{A}$ which covers Y. ■

<u>Theorem</u> The image of a compact space under a continuous function is compact.

<u>Proof</u>: Let $f: X \to Y$ continuous, X compact. W.T.S $f(X)$ compact

Let $\mathcal{A}$ be an open covering of $f(X)$ by open sets in Y.

$X = f^{-1}(Y)$ has an open cover by $\{ f^{-1}(A) \mid A \in \mathcal{A} \}$

$= f^{-1}(f(X))$

$\underset{X \text{ compact}}{\Longrightarrow}$ finite $f^{-1}(A_1), \ldots, f^{-1}(A_n)$ form an open cover of X.

$\Rightarrow A_1, \ldots, A_n$ finite cover of $f(X)$ ■

Theorem: Every compact subspace of a Hausdorff space is closed.

Proof: $Y \subseteq X$, X Hausdorff. Y compact. WTS $X \setminus Y$ open.

Let $x_0 \in X \setminus Y$. We will show $\exists$ nbd of x_0 which is disjoint from Y.

For each $y \in Y$. $\exists$ nbds $U_y \ni x_0$ & $V_y \ni y$ that are disjoint.

$\{V_y\}_{y \in Y}$ open sets in X which covers Y.

$\underset{Y \text{ compact}}{\Rightarrow}$ $V_{y_1} \ldots V_{y_n}$ covers Y.

The open set $V = V_{y_1} \cup \ldots \cup V_{y_n} \supseteq Y$ is disjoint from

$U = U_{y_1} \cap \ldots \cap U_{y_n} \ni x_0$. ∎

ex: In $\mathbb{R}$, (a,b) and $(a,b]$ not compact (since not closed).

Theorem: Let $f: X \to Y$ be a bijection & continuous fn.

If X is compact & Y is Hausdorff, the f is a homeomorphism

Proof: We show f is a closed map. If A is closed in X,

A is also compact $\underset{f \text{ cont.}}{\Rightarrow}$ $f(A)$ compact $\underset{\text{above Thm}}{\Rightarrow}$ $f(A)$ is closed. ∎

Lec

Thm: The finite product of compact spaces is also compact.

Pf sketch: X, Y compact, A open covering of $X \times Y$. Show A admits a finite subcover. Fix $x_0 \in X$. Since the "slice" $x_0 \times Y$ (homeo to Y) is compact, a finite subcollection of sets in A which covers $x_0 \times Y$. These sets actually cover a "tubular nbd" of $x_0 \times Y$, i.e. a thickened slice $x_0 \times Y$, where U_{x_0} is a nbd of x_0.

Since $\{U_x\}_{x \in X}$ open cover of X (compact).

$\Rightarrow U_{x_1}, \ldots, U_{x_n}$ covers X.

Then a finite subcollection of sets in A covers $\bigcup_{i=1}^{n} x \times y = X \times Y$.

Proof follows from induction ∎

§27 Compact Subspaces of the Real line.

We will show $[a,b] \subseteq \mathbb{R}$ is compact. As an application, $f: [a,b] \to \mathbb{R}$ continuous satisfy Extreme Value Theorom (EVT)

Thm (EVT): Let $f: X \to Y$ continuous, Y order topology. If X is compact, $\exists\ c, d \in X$ s.t $f(c) \le f(x) \le f(d)$ $\forall x \in X$.

Pf:
$A = f(X)$ compact. Suppose A has no largest elt.

Then A has an open covering by $\{(-\infty, a) \mid a \in A\}$.

$\Rightarrow (-\infty, a_1), \dots, (-\infty, a_n)$ cover A. Let $a = \max\{a_1, \dots, a_n\}$

is in A but not covered. (Pf for min similar) ∎

Recall $\mathbb{R}$ is a linear continuum, satisfies least upper bound property (LUB).

- A closed $\subseteq X$ compact $\Rightarrow A$ compact.
- B compact $\subseteq Y$ Hausdorff $\Rightarrow B$ closed.

Thm: X = simply ordered set w. least upper bound property (LUB)

In the order top. on X, each closed interval in X is compact.

Pf Outline: ① $x \in [a,b) \Rightarrow \exists\, y > x$ s.t. $[x,y]$ can be covered by ≤ 2 elts. of $\mathcal{A}$.

convex

[Show $[a,b]$ is compact. Let $\mathcal{A}$ open cover of $[a,b]$. (Subspace top = order top).]

② $C = \{y \in [a,b] \mid [a,y]$ has a finite subcover by sets in $\mathcal{A}\}$ is non empty.

③ Let c = LUB of C; show $c \in C$ &

④ $c = b$.

Proof ① Let $x \in [a,b)$.

- If x has an immediate successor y in X, then

$[x, y] = \{x, y\}$ (covered by ≤ 2 elts of A).

- Else, choose $U \in A$ s.t. $x \in U$.

$\underset{\text{open}}{\overset{U}{\Rightarrow}}$ U contains some $[x, c)$, where $c \in [a, b]$

Choose $y \in (x, c) \Rightarrow [x, y] \subseteq U$ (covered by 1 elt. of A)

② $C \neq \emptyset$ follows by ① applied to $x = a$, $\exists y \in C$.

③ $c = lub(C)$ (exists b/c X LUB, C bounded) Show $c \in C$.

i.e. $[a, c]$ has a finite # elts in A covering $[a, c]$.

Choose $A_c \in A$ containing c. $(a < c \leq b)$

A_c open $\Rightarrow \exists (d, c] \subseteq A_c$. Since $d \in [a, b]$.

Suppose $c \notin C$. $\exists z \in C$. s.t $z \in (d, c)$ (else z is a lower upper bound for C).

Since $z \in C$, $[a, z]$ is covered by finite $\underset{(=n)}{\#}$ elts in A.

$[a, c] = [a, z] \cup \underbrace{[z, c]}_{\subseteq A_c}$ $\Rightarrow$ covered by $n+1$ elts of A.

$\Rightarrow c \in C$.

④ Show $c = b$, suppose $c < b$.

Let $x = c$ in ① $\Rightarrow \exists y > c$ s.t $[c, y]$ is covered by 2 elts of A. $[a, y] = [a, c] \cup [c, y]$ covered by finite # of elts.

$\Rightarrow y \in C$ (but $c = lub(C)$) contradiction ∎

Lec §27 (contd.)

Thm: $A \subseteq \mathbb{R}^n$ is compact iff A is closed and bounded in $(\mathbb{R}^n, d)$ or $(\mathbb{R}^n, \rho)$.

Recall: $d(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$

$\rho(\vec{x}, \vec{y}) = \max_i \{ |x_i - y_i| \}$

Rmk: Not true for any arbitrary metric space.

Pf ($\Rightarrow$) $A \subseteq \mathbb{R}^n$ compact. Since $\mathbb{R}^n$ is Hausdorff, $\Rightarrow A$ is closed.

Recall $\rho(\vec{x}, \vec{y}) \leq d(\vec{x}, \vec{y}) \leq \sqrt{n}\, \rho(\vec{x}, \vec{y}) \Rightarrow$ bdedness in $(\mathbb{R}^n, d) \underset{\text{equivalent}}{\Longleftrightarrow}$ bdedness in $(\mathbb{R}^n, \rho)$.

$\{ B_\rho(0, m) \mid m \in \mathbb{Z}_+ \}$ open cover of A

$B_\rho(0, m_1), \dots, B_\rho(0, m_j)$ cover $A \Rightarrow m = \max_{i=1,\dots,j} \{ m_i \}$

$\Rightarrow A \subseteq B_\rho(0, m) \Rightarrow \rho(\vec{x}, \vec{y}) \leq 2m \ \forall \vec{x}, \vec{y} \in A. \Rightarrow A$ bded

($\Leftarrow$) A closed, bounded in $(\mathbb{R}^n, \rho)$. Suffices to show $\underset{(\text{closed})}{A \subset}$ some compact space.

$\Rightarrow \rho(\vec{x}, \vec{y}) \leq N$ for some $N > 0$ $\forall \vec{x}, \vec{y} \in A$.

Choose $x_0 \in A$, call $\rho(\vec{x_0}, 0) = b$.

For $\vec{x} \in A$, $\rho(\vec{x},\vec{0}) \leq \rho(\vec{x},\vec{x_0}) + \rho(\vec{x_0},\vec{0})$

$\Rightarrow A \subseteq \overline{B_\rho(0, N+b+1)} \Rightarrow A$ compact ■

ex Are the spaces compact?

(1) $S^{n-1} \subseteq \mathbb{R}^n$ unit sphere. ✓

(2) $\overline{B^n} \subseteq \mathbb{R}^n$ closed unit ball ✓

(3) $A = \{x \times (\frac{1}{x}) \mid 0 < x \leq 1\} \subseteq \mathbb{R}^2$ ✗

Uniform Continuity Theorem.

Def: $f: X \to Y$, (X, d_X), (Y, d_Y) metric spaces is uniformly continuous if $\forall \epsilon > 0$, $\exists \delta > 0$ S.T $d_X(x_0, x_1) < \delta$ $\Rightarrow d_Y(f(x_0), f(x_1)) < \epsilon$ $\forall x_0, x_1 \in X$.

Thm: (Uniform Continuity Theorem).

Let $f: X \to Y$ be continuous, (X, d_X) compact metric space, (Y, d_Y) metric sp. Then f is uniformly continuous.

Def: (X, d_X) metric space and $A \subseteq X$ nonempty subset. For any $x \in X$, $d(x, A) = \underset{(glb)}{\inf} \{d(x,a) \mid a \in A\}$

Claim: $d = d(-, A): X \to \mathbb{R}$ (for fixed A) is a continuous function of x.

For any $x, y \in X$, $a \in A$; $d(x,A) \leq d(x,a) \leq d(x,y) + d(y,a)$

$$\Rightarrow d(x,A) - d(x,y) \leq d(y,A)$$

$$\Rightarrow d(x,A) - d(y,A) \leq d(x,y)$$

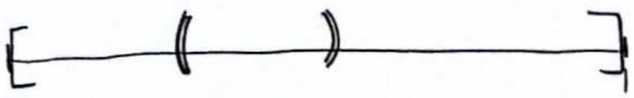
$$\Rightarrow |d(x,A) - d(y,A)| \leq d(x,y)$$

$\Rightarrow$ d is continuous.

Recall,
diam $A = \sup\{d_X(a_1, a_2) \mid a_1, a_2 \in A\}$ for any bded subset $A \subseteq (X, d_X)$.

<u>Lemma</u> (Lebesgue Number Thm). (Proof omitted).
Let $\mathcal{A}$ be an open covering of a metric space (X,d).
If X compact, there is a $\delta > 0$ s.t. for each subset of X of diam $< \delta$, there exists an elt of $\mathcal{A}$ that contains it.
(Def: δ is called a Lebesgue number of $\mathcal{A}$).

<u>ex</u>: $[0,1]$ has covering $[0, 0.6)$, $(0.4, 1]$, $(0.4, 0.6)$



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What is the Lebesgue number for this covering? Any no. ≤ 2.

Proof of Uniform Continuity thm.

Given $\epsilon > 0$: Choose $_{\wedge}^{\text{(open)}}$ covering of Y by $\frac{\epsilon}{2}$-radius balls

$\{B_{d_Y}(y, \frac{\epsilon}{2})\}_{y \in Y}$. Then $\mathcal{A} = \{f^{-1}(B_{d_Y}(y, \frac{\epsilon}{2}))\}$ open cover of X.

Choose a Lebesgue number δ for $\mathcal{A}$. If $d_X(x_1, x_2) < \delta$ then $\{x_1, x_2\}$ has diam $< \delta$. By lemma, $\{x_1, x_2\}$ lies in some set $A \in \mathcal{A}$.

$\Rightarrow \{f(x_1), f(x_2)\} \subseteq f(A) = B_{d_Y}(y, \frac{\epsilon}{2}) \Rightarrow d_Y(f(x_1), f(x_2)) \underset{\text{by triang ineq.}}{<} \epsilon$ ∎

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midterm

REVIEW

Thm A space X is locally path-connected iff for every open set U of X, each path component of U is open in X.

Rem. Analagous for connected $\leftrightarrow$ component.

Proof of Thm ($\Rightarrow$) X locally path connected, Let U open set in X, and C be a path component of U. WTS C is open.

Let $x \in C \subseteq U$. By local path conn. condition, $\exists$ a path conn. nbd V_x of x s.t $x \in V_x \subseteq U$. Because V_x is path conn, $V_x \subseteq C$.

$\Rightarrow C$ is open. (every pt. in C has an $_{\wedge}^{\text{open}}$ nbd in C)

($\Leftarrow$) Suppose path components of open set are open in X.

Let $x \in X$ and U open set containing x.

Let C be the path component of U containing x.

$\Rightarrow$ C is open & $x \in C \subseteq U$. $\Rightarrow$ X is locally path-conn ■

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§28 Limit Point Compactness.

Def: A space X is called limit point compact if every infinite subset of X has a limit point.

Recall: $A \subseteq X$, then $x \in X$ is a limit pt. of A if every nbd of x intersects A in some pt. other than itself. (i.e. $x \in \overline{A - \{x\}}$).

Thm: Compactness implies limit pt. compactness, but not conversely.

Proof: ($\not\Leftarrow$) by counterex.

$$Y = \{a, b\} \quad T = \{\phi, \{a,b\}\}, \quad X = \underset{\hookrightarrow \text{discrete top.}}{\mathbb{Z}_+} \times Y$$

X is limit pt compact but not compact. Every non empty subset of X has a limit pt.

$(n, a) \in A$ subset of X

Let U be nbd of (n,a) $\Rightarrow$ $(n,a) \in \{n\} \times Y$ open (basis elt).

$\Rightarrow$ (n,b) is a limit pt. of A.

But X is not compact, $\{\{n\} \times Y\}$ is open cover, no finit subcover.

($\Rightarrow$) X compact. If $A \subseteq X$ has no limit pt., then

WTS A is finite

If A has no limit pts, then A contains all its limit pts. $\Rightarrow A$ closed.

For each $a \in A$, choose a nbd $U_a \ni a$ s.t $U_a \cap A = \{a\}$.

$\{U_a\}_{a \in A} \cup (X \setminus A)$ open cover of X.

$\Rightarrow U_{a_1}, \ldots, U_{a_n}$; $X \setminus A$ open cover of X.

