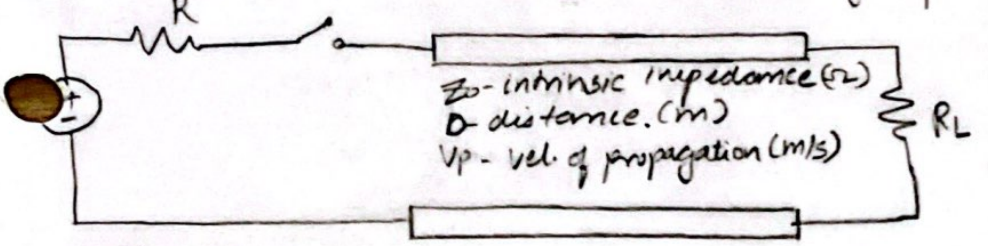
Transmission Lines - The Telegrapher's Equations!



In the line, the governing equations for V & I are:-

Telegrapher's Eqns

$$\frac{\partial^2 V(z,t)}{\partial z^2} - LC\frac{\partial^2 V(z,t)}{\partial t^2} = 0$$

$$\frac{\partial^2 i(z,t)}{\partial z^2} - LC\frac{\partial^2 i(z,t)}{\partial t^2} = 0$$

The solution to these equations give the V & i that propagate at a constant velocity in the forward & backward direction.

$$\left(\frac{\partial^2}{\partial z^2} - LC\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix} V \\ i \end{pmatrix} = 0$$

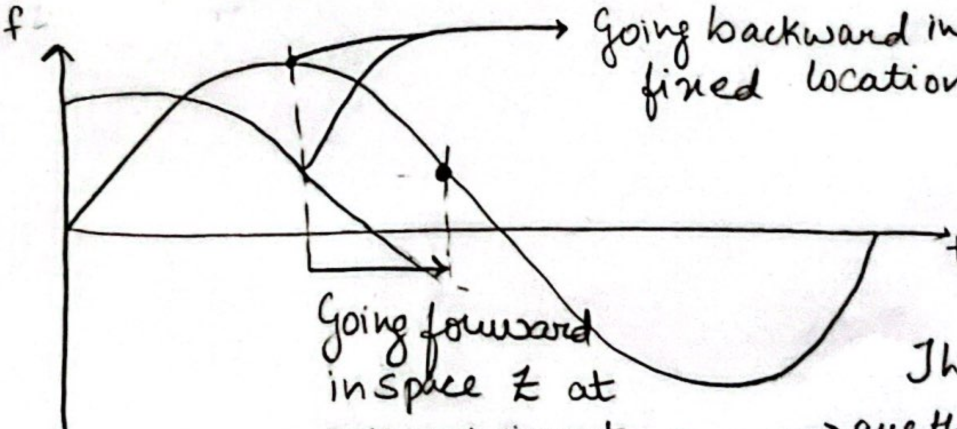
General solution

$$V(z,t) = V^+ f\left(t - \frac{z}{V_p}\right) + V^- g\left(t + \frac{z}{V_p}\right)$$

V^+ → Amplitude of voltage.

$f\left(t - \frac{z}{V_p}\right)$: Eqn. describing a travelling wave in $\xrightarrow{+ve}$ z — forward propagating wave

$g\left(t + \frac{z}{V_p}\right)$: Eqn. describing travelling wave in $\xleftarrow{-ve}$ z. Backward propagating wave.



→ Going backward in time at a fixed location z.

Going forward in space z at a fixed time t. →

These two are same, are therefore a forward wave is given by $f\left(t - \frac{z}{V_p}\right)$

> The corresponding current is:

The $+z$ direction is chosen for the current & therefore we need this $-$ve sign to get proper current direction

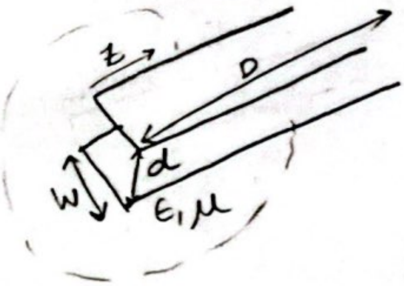
$$i(z,t) = \frac{V^+}{Z_0} f\left(t - \frac{z}{V_p}\right) - \frac{V^-}{Z_0} g\left(t + \frac{z}{V_p}\right)$$

> Velocity of propagation $V_p = \frac{1}{\sqrt{LC}}$ where L, C are per meter

> Intrinsic impedance $Z_0 = \sqrt{\frac{L}{C}}$. It describes the voltage to current ratio at any point for a given line.

> These are lossless lines => energy is not lost.

① Parallel plate Transmission line



For vacuum

$\epsilon = \epsilon_0$ 8.85×10^{-12} F/m
permittivity

$\mu = \mu_0$ $4\pi \times 10^{-7}$ H/m   (μ_r is close to 1 for most materials)
permeability

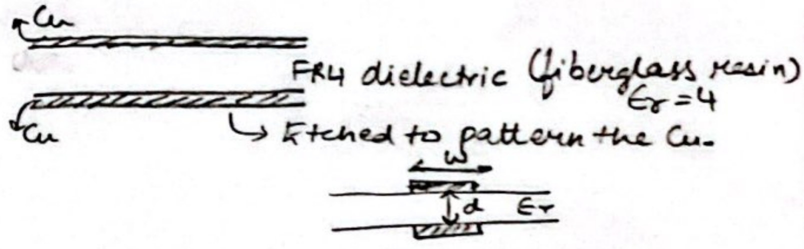
permittivity:- How much E energy can a material store

permeability:- How much M energy can a material store

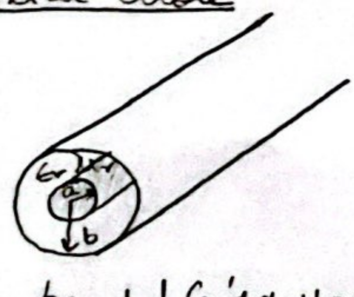
$L_{perm} = \mu \frac{d}{W}$ $C_{perm} = \epsilon \frac{W}{d}$

$Z_0 = \sqrt{\frac{L}{C}} = \frac{d}{W}\sqrt{\frac{\mu}{\epsilon}}$ & $V_p = \frac{1}{\sqrt{\mu\epsilon}}$ → In this case geometry doesn't change change V_p.

→ PCB traces use these models.



② Coaxial cable



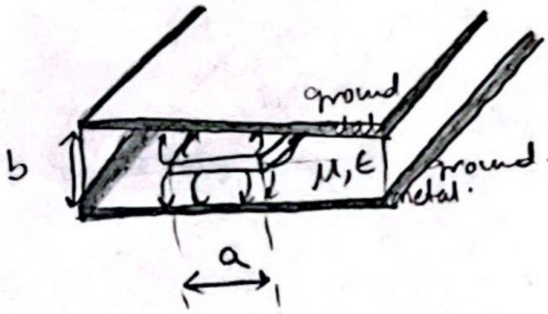
$L = \frac{\mu}{2\pi} \ln(b/a)$ $C = \frac{2\pi\epsilon}{\ln(b/a)}$

$V_p = \frac{1}{\sqrt{\mu\epsilon}}$ $Z_0 = \frac{1}{2\pi} \ln\left(\frac{b}{a}\right)\sqrt{\frac{\mu}{\epsilon}}$

→ It can bend! Can't do it on PCB

→ The fields are all } No coupling!

(3) Symmetrical stripline.



→ Fields are confined.

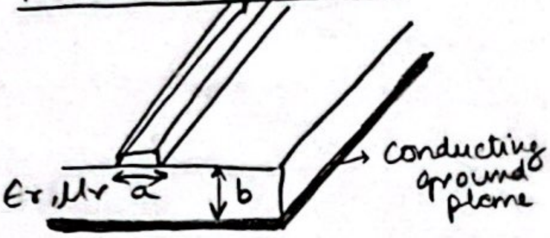
→ $V_p = \dfrac{1}{\sqrt{\mu\epsilon}}$

Empirical

$$Z_0 = \frac{30\pi}{\sqrt{\epsilon_r}} \frac{b}{a_{eff} + 0.441b}$$

$$a_{eff} = \begin{cases} a & a > 0.35b \\ a - \left(0.33 - \frac{a}{b}\right)^2 b & a < 0.35b \end{cases}$$

(4) Microstrip line

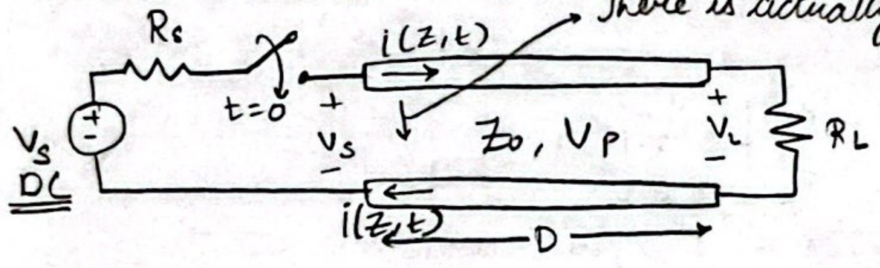


$$\epsilon_{eff} = \epsilon_0 \left[\frac{\epsilon_r + 1}{2} + \left(\frac{\epsilon_r - 1}{2}\right) \cdot \frac{1}{\sqrt{1 + 12\, b/a}} \right]$$

$$V_p = \frac{1}{\sqrt{\mu \epsilon_{eff}}}$$

$$Z_0 = \begin{cases} \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon_{eff}}} \ln\left(\frac{8b}{a} + \frac{a}{4b}\right) & a < b \\ \sqrt{\frac{\mu}{\epsilon_{eff}}} \frac{1}{\frac{a}{b} + 1.393 + 0.667 \ln\left(\frac{a}{b} + 1.444\right)} & \text{when } a > b \end{cases}$$

Transients



There is actually a displacement current here

Assume at $t<0$, it is a "deadline" → "no caps are charged" ⇒ No voltages or currents exist

$$V(z,t) = V^+ f\left(t - \frac{z}{V_p}\right) + V^- g\left(t + \frac{z}{V_p}\right)$$

$$i(z,t) = \frac{V^+}{Z_0} f\left(t - \frac{z}{V_p}\right) - \frac{V^-}{Z_0} g\left(t + \frac{z}{V_p}\right)$$

$$V_s = V_0 u(t)$$

$T > t > 0$

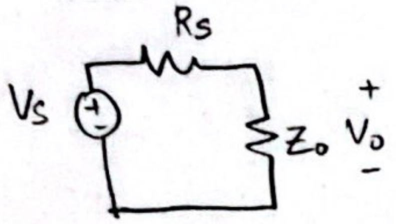
$V(z,t) = V_0 u\left(t - \frac{z}{V_p}\right)$ where $T = \dfrac{D}{V_p}$. When $t < T$, the wave is still travelling forward.

$i(z,t) = \frac{V_0}{Z_0} u\left(t - \frac{z}{V_p}\right)$ T - Transit time

① Dead line model

This model can only be used before two transit times or until reflection comes back.

$\Rightarrow$ $t < T$

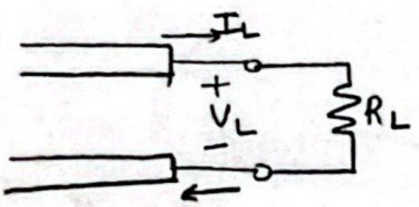


$$V_0 = V_S \frac{Z_0}{R_S + Z_0}$$

At the end of the line $z = D$, we have some backward wave,

$$V(z,t) = V_0 u\left(t - \frac{D}{V_p}\right) + V^- g\left(t + \frac{D}{V_p}\right)$$

$$i(z,t) = \frac{V_0}{Z_0} u\left(t - \frac{D}{V_p}\right) - \frac{V^-}{Z_0} g\left(t + \frac{D}{V_p}\right)$$



$$\frac{V(D,t)}{i(D,t)} = \frac{V_L}{I_L} = R_L = \frac{V_0 u(t - \frac{D}{V_p}) + V^- g(t + \frac{D}{V_p})}{\frac{V_0}{Z_0} u(t - \frac{D}{V_p}) + \frac{V^-}{Z_0} g(t + \frac{D}{V_p})}$$

Rearranging :- $V^+ u\left(t - \frac{D}{V_p}\right)\left[1 - \frac{R_L}{Z_0}\right] = -\left[1 + \frac{R_L}{Z_0}\right] V^- g\left(t + \frac{D}{V_p}\right)$
$(V^+ \equiv V_0)$

The boundary condition must equate the two functional time dependencies ①

$\Rightarrow$ $g\left(t + \frac{D}{V_p}\right) = u\left(t - \frac{D}{V_p}\right)$ $\Rightarrow$ $g(t) = u\left(t - \frac{2D}{V_p}\right)$

for $t > T$

$g(t) = u\left(t - \frac{2D}{V_p} + \frac{z}{V_p}\right)$ $\Rightarrow$ until $t = \frac{D}{V_p}$ & $z = D$, the reverse wave does not exist.