$A \subseteq (U_{a_1} \cap A) \cup \ldots \cup (U_{a_n} \cap A)$

$\Rightarrow A = \{a_1, \ldots, a_n\} \Rightarrow A$ is finite. ∎

<u>Def</u> X space. If (x_n) sequence of pts. in X, and if $n_1 < n_2 < \ldots < n_i < \ldots$ is an increasing seq. of pts. in $\mathbb{Z}_+$, then $y_i = x_{n_i}$ is called a subsequence of (x_n).

<u>Def</u>. X is <u>sequentially compact</u> if every sequence of pts. of X has a convergent subsequence.

<u>Recall</u>: y_n converges to y $(y_n \to y)$ if $\forall$ nbd. U of y, $\exists N$ s.t $y_n \in U$ $\forall n \geq N$.

ex: In $\mathbb{R}$, $(1,0,1,0,\dots)$ does not converge but has a convergent subseq $(1,1,1\dots)$

Thm: Let X metrizable space. Then TFAE.

1) X is compact
2) X is limit pt. compact
3) X is sequentially compact

Proof: (1) $\Rightarrow$ (2) True in general.

(2) $\Rightarrow$ (3)

Given (x_n), WTS $\exists$ convergent subseq.

Let $A = \{x_n \mid n \in \mathbb{Z}_+\}$. If A is finite, the proof is easy
unique values that show up in (x_n) since the seq. is eventually const.

Assume A is infinite, by limit pt. compactness, A has a limit pt. x.

Construct a convergent subseq of (x_n) by choosing:

$x_{n_1} \in B(x,1)$, $x_{n_2} \in B(x,\frac{1}{2})$, $\dots$, $x_{n_i} \in B(x,\frac{1}{i})$, $\dots$

(can guarantee $n_{i+1} > n_i$ at each step since $B(x,\frac{1}{i})$ intersects A in infinitely many points).

(x_{n_i}) is a convergent sub. $\to x$.

(3) $\Rightarrow$ (1) This is a hefty analytical proof (book) ■

Practice Question

- Let $X = \{a, b, c\}$

For each of the following topol, write down a path from a to c if one exists, or show "DNE".

(1) $\mathcal{T} = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}\}$

(2) $\mathcal{T} = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}\}$

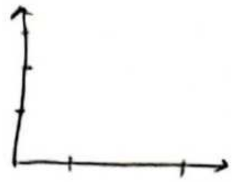
(3) $\mathcal{T} = \{\phi, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}\}$

Sol: My attempt

- $f : [a, b] \to X$ continuous.

${}_{1}{}_{3}$

$f(a) = \{a\}$ $\quad f(b) = \{c\}$



(1) $\{a\}, \{b, c\}$ seperation of X.

$f : [0, 1] \to X$ $\quad f(0) = a$, $f(1) = c$

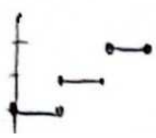
$f^{-1}(\{a\})$, $f^{-1}(\{b, c\})$ open, disjoint & $f^{-1}(\{a\}) \cup f^{-1}(\{b, c\}) = [0, 1]$

Contradiction since $[0, 1]$ is connected

(2) $f(x) = \begin{cases} a & 0 \le x < \frac{1}{3} \\ b & \frac{1}{3} \le x < \frac{2}{3} \\ c & \frac{2}{3} \le x \le 1 \end{cases}$ $\quad f^{-1}(\{a, b\}) = [0, \frac{2}{3})$ open

- $f^{-1}(\{a\}) = [0, \frac{1}{3})$ open

(3) $f(x) = \begin{cases} a & 0 \le x < \frac{1}{3} \\ b & \frac{1}{3} \le x \le 2/3 \\ c & \frac{2}{3} < x \le 1 \end{cases}$



§29 Local Compactness.

Def: A space X is locally compact at a pt. if there is some compact subspace C of X that contains a nbd of x.

If X locally compact at each of its points, then we say X is locally compact.

ex: Any compact space is locally compact.

ex: $\mathbb{R}$ is locally compact. $x \in (a,b) \subseteq [a,b]$.

ex: $\mathbb{R}^n$ is locally compact. $\vec{x} \in (a_1, b_1) \times \dots \times (a_n, b_n)$

$$\subseteq \prod_{i=1}^{n} [a_i, b_i]$$

ex: Any simply ordered set X having LUB is locally compact.
Given $x \in X$ & $x \in$ basis elt for X, its closure is closed.
From Thm it is also compact.

ex: $\mathbb{R}^\omega$ not locally compact. (product top)

$B = (a_1, b_1) \times (a_2, b_2) \times \mathbb{R} \times \mathbb{R} \times \dots$

If $B \subseteq$ compact C, then $\overline{B} = [a_1, b_1] \times [a_2, b_2] \times \mathbb{R} \times \mathbb{R} \times \dots$
not compact since $\mathbb{R}$ is not compact.

C is closed since $\mathbb{R}^\omega$ is Hf. so $\overline{B} \subseteq C \overset{\text{Thm}}{\Rightarrow} \overline{B}$ is compact $\divideontimes$.

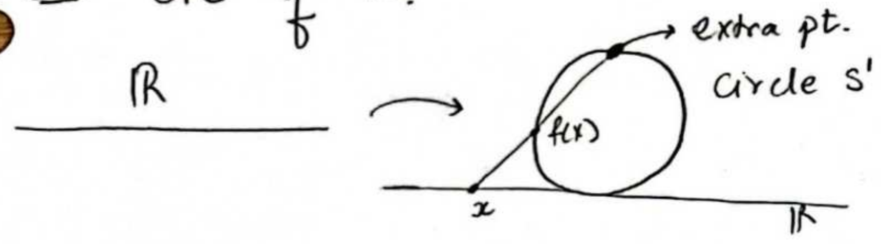
Thm X is locally compact Hausdorff iff there exists a space Y satisfying the following properties:

Existence
(1) X is a subspace of Y
(2) Y is a compact Hf space
(3) $Y \setminus X$ is a single point.

Uniqueness
If Y, Y' are two spaces satisfying these conditions, then there is a homeo $Y \to Y'$ that equals the identity map on X.

Def If Y is compact Hausdorff; $X \subsetneq Y$ s.t $Y \setminus X$ is a single point we call Y the <u>one point compactification of X</u>.

ex: OPC of $\mathbb{R}$?



ex: $\mathbb{R}^2 \longrightarrow S^2$

Lec
Proof of Thm
(Uniqueness) Y, Y' satisfying ①-③.

Def $h : Y \longrightarrow Y'$. If $Y \setminus X = \{p\}$, $Y' \setminus X = \{q\}$,

then let $h|_X = id_X \to$ identity map & $h(p) = q \Rightarrow h$ bijective.

We'll show if $U \subseteq Y$ open, $h(U) \subseteq Y'$ is open (By symmetry, h is homeo

$\underline{\text{Case 1}}$ $p \notin U$. Then $h(U) = U$, $U \subseteq X \subseteq Y$

$\Rightarrow$ U open in X $\Rightarrow$ $U \subseteq X \subseteq Y'$ & U open in Y' since X is
U open in Y

open in Y' (since X^c is $\underset{\hookrightarrow \text{ompt.}}{\text{finite}}$ & Y' being Hf $\Rightarrow$ X^c is closed).

$\underline{\text{Case 2}}$ $p \in U$. Then $C = Y \setminus U$ closed $\underset{Y\text{compact}}{\Longrightarrow}$ C is compact in Y.

Also $C \subseteq X \Rightarrow C$ is also compact in X.

$X \subseteq Y' \Rightarrow C$ is also compact in Y'. $\underset{Y' \text{Hf}}{\Longrightarrow}$ C is closed in Y'

$\Rightarrow h(U) = Y' \setminus C$ is open.

$\underline{\text{Existence.}}$

$\Rightarrow$) Suppose X is locally compact, Hausdorff.

① Form $Y = X \cup \{\infty\}$ $\hookrightarrow$ a pt. not in X

② Topology on Y:

$$T_y = T_X \cup \{ Y - C \mid C \text{ compact in } X \}$$

Check T_y is a topology (case work)

Intersection of 2 opens is open:

(1) $U_1 \cap U_2 = U$ open.

(2) $U_1 \cap (Y - C_1) = U_1 \cap \underset{\hookrightarrow \text{closed in } X}{(X - C_1)}$ $\Rightarrow$ $U_1 \cap (X - C_1)$ open in Y.

since extra pt. does not lie in U_1.

(3) $(Y - C_1) \cap (Y - C_2) = Y - (\underbrace{C_1 \cup C_2}_{\text{compact in } X})$ $\Rightarrow$ open in Y.

Unions of arb. opens is open.

1) $\bigcup U_\alpha = U$ open in X $\Rightarrow$ open

2) $\bigcup (Y - C_\beta) = Y - \underbrace{\bigcap C_\beta}_{\text{closed} \subseteq Y \hookrightarrow \text{compact}}$ $\Rightarrow \bigcap C_\beta$ is compact

$\Rightarrow$ open (of the 2nd type)

3) $\underbrace{\left(\bigcup U_\alpha\right)}_{\substack{\text{open} \\ U}} \cup \underbrace{\left(\bigcup (Y - C_\beta)\right)}_{Y - C \hookrightarrow \text{compact in } X} = \underbrace{Y - (\underbrace{C - U}_{\substack{\text{closed in} \subseteq C \text{ compact} \\ \Rightarrow C - U \text{ compact}}})}_{\text{open (of 2nd type)}}$ where U is open

③ $X \subseteq Y$ subspace

Check $T_X = \{V \cap X \mid V \text{ open in } Y\}$ (Check $\supseteq$ & $\subseteq$)

($\subseteq$) $T_X \subseteq T_Y$ implies ($\subseteq$)

($\supseteq$) V open in Y

- If $V = U$ open in X, $V \cap X = V$ open in X
- If $V = Y - C$ (C compact in X), then $V \cap X = (Y - C) \cap X = (X - C) \cap X = X - C$ (C closed, so $X - C$ open in X)

④ Show Y is compact. Let A be open cover of Y.

A must contain a set $Y - C$ (C compact in X) which contains ∞.

C compact in X $\Rightarrow$ $\mathcal{A}$ has a finite subcover of C.

$\Rightarrow$ together with $Y-C$, forms finite subcover of Y.

(5) Y is Hausdorff.

Let $x, y \in Y$ $(x \neq y)$.

- If $x, y \in X$, done since X Hf.
- If $x \in X$, $y = \infty$. Use local compactness of X to find a compact $C \supseteq U \ni x$. Then $Y - C \ni \infty$, $U \ni x$ open disjoint nbds.

($\Leftarrow$) Converse. (Show X is locally compact & Hf).

- X Hausdorff since $X \underset{\text{subspace}}{\subseteq} Y$ Hausdorff.
- $x \in X$. Choose U, V open $\underset{\text{in } Y}{\text{nbds}}$ $U \ni x$, $V \ni \infty$ disjoint.

  $C = Y - V$ closed $\subseteq Y$ compact $\Rightarrow$ C compact in Y.

  $C \subseteq X \Rightarrow C$ compact in X.

  $\Rightarrow x \in U \subseteq C$ compact.

$\blacksquare$

Ch 4 Countability and Seperation Axioms.

Motivation

> When does a space X imbed in a metric space?

> When does a space X imbed in $\mathbb{R}^n$, some n?

embedding : map $f : X \to Y$ f is homeo onto $f(X)$.
$\Rightarrow X$ is embedded into Y.

Urysohn metrization theorem.

If X is second countable and regular then X can be imbedded in a metric space.

§30 Countability Axioms

Def: A space X has a countable basis at $x \in X$ if there is a countable collection $\mathcal{B}$ of nbds of x s.t every nbd of x contains a $B \in \mathcal{B}$.

If X has a countable basis at each of it's pts. we say X is first countable (or satisfies the 1^{st} countability axiom).

eg: If (X, d) is a metric space, $\{B_d(x, \frac{1}{n}) \mid n \in \mathbb{Z}_+\}$ countable basis at x.

eg: $\mathbb{R}_\ell$ first-countable. Given $x \in \mathbb{R}_\ell$, $\{[x, x+\frac{1}{n}) \mid n \in \mathbb{Z}_+\}$

Def: A space X has a countable basis for its topology, then X is <u>second-countable</u> (or satisfies 2^{nd} countability axiom)

Rmk: Second countable $\Rightarrow$ first countable.

ex: $\mathbb{R}$ is second countable, w. basis $\{(a,b) \mid a, b \in \mathbb{Q}\}$

ex: $\mathbb{R}^n$ is also second countable w. basis $\{(a_1, b_1) \times \cdots \times (a_n, b_n) \mid a_i, b_i \in \mathbb{Q}\}$

ex: $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \cdots$ w. product topology has countable basis (same as above)

$$\left\{ \prod_{i \in \mathbb{Z}_+} U_i \;\middle|\; \begin{array}{l} U_i = (a_i, b_i) \quad a_i, b_i \in \mathbb{Q} \text{ for finitely many } i \text{ and} \\ U_i = \mathbb{R} \text{ for all other } i \end{array} \right\}$$

ex: In the uniform topology on $\mathbb{R}^\omega$, $\bar{\rho}(\vec{x}, \vec{y}) = \sup(\bar{d}(x_i, y_i))$

metrizable $\Rightarrow$ first countability. but not second countable.

seqs in $\mathbb{R}^\omega$ of 0's & 1's $\Rightarrow \underbrace{\{0,1\}^\omega}_{\text{uncountable}} \subseteq \mathbb{R}^\omega$ discrete top.

$a \neq b \quad \bar{\rho}(a,b) = 0$

$\Rightarrow (\mathbb{R}^\omega, \bar{\rho})$ uncountable basis.

Thm: ① If $A \subseteq X$, X is 1^{st} or 2^{nd} countable then A is also 1^{st} or 2^{nd} countable.

② A countable product of 1^{st}, 2^{nd} countable spaces is also 1^{st} or 2^{nd} countable.

Dense subsets

Def A subset $A \subseteq X$ is dense in X if $\overline{A} = X$.

ex: $\mathbb{Q}$ dense in $\mathbb{R}$

Thm: Suppose X second-countable. Then

(a) Every open cover of X has a countable subcollection covering X. (X is Lindelof)

(b) There exists a countable subset of X that is dense in X (X is seperable) weaker than second-countable.

Proof Let $\{B_n\}$ be a countable basis for X.

a) Let $\mathcal{A}$ open cover of X, choose $n \in \mathbb{Z}_+$. Choose $A_n \in \mathcal{A}$ s.t $B_n \subseteq A_n$ (if possible).

$\mathcal{A}' = \{A_n\} \subseteq \mathcal{A}$ is a countable subcollection.

$\mathcal{A}'$ covers X: Given $x \in X$, $\exists\ A \in \mathcal{A}$ s.t $x \in A$.

But A open $\Rightarrow \exists$ basis elt B_n s.t $x \in B_n \subseteq A$.

$\Rightarrow B_n$ lies in $A_n \Rightarrow x \in B_n \subseteq A_n$.

b) Choose $x_n \in B_n$. Let $D = \{x_n\}$. Then D is dense in X

Given any $x \in X$, every basis elt containing x intersects D

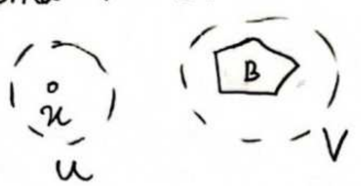
So $x \in \overline{D}$. ∎

§31 <u>Seperation Axioms</u> (Not seperation from connectedness)

> Recall, Hausdorff space X: $x \neq y$ $\exists$ open disjoint U, V containing x & y respectively.

<u>Def</u>: Suppose one-pt. sets are closed in X. (T_1 is satisfied).

- X is <u>regular</u> if: for each pair consisting of a point $x \in X$ and a closed set B disjoint from x. There are open disjoint sets $U \ni x$ and $V \supset B$.



- X is <u>normal</u> if: for each pair of disjoint closed sets A & B, $\exists$ open disjoint sets $U \supset A$ & $V \supset B$.

<u>Rem</u>: Normal $\Rightarrow$ Regular $\Rightarrow$ Hausdorff $\Rightarrow$ T1

<u>Lemma</u>. Let X be a T_1-space.

(a) X is <u>regular</u> iff given $x \in X$ and nbd U of x, there exists a nbd V of x s.t $\overline{V} \subseteq U$.

(b) X is <u>normal</u> iff given any closed set A and open set U containing A, there is an open set $V \supseteq A$ s.t $\overline{V} \subseteq U$.

<u>Proof</u>: (a) ($\Rightarrow$) X is reg. Let $x \in X$ and a nbd U of x, $X \setminus U$ closed. By regular cond. of X, $\exists$ open disjoint sets $V \ni x$ and $W \supseteq (X \setminus U)$. Then $\bar{V} \subset U$ since $V \subseteq U$ and if $y \in X \setminus U$, then W is a nbd of y disjoint from V.

($\Leftarrow$): Let $x \in X$, B closed set in X not containing x. $X \setminus B$ open nbd of x. By hypothesis, $\exists$ nbd V of x s.t $\bar{V} \subseteq (X \setminus B)$.

The open sets V and $X \setminus \bar{V}$ are disjoint sets containing x and B resp.

(b) Same; replace the pt. x with closed set A. ∎

<u>Thm</u> (a) A subspace of a regular space is regular.
(b) A product of regular spaces is also regular.

<u>Rem</u>. Thm is also true for Hf instead of regular, but not for normal spaces.

<u>Proof of Thm</u>: (a) $Y \subseteq X$, X regular. Let $x \in Y$ and B closed set in Y not containing x.

$B = \bar{B} \cap Y$ where $\bar{B}$ is closure of B in X, and $x \notin \bar{B}$.

By reg. of X, $\exists$ open (in X) disjoint sets $U \ni x$ & $V \supset \bar{B}$.

Then $U \cap Y \ni x$ and $V \cap Y \supset B$, open disjoint in Y.

(b) X_α reg. WTS $X = \prod X_\alpha$ is reg.

Let $x = (x_\alpha)$ point of X, U nbd of x. We use the prec. lemma.

Choose a basis elt. $x \in \prod U_\alpha \subseteq U$, each U_α open in X_α, nbd of x_α.

By lemma, choose a nbd $V_\alpha \ni x_\alpha$ s.t $\overline{V_\alpha} \subseteq U_\alpha$.

(if $U_\alpha = X_\alpha$ then $V_\alpha = X_\alpha$). Then $V = \prod V_\alpha$ is nbd of x.

Since $\overline{V} = \prod \overline{V_\alpha}$, $\overline{V} \subseteq \prod U_\alpha \subseteq U$. X is regular ■

ex: $\mathbb{R}_K$ basis $\begin{cases} (a,b) \\ (a,b) - K, \quad K = \{\frac{1}{n}, n \in \mathbb{Z}_+\} \end{cases}$

- $\mathbb{R}_K$ is Hausdorff. ($\mathbb{R}_K$ is finer than $\mathbb{R}_{std}$ which is Hf)
- $\mathbb{R}_K$ is not regular.

  K is closed in $\mathbb{R}_K$ & does not contain 0.

  Suppose $\exists$ open disjoint sets $U \ni 0$ & $V \supset K$.

  A basis elt containing 0 & lying in U must be of the form $(a,b) - K$ for it to be disjoint from K.

  Choose $\frac{1}{n} \in (a,b)$ some n. Since $\frac{1}{n} \in V$ open, $\exists$ a basis elt (c,d) s.t $\frac{1}{n} \in (c,d) \subset V$.

Can find $z \in U \cap V$: choose $\max(c, \frac{1}{n+1}) < z < \frac{1}{n}$!

ex: $\mathbb{R}_\ell$ lower limit topology, basis $[a, b)$

$\mathbb{R}_\ell$ normal, but the product $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not normal.
↳ not shown.

- Normal
  - one-pt sets closed ($\mathbb{R}_\ell$ is finer than $\mathbb{R}$)
  - A, B disjoint closed sets in $\mathbb{R}_\ell$

    $A = \overline{A}$, $B = \overline{B}$

    $\Rightarrow \forall a \in A$, choose a basis elt $[a, x_a)$ disjoint from B.

    $\forall b \in B$, " " " $[b, x_b)$ " A.

    $U = \bigcup_{a \in A} [a, x_a)$ & $V = \bigcup_{b \in B} [b, x_b)$

    open, contain A & B & disjoint.

Lec **Motivating Theorems**

> Urysohn's metrization theorem: Every regular space X with a countable basis is metrizable.

> Tietze extension theorem: Let X be a normal space, $A \subseteq X$ closed subspace:
>
> a) Any continuous map $f: A \longrightarrow [a, b] \subseteq \mathbb{R}$ can be extended to a continuous $\tilde{f}: X \longrightarrow [a, b]$.
>
> b) Any continuous map $g: A \rightarrow \mathbb{R}$ can be extended to a continuous $\tilde{g}: X \rightarrow \mathbb{R}$.

Imbedding of manifolds in $\mathbb{R}^n$

Recall f imbedding: $f: X \to Y$ continuous, injective s.t. f homeo onto its image.

m-dim. manifold: Hausdorff space with a countable basis s.t. each point has a nbd homeo with open subsets of $\mathbb{R}^m$. m = dim. of manifold.

m: If X is a compact m-dim manifold, then X can be imbedded in $\mathbb{R}^N$ for some positive integer N.

ex:  S^2 is a 2dim manifold, can be imbedded in $\mathbb{R}^3$.

---

§ 32 Normal Spaces

Thm: Every metrizable space is normal.

Pf(sketch): Let $A, Z \subseteq (X, d)$ disjoint closed sets,

$\forall a \in A$, $a \in X \setminus Z$ open, choose ϵ_a s.t. $B(a, \epsilon_a) \cap Z = \phi$

$\forall z \in Z$, $z \in X \setminus A$ open, choose ϵ_z s.t. $B(z, \epsilon_z) \cap A = \phi$

Let $U = \bigcup_{a \in A} B(a, \frac{\epsilon_a}{2})$, $V = \bigcup_{z \in Z} B(z, \frac{\epsilon_z}{2})$ open sets

Then triangle inequality $\Rightarrow U \cap V = \phi$ ■

Thm: Every regular space with a countable basis is normal.

Pf: X regular, $\mathcal{B} = \{B_n\}$ countable basis. Let A & Z be distinct closed sets in X.

$\forall a \in A$,

- choose nbd $U_a \ni a$ s.t $U_a \cap Z = \phi$
- choose nbd $W_a \ni a$ s.t $\overline{W_a} \subseteq U_a$ (regularity)
- choose $B_a \in \mathcal{B}$ s.t $B_a \subseteq W_a$

Then $\{B_a \mid a \in A\}$ is a countable covering of A
Relabelled as $\{U_n\}$.
Similarly, $\forall z \in Z$, do same thing. Obtain a countable covering of Z, $\{B_z \mid z \in Z\}$, relabelled as $\{V_n\}$ s.t each $\overline{V_n} \cap A = \phi$.

Now $U = \bigcup U_n$, $V = \bigcup V_n$, open sets containing A & Z, but might not be disjoint. Make an alteration:

For each n, define:

$$U_n' = U_n - \underbrace{\bigcup_{i=1}^{n} \overline{V_i}}_{\text{closed}} \qquad V_n' = V_n - \underbrace{\bigcup_{i=1}^{n} \overline{U_i}}_{\text{closed}}$$

Observe: Each U_n' is open, V_n' open.

- $\{U_n'\}$ still contains A, $\{V_n'\}$ covers Z.

The opens $U' = \bigcup U_n'$ & $V' = \bigcup V_n'$ are disjoint: If $x \in U' \cap V'$, then $x \in U_j' \cap V_k'$, some j, k. WLOG, $j \leq k$. $x \in U_j$ by def of U_j'. But $j \leq k \Rightarrow$ by def V_k', $x \in \overline{U_j} \Rightarrow x \notin V_k'$. $\ast$
Contradiction ■

<u>Thm:</u> Every compact Hf space is normal.

<u>Thm</u>: Every <u>well ordered set</u> is normal in the order topology.

↳ every nonempty subset has a smallest elt.

for generalization of Thm, replace second countable with Lindelöf.

<u>Lemma:</u> A closed subspace Y of a Lindelöf space X is Lindelöf.

Ch 9: Fundamental Group

Q: Given X, Y top. spaces, are X & Y homeomorphic?

ex: $[0,1]$ and $(0,1)$ are not homeo (compact vs. non compact)

ex: $\mathbb{R}$ & $\mathbb{R}^2$ are not homeo (deleting a pt. gives a non-conn vs. conn. space).

ex: $\mathbb{R}^2$ & $\mathbb{R}^3$ are not homeo. Why?

ex: S^2 sphere & Torus are not homeo. Why?

} Need new techniques. ex: Simply connectedness

Simply connected (roughly speaking)

(X is simply connected if every closed curve on X can be shrunk to a constant loop eg. a pt.)

(Torus → not ; S^2 → yes)

* The fundamental group generalizes the simply connectedness property

Applications

- Show two spaces are not homeomorphic.
- maps of spheres & fixed points.
- Fundamental thm of Algebra.

§51 Homotopy of Paths

Def: If f and f' are continuous maps $f: X \to Y$, $f': X \to Y$, we say f is homotopic to f' if there is a continuous map

$F: X \times I \to Y$ $(I = [0,1])$ s.t. $F(x,0) = f(x)$, $F(x,1) = f'(x)$ $\forall x$.

The map F is called a <u>homotopy</u> b/w f and f'.

<u>Notation:</u> $f \simeq f'$ denotes f is homotopic to f'.

If $f \simeq$ (a constant map) then we say f is <u>null homotopic</u>.

<u>Special Case</u> $f: [0,1] \to X$ is a path in X from $f(0) = x_0$ ← initial pt. to

$f(1) = x_1$
↑ final pt.

<u>Stronger relation</u>

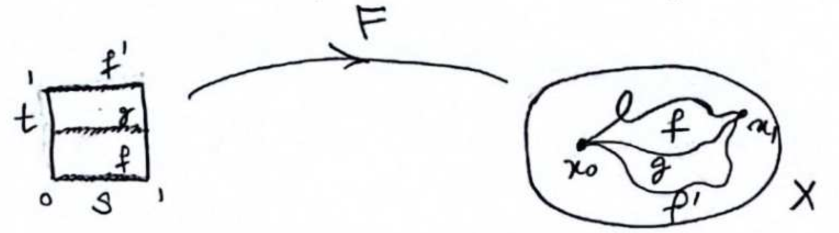
<u>Def</u>: Two paths f and f' mapping $I = [0,1]$ to X are path-homotopic, denoted $f \simeq_p f'$, if:

- the paths f and f' have same initial point x_0 and same final pt. x_1.
- there exists a continuous map $F: I \times I \to X$ s.t

$$\begin{cases} F(s,0) = f(s) \quad \text{and} \quad F(s,1) = f'(s) & \leftarrow (F \text{ is homotopy}) \\ F(0,t) = x_0 \quad \text{and} \quad F(1,t) = x_1 & \leftarrow (\forall t, \text{ it is a path from } x_0 \text{ to } x_1). \end{cases}$$

$\forall\, s \in I,\ t \in I$.

We call F a path homotopy b/w f and f'.

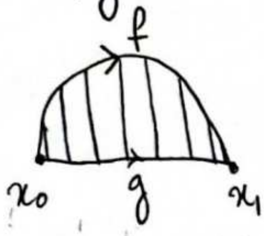


ex: If $f : X \to \mathbb{R}^2$ and $g : X \to \mathbb{R}^2$ are continuous maps that are homotopic, then

$$F(x,t) = (1-t) f(x) + t\, g(x)$$ is a "straight line" homotopy b/w f & g, because for each x it is the straight line segment joining $f(x)$ & $g(x)$.

If f, g are paths ($X = I = [0,1]$), then F is a path homotopy.

ex:



Lemma: The relations $\simeq$ and $\simeq_p$ are equivalence relations

Proof ① $f \simeq f$ (since $F(x,t) = f(x)$)

② If $f \simeq f'$ then want $f' \simeq f$: Let F be a homotopy b/w f & f', define $G : X \times I \to Y$ by $G(x,t) = F(x, 1-t)$ is a homotopy b/w f' and f.

③ If $f \simeq f'$, $f' \simeq f''$, then want $f \simeq f''$.