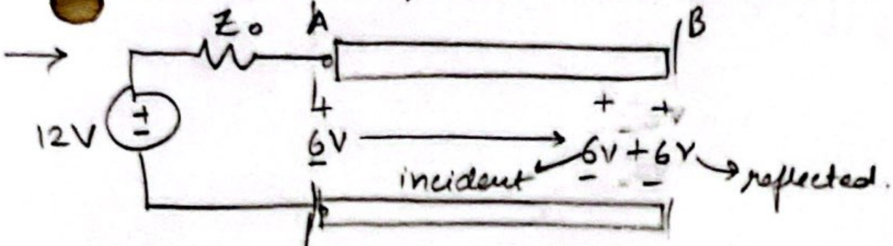
* The reverse wave has the same shape when load is purely resistive

From ①, the two unit steps are same. $\Rightarrow$ $\boxed{\frac{V^-}{V^+} = \frac{R_L - Z_0}{R_L + Z_0} = \Gamma}$

Reflection coefficient.

→ $|\Gamma| \leq 1$ → $-1 \leq \Gamma \leq 1$

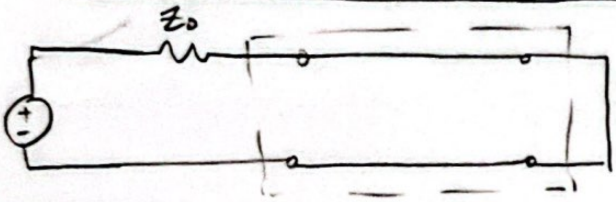
Transients of an open circuit!



→ Initially it is a deadline & from the deadline model we get a voltage divider ⇒ 6V across A. This travels to B and another 6V is produced which travels back. So now we get a 12V signal propagating back. When it reaches 'A' the load is matched and we see 12V all across the line and no drop across the Z_0 (both sides of Z_0 are at 12V)

→ In short circuit, 6V goes from A → B and −6V comes back ⇒ After 2T, there is no more voltage across the line!

(2) Model when $t >> T$ and a DC source is connected for a long time



→ $R_L < Z_0$ ⇒ small load ⇒ Reflected wave has opposite sign.

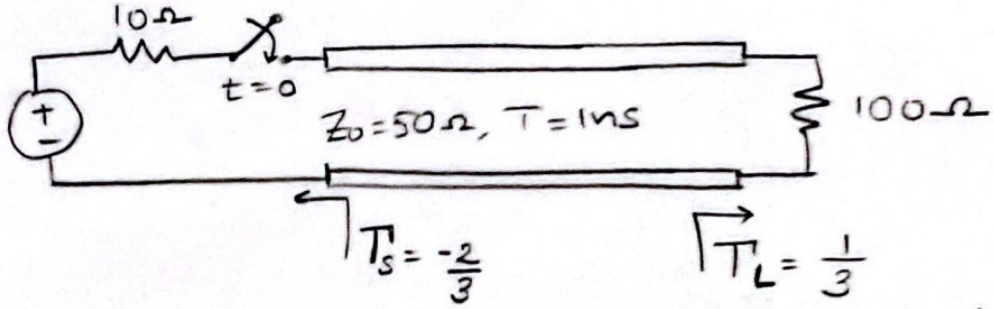
$R_L > Z_0$ ⇒ large load ⇒ Reflected wave has same sign.

→ So we could end up with higher voltages than what we put it, but power is conserved because a large load draws lesser currents.

→ $\Gamma = \frac{I^-}{I^+}$; When open :- $\left.\begin{array}{l}\Gamma = 1 \\ \Rightarrow -\Gamma = -1\end{array}\right\}$ ⇒ Returning current wave must cancel the incident

When short :- $\left.\begin{array}{l}\Gamma = -1 \\ \Rightarrow -\Gamma = 1\end{array}\right\}$ ⇒ Returning current doubles the incident!

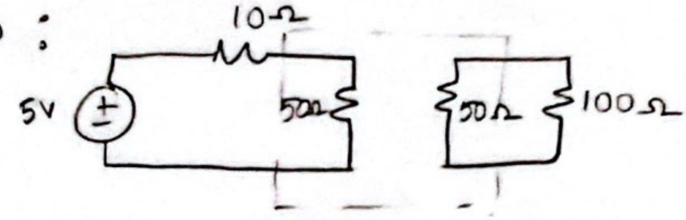
Problem



Visualization of transients : Time to play tennis (:P)

$t \leq 0$: Uncharged line

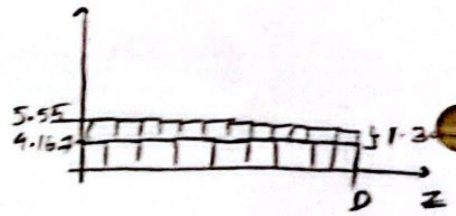
$1ns > t > 0$:



$$V^+ = \frac{50}{60} \cdot 5V = 4.167V$$

$2ns > t > 1ns$:

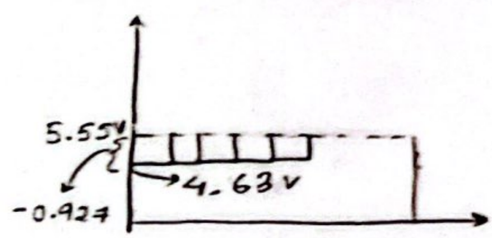
$$V^- = V^+ \Gamma_L = (4.167)(\tfrac{1}{3}) = 1.39V$$



Now, we have 5.55V in the line.

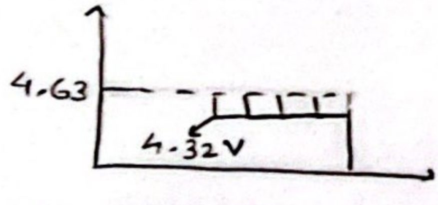
$3ns > t > 2ns$:

$$V^{++} = -\frac{2}{3} \cdot 1.39 = -0.927$$

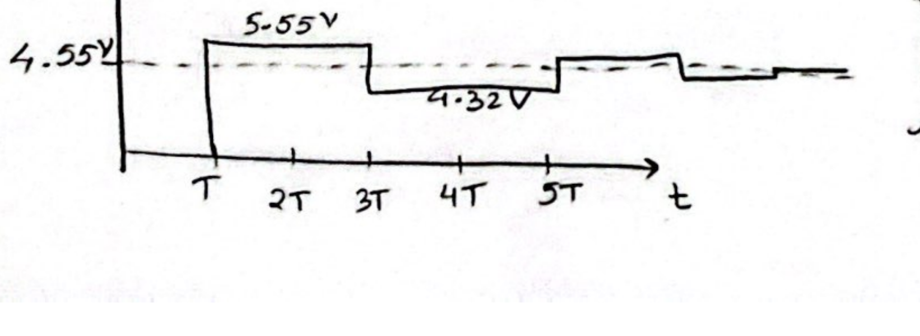


$4ns \geq t > 3ns$:

$$V^{--} = -0.31V$$



→

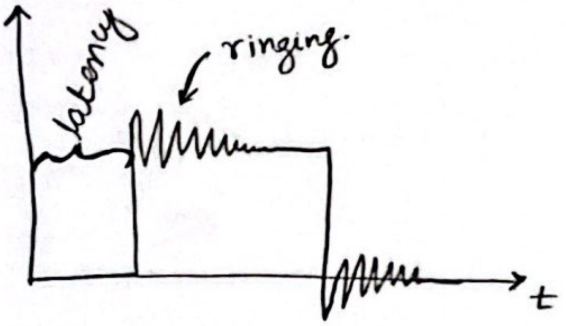


} Where 4.55V is the load voltage when we use the second (steady state) model of a transmission line. Just a voltage divider!

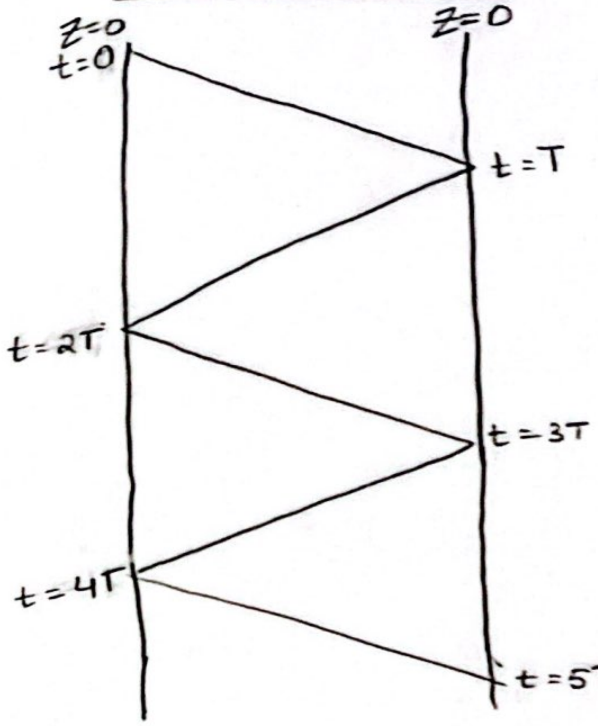
→ We could also define a transmission coefficient τ_L

$\boxed{\tau_L = 1 + \Gamma_L}$ $\boxed{\tau_L = \frac{V_L}{V^+}}$ => $V_L = V^+ + V^-$

→



Bounce diagram



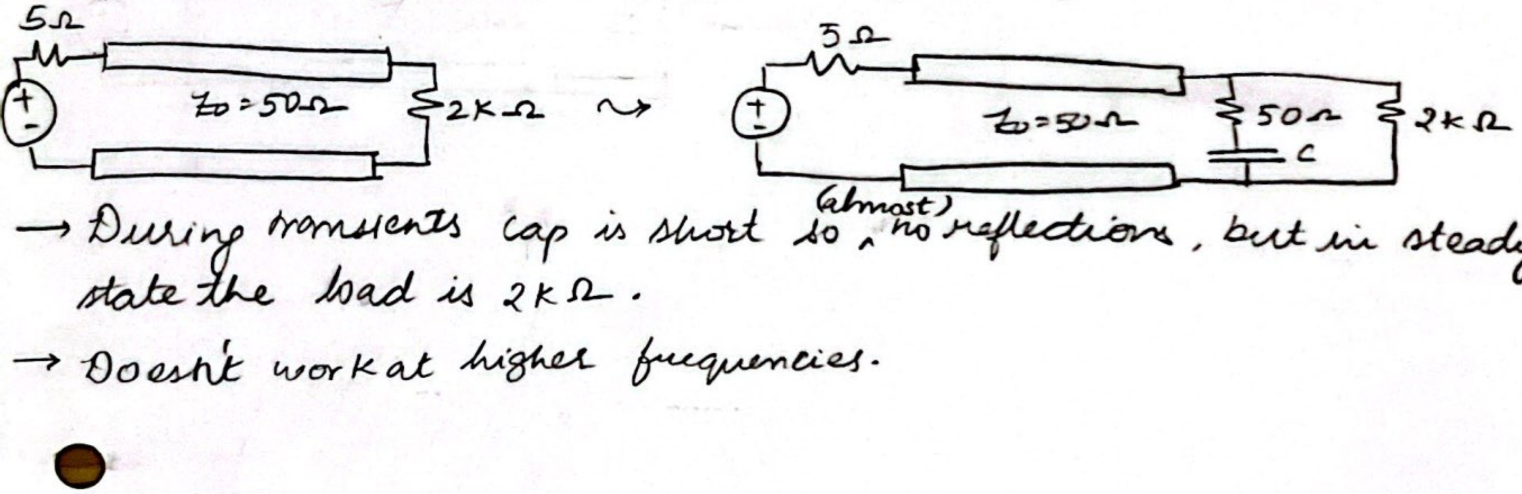
→ To visualize this stuff go to:
propagation.ece.gatech.edu/ECE3025/tutorials/Tlines/Tlines.htm.