Let F be a homotopy b/w f and f'.

F' is a homotopy b/w f' and f''.

Define $G : X \times I \to Y$ by $G(x,t) = \begin{cases} F(x, 2t) & t \in [0, \frac{1}{2}] \\ F'(x, 2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$

- G well defined, continuous (pasting lemma). G is homotopy b/w f and f''.

- In all cases, if F and F' path homotopies, then so is G.

$\simeq_p$ is also an equivalence rel'n ■

<u>Notation</u>: If f is a path in X, denote it's equivalence class under $\underset{\text{path homotopy}}{\simeq_p}$ by $[f]$.

> <u>An operation $*$ on paths</u>:

<u>Def</u>: If f is a path in X from x_0 to x_1 and if g is a path in X from x_1 to x_2, then the <u>product $f*g$</u> of f and g is the path h:

$$h(s) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ g(2s-1) & s \in [\frac{1}{2}, 1] \end{cases}$$

- h well defined & continuous (pasting lemma); h is a path from x_0 to x_2.

$*$ induces an operation on path-homotopy classes:

$$[f] * [g] := [f * g]$$

Lec (§51 contd.)

Last time: f, g paths in X. from x_0 to x_1

$f \simeq_p g$ if $\exists$ path homotopy $F : I \times I \to X$

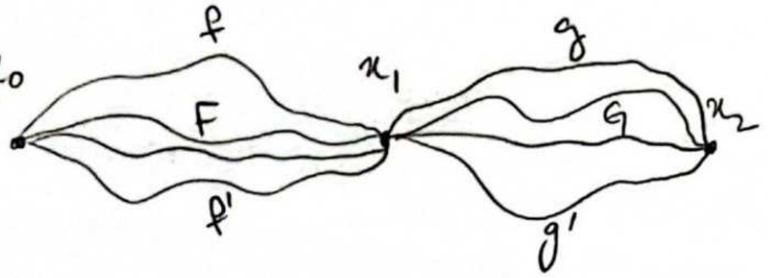
$F(s,0) = f(s)$, $F(0,t) = x_0$
$F(s,1) = g(s)$, $F(1,t) = x_1$

Defined $*$ on paths, $*$ on path homotopy classes $[f]$

Check: $[f] * [g]$ well-defined

$$[f] * [g] = [f * g]$$

If $f \simeq_p f'$ and $g \simeq_p g'$, then check $[f] * [g] = [f'] * [g']$.



$$H(s,t) = \begin{cases} F(2s,t) & t \in [0, \tfrac{1}{2}] \\ G(2s-1, t) & t \in [\tfrac{1}{2}, 1] \end{cases}$$

- H is well-defined
- H is also continuous
- H is a path homotopy b/w $f * g$ and $f' * g'$.

Properties of $[f] * [g]$

- Only defined for paths where $f(1) = g(0)$.

Thm. ① If $[f] * ([g] * [h])$ is defined, so is $([f] * [g]) * [h]$ and they're equal.

② $\forall x \in X$, let e_x denote the constant path $e_x : I \to X$ $e_x(t) = x$ $\forall t$. Then $[f] * [e_{x_1}] = [f]$
$[e_{x_0}] * [f] = [f]$ $\forall f$ path from x_0 to x_1.

③ Given a path in X, from x_0 to x_1, let $\bar{f}$ be the path defined by $\bar{f}(s) = f(1-s)$. $\bar{f}$ is called the reverse of f.

Then $[f] * [\bar{f}] = [e_{x_0}]$

$[\bar{f}] * [f] = [e_{x_1}]$.

Pf: ①

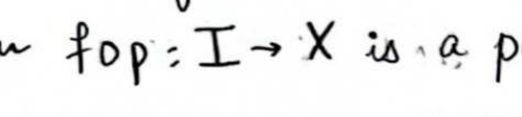
$f * (g * h)$ $\qquad$ $(f * g) * h$

Lemma: If $f : I \to X$ is a path and $p : I \to I$ path from 0 to 1, then $f \circ p : I \to X$ is a path s.t $f \simeq_p f \circ p$ ← called a reparametrization

Pf of Lemma: $H(s,t) = f((1-t)s + t\,p(s))$ $\quad$ $t=0: f(s)$ $\quad$ $t=1: f(p(s))$ $\quad$ $s=0: f(t.0) = f(0) = x_0$ $\quad$ $s=1: f(1-t+t) = f(1) = x_1$.

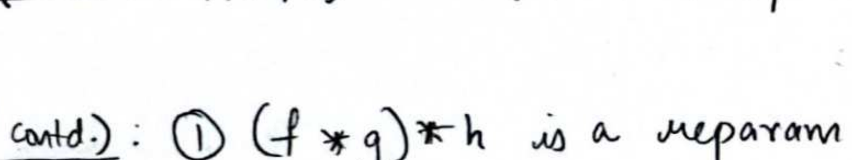
(Pf contd.): ① $(f * g) * h$ is a reparam of $f * (g * h)$.

②



Again, a reparameterization.

③



Traverse partway & reverse direction to get a continuous family of paths from x_0 to x_0

$H(s,t) = f_t(s) * \bar{f}_t(s)$

$$f_t(s) = \begin{cases} f(s) & s \in [0, 1-t] \\ f(1-t) & s \in [1-t, 1] \end{cases}$$ ■

§52 Fundament Group

Def: X space, $x_0 \in X$. A path that begins and ends at x_0 is called a loop based at x_0. The fundamental group of X w.r.t base pt. x_0 is defined

$\pi_1(X, x_0) = \{$ Path homotopy classes of loops based at x_0 $\}$ with group operation $*$ (concatenation).

The previous thm $\Rightarrow \pi_1(X, x_0)$ is a group

- $[f] * [g]$ is always defined for loops $[f]$, $[g]$.
- identity, e_{x_0}, associativity, inverses $[\bar{f}]$ is inverse for $[f]$

ex: $\pi_1(\mathbb{R}^n, x_0)$, $x_0 \in \mathbb{R}^n$ any point, is the trivial group (consisting of only e_{x_0}).
↳ identity

Any f, loop based at x_0, is path homotopic to e_{x_0} (via the straight line path homotopy).

ex: $\pi_1(B^n, x_0)$ B^n = ball of radius 1 in $\mathbb{R}^n$, $x_0 \in B^n$ is again a trivial group.

Q: How does the fundamental group of X depend on the choice of base point?

Thm: If there is a path in X from x_0 to x_1, then $\pi_1(X, x_0)$ is isomorphic ($\cong$) to $\pi_1(X, x_1)$

Pf: Let α be a path from x_0 to x_1.

Define $\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ by

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$$

Check that α is a group homomorphism.

Check: $\hat{\alpha}([f]) * \hat{\alpha}([g]) = \hat{\alpha}([f * g])$

$$\hat{\alpha}([f]) * \hat{\alpha}([g]) = ([\bar{\alpha}] * [f] * [\alpha]) * ([\bar{\alpha}] * [g] * [\alpha])$$
$$= [\bar{\alpha}] * [f] * [g] * [\alpha]$$
$$= \hat{\alpha}([f] * [g])$$

To check $\hat{\alpha}$ is an isomorphism, we construct an inverse map $\hat{\beta}$ for $\hat{\alpha}$, where β = reverse of $\alpha = \bar{\alpha}$.

$\hat{\alpha}(\hat{\beta}[f]) = [f]$ & $\hat{\beta}(\hat{\alpha}[f']) = [f'] \ \forall f \in \pi_1(X, x_0)$

∎

<u>Remark</u> : If X path-connected, then $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$ $\forall\, x_0, x_1 \in X$.

If C is a path component of X containing x_0, then

$$\pi_1(C, x_0) \simeq \pi_1(X, x_0)$$

No natural way of identifying $\pi_1(X, x_0)$ with $\pi_1(X, x_1)$ (if X is path connected) (even though isomorphic).

<u>Def</u> <u>Simply connected</u>

A space X is simply connected if it is a path connected space and if $\pi_1(X, x_0)$ is the trivial (one-element) group for some $x_0 \in X$, and hence for every $x_0 \in X$.

<u>Notation</u>: $\pi_1(X, x_0) = 0$

The fundamental group is a <u>topological invariant</u> of space X.

<u>Notation:</u> $h : (X, x_0) \longrightarrow (Y, y_0)$ is a continuous map h from $X \to Y$ that carries x_0 to y_0 i.e. $h(x_0) = y_0$.

<u>Def</u>: <u>Homomorphism induced by h</u> (moves loops based at x_0 to loops based at y_0)

Let $h : (X, x_0) \longrightarrow (Y, y_0)$ continuous map. Then define $h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ by $\boxed{h_*([f]) = [h \circ f]}$

The map h_* is:

- well defined: F path homotopy b/w f & f'
  $\Rightarrow h \circ F$ path homotopy b/w $h \circ f$ & $h \circ f'$.

- a homomorphism: $(h \circ f) * (h \circ g) = h \circ (f * g)$.

Functorial Properties of h_*

Thm: If $h: (X, x_0) \to (Y, y_0)$ and $k: (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$.

If $i: (X, x_0) \longrightarrow (X, x_0)$ is the identity map, then i_* is the identity homomorphism.

Pf: By def

$$(k \circ h)_* ([f]) = [(k \circ h) \circ f]$$

$$k_* \circ h_* ([f]) = k_* (h_* [f]) = k_* ([h \circ f]) = [k \circ (h \circ f)]$$

Similarly, $i_* ([f]) = [i \circ f] = [f]$ ∎

Topological invariance of π_1

Cor: If $h: (X, x_0) \longrightarrow (Y, y_0)$ is a homeomorphism of X with Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.

Proof: Let $K: (Y, y_0) \to (X, x_0)$ inverse of h.

$$\left.\begin{array}{l} K_* \circ h_* = (K \circ h)_* = i_{x*} \\ h_* \circ K_* = (h \circ K)_* = i_{y*} \end{array}\right\} \text{identity homomorphism induced by identity maps.}$$

h_* K_* homomorphisms $\pi_1(X, x_0) \underset{K_*}{\overset{h_*}{\rightleftarrows}} \pi_1(Y, y_0)$ K_* is inverse of h_* ∎

We will develop techniques to compute the fundamental groups of spaces.

ex: $\pi_1(S^1, x_0)$ is isomorphic to $(\mathbb{Z}, +)$ (additive group of integers)

ex: S^n is simply connected for $n \geq 2$

$\uparrow$ n-sphere $\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$

ex: $\pi_1(S^1, x_0) \cong \pi_1(\mathbb{R}^2 \setminus \{0\}, y_0)$

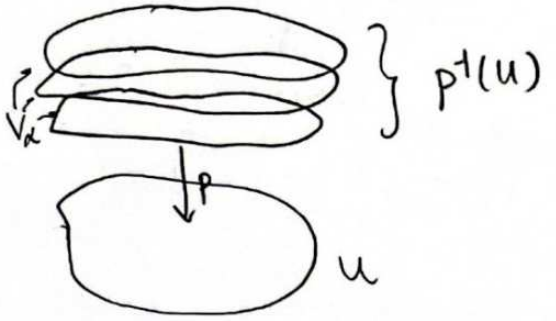
Q? Fundament groups of solid torus $B^2 \times S^1$, torus $S^1 \times S^1$, infinite cylinder $S^1 \times \mathbb{R}$, $\mathbb{R}^3 \setminus \{x\text{ axis}, y\text{ axis}, z\text{ axis}\}$? etc...

Lec : Sec 53 Covering Spaces

Def. Let $p: E \to B$ continuous surjective map. The open set U of B is evenly covered by p if

- $p^{-1}(U) = \bigcup V_\alpha$, where V_α disjoint open set in E
- $p|V_\alpha : V_\alpha \to U$ is a homeo for each α.

$\{V_\alpha\}$ partitions $p^{-1}(U)$ into slices.



Def: Let $p: E \to B$, continuous surjective map. If every pt. b of B has a nbd U that is evenly covered by p, then p is called a covering map.

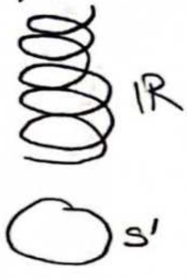
ex: (Trivial examples)

- The identity $Id_X : X \to X$ is a covering map.
- The map $p: X \times \{1, \dots, n\} \to X$ is a covering map.
  ↳ n disjoint copies of X

Thm: The map $p: \mathbb{R} \to S^1$ given by

$$p(x) = (\cos 2\pi x, \sin 2\pi x) \text{ is a covering map.}$$

Picture p as a function that wraps $\mathbb{R}$ around S^1 & maps $[n, n+1]$ onto S^1.



Pf:

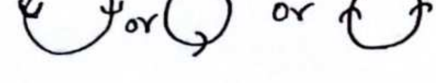
$U \subset S^1$ consisting of points $(x_1, x_2) \in S^1 \subset \mathbb{R}^2$; $x_1 > 0$ (pts with the first coord)

$$p^{-1}(U) = \{x \in \mathbb{R} \mid \cos(2\pi x) > 0\} = \bigcup_{n \in \mathbb{Z}} V_n$$

where $V_n = \left(n - \frac{1}{4}, n + \frac{1}{4}\right)$

Check: $p|_{V_n} : V_n \to U$ is a homeo. $\forall n \in \mathbb{Z}$

- $p|_{\overline{V_n}}$ injective since $\sin(2\pi x)$ strictly monotonic on $\overline{V_n}$
- $p|_{\overline{V_n}} : \overline{V_n} \to \overline{U}$ surjective, $V_n \to U$ surjective } Intermediate Value Thm.
- By thm 26.6, $\overline{V_n}$ compact, $p|_{\overline{V_n}}$ bijective continuous.

$\underset{HF}{\Rightarrow}$ $p|_{\overline{V_n}}$ homeo. Similar args for  &

these open sets cover S^1 & each is evenly covered by p.

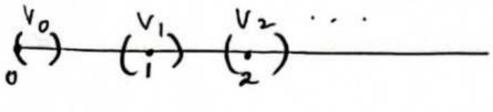
∎

Def: Local homeomorphism

Each $e \in E$ has a nbd that is mapped homeomorphically by p onto an open subset of B.

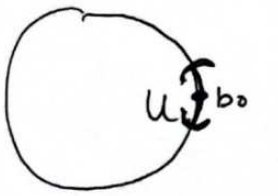
If $p : E \to B$ is a covering map it is a local homeo but not conversely.

<u>Nonex</u>. $p : \mathbb{R}_+ \to S^1$ $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is <u>not</u> a covering map despite being surjective & a local homeo.



The nbd U of $b = (1,0)$ in S^1 has preimage

$$p^{-1}(U) = \bigcup_{n \in \mathbb{Z}_{\geq 0}} V_n$$



$p|_{V_0} : V_0 \to U$ not ~~surjective~~ homeo.

<u>Thm</u>: Let $p : E \to B$ covering map. If $B_0 \subset B$ subspace and $E_0 = p^{-1}(B_0)$, then $p_0 : E_0 \to B_0$ obtained by restricting p is a covering map.

<u>Pf</u>: Let $b_0 \in B_0$. Let U open set in B containing b_0 that is evenly covered by p. $p^{-1}(U) = \bigcup V_\alpha$ ← disjoint open sets in B.

Then $p^{-1}(U \cap B_0) = \bigcup (V_\alpha \cap E_0)$ ← disjoint open sets in E_0

$U \cap B_0$ ← nbd of b_0 in B.

Each $V_\alpha \cap E_0$ is mapped homeo onto $U \cap B_0$ b...

<u>Thm</u>: If $p: E \to B$ and $p': E' \to B'$ are covering maps, then

$p \times p': E \times E' \longrightarrow B \times B'$ is a covering map.

<u>Pf</u>: Let $(b, b') \in B \times B'$. Let U, U' nbds of b, b' evenly covered by p and p' respectively.

$p^{-1}(U) = \bigcup V_\alpha \leftarrow$ disjoint opens in E.

$p'^{-1}(U') = \bigcup V_\alpha' \leftarrow$ disjoint opens in E'.

Then $(p \times p')^{-1}(U \times U') = \bigcup (V_\alpha \times V_\beta') \leftarrow$ disjoint opens in $E \times E'$

Each $V_\alpha \times V_\beta'$ is mapped homeomorphically onto $U \times U'$ by $p \times p'$. ∎

<u>ex</u>: $S^1 \times S^1$ is a torus

$p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ is a covering map, where $p: \mathbb{R} \to S^1$ is the map from Thm before. Each square $[n, n+1] \times [m, m+1]$ gets wrapped around the entire torus.

§54 Fundamental group of the Circle

Def: Lifting

Let $p: E \to B$ be a map. If $f: X \to B$ continuous, a lift of f is a map $\tilde{f}: X \to E$ s.t. $p \circ \tilde{f} = f$.

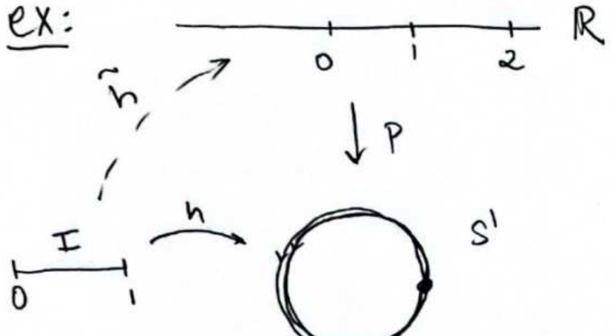
$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Key Ideas

* When p is a covering map, $p: E \to B$, paths in B can be lifted to paths in E.
covering space

* Path homotopies in B can also be lifted.

ex:



$h: I \to S^1$

$h(s) = (\cos 4\pi s, \sin 4\pi s)$

h: path that wraps around S^1 twice can be lifted to $\tilde{h}: I \to \mathbb{R}$ (lift of h)

$\tilde{h}(s) = 2s$ path from 0 to 2 in $\mathbb{R}$.

Lemma: Let $p: E \to B$ covering map, and $p(e_0) = b_0$. Any path $f: [0,1] \to B$ beginning at b_0 has a unique lift to a path $\tilde{f}$ in E beginning at e_0.

Pf: Cover B by open sets U which are evenly covered by p.

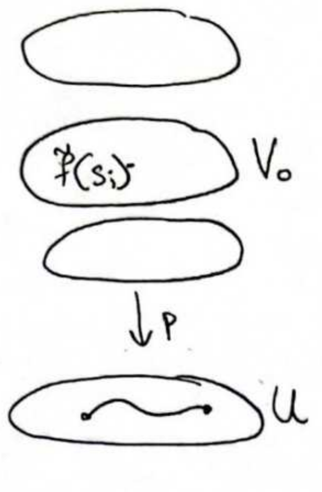
Subdivide $[0,1]$ into intervals $[s_0, s_1], \ldots, [s_{n-1}, s_n]$ (↗ compact metric space)

Such that each set $f([s_i, s_{i+1}])$ lies in one of the open sets U.

(Here we used Lebesgue number lemma) $\{p^{-1}(U)\}$ covers $[0,1]$}

Define $\tilde{f}$ by: • $\tilde{f}(0) = e_0$

• Suppose $\tilde{f}(s)$ is defined for all $0 \le s \le s_i$, then we define $\tilde{f}$ on $[s_i, s_{i+1}]$ as:



Note: $f([s_i, s_{i+1}]) \subseteq U$.

$p^{-1}(U) = \bigcup V_\alpha$ ↳ slices $\forall \alpha$

$p|V_\alpha$ homeo onto U

$\tilde{f}(s_i) \in V_0$ ← one of the V_α.

Define $\tilde{f}(s) = (p|V_0)^{-1}(f(s)) \quad \forall\, s \in [s_i, s_{i+1}]$

Since $p|V_0$ homeo, $\tilde{f}$ continuous on $[s_i, s_{i+1}]$

Thus, $\tilde{f} : [0,1] \to E$ • is defined

• continuous by pasting lemma

• $p \circ \tilde{f} = f$.

Uniqueness: Supp. $\tilde{g}$ is another lift of f beginning at $e_0 \Rightarrow \tilde{g}(0) = e_0 = \tilde{f}(0)$

$\tilde{g}$ lifts $f \Rightarrow \tilde{g}([s_0, s_1]) \subseteq p^{-1}(U) \subseteq \bigcup V_\alpha$

Suppose $\tilde{f}(s) = \tilde{g}(s) \quad \forall s \in [0, s_i]$

- V_α's open disjoint
- $\tilde{g}([s_i, s_{i+1}])$ connected $\Rightarrow$ $\tilde{g}([s_i, s_{i+1}])$ lies in exactly one V_α.

Since $\tilde{f}(s_i) = \tilde{g}(s_i) \in V_0 \Rightarrow \tilde{g}([s_i, s_{i+1}]) \subseteq V_0$.

$$\forall s \in [s_i, s_{i+1}], \quad \tilde{g}(s) \in \underbrace{V_0 \cap p^{-1}(\{f(s)\})}_{(p|_{V_0})^{-1}(\{f(s)\}) = \{\tilde{f}(s)\}}$$

$\Rightarrow \tilde{g}(s) = \tilde{f}(s) \quad \forall s \in [s_i, s_{i+1}]$ ∎

<u>Lemma</u>: Let $p: E \to B$ covering map with $p(e_0) = b_0$. Let $F: I \times I \to B$ be continuous, with $F(0,0) = b_0$. There is a unique lift of F to a continuous map $\tilde{F}: I \times I \to E$. S.T $\tilde{F}(0,0) = e_0$. If F path homotopy, So is $\tilde{F}$.

<u>Pf</u>: Same idea. Define $\tilde{F}(0,0) = e_0$. Use preceeding lemma to define $\tilde{F}$ on $0 \times I$ and $I \times 0$ subdivide $I \times I$

$I_i \times J_j = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$

each rectangle mapped into an open set of B evenly covered by p.

Supp. $\tilde{F}$ defined on set $A = 0 \times I \cup I \times 0 \cup$ all rects. "previous" to $I_{i_0} \times J_{j_0}$.

$F(I_{i_0} \times J_{j_0}) \subseteq U$ evenly covered by p.

$p^{-1}(U) = \bigcup V_\alpha$, $p|V_\alpha$ homeo onto U.

$\tilde{F}$ defined already on $C = A \cap (I_{i_0} \times J_{j_0})$, connected

$\tilde{F}(C) \subseteq V_0 \leftarrow$ one of V_α's.

Let $p_0 : V_0 \to U$ denote $p|V_0$. Define $\tilde{F}(x) = p_0^{-1}(F(x))$ $\forall x \in I_{i_0} J_{j_0}$. ∎

$\boxed{\text{Check: Uniqueness, path homotopy..}}$

---

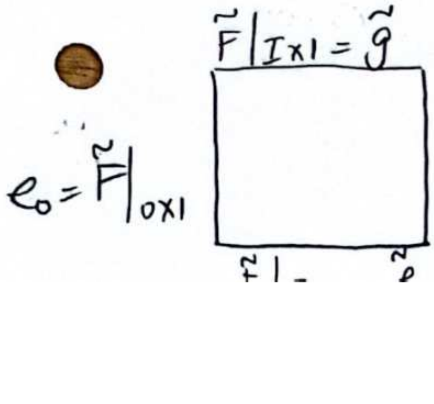
§54 contd.

Thm: Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$. Let f and g be two paths in B from b_0 to b_1. Let $\tilde{f}$ & $\tilde{g}$ be lifts of f and g to paths in E beginning at e_0. If f and g are path homotopic then $\tilde{f}$ and $\tilde{g}$ end at the same pt. of E and are path homotopic.

Pf. $F : I \times I \to B$ path homotopy b/w f and g. lifts to

$\tilde{F} : I \times I \to E$ path homotopy (by lemma)

$\Rightarrow \tilde{F}(0 \times I) = e_0$, $\tilde{F}(1 \times I) = \{e_1\}$.



$\tilde{F}|_{I\times 0}$ path in E beginning at e_0 lifting $F|_{I\times 0} = f$. By uniqueness of path liftings

$\tilde{F}|$ $\tilde{f}$

Similarly, $\tilde{F}|_{I\times 1} = \tilde{g}$.

Thus, $\tilde{f}$ and $\tilde{g}$ end at e_1 and $\tilde{F}$ is path homotopy b/w them.

Def: $p: E \to B$ covering map, $b_0 \in B$. Choose e_0 s.t. $p(e_0) = b_0$. Define the set map

$$\phi = \phi_{e_0} : \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$$
$$[f] \longmapsto \tilde{f}(1)$$

lift of f beginning at e_0.

called the <u>lifting correspondance</u> (ϕ well defined by prev. thm).

<u>Thm</u>: If E is path connected, then the lifting correspondance $\phi = \phi_0 : \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$ is surjective. If E simply connected, ϕ is bijective.

<u>Pf</u>: If E path connected, given $e_1 \in p^{-1}(b_0)$, there is a path $\tilde{f}$ in E from e_0 to e_1. Then $f = p \circ \tilde{f}$ is a loop in B based at b_0, so $\phi([f]) = e_1$.

Suppose E simply connected, and $\phi([f]) = \phi([g])$ for $[f], [g] \in \pi_1(B, b_0)$. Let $\tilde{f}$ and $\tilde{g}$ lifts to paths in E that begin at e_0. Then $\tilde{f}(1) = \tilde{g}(1)$.

E simply connected $\Rightarrow$

$[\tilde f * \tilde g^{-1}] = [e_{e_0}]$ → Constant loop at e_0

$[\tilde f] = [\tilde f * \tilde g^{-1}][\tilde g] = [\tilde g]$

$\tilde f$ path homotopic to $\tilde g$ via a path homotopy $\tilde F$.

f path homotopic to g via a path homotopy $p \circ \tilde F$.

$\Rightarrow [f] = [g] \Rightarrow$ (ϕ injective) ∎

Last time: $p: E \to B$ covering map $\quad p(e_0) = b_0$

- Defined the lifting correspondence

$$\phi : \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$$
$$[f] \longmapsto \tilde{f}(1) \leftarrow \text{end pt. of lifted path } \tilde{f} \text{ in } E.$$

Also last time. E simply connected $\Rightarrow$ ϕ is bijective.

<u>Thm:</u> The fundamental group of S^1 is isomorphic to $(\mathbb{Z}, +)$.

<u>Pf:</u> Let $p: \mathbb{R} \to S^1$ be the standard covering map.

Let $e_0 = 0 \in \mathbb{R}$ and $p(e_0) = b_0 = (1,0) \in S^1$.

- We have $p^{-1}(b_0) = \mathbb{Z}$.

Since $\mathbb{R}$ is simply connected, $\phi: \pi_1(S^1, b_0) \longrightarrow \mathbb{Z}$ is bijective.

Check moreover that ϕ is a homomorphism.

If $[f], [g] \in \pi_1(S^1, b_0)$, show $\phi([f]*[g]) = \phi([f]) + \phi([g])$

<u>Def:</u> $\tau_n : \mathbb{R} \to \mathbb{R}$, $\tau_n(x) = n + x$. Then $\tau_n \circ \tilde{f}$ is a lift of f starting at n (if $\tilde{f}$ lift of f starting at 0).

Then $\tau_n \circ \tilde{f}(1) = n$.

The lift of $f * g$ to a path in $\mathbb{R}$ starting at 0

- is $\widetilde{f * g} = \tilde{f} * (\tau_n \circ \tilde{g})$

$n = \phi([f]) = \tilde{f}(1)$

with endpoint $\tilde{f} * (\tau_n \circ \tilde{g})(1) = \tilde{f}(1) + \tilde{g}(1) = \phi([f]) + \phi([g])$.

and $\phi([f] * [g]) = \widetilde{f * g}(1)$ ■

§55 Retractions and Fixed Points

Def Retraction

If $A \subset X$, a retraction of X onto A is a continuous map

$r: X \to A$ s.t $r|_A = id_A$. Say: A is a retraction of X.

ex: $x_0 \in X$, $r: X \to \{x_0\}$ retraction.

ex: $r: S^1 \times S^1 \to S^1 \times \{b_0\}$ retraction
$x \times y \to x \times b_0$

ex: $r: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \{0\}$ retraction.

Lemma If A is a retract of X, then the inclusion map
$j: A \to X$ induces an injective homomorphism
$j_*: \pi_1(A, a) \to \pi_1(X, a)$.