Termination Schemes

{Resistive matches are lame :P}

→ (To avoid ringing & reflections)

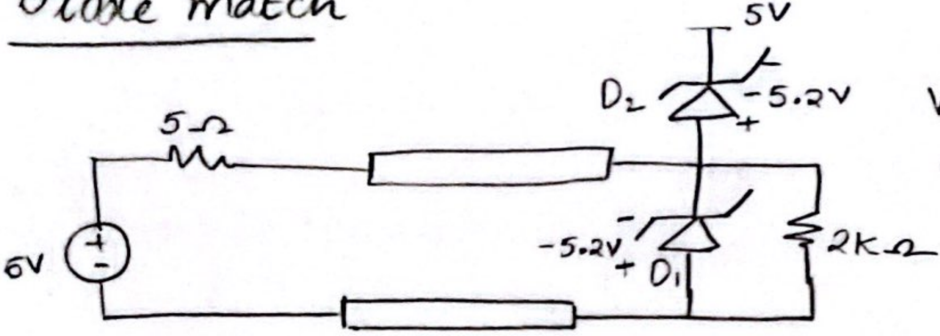
Capacitive termination



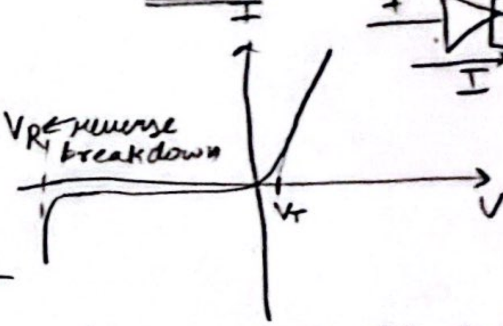
→ During transients cap is short so (almost) no reflections, but in steady state the load is 2KΩ.

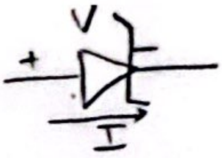
→ Doesn't work at higher frequencies.

Diode match



Zener





Zener ⇒ Well calibrated V_R.

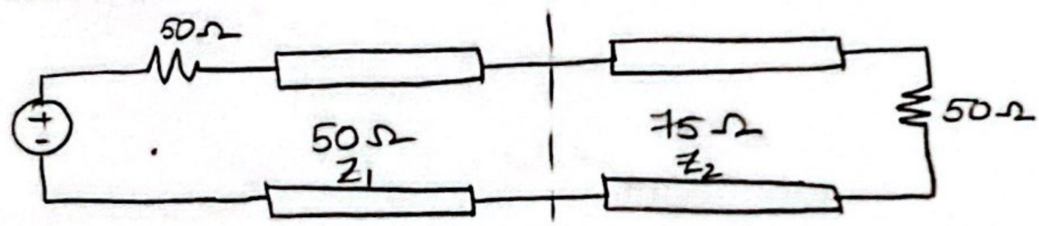
→ D_1 kicks in when the (almost) 5V step reaches the end and tries to "double". It saws off the waveform ripple at 5.2V. D_2 saws off the ringing when the voltage drops down to -5V and ringing starts.

→ Diodes are easy to integrate on chip.

Cascades and Fan outs

Cascade

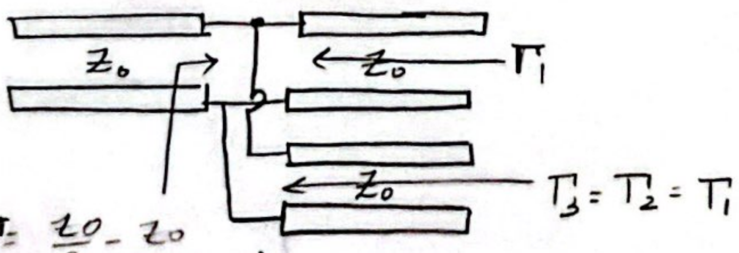
$\Gamma_{21} = -\frac{1}{5} = -\Gamma_{12}$



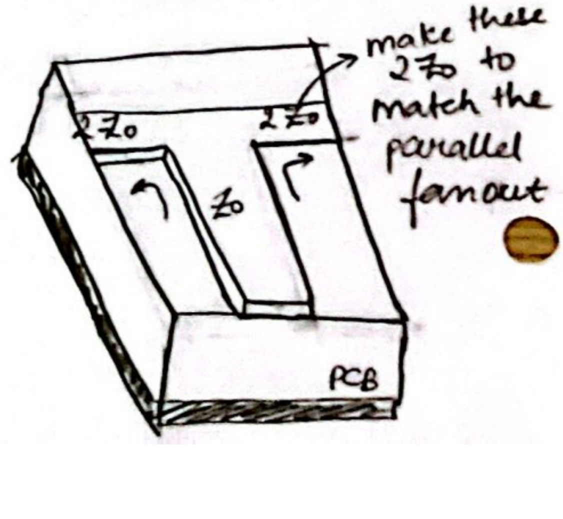
$$\Gamma_{12} = \frac{75-50}{75+50} = \frac{1}{5}$$

$$\Gamma_{12} = \frac{Z_2 - Z_1}{Z_2 + Z_1} = -\Gamma_{21}$$

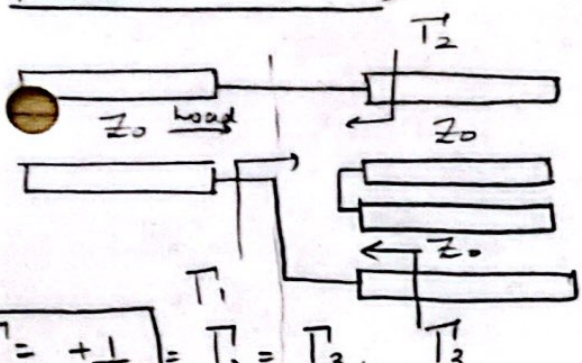
Fanouts (Parallel)



$$\Gamma_1 = \frac{\frac{Z_0}{2} - Z_0}{\frac{Z_0}{2} + Z_0} = \frac{-1}{3}$$

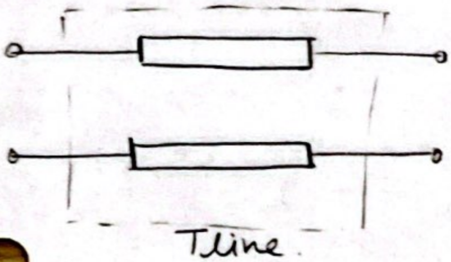


Fanout (Series)

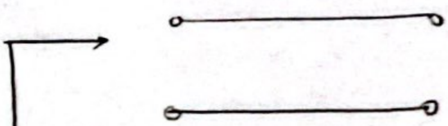


$$\boxed{\Gamma_1 = +\frac{1}{3}} = \Gamma_2 = \Gamma_3.$$

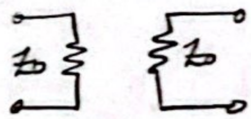
Initially charged Transmission lines



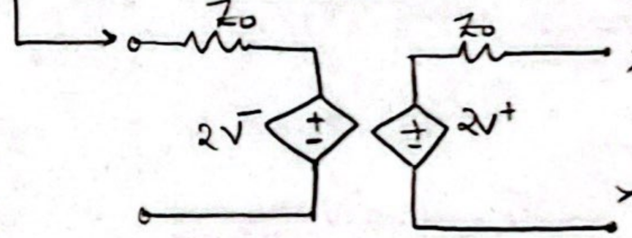
DC - Steady state



Discharged line



In General

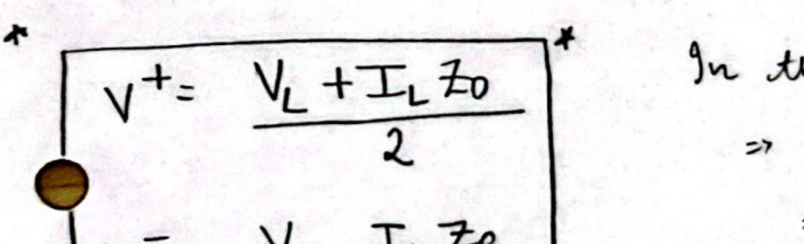


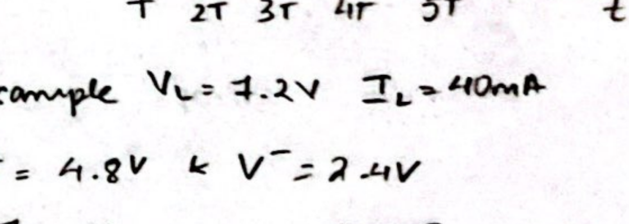
**

> V^- is the reflected Voltage striking the source side

> V^+ is incident voltage striking load side

Problem





*

$$\boxed{V^+ = \frac{V_L + I_L Z_0}{2} \qquad V^- = \frac{V_L - I_L Z_0}{2}}$$

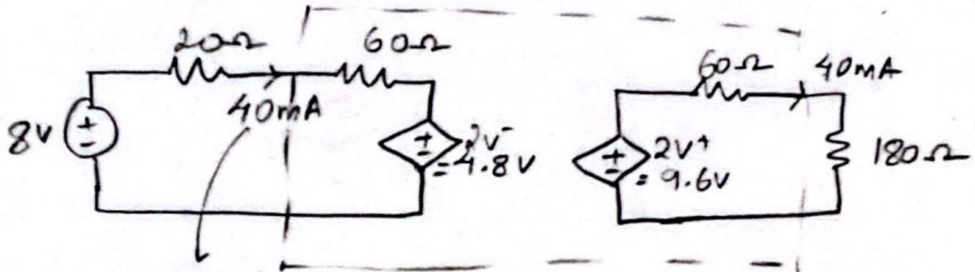
In this example $V_L = 7.2V$ $I_L = 40mA$

$\Rightarrow$ $V^+ = 4.8V$ & $V^- = 2.4V$

$\Rightarrow$ $\Gamma_L = +\frac{1}{2}$ & $V_L = V^+ + V^- = 7.2V$.

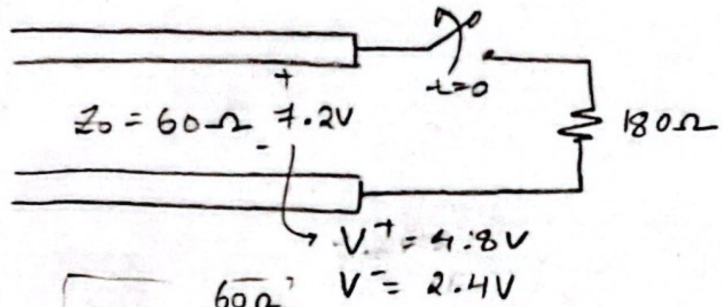
Verified!

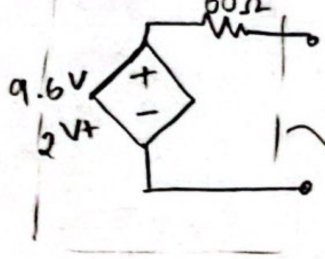
→ Let's check the general model.



Verified

→





→ When the switch is broken, the new $\Gamma_{new} = +1 \Rightarrow V^-$ changes from 2.4V to 4.8V.