Pf: $r: X \to A$ retraction; $r \circ j: A \to A$ equals the identity
$id_A: A \to A$. Then $r_* \circ j_* = id_{A*}: \pi_1(A, a) \to \pi_1(A, a)$.

$\Rightarrow j_*$ injective. ■

Thm: There is no retraction of B^2 onto S^1

2-dim ball $B^2 \subseteq \mathbb{R}^2$.

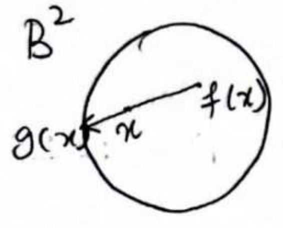
Pf: If S^1 were a retract of B^2, then the inclusion map $j : S^1 \to B^2$ would induce an injective $j_* : \underset{\underset{\mathbb{Z}}{\cong}}{\pi_1(S, b_0)} \to \underset{\underset{0}{\cong}}{\pi_1(B^2, b_0)}$

Contradiction ▨

☆ Thm: (Brouwer Fixed Pt. Thm for B^2)

If $f : B^2 \to B^2$ continuous, then there exists some $x \in B^2$ such that $f(x) = x$.

Pf: Supp. $f(x) \neq x \quad \forall\, x \in B$?



For each $x \in B^2$, define $g(x)$ = end pt. on S^1 of ray from $f(x)$ to x.

Then $g : B^2 \to S^1$. • g is continuous (since small perturbations of x produce small perturbations in $f(x)$ + also in the rays).

• $g|_{S^1} = id|_{S^1}$, $g(x) = x$ if $x \in S^1$

So g is a retraction, contradicts previous thm ▨

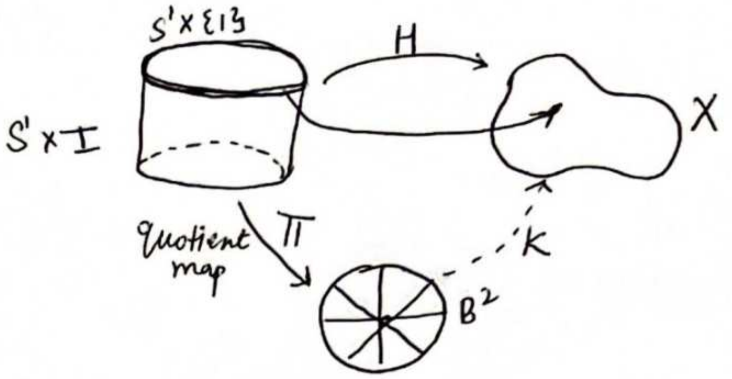
Lemma Let $h: S^1 \longrightarrow X$ continuous map. TFAE

1) h is nulhomotopic
2) h extends to a continuous map $K: B^2 \to X$ $\left(K|_{S^1 = bdry(B^2)} = h\right)$
3) h_* is the trivial homomorphism on the fundamental group.

Pf. (1) $\Rightarrow$ (2) There exists a continuous map $H: S^1 \times I \longrightarrow X$
↳ (homotopy b/w h and a constant map)

$H|_{S^1 \times \{1\}}$ is constant map.

$\pi|_{S^1 \times \{1\}} : x \longrightarrow (0,0)$ in B^2



$\pi: S^1 \times I \to B^2$ is defined by $\leftarrow$ Continuous surjective quotient map
$\pi(x, t) = (1-t)x$

$\exists$ $k: B^2 \longrightarrow X$ (see section on quotient topol.) which extends h
(b/c the diagram $S^1 \times I \xrightarrow{H} X$ commutes)
$$\begin{tikzcd} S^1 \times I \arrow[rr, "H"] \arrow[dr, "\pi"'] & & X \\ & B^2 \arrow[ur, "k"'] & \end{tikzcd}$$

(2) $\Rightarrow$ (3). $K: B^2 \longrightarrow X$ continuous, $k|_{S^1} = h$.

If $j: S^1 \hookrightarrow B^2$ inclusion, then $h = k \circ j \Rightarrow h_* \Rightarrow k_* \circ j_*$.

But $j_* : \pi_1(S^1, b_0) \longrightarrow \pi_1(B^2, b_0) \leftarrow$ Trivial $\Rightarrow j_*$ trivial $\Rightarrow h_*$ trivial. ✓

(3) $\Rightarrow$ (1) (Omitted).

Cor: The inclusion $j: S^1 \to \mathbb{R}^2 \setminus \{0\}$ is not null-homotopic.

The identity map $id: S^1 \to S^1$ is not nulhomotopic.

Proof: There is a retraction $r: \mathbb{R}^2 \setminus \{0\} \to S^1$

$\left(r(\vec{x}) = \frac{\vec{x}}{|\vec{x}|} \right)$.

Thus, j_* injective $\Rightarrow$ ~~[illegible]~~ $j_*: \pi_1(\underset{\substack{\text{S}\| \\ \mathbb{Z}}}{S^1, b_0}) \to \pi_1(\mathbb{R}^2 \setminus \{0\})$,

j_* is nontrivial.

By lemma, j is not nulhomotopic.

Similarly $id_*: \pi_1(S^1, b_0) \to \pi_1(S^1, b_0)$ is the identity homomorphism, and thus nontrivial. ∎

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§56 Fundamental Theorem of Algebra.

Thm: A polynomial equation (w $\mathbb{R}$ or $\mathbb{C}$ coefficients)

$$x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0 \quad , \; n > 0$$

has atleast one (real or complex) root.

Proof Step 1 $f: S^1 \to S^1$ $f(z) = z^n$ induces "multiplication by n" map on $\pi_1(S^1, b_0)$ (HW) $\Rightarrow$ f_* injective.

Step 2 If $g: S^1 \to \mathbb{R}^2 \setminus \{0\}$ given by $g(z) = z^n$, then g is not nulhomotopic.

- $g = j \circ f$, $f: S^1 \to S^1$ from Step 1, $j: S^1 \to \mathbb{R}^2 \setminus \{0\}$ inclusion.

$g_* = j_* \circ f_*$, j_* injective (as in pf of prev. cor.).

f_* injective (Step 1).

$\Rightarrow g_*$ injective. $\Rightarrow g_* : \mathbb{Z} \to \pi_1(\mathbb{R}^2 \setminus \{0\}, b_0)$ is nontrivial.

$\Rightarrow g$ is not nulhomotopic.

Step 3

Special case: $|a_{n-1}| + \dots + |a_1| + |a_0| < 1$.

Show poly eqn has a root in unit ball B^2.

If not, define $K : B^2 \to \mathbb{R}^2 \setminus \{0\}$.

$K(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$.

If $h = K|_{S^1}$, h is nulhomotopic (b/c exists extn. $K : B^2 \to \mathbb{R}^2 \setminus \{0\}$)

However, we can define a homotopy F b/w h and g from Step 2, a contradiction.

$\underset{\text{nulhomotopic}}{h}$ $\underset{\text{not nulhomotopic}}{g}$

$F : S^1 \times I \to \mathbb{R}^2 \setminus \{0\}$, $F(z,t) = z^n + t(a_{n-1} z^{n-1} + \dots + a_1 z + a_0)$.

$$\Big(F(z,t) \neq 0 \text{ since } |F(z,t)| \geq |z^n| - |t| \, |a_{n-1} z^{n-1} + \dots + a_1 z + a_0|$$
$$= 1 - t(|a_{n-1}| + \dots + |a_1| + |a_0|) > 0 \Big)$$

Step 4 General Case.

Let $x = cy$ $(c \in \mathbb{R}_+)$

$(cy)^n + a_{n-1}(cy)^{n-1} + \dots + a_1(cy) + a_0 = 0$

$\Rightarrow y^n + \frac{a_{n-1}}{c} y^{n-1} + \dots + \frac{a_1}{c^{n-1}} y + \frac{a_0}{c^n} = 0$ → if this has a root $y = y_0$ then the original equation has root $x_0 = cy_0$. Choose c large enough s.t $\left|\frac{a_{n-1}}{c}\right| + \dots + \left|\frac{a_1}{c^{n-1}}\right| + \left|\frac{a_0}{c^n}\right| < 1$ to reduce to Step 3 ∎

Lec §58 Deformation Retracts and Homotopy Type

Lemma: Let $h, k : (X, x_0) \to (Y, y_0)$ continuous. If h, k are homotopic, and the image of $x_0 \in X$ remains fixed at y_0 during the homotopy, then h_* and k_* are equal.

Pf: $H: X \times I \to Y$ homotopy b/w h and K.

$H(x_0, t) = y_0 \quad \forall t \qquad (H(x,0) = h(x),\ H(x,1) = k(x))$

If $[f] \in \pi_1(X, x_0)$, f loop $f: I \to X$ then

$I \times I \xrightarrow{f \times id} X \times I \xrightarrow{H} Y$ is a homotopy b/w $h \circ f$ & $k \circ f$.

(Check: $(x,0) \xrightarrow{f \times id} (f(x), 0) \xrightarrow{H} h(f(x))$
$(x,1) \xrightarrow{f \times id} (f(x), 1) \xrightarrow{H} h(f(x))$)

Path homotopy b/c f loop at x_0, $H(\{x\} \times I) \to \{y\}$

$[h \circ f] = [k \circ f]$ same path homotopy class

$h_*([f]) = k_*([f])$ ∎

Thm: The inclusion map $j: S^n \to \mathbb{R}^{n+1} \setminus \{\vec{0}\}$ induces an isomorphism of fundamental groups.

Idea: Deform the identity map on $\mathbb{R}^{n+1} \setminus \{\vec{0}\}$ to a map that collapses all of $\mathbb{R}^{n+1} \setminus \{0\}$ onto S^n. where each pt. of S^n remained fixed during deformation.

$n=1$ can "collapse" $\mathbb{R}^2 \setminus \{\vec{0}\}$ onto S^1

<u>Pf</u>: Let $X = \mathbb{R}^{n+1} - \vec{0}$; Let $b_0 = (1, 0, 0, \ldots 0)$. Let $r : X \to S^n$ be $r(x) = x/\|x\|$. Then $r \circ j : S^n \to S^n$ is the identity map, So $r_* \circ j_* =$ identity homomorphism on $\pi_1(S^n, b_0)$. $\Rightarrow j_*$ injective.

Now, show $j_* \circ r_*$ is also identity on $\pi_1(X, b_0)$,

$$X \xrightarrow{r} S^n \xrightarrow{j} X$$

$j \circ r$ is not identity, but is homotopic to identity map.

$$H : X \times I \to X$$

$$H(x, t) = (1-t)x + t\frac{x}{\|x\|}$$

It is a homotopy b/w id_X and $j \circ r$.

Note $H(b_0, t) = (1-t)\, b_0 + t\, b_0 = b_0$. fixed by H.

By lemma, $(j \circ r)_* = j_* \circ r_*$ is identity homomorphism on $\pi_1(X, b_0) \Rightarrow j_*$ surjective ∎

More generally:

<u>Def</u>: $A \subset X$, A is a deformation retract of X if $id_X : X \to X$ is homotopic to a map that carries all of X into A s.t each point of A remains fixed during the homotopy.

that is, $\exists$ continuous map $H : X \times I \to X$ s.t

$$H(x,0) = x \qquad \forall x \in X$$
$$H(x,1) \in A \qquad \forall x \in X$$
$$H(a,t) = a \qquad \forall a \in A.$$

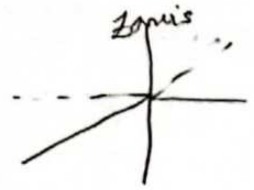
H is called a <u>deformation retraction</u> of X onto A.

<u>Note</u>: $r : X \to A$ defined by $r(X) = H(x,1)$. is a retraction of X onto A, and H is a homotopy b/w $id_X : X \to X$ and $j \circ r : X \to X$ where $j : A \to X$ is inclusion.

The proof of the previous thm generalizes.

<u>Thm</u>: Let A be a deformation retract of X, let $x_0 \in A$. Then the inclusion $j : (A, x_0) \longrightarrow (X, x_0)$ induces an isomorphism of fundamental groups.

ex:



$\mathbb{R}^3 \setminus \{z\text{ axis}\} = X$

$\mathbb{R}^2 \setminus \vec{0} \subseteq X$

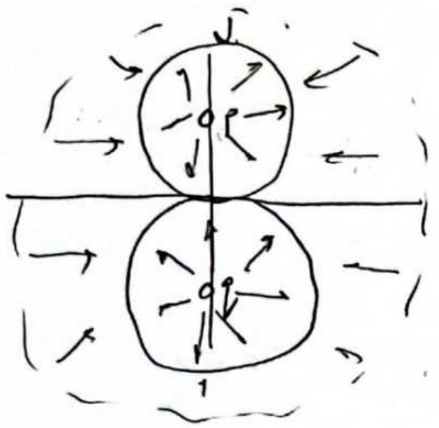
$\mathbb{R}^2 - \vec{0}$ is a deformation retract of X.

$H: X \times I \to X$ deformation retraction

$(x, y, z, t) \longleftrightarrow (x, y, z(1-t))$

$\Rightarrow \pi_1(\mathbb{R}^3 - \{z\text{ axis}\}) \simeq \mathbb{Z}$.

ex: $\mathbb{R}^2 \setminus p \setminus q$ doubly punctured plane.

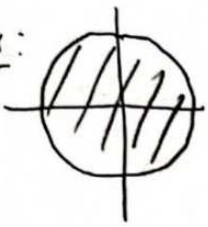


The "figure eight" space $\subseteq \mathbb{R}^2 - p - q$

These spaces also have isomorphic fundamental groups.

Lec: §58 Homotopy Type

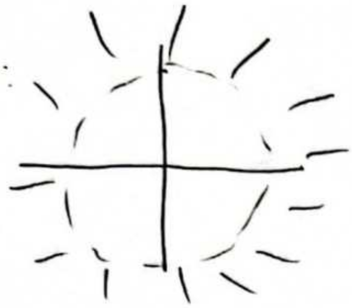
Recall: Last time, deformation retracts have isomorphic π_1.

ex: 

B^2 def. retracts onto a point

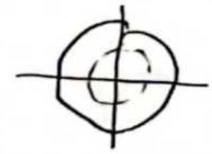
↑ closed unit ball in $\mathbb{R}^2$.

$\subseteq \mathbb{R}^2$

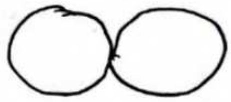
ex: 

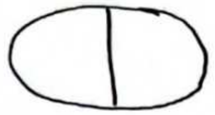
$\{x \in \mathbb{R}^2 \mid \|x\| > 1\}$ def retracts onto a circle $x^2 + y^2 = 4$.

(similar to $\mathbb{R}^2 \setminus \vec{0}$ defretracts to S^1)



ex

$\xrightarrow{\text{defretracts}}$  fig eight

$\xrightarrow{\text{defretracts}}$  theta space

Neither fig 8 or theta space is a def retract of the other, but have isomorphic π_1.

Can we generalize?

Def: Let $f: X \to Y$ and $g: Y \to X$ continuous maps.

Suppose $f \circ g$ is homotopic to $id_Y : Y \to Y$

$g \circ f$ is homotopic to $id_X : X \to X$.

Then, f and g are called <u>homotopy equivalences</u> and f is a <u>homotopy inverse</u> of g (& vice versa).

<u>Note</u>: If $f: X \to Y$ is a homotopy equiv, and $h: Y \to Z$ is a homotopy equivalence, then $h \circ f: X \to Z$ is a homotopy equivalence.

<u>Thm</u>: The relation of homotopy equivalence (on top. spaces) is an equivalence relation.

<u>Proof</u>: exercise.

<u>ex</u>: If r is def. retract, then r, j are homotopy inverses

<u>Thm</u>: If $f: X \to Y$ is a homotopy equivalence, then $\forall x_0 \in X$, $f_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$ is an isomorphism.

<u>Lemma</u>: Let $h, k: X \to Y$ continuous and $h \simeq k$ via homotopy $H: X \times I \to Y$. Let $x_0 \in X$. Then $\exists$ a path α in Y from $h(x_0)$ to $k(x_0)$ s.t. $k_* = \hat{\alpha} \circ h_*$ (h_*, k_* differ by basept. change map). Indeed α is the path $\alpha(t) = H(x_0, t)$

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "h_*"] \arrow[dr, "k_*"'] & \pi_1(Y, h(x_0)) \arrow[d, "\hat{\alpha}"] \\ & \pi_1(Y, k(x_0)) \end{tikzcd} \qquad \begin{tikzcd} {[\gamma]} \arrow[d, "\hat{\alpha}"] \\ {[\bar{\alpha}] * [\gamma] * [\alpha]} \end{tikzcd}$$

Pf: Show $k_*([f]) = \hat{\alpha}(h_*[f]) \Leftrightarrow [k \circ f] = [\bar{\alpha}] * [h \circ f] * [\alpha]$
(Omitted)

Pf. of Thm

Let $g : Y \to X$ homotopy inverse of f.

$$X \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{f} Y$$

$$\pi_1(X, x_0) \xrightarrow{(f_{x_0})_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, g(f(x_0))) \xrightarrow{(f_{x_1})_*} \pi_1(Y, f(g(f(x_0))))$$

$f \circ g \simeq id_Y \Rightarrow (f \circ g)_* = \hat{\alpha} \circ (id_Y)_* = \hat{\alpha}$ (Some path α in Y)

$\hat{\alpha}$ isomorphism $\Rightarrow f_* \circ g_*$ isomorphism ②

$g \circ f_{x_0} \simeq id_X \Rightarrow g_* \circ f_{x_0 *}$ isomorphism ①

$\Rightarrow f_*$ injective ①, $\left\{ \begin{array}{l} g_* \text{ injective ②} \\ g_* \text{ surjective ①} \end{array} \right\} \Rightarrow f_*$ surjective ∎

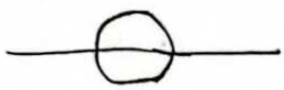
Simplest example of homotopy type:

<u>Homotopy Types of a one pt. space</u>

Recall X is <u>contractible</u> if $id_X : X \to X$ is null homotopic.

<u>Exer</u>: X is contractible $\Leftrightarrow$ X has homotopy type of one pt. space

<u>ex</u>: $S^1 \cup (\mathbb{R} \times \{0\}) \subseteq \mathbb{R}^2$

 has a def. retract to  theta space.

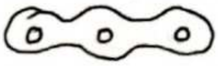
which is homotopy equiv. to 8 "fig-eight".

<u>ex</u>: $S^1 \cup (\mathbb{R}_+ \times \mathbb{R})$ $(\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\})$.

 has homotopy type of S^1.

<u>ex</u> fig 8 and theta have same homotopy type.

Lec §59 Fundamental Group of S^n

Can use π_1 to show S^2, T^2,  , ... are topolo-gically distinct.

Thm: Let $X = U \cup V$, where U and V open in X. Supp. $U \cap V$ path connected, $x_0 \in U \cap V$. Let $i_u : U \to X$, $i_V : V \to X$ inclusion maps. Then the images of

$$i_{u*} : \pi_1(U, x_0) \longrightarrow \pi_1(X, x_0)$$

$$i_{V*} : \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

generate $\pi_1(X, x_0)$.

Pf: WTS: If f loop in X at x_0, f is path homotopic to $g_1 * g_2 * \cdots * g_n$ where each g_i is a loop in U or V at x_0.

Step ① $\exists\ a_0 < a_1 < \cdots < a_n$ subdivision of $I = [0,1]$ s.t. $f(a_i) \in U \cap V$ and $f([a_i, a_{i+1}]) \subseteq$ either U or V.

($f^{-1}(U)$, $f^{-1}(V)$ cover $[0,1]$).

By Lebesgue number lemma, $\exists\, b_0 < \cdots < b_m$ subdivision of $[0,1]$ s.t each $f([b_i, b_{i+1}])$ is contained in either U or V.

If $f(b_i) \in U$, but not V, then $f(b_{i-1} \overline{}_{b_i} \overline{} b_{i+1}) \subseteq U$

remove b_i from the subdivision.

Similarly, if $f(b_j) \in V$ but not U,

remove b_j. Repeat until all remaining b_* have

$f(b_*) \in U \cap V$.

Step ② Define $f_i : I \to X$ by $I \xrightarrow[\text{linear map}]{} [a_{i-1}, a_i] \xrightarrow{f} X$

Then f is path is either U or V, by choice of a_i's.

Also, $[f] = [f_1] * [f_2] * \cdots * [f_n]$

Need to replace f_i with loops g_i based at x_0.

$g_i = \alpha_{i-1} * f_i * \overline{\alpha_i}$

α_i path in $\underset{\uparrow \text{path conn.}}{U \cap V}$ from x_0 to $f(a_i)$.

α_0, α_n constant paths at $x_0 = f(a_0) = f(a_n)$.

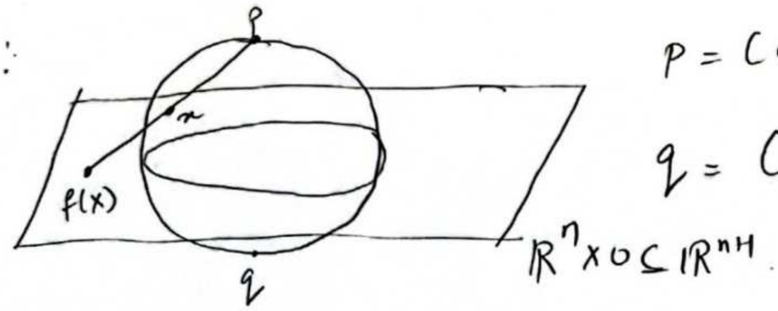
g_i loop in either U or V, based at x_0.

$$[g_1] * [g_2] * \cdots * [g_n] = [f_1] * \cdots * [f_n] \quad \blacksquare$$

Cor: Suppose $X = U \cup V$, where U, V open in X, $U \cap V \neq \phi$, path conn., If U & V simply connected, then X is simply connected.

Thm: If $n \geq 2$, the n-sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ is simply connected.

Pf:



$p = (0, 0, \ldots, 0, 1) \in S^n$ north pole

$q = (0, 0, \ldots, 0, -1) \in S^n$ south pole.

① If $n \geq 1$, $S^n - p$ is homeo to $\mathbb{R}^n$. Define $f: (S^n - p) \to \mathbb{R}^n$ by stereographic projection: $f(x) = f(x_1, \ldots, x_{n+1}) = \dfrac{(x_1, \ldots, x_n)}{1 - x_{n+1}}$

f is a homeo: $\exists$ inverse map $g: \mathbb{R}^n \longrightarrow (S^n - p)$

$$g(y) = g(y_1, \ldots, y_n) = (\underbrace{t(y)\, y_1, \ \ldots, \ t(y)\, y_n, \ 1 - t(y)}_{\text{norm } 1})$$

where $t(y) = \dfrac{2}{1 + \|y\|^2}$

Since $S^n - p$ is homeo w. $S^n - q$ via reflection.

$(x_1, \ldots, x_n, x_{n+1}) \longmapsto (x_1, \ldots, x_n, -x_{n+1})$. $S^n - q \cong \mathbb{R}^n$

② $U = S^n - p$. $V = S^n - q$, $S^n = U \cup V$.

U, V path conn. since $\underset{\text{homeo}}{\cong} \mathbb{R}^n$ and have pt. in common.

$\Rightarrow S^n$ path conn.

U, V simply conn (since $\cong \mathbb{R}^n$).

$U \cap V = S^n - p - q$ homeo to $\mathbb{R}^n - \vec{0}$

using stereographic proj.

$\Rightarrow U \cap V$ path conn. (Since $\mathbb{R}^n - \vec{0}$ path conn. as every pt. can be joined to a pt. on S^{n-1} via via straight line path, and S^{n-1} is path connected if $n \geq 2$).

Apply Cor. ▪

Lec §60 Fundamental Groups of some surfaces

-  S^2 $\pi_1(S^2, b_0) = 0$

-  $S^1 \times S^1$ $\pi_1(S^1 \times S^1, b_0) = \mathbb{Z} \times \mathbb{Z}$.

Thm: $\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$

Recall group structure on $A \times B$ is $(a \times b) \cdot (a' \times b') = (a \cdot a') \times (b \cdot b')$

Proof: $p : X \times Y \to X$
$q : X \times Y \to Y$ projection maps

$$p_* : \pi_1(X \times Y, x_0 \times y_0) \longrightarrow \pi_1(X, x_0)$$

$$q_* : \pi_1(X \times Y, x_0 \times y_0) \longrightarrow \pi_1(Y, y_0)$$

Define homomorphism,

$$\Phi : \pi_1(X \times Y, x_0 \times y_0) \longrightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

by $\Phi([f]) = p_*([f]) \times q_*([f]) = [p \circ f] * [q \circ f]$

Φ is an isomorphism

Surjective: Let $[g] \in \pi_1(X, x_0)$, $[h] \in \pi_1(Y, y_0)$

Then $f : I \to X \times Y$, $f(t) = g(t) \times h(t)$.

$$\Phi([f]) = [p \circ f] \times [q \circ f] = [g] \times [h]$$

Injective: Let $\Phi([f]) = [p \circ f] \times [q \circ f]$ be the identity elt. $e_{x_0} \times e_{y_0}$ ← constant loops.

$p \circ f \simeq e_{x_0}$ via path homotopy G

$q \circ f \simeq e_{y_0}$ via path homotopy H

Then $F : I \times I \to X \times Y$, $F(s,t) = G(s,t) \times H(s,t)$

is a path homotopy b/w f and the constant loop $e_{x_0 \times y_0}$ ∎

<u>Cor:</u> $\pi_1(S^1 \times S^1, b_0) \cong \mathbb{Z} \times \mathbb{Z}$

<u>Def</u>: The projective plane P^2 is the quotient space obtained from S^2 by identifying each pt. x of S^2 with it's antipodal point $-x$.

$$P^2 = S^2 /_{x \sim -x \ \forall x}$$

<u>Thm</u>: $\pi_1(P^2, y)$ is a group of order 2 ($\cong \mathbb{Z}/2\mathbb{Z}$)

<u>Proof</u>: The projection $p : S^2 \to P^2$, $x \to [x]$ is a covering map (shown below)

Since S^2 is simply conn., the lifting correspondance

$\pi_1(P^2, y) \longrightarrow p^{-1}(y)$ is a bijection. But $p^{-1}(y) = \{y, -y\}$ two elements

$\Rightarrow \pi_1(P^2, y)$ is a group of order 2. $\Rightarrow \pi_1 \cong \mathbb{Z}/2\mathbb{Z}$.

→ <u>p Covering map</u>: Let $[y] \in P^2$, $p^{-1}([y]) = \{y, -y\}$.

Let $U = \epsilon$-nbd of y in S^2, $\epsilon < 1$ using Euclidean metric of $\mathbb{R}^3$

Then if $z \in U$, $-z \notin U$, since $d(z,-z) = 2 \Rightarrow U$ contains no pair $\{z, -z\}$. Thus, $p: U \to p(U)$ is bijective.

- p is also continuous
- p open map

$\left.\right\}$ $p|_U$ is homeomorphism, $p|_{a(U)}$ is homeo

$a: S^2 \to S^2$, $a(x) = -x$ is a homeo

If U open in S^2, $a(U)$ open too.

$p^{-1}(p(U)) = U \cup a(U)$ is open in S^2 in quotient top.