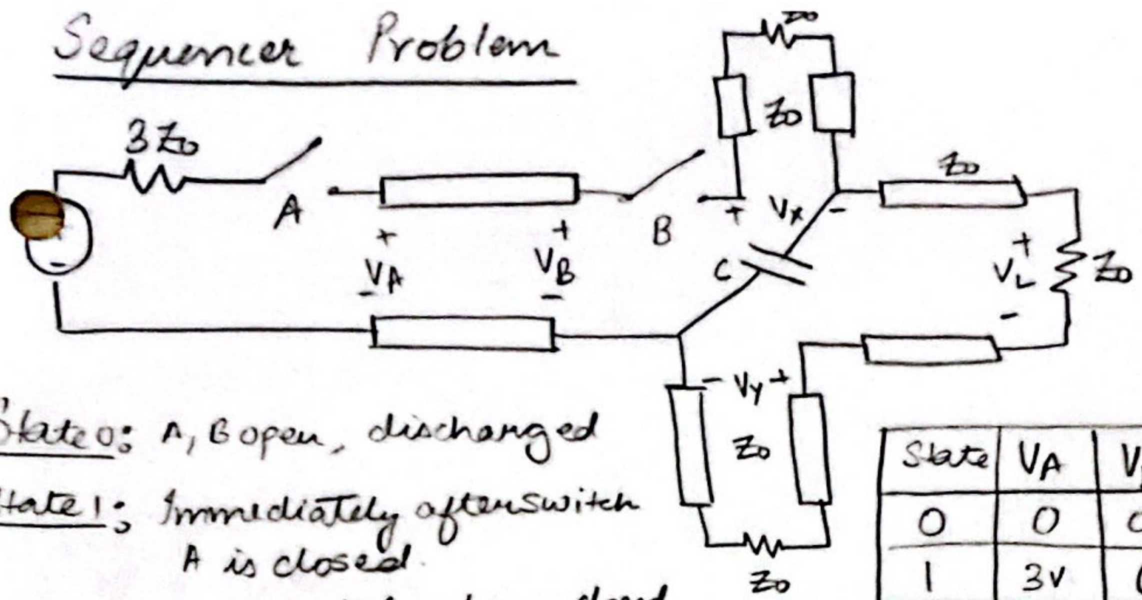
$V^+_{new} = 4.8V$

$V^-_{new} = 4.8V$

→ V_L goes from 7.2V → 9.6V!

This shows the same result!

Sequencer Problem



State 0: A, B open, discharged

State 1: Immediately after switch A is closed.

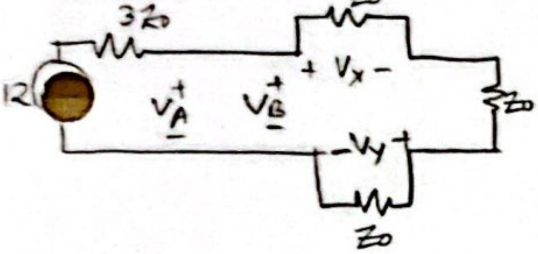
State 2: Switch A has been closed for a long time

State 3: Immediately after B closes

State 4: B closed for long!

State	V_A	V_B	V_L	V_x	V_y
0	0	0	0	0	0
1	3V	0	0	0	0
2	12V	12V	0	0	0
3	12V	6V	0	6V	0
4	6V	6V	2V	2V	2V

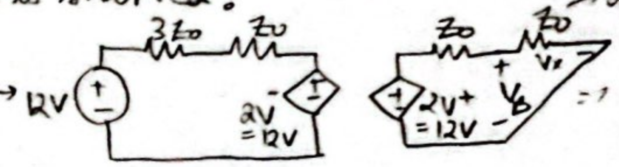
State 4



State 3

Cap is shorted!

→ deadline!



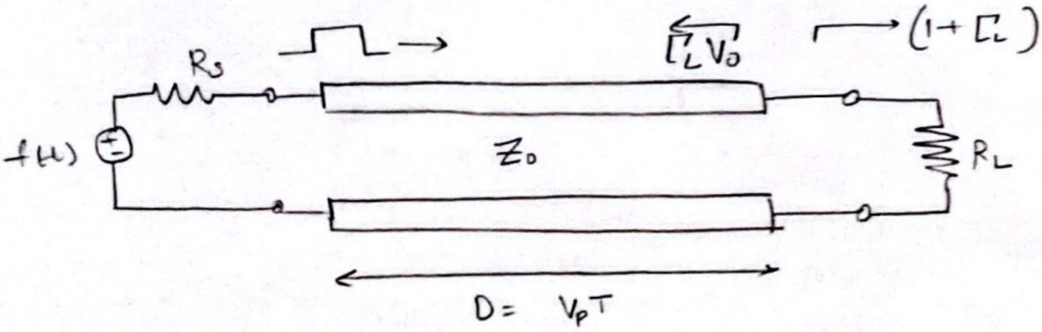
$\Rightarrow V_B = V_x = 6V$

$$V^+ = \frac{12V + \cancel{I_L Z_0}^{\,0}}{2} = 6V$$

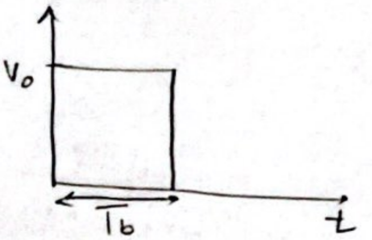
$$V^- = \frac{12V - \cancel{I_L Z_0}^{\,0}}{2} = 6V$$

Also remember V^- changes after one T & V^+ after 2T. Since V^+ is taken on left port.

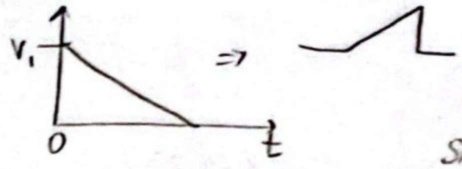
Short pulses on Transmission Lines



Input pulse

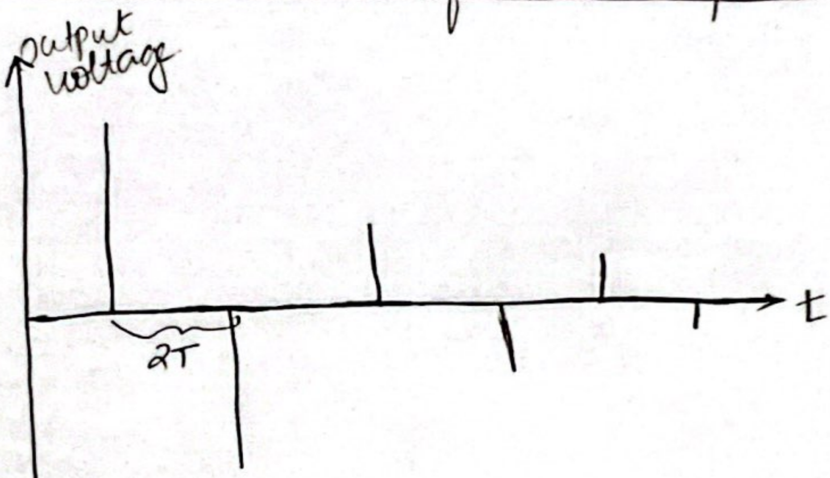


Keep in mind the shape of the pulse.

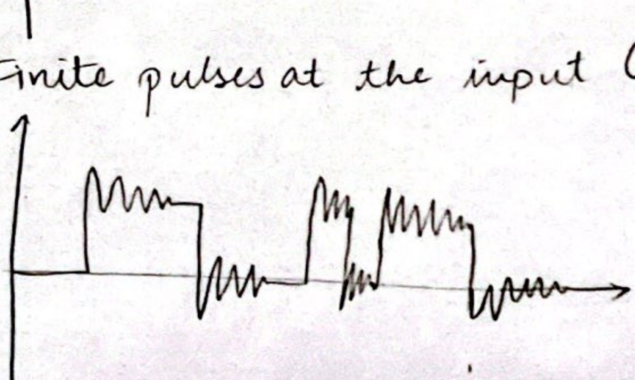


→ This is the pulse travelling on the line. Since at $t=0$ it jumps ups then settles.

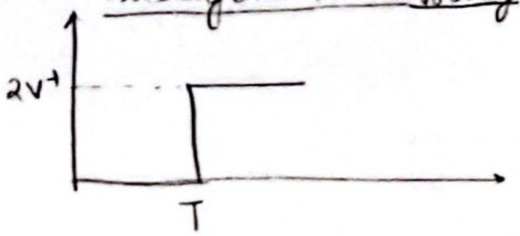
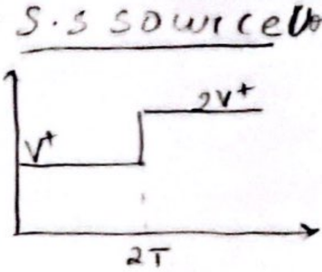
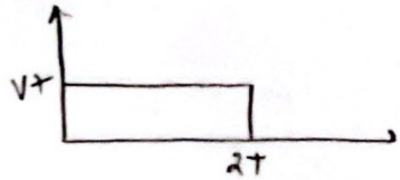
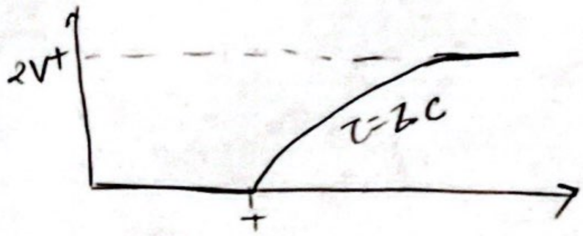
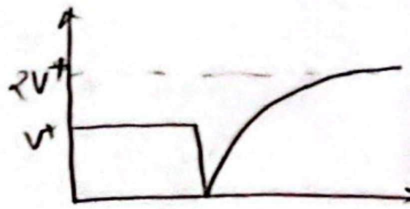
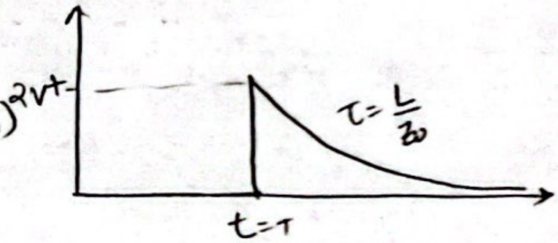
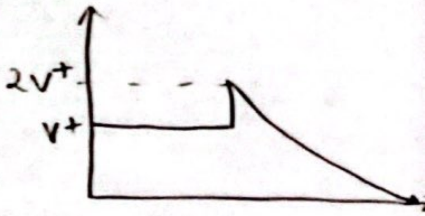
Output load voltage for an impulse at i/p



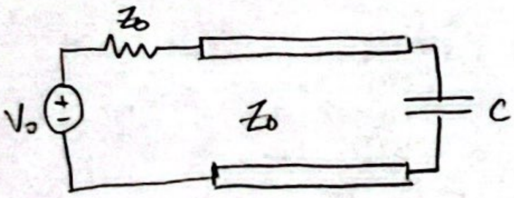
→ Finite pulses at the input (When the ringing is leaking onto the next bit we get inter symbol interference)



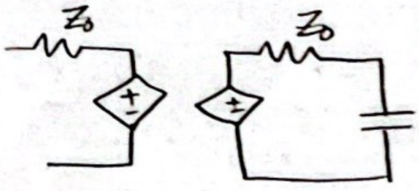
Reactive Termination on T-Lines

Termination	Steady state load voltage	S.S source voltage
Open circuit ($R_L = +\infty$, $\Gamma = +1$)		
Short circuit ($R_L = 0$, $\Gamma = -1$)		
Capacitor (Combination of short at $t = T$ & open at $t = \infty$)		
Inductor (Combination of open at $t = T$ & short at $t = \infty$)		

RC



What is the time constant?



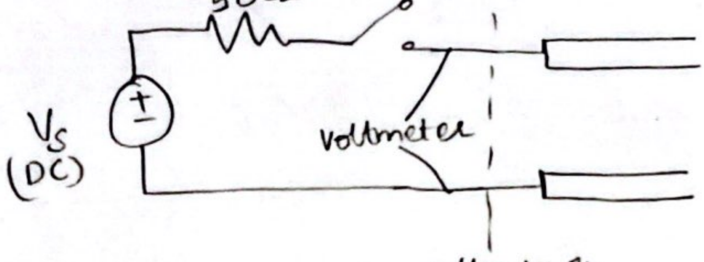
$$\boxed{\tau = Z_0 C}$$

RL

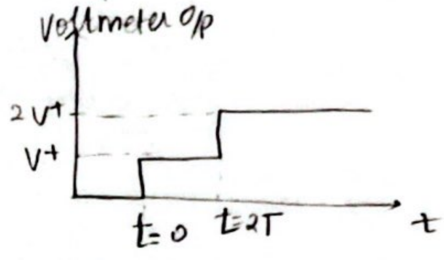
$$\boxed{\tau = \frac{L}{Z_0}}$$

Time Domain Reflectometer (TDR) → Used to test cable.

"Troubleshooting for shorts and opens in a T-line"

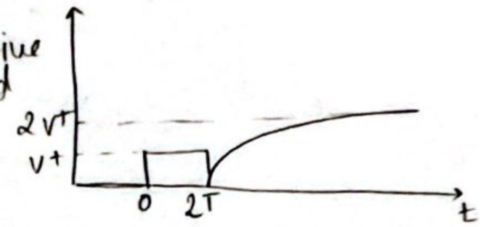


Open circuit

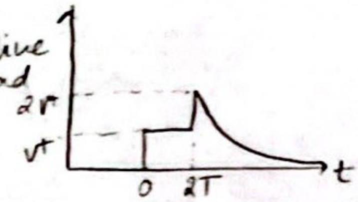


Short circuit

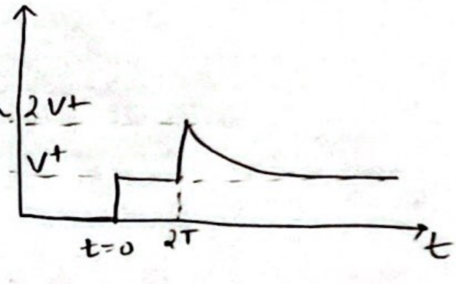
Capacitive load



Inductive load

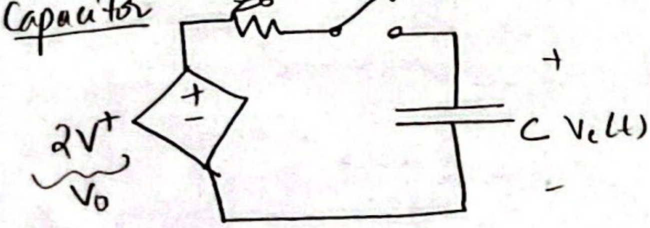


Inductor in series with 50Ω



Analysis

Capacitor

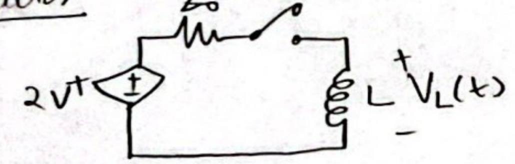


$$V_c(t) = V_0 \left[1 - \exp\left(-\frac{t}{\tau}\right)\right] u(t)$$

$\tau = RC$

$= Z_0 C$

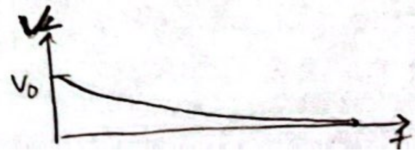
Inductor



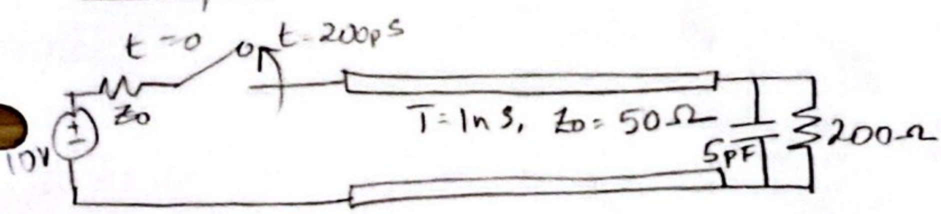
$$V_L(t) = V_0 \exp\left(-\frac{t}{\tau}\right) u(t)$$

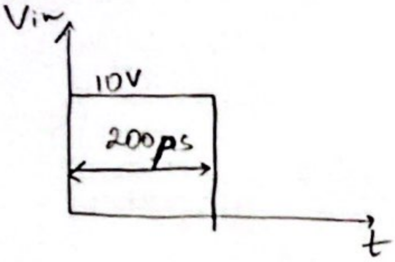
$\tau = \frac{L}{R}$

$= \frac{L}{Z_0}$



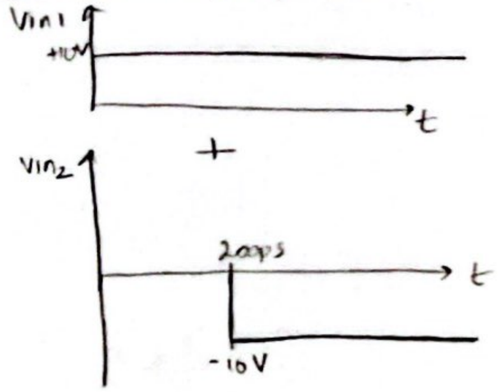
Example



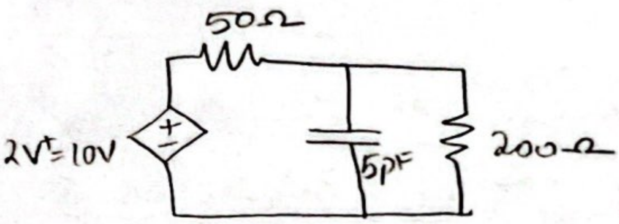


=

Linear superposition



End of the line



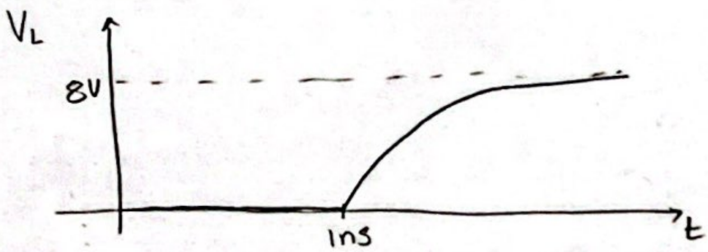
$V_{ss} = 10V \cdot \frac{200}{250} = \underline{\underline{8V}}$

Timeconstant : $R_{eq} \cdot C$

where $R_{eq} = 200 \| 50$

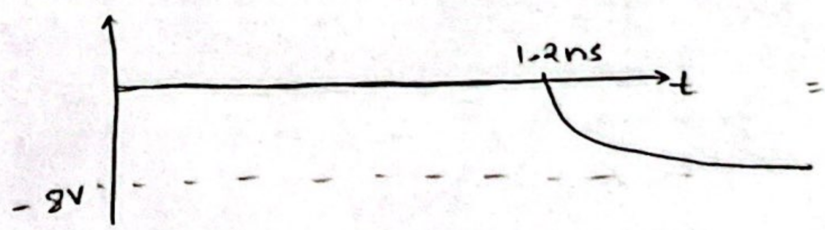
$\Rightarrow \tau = 200ps$

Solution ①

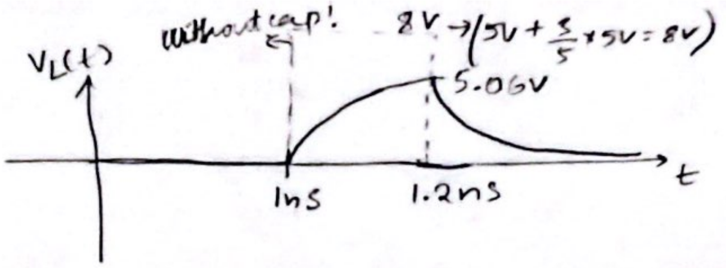


$= 8V \left[1 - \exp\left(\frac{-(t-1ns)}{\tau}\right)\right] u(t-1ns)$

Solution ②

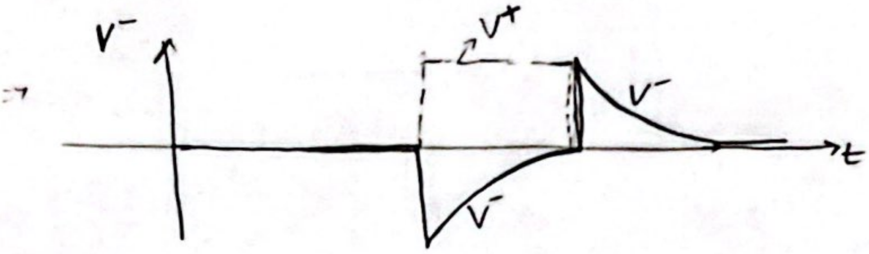


$= 8V \exp\left(-\frac{(t-1.2ns)}{\tau}\right) u(t-1.2ns)$

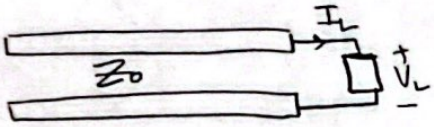


$$V_L(t) = \begin{cases} 0 & t < 1ns \\ 8\left[1 - \exp\left[-\frac{t - 1ns}{200ps}\right]\right] & 1ns < t < 1.2ns \\ 8\left[\exp\left(-\frac{t-1ns}{200ps}\right) - \exp\left(-\frac{t - 1.2ns}{200}\right)\right] & t > 1.2ns \end{cases}$$

$$V_L = V^+ + V^- \quad = \quad V_L - V^+ = V^-$$



Nonlinear loads on Transmission Lines



$$V^+ = \frac{V_L + I_L Z_0}{2} \qquad I_L = \frac{2}{Z_0} V^+ - \frac{V_L}{Z_0}$$

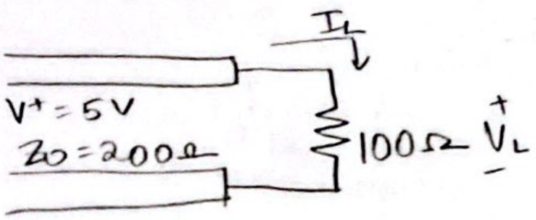
$$V^- = \frac{V_L - I_L Z_0}{2} \qquad (I_L)_{new} = \frac{2}{Z_0} V^+ - \frac{1}{Z_0} f(I_{L\,old})$$

$$(I_L)_{new} = \frac{2}{Z_0} V^+ - (I_L)_{old} \frac{R_L}{Z_0}$$

Iteration

1. make a guess for $(I_L)_{old}$
2. Calculate $(I_L)_{new}$ from $(I_L)_{old}$
3. If $(I_L)_{new} \neq (I_L)_{old}$, repeat with $(I_L)_{old} = (I_L)_{new}$.

Example (A simple example)



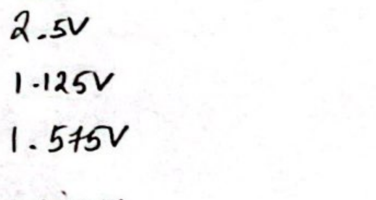
$$(I_L)_{new} = 0.05 - \frac{1}{2}(I_L)_{old}$$

Guess #	I_L^{old}	I_L^{new}
1	0A	0.05A
2	0.05A	0.025A
3	0.025A	0.0375A
4	0.0375A	0.03125A
⋮	⋮	⋮
9	0.033A	0.033A ✓

→ The formulation we have :- $(I_L)_{new} = \frac{2}{Z_0} V^+ - \frac{1}{Z_0} f((I_L)_{old})$

- A different formulation : (when I_L doesn't converge) $(V_L)_{new} = 2V^+ - Z_0 f^{-1}((V_L)_{old})$

Example



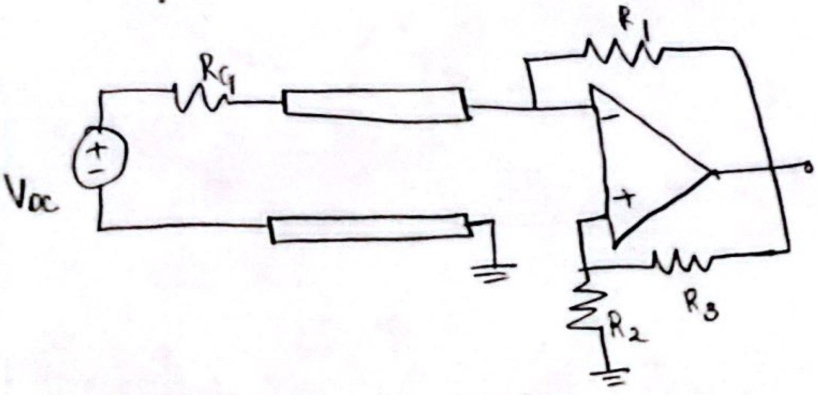
$$I_L = \sqrt{\frac{V_L}{50}} \Rightarrow \frac{(V_L)_{new} - 2V^+}{Z_0} = \sqrt{\frac{(V_L)_{old}}{50}}$$

$$\Rightarrow (V_L)_{old} = \frac{[2V^+ - (V_L)_{new}]^2}{50}$$

$$\Rightarrow (V_L)_{new} = \frac{[2V^+ - (V_L)_{old}]^2}{50}$$

Iteration #	V_L
1	2.5V
2	1.125V
3	1.575V
4	1.420V
⋮	
10	1.459V → final!