$p^{-1}(p(U))$ open $\Leftrightarrow p(U)$ open

Thus, $\forall\, [y] \in P^2$, $p(U)$ is nbd of $[y] = p(y)$ evenly covered by p.

$$p^{-1}(p(U)) = \underbrace{U \cup a(U)}_{\text{disjoint open sets each homeo to } p(U).}$$

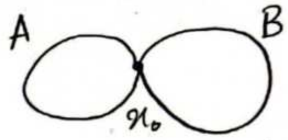
■

Similarly define P^n: projective n-space

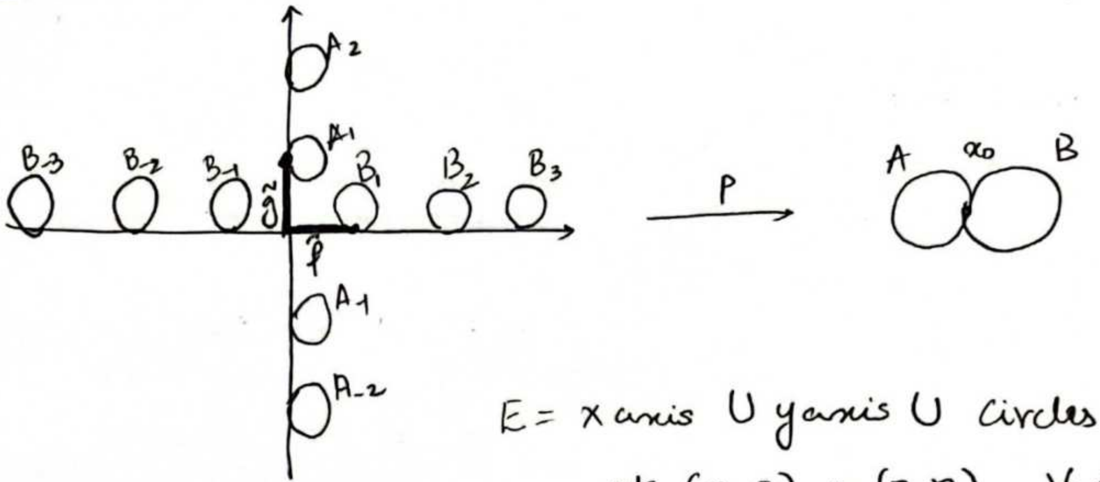
$P^n = S^n / x \sim -x$

$p: S^n \to P^n$ Covering map. S^n simply conn. $\forall n \geq 2 \Rightarrow \pi_1(P^n, y) \cong \mathbb{Z}/2$ ($n \geq 2$)

Lec Fundamental group of Fig 8

- X = Figure eight space  in $\mathbb{R}^2$

Lemma: The fundamental group of X is not abelian.



E = x axis $\cup$ y axis $\cup$ circles tangent to axis at $(n,0)$ & $(0,n)$, $\forall n \in \mathbb{Z} \setminus \{0\}$

- We describe a covering map $p : E \to X$.

$$\left.\begin{array}{l}
\mathbb{R} \approx x\text{-axis} \xmapsto{\ p\ } A \approx S^1 \\
\qquad (n,0) \xmapsto{p} x_0 \\
\mathbb{R} \approx y\text{-axis} \xmapsto{\ p\ } B \approx S^1 \\
\qquad (0,n) \xmapsto{p} x_0 \\
\text{circle along xaxis} \xrightarrow[\text{homeo}]{\approx p} B \\
\text{circle along yaxis} \xrightarrow[\text{homeo}]{\approx p} A
\end{array}\right\} \quad P \text{ is a covering map.}$$

- We find loops in X based at x_0 s.t $f * g$ and $g * f$ are not path homotopic.

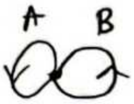
Let $\tilde{f} : I \to E$ path from 0×0 to 1×0, $\tilde{f}(s) = s \times 0$

$\tilde{g} : I \to E$ path from 0×0 to 0×1, $\tilde{g}(s) = 0 \times s$.

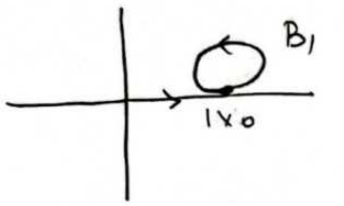
Let $f = p \circ \tilde{f}$ and $g = p \circ \tilde{g}$ loops in X based at x_0.

f wraps around A. g wraps around B.

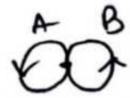
The lifting correspondance $\phi : \pi_1(X, x_0) \to p^{-1}(x_0)$ maps $f * g$ and $g * f$ to different points:

$\widetilde{f*g}$ lift of $f * g$ 

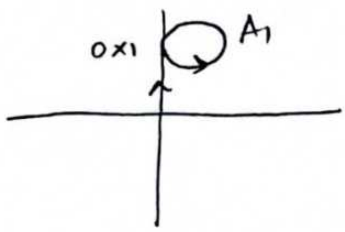
(loop is A then B).



$\widetilde{f*g}$ path from 0×0 to 1×0 (goes along x-axis, then around the circle at 1×0)

$\widetilde{g*f}$ lift of $g * f$ 

(loop is B then A)



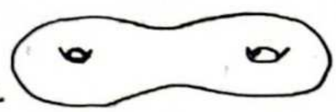
$\widetilde{g*f}$ path from 0×0 to 0×1 (along y axis, then around circle at 0×1)

$$\phi([f*g]) = \widetilde{f*g}(1) = 1 \times 0$$

$$\phi([g*f]) = \widetilde{g*f}(1) = 0 \times 1$$

$$\Rightarrow [f*g] \neq [g*f]$$

<u>Thm</u>: The fundamental group of the genus 2 surface is not abelian. Σ_2



<u>Pf</u>: The fig 8 space X is a retract of Σ_2.

Thus, $j: X \to \Sigma_2$ induces an injection map j_*, so

$\pi_1(\Sigma_2, x_0)$ is not abelian ■

The fundamental group of fig 8 is the free group on two generators $\mathbb{Z} * \mathbb{Z}$ (can be proven by Van Kampen's Thm §70)

<u>Free Groups</u>:

<u>Def</u>: Let G be a group; let $\{a_\alpha\}$ be a set of elts of G.

$\{a_\alpha\}$ <u>generate</u> G if every elt of G can be written as a product of powers of a_α.

$\mathbb{Z} * \mathbb{Z}$ <u>free group</u> on two gens a, b.

$\mathbb{Z} * \mathbb{Z}$ is the free product of $\mathbb{Z}$.

Every elt. is of the form $a^{n_1} b^{m_1} a^{n_2} b^{m_2} \dots a^{n_j} b^{m_j}$ an alternating product of powers of a with powers of b.

$\pi_1(\infty, x_0)$ elts are going around loops in any orientation, any order.

$\mathbb{Z} * \mathbb{Z} = \{\text{strings of } a^{\pm 1}\text{'s}, b^{\pm 1}\text{'s}\} / \sim$

$$w a a^{-1} z \sim w z \sim w a^{-1} a z \qquad \forall\, w, z \in \mathbb{Z} * \mathbb{Z}$$

$$w b b^{-1} z \sim w z \sim w b^{-1} b z$$

- group operation: concatenation. $(a b a^{-1} b^{-1})^{-1} = b a b^{-1} a^{-1}$
- Every elt. has reduced form: no aa^{-1}, bb^{-1}, $a^{-1}a$, $b^{-1}b$.

"Free": If G group, $g, h \in G$, $\exists!$ homomorphism

$f : \mathbb{Z} \times \mathbb{Z} \to G$ s.t $f(a) = g$, $f(b) = h$.

---

§70 The Seifert-van Kampen theorem.

Let $X = U \cup V$, where U and V are open in X, assume U, V and $U \cap V$ are path connected; let $x_0 \in U \cap V$. Let H be a group and let

$$\phi_1 : \pi_1(U, x_0) \longrightarrow H \quad \text{and} \quad \phi_2 : \pi_1(V, x_0) \longrightarrow H.$$

be homomorphisms. Let i_1, i_2, j_1, j_2 be the homomorphisms indicated inthe following diagram, each induced by inclusion.

$$\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[dr, "\phi_1"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r] \arrow[dr, "i_2"'] & \pi_1(X, x_0) \arrow[u, "j_1"'] \arrow[r] & H \\
 & \pi_1(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

If $\phi_1 \circ i_1 = \phi_2 \circ i_2$, then there is a unique homomorphism $\phi : \pi_1(X, x_0) \longrightarrow H$ such that $\phi \circ j_1 = \phi_1$ and $\phi \circ j_2 = \phi_2$.

If ϕ_1 and ϕ_2 are arbitrary homomorphisms that are "compatible on $U \cap V$", then they induce a homomorphism of $\pi_1(X, x_0)$ into H.

---

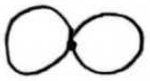
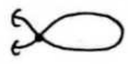
ex: $S^2 = B^2 \cup B^2$

$= U \cup V$

$U \cap V$ = nbd of equator

$\pi_1(U) = \pi_1(V) = 0 \Rightarrow \pi_1(U) * \pi_1(V) = 0 \Rightarrow \pi_1(S^2) = 0.$

ex: Fig eight space X or $S^1 \vee S^1$ wedge of 2 circles

 $U =$  $V =$ 

U, V same homotopy type as a circle

$\pi_1(U) = \pi_1(V) = \pi_1(S^1) = \mathbb{Z}$

$U \cap V$ is contractible (homotopy type of a point).

$\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$. (Since $\pi_1(U \cap V) = 0$).

If a, b are the two generators of $\mathbb{Z} \times \mathbb{Z}$, then for ex. $a^2 b^{-1} a^3 b a$ is an elt of $\mathbb{Z} \times \mathbb{Z}$ corresponding to loop in $\pi_1(X)$.

Last time: Seifert Van Kampen Thm.

$X = U \cup V$; $U, V, U \cup V$ path-conn.

$$\begin{tikzcd} & \pi_1(U) \arrow[dr, "j_1"] \arrow[d] & \\ \pi_1(U\cap V) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \pi_1(U) * \pi_1(V) \arrow[r, dashed, "j"] & \pi_1(X) \\ & \pi_1(V) \arrow[ur, "j_2"'] \arrow[u] & \end{tikzcd}$$

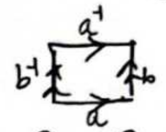
j is surjective and it's kernel is the least normal subgroup containing all elts $i_1(g)^{-1} i_2(g)$ $\forall g \in \pi_1(U\cap V)$.

i.e. $\pi_1(X) \cong \pi_1(U) * \pi_1(V) / \ker j$

ex: $S^1 \times S^1 = U \cup V$

 $\to U$ =  $\cup$  ; $U \cap V$ =  $\pi_1(U\cap V) \cong \mathbb{Z}$

V   U

$\pi_1(U) \cong 0$ ↳ def retr. onto it's bdy  $\Rightarrow \pi_1(V) \cong \pi_1(\text{fig } 8)$

$\cong \mathbb{Z} * \mathbb{Z}$

$\langle c \rangle = \mathbb{Z} \to$ generator c

$i_1 : \pi_1(U\cap V) \longrightarrow \pi_1(U) = 0$ is trivial

$i_2 : \pi_1(U \cap V) \longrightarrow \pi_1(V) \cong \langle a, b \rangle$

$c \overset{}{\underset{i_2}{\longmapsto}} aba^{-1}b^{-1}$

generator

$\Rightarrow \pi_1(S^1 * S^1) \cong \langle a, b \rangle / aba^{-1}b^{-1} \cong \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle$

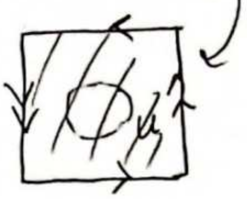
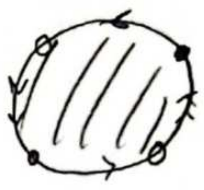
$\updownarrow$

$ab = ba$

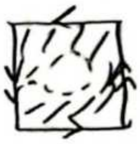
$\cong \mathbb{Z} \oplus \mathbb{Z}$ ↳ free abelian group on two generators

↳ Group Thry

ex: P^2 projective plane. $P^2 = S^2 / x \sim -x \;\; \forall x \in S^2 = B^2 / y \sim -y \;\; \forall y \in S^1$ (bdy of B^2)

 $\approx$ B^2  $\approx P^2$

$P^2 = U \cup V$

$U =$  open disk $\quad V =$  def ret to bdry  P^1 $\approx S^1$ HW

points are the same

$U \cap V =$ 

$\pi_1(V) \cong \langle ab \rangle \cong \mathbb{Z}$

$\pi_1(U) = 0$

$\pi_1(U \cap V) \cong \mathbb{Z}$

$i_1 : \pi_1(U \cap V) \longrightarrow \pi_1(U) = 0$ trivial

$i_2 : \pi_1(U \cap V) \longrightarrow \pi_1(V) = \langle ab \rangle$

$\mathbb{Z} \cong \langle c \rangle$ $\quad i_2(c) = (ab)^2$

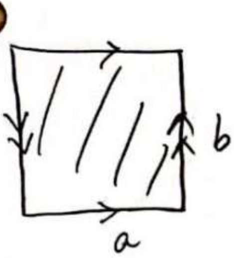
generator

$\Rightarrow \pi_1(P^2) \cong \langle ab \mid (ab)^2 = 1 \rangle$

$\cong \mathbb{Z}/2\mathbb{Z}$

from Group theory

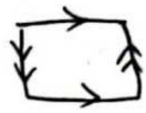
ex: Klein bottle



$K = U \cup V$

$U =$  as before $\qquad U \cap V =$ 

$V =$  as before

defret onto 

$\pi_1(U) = 0$

$\pi_1(V) = \mathbb{Z} * \mathbb{Z}$ (fig 8 space) $= \langle a, b \rangle$

$\pi_1(U \cap V) \cong \mathbb{Z} \cong \langle c \rangle$

i_1 : trivial ; $i_2 : \pi_1(U \cap V) \longrightarrow \pi_1(V) \cong \langle a, b \rangle$

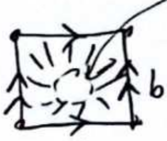
$c \longmapsto aba^{-1}b$

$$\boxed{\pi_1(K) \cong \langle a, b \mid aba^{-1}b \rangle}$$

ex:  $=$  $\qquad U \cap V =$

Σ_2 double torus

$\Sigma_2 = U \cup V$

$\pi_1(U) =$  defrets to bdy $\cong$  $\pi_1(U) \cong \langle a, b \rangle$

$\pi_1(V) =$ same  $= \langle c, d \rangle$

$\pi_1(U \cap V) \cong \mathbb{Z} = \langle c \rangle$

$i_1 : \pi_1(U \cap V) \to \pi_1(U)$

$c \longrightarrow aba^{-1}b^{-1}$

$i_2 : \pi_1(U \cap V) \to \pi_1(V)$

$c \to cdc^{-1}d^{-1}$

$$\Rightarrow \pi_1(\Sigma_2) = \frac{\langle a, b, c, d \rangle}{aba^{-1}b^{-1} = cdc^{-1}d^{-1}}$$

$$= \langle a, b, c, d \mid aba^{-1}b^{-1} = cdc^{-1}d^{-1} \rangle$$ not abelian