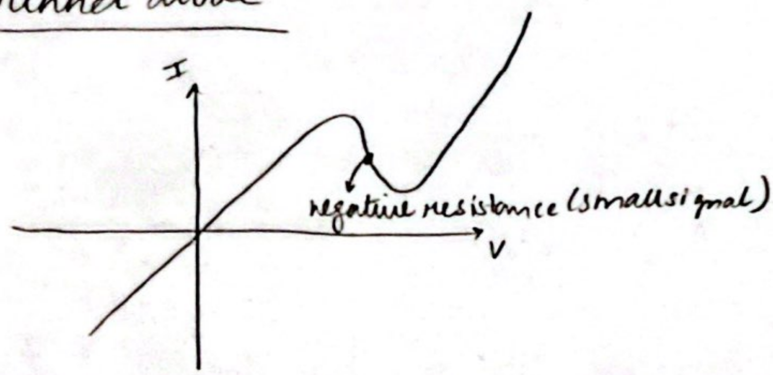
Example

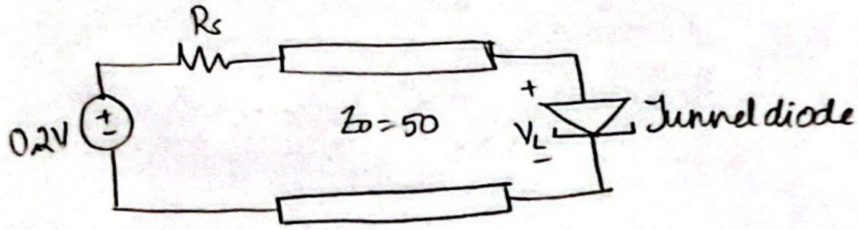


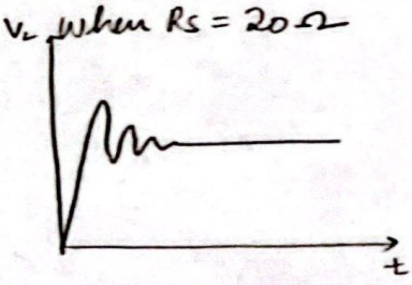
$$R_L = -\left(\frac{R_2 R_3}{R_1 - R_3}\right)$$

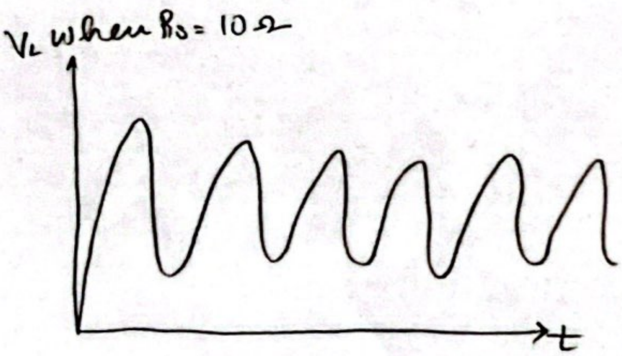
→ $|\Gamma_L|$ could be greater than 1.

Tunnel diode





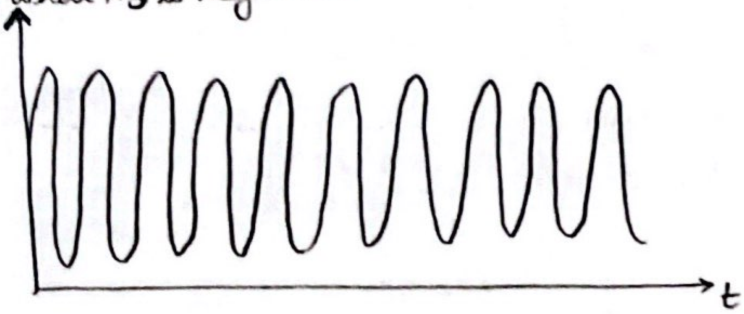




→ You get an oscillator that can have very high frequency that depends on the length of the line.

→ We can create new frequency content with non linear circuits.

→ V_L when R_S is negative.

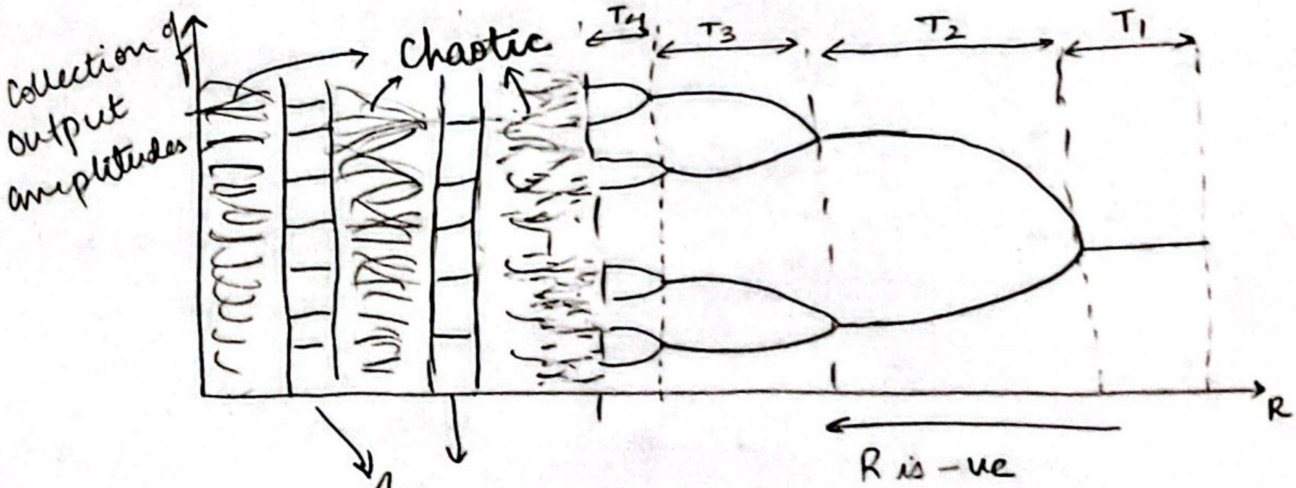


→ V_L when R_S is $-20\,\Omega$

⇒ 4 points of reflections

→ When $R_S = -22\,\Omega$ we get 8 points of reflections

→ At $R_S = -25\,\Omega$, system is chaotic & doesn't oscillate but it is bounded.

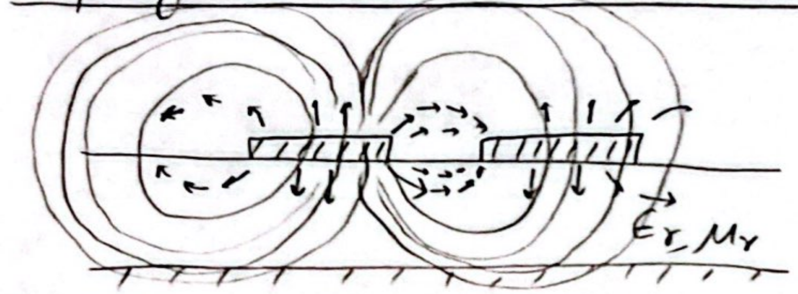


Islands of stability where the system can have 3, 5, 7 .. odd no. of stable modes.

→ $\frac{T_1}{T_2} = \frac{T_2}{T_3} = \frac{T_3}{T_4} = \ldots$ alway approaches a number 4.66 for any nonlinear chaotic system. Feigenbaum constant.

→ For chaotic systems you need atleast a third order differential equation => atleast 3 Ls or Cs. Need some damping => R. Some active circuit to sustain it.

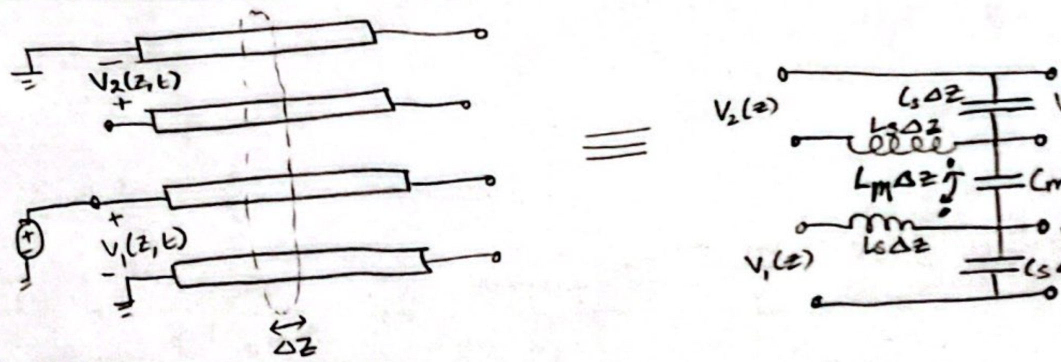
Coupling in Transmission Lines



Coupling

Sharing of fields between T-lines leads to crosstalk.

circuit model



L_s - single line unit length inductance (H/m)

C_s - cap/length. (F/m)

C_m - mutual cap per length

Let,

$L = L_s$; $M = L_m$; $C = C_s + C_m$, $E = C_m$.

$\Rightarrow V_p = \frac{1}{\sqrt{LC}}$; $Z_0 = \sqrt{\frac{L}{C}}$

If the lines are embedded in a homogeneous medium-

$\Rightarrow \frac{M}{L} = \frac{E}{C} = \Delta$ always lies b/w 0 & 1 => 0 → no coupling. } Realistically,
1 → perfect coupling. } Δ is close to 0.

Crosstalk coefficients

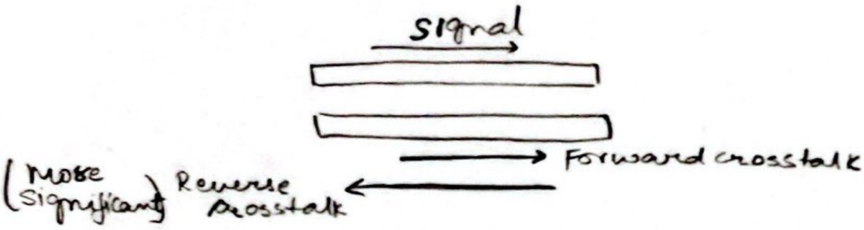
Weakly coupled assumption: $\Delta \ll 1$

> Forward crosstalk coefficient

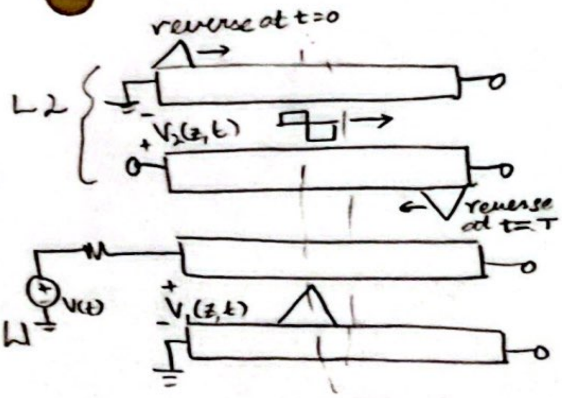
$$K_F = \frac{\sqrt{LC}}{2}\left[\frac{E}{C} - \frac{M}{L}\right] \approx 0 \text{ (ideally should be 0) units} \rightarrow \frac{\text{sec}}{\text{mtr}}$$

> Reverse crosstalk coefficient

$$K_R = \frac{1}{4}\left[\frac{E}{C} + \frac{M}{L}\right] = \frac{\Delta}{2} \text{ (unitless)}$$



→ The units are different because they manifest differently.



Increase in forward crosstalk voltage

$$\Delta V_2^+ = K_F \Delta z \frac{\partial V_1(z,t)}{\partial t}$$

Increase in reverse crosstalk.

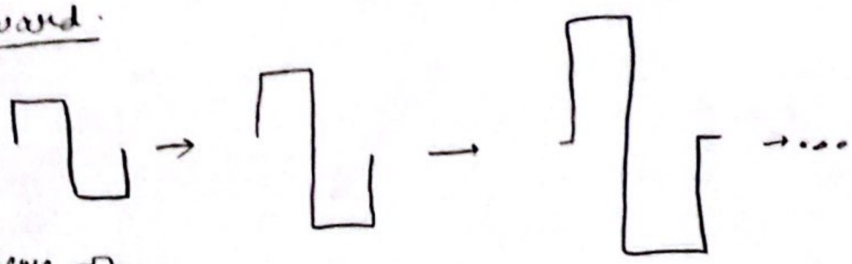
$$\Delta V_2^- = K_R \, 2\sqrt{LC} \, \Delta z \frac{\partial V_1(z,t)}{\partial t}$$

→ The derivative in the equation's are due to the fact that a changing electric field V_1 produces a changing magnetic field and only a changing magnetic field can produce an emf in Line L1.

→ Because of the derivative if there is a forward propagating triangle wave in L1 The coupled forward wave would be a square wave. which grows in amplitude as the waveform travels through the line.

→ However, in the reverse direction each of the square waves going backwards are integrated again due to the direction of formation & propagation being opposite. The integrations end up cancelling except at the ends of the transmission line. So reverse waves are produced only at t=0 & t=T. They show up at the source at t=0 & t=2T. The one produced at the end of the line is flipped upside down.

→ <u>Forward</u>:



This is relatively small due to the K_F. Despite growing in amplitude

<u>Reverse</u>

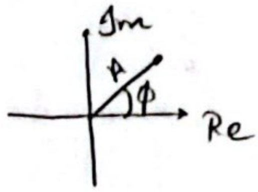
← 

→ Used to make directional couplers!

Phasors

$$x(t) = A\cos(2\pi f t + \phi)$$

- In LTI systems f does not change. So we only care about A & ϕ. So we use complex numbers



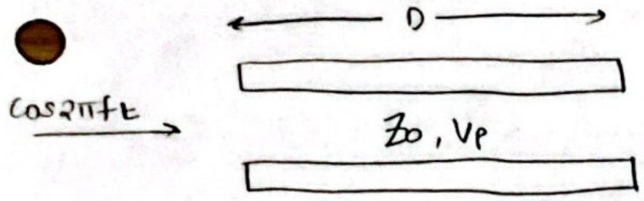
polar $\tilde{X} = A\exp(j\phi)$

Cartesian $\tilde{X} = A\cos\phi + jA\sin\phi$

Phasor to time domain conversion

$$x(t) = \mathrm{Re}\left\{\tilde{X}\exp(j2\pi f t)\right\}$$

Sinusoids on Transmission lines



$$v(z,t) = V^+ f\left(t - \frac{z}{V_p}\right) + V^- g\left(t + \frac{z}{V_p}\right)$$

$$i(z,t) = \frac{V^+}{Z_0} f\left(t - \frac{z}{V_p}\right) - \frac{V^-}{Z_0} g\left(t + \frac{z}{V_1}\right)$$

$\downarrow$

wave number $\boxed{\beta = \frac{2\pi f}{V_p} = \frac{2\pi}{\lambda} = \frac{\text{rad}}{m}}$

$\tilde{V}(z) = \tilde{V}^+\exp\left(-j\frac{2\pi f}{V_p}z\right) + \dots$ ← Phasor

$$v(z,t) = V^+\cos\left[2\pi f\left(t - \frac{z}{V_p}\right)\right] + V^-\cos\left[2\pi f\left(t + \frac{z}{V_p}\right)\right]$$

$$i(z,t) = \frac{V^+}{Z_0}\cos\left[2\pi f\left(t - \frac{z}{V_p}\right)\right] + \frac{V^-}{Z_0}\cos\left[2\pi f\left(t + \frac{z}{V_p}\right)\right]$$

$\Downarrow$

$$\tilde{V}(z) = \underbrace{\tilde{V}^+\exp(-j\beta z)}_{\text{constant velocity in the +ve z direction}} + \underbrace{\tilde{V}^-\exp(j\beta z)}_{\text{Constant velocity in } -z \text{ direction}}$$

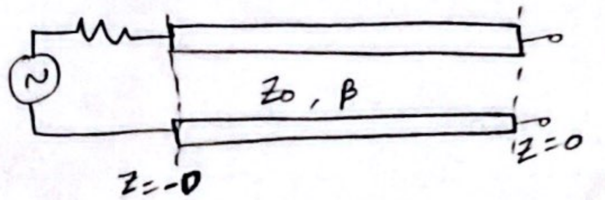
$$\tilde{I}(z) = \overbrace{\frac{\tilde{V}^+}{Z_0}\exp(-j\beta z)} - \overbrace{\frac{\tilde{V}^-}{Z_0}\exp(j\beta z)}$$

unclear!

→ In physics $i = \sqrt{-1}$

→ In physics phasors are of the form $\exp(+i\beta z)$ for forward propagation since $i = -j$ where both i, j are the 2 solutions to $x^2 = -1$. → This convention is in EM. The value of $j = \sqrt{-1}$. In circuits i & j are same.

Open and Short Circuit Loads on T-lines



$$\tilde{V}(z) = \tilde{V}^+ \exp(-j\beta z) + \tilde{V}^- \exp(+j\beta z)$$

$$\tilde{I}(z) = \frac{\tilde{V}^+}{Z_0} \exp(-j\beta z) - \frac{\tilde{V}^-}{Z_0} \exp(+j\beta z)$$

Case ① OPEN

At an open circuit

$$\underbrace{\Gamma_L}_{=1} \tilde{V}^+ = \tilde{V}^-$$

$\Rightarrow$ Voltage and current when $\Gamma_L = +1$

$$\tilde{V}(z) = 2\tilde{V}^+ \left[\frac{\exp(-j\beta z) + \exp(+j\beta z)}{2} \right]$$

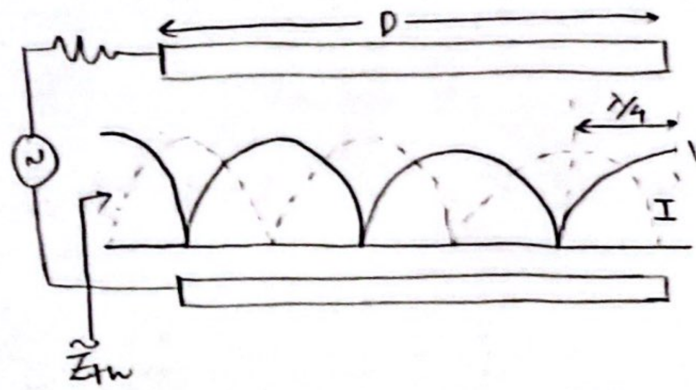
$$\tilde{V}(z) = 2\tilde{V}^+ [\cos\beta z]$$ $\Rightarrow$ phase is constant but amplitude is varying.

$\rightarrow$ at $z=0$ we get a peak $\Rightarrow$ at the end of the line.

$$\tilde{I}(z) = -2j \cdot \frac{\tilde{V}^+}{Z_0} \left[\frac{-\exp(-j\beta z) + \exp(+j\beta z)}{+2j} \right]$$

$$\tilde{I}(z) = -2j \frac{\tilde{V}^+}{Z_0} [\sin\beta z]$$ $\Rightarrow$ current phase is also constant but it is out of phase w.r.t voltage by 90°

$\rightarrow$ at $z=0$ we get a null



→ VSWR :- Voltage Standing wave ratio

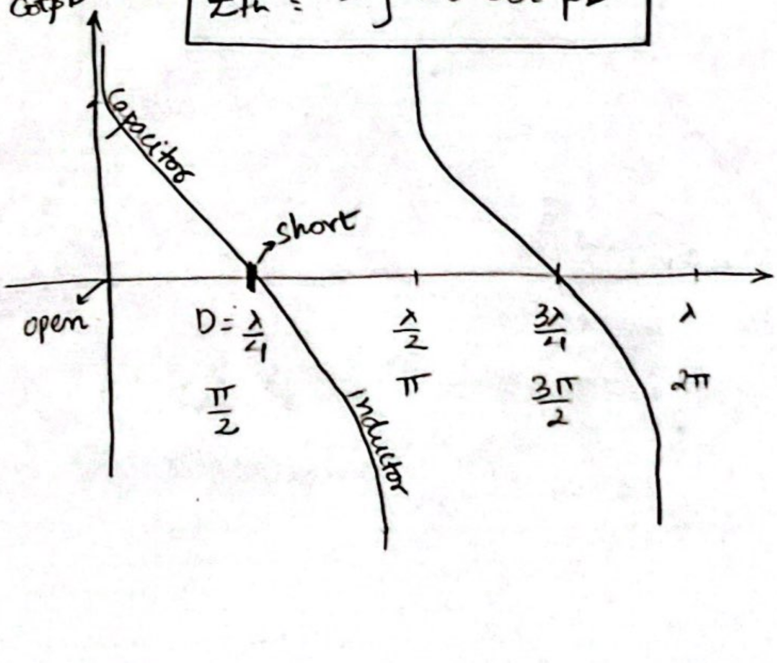
$$VSWR = \frac{V_{max}}{V_{min}} \text{ on the line} \quad \{\text{always b/w 1 and } \infty\}$$

$$= \frac{2|\tilde{V}^+|}{0} = +\infty \text{ for open circuit}$$

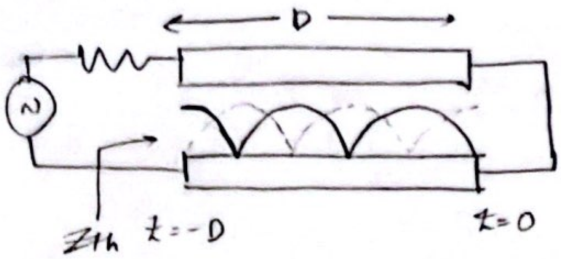
Also represented in $20\log_{10} VSWR$ (dB) { b/w 0 to ∞}

$$\rightarrow \tilde{Z}_{th} = \frac{\tilde{V}(z)|_{z=-D}}{\tilde{i}(z)|_{z=-D}} = \frac{2\tilde{V}^+ \cos\beta(+D)}{+2j\frac{\tilde{V}^+}{Z_0}\sin\beta(+D)} = -jZ_0\cot(\beta D)$$

$$\boxed{\tilde{Z}_{th} = -jZ_0 \cot\beta D} \qquad \beta D = \frac{2\pi}{\lambda}\cdot D$$



Case ② SHORT



$\Gamma_L = -1$

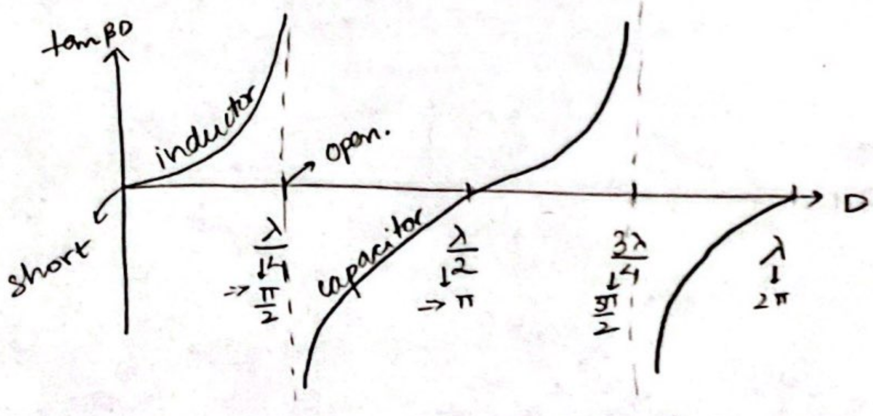
$\tilde{V}^- = -\tilde{V}^+$

$$\tilde{V}(z) = -2j\,\tilde{V}^+ \left[\frac{-\exp(-j\beta z) + \exp(+j\beta z)}{2j}\right] = -2j\,\tilde{V}^+ \sin\beta z$$

$$\tilde{i}(z) = 2\,\tilde{V}^+ \left[\frac{\exp(-j\beta z) + \exp(+j\beta z)}{2}\right] = 2\,\tilde{V}^+ \cos\beta z$$

$$\tilde{Z}_{in} = \frac{\tilde{V}(z)\,|_{z=-D}}{\tilde{i}(z)\,|_{z=-D}} = \frac{+j\cancel{2}\,\cancel{\tilde{V}^+}\sin\beta(+D)}{\cancel{2}\,\cancel{\tilde{V}^+}\cos\beta(-D)} = j Z_0 \tan\beta D$$

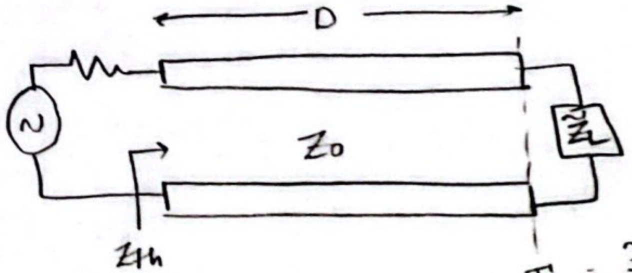
$$\boxed{\tilde{Z}_{in} = j Z_0 \tan\beta D}$$



→ Short circuit can be used for biasing and still get high impedance Z_{in}. Better for inductors (small line)

→ Open circuit is better to avoid vias. (Also better for capacitor)

Sinusoids on T-Lines with Arbitrary Loads.



→ When Z_L is short or open or purely reactive all the power is reflected back.

$$\Gamma_L = \frac{\tilde{Z}_L - Z_0}{\tilde{Z}_L + Z_0}$$

→ For reactive loads, $\tilde{Z}_L$ is imaginary ⇒ Γ_L can be complex.

→ $$\boxed{VSWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}}$$

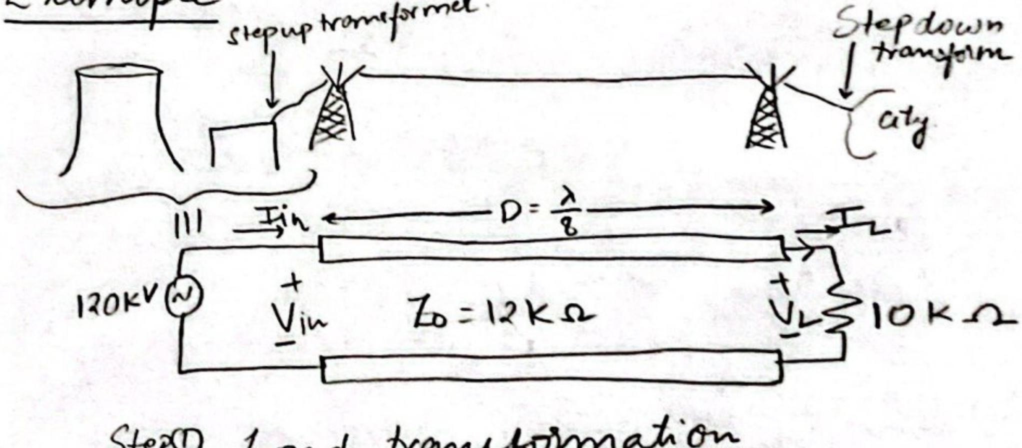
VSWR = ∞ for open, short, reactive loads.
VSWR = 1 for matched case.

→ $$\boxed{\tilde{Z}_{th} = Z_0 \cdot \frac{Z_L + jZ_0 \tan\beta D}{Z_0 + jZ_L \tan\beta D}}$$

Impedance transformation.

Repeats itself every half a wavelength (tan repeats every π)

Example

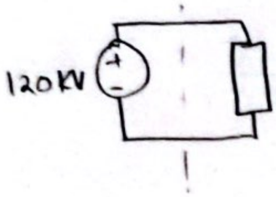


Step ① Load transformation

characterize $\tilde{Z}_{in}$; $Z_{in} = Z_0 \left[\frac{\tilde{Z}_L + jZ_0 \tan\beta D}{Z_0 + j\tilde{Z}_L \tan\beta D}\right] = Z_0\left[\frac{\tilde{Z}_L + jZ_0}{Z_0 + jZ_L}\right]$

$= 12\angle 10^\circ K\Omega = 11.8 + j2.2 K\Omega$

Step2 Find V_{in}, I_{in}.



$\tilde{Z}_{in} = 11.8 + j2.2 k\Omega$

$$\tilde{V}_{in} = 120KV \angle 0^\circ$$

$$\tilde{I}_{in} = \frac{\tilde{V}_{in}}{11.8 + j2.2k\Omega} = 10 \angle{-10^\circ} \text{ A}$$

Step3 Enforce the source side continuity

$$\tilde{V}(z) = \tilde{V}^{+} \exp(-j\beta z) + \tilde{V}^{-} \exp(+j\beta z)$$

$$\tilde{i}(z) = \frac{\tilde{V}^{+}}{Z_0} \exp(-j\beta z) - \frac{\tilde{V}^{-}}{Z_0} \exp(j\beta z)$$

$z=0$ is front of the line

Step4 solve for $\tilde{V}^{+}$ & $\tilde{V}^{-}$

$$\left.\begin{aligned} \tilde{V}(0) &= \tilde{V}^{+} + \tilde{V}^{-} = 120\angle 0^\circ \text{ KV} \\ \tilde{i}(0) &= \frac{\tilde{V}^{+}}{Z_0} - \frac{\tilde{V}^{-}}{Z_0} = 10 \angle{-10^\circ} \text{A} \end{aligned}\right\}$$

Solve $\tilde{V}^{+}$ & $\tilde{V}^{-}$:

$\tilde{V}^{+} = 119.5 \angle{-1^\circ}$ kV

$\tilde{V}^{-} = 11 \angle 85^\circ$ KV → travelling to the generator.

Step5 Solve V_L, I_L

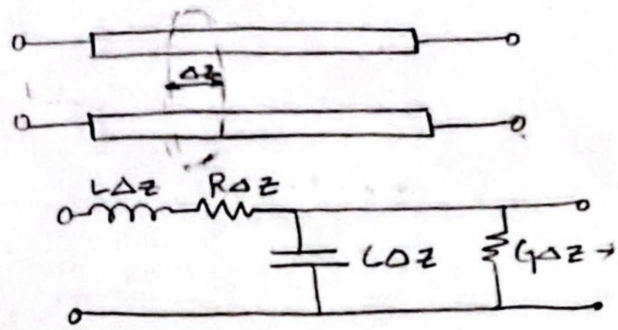
$$\tilde{V}(D) = \tilde{V}^{+} \exp(-j \cdot \tfrac{\pi}{4}) + \tilde{V}^{-} \exp(+j\tfrac{\pi}{4}) = 109 \angle{45.2^\circ} \text{ kV}$$

$$\tilde{i}(D) = \frac{\tilde{V}^{+}}{Z_0} \exp(-j\tfrac{\pi}{4}) - \frac{\tilde{V}^{-}}{Z_0} \exp(+j\tfrac{\pi}{4}) = 10.9 \angle{45.7^\circ} \text{ A}$$

Step6 Verify answer with Ohm's law!

$$\frac{\tilde{V}_L}{I_L} = \frac{109 \angle 45.2}{10.9 \angle 45.7} = 10k\Omega \angle \cancel{-0.5}^{\circ} \; 0$$

Lossy Transmission Lines



$\rightarrow$ dielectric leakage & dielectric relaxation losses.

Telegrapher's equations solutions

$$\tilde{V}(z) = \tilde{V}^{+} \exp(-\gamma z) + \tilde{V}^{-} \exp(+\gamma z)$$

$\gamma \rightarrow$ propagation constant

$$\tilde{i}(z) = \frac{\tilde{V}^{+}}{Z_0} \exp(-\gamma z) - \frac{\tilde{V}^{-}}{Z_0} \exp(+\gamma z)$$

Where $Z_0 = \sqrt{\dfrac{R + j2\pi f L}{G + j2\pi f C}}$ $\rightarrow$ Complex Z_0 $\Rightarrow$ Lossy line

$\Rightarrow$ Current and Voltage are out of phase.

$$\begin{cases} \text{Real } Z_0 \Rightarrow \text{lossless} \\ \text{Complex } Z_0 \Rightarrow \text{lossy} \end{cases}$$

Also, $\gamma = \underbrace{\alpha}_{\text{attenuation coefficient}} + j\underbrace{\beta}_{\text{wave number}} = \sqrt{(R + j2\pi f L)(G + j2\pi f C)}$

α +ve $\Rightarrow$ $\tilde{V}^{+}$ experiences exponential attenuation in +ve z direction

α +ve $\Rightarrow$ $\tilde{V}^{-}$ experiences exponential attenuation in -ve z direction.

→ Considering the forward wave

$$\tilde{V}(z) = \tilde{V}^+ \exp(-\gamma z)$$

$$= \tilde{V} \exp(-j\beta z - \alpha z)$$

$$V(z,t) = \mathrm{Re}\{\tilde{V}(z)\exp(j2\pi f t)\} \quad -①$$

Note:-

$$|\tilde{V}(z)| = \underbrace{|\tilde{V}^+|}_{V^+}\ \underbrace{|\exp(-\alpha z)|}_{\exp(-\alpha z)}\ \underbrace{|\exp(-j\beta z)|}_{0}$$

$$= V^+ \exp(-\alpha z) \rightarrow \text{amplitude has exponential taper.}$$

Continuing from Eq. ①

$$V(z,t) = \mathrm{Re}\{\tilde{V}(z)\exp(j2\pi f t)\}$$

$$= \mathrm{Re}\{\underbrace{\tilde{V}^+}_{V^+\exp(j\phi)} \exp[-[\alpha + j\beta]z + j2\pi f t\}$$

$$= \mathrm{Re}\{V^+ \exp\{-(\alpha + j\beta)z + j\phi + j2\pi f t\}\}$$

$$= V^+ \exp(-\alpha z)\ \mathrm{Re}\{\exp(-j\beta z + j\phi + j2\pi f t)\}$$

$$\boxed{V(z,t) = \overbrace{\underbrace{V^+ \exp(-\alpha z)}_{\text{exponential attenuation of amplitude.}}}^{\text{envelope!}} \cos\underbrace{\{2\pi f t - \beta z + \phi\}}_{\text{travelling wave.}}} \text{ Volts!}$$

Power

$$P(z) = \frac{1}{2} \, Re\left\{ \tilde{V}(z) \, \tilde{I}^*(z) \right\} \quad \text{Watts}$$

$$= \frac{1}{2} \, Re\left\{ \tilde{V}(z) \, \frac{\tilde{V}^*(z)}{Z_0^*} \right\}$$

$$= \frac{1}{2} \, Re\left\{ \frac{|\tilde{V}(z)|^2}{Z_0^*} \right\}$$

$$P(z) = \frac{1}{2} \, |\tilde{V}(z)|^2 \, Re\left\{ \frac{1}{Z_0^*} \right\}$$

$$P(z)\Big|_{z=0} = \frac{1}{2} V^{+2} \overset{1}{\cancel{\exp(-2\alpha z)}} \, Re\left\{ \frac{1}{Z_0^*} \right\} = \frac{1}{2} V^{+2} Re\left\{ \frac{1}{Z_0^*} \right\}$$

$$P(z)\Big|_{z=1m} = \frac{1}{2} V^{+2} \exp(-2\alpha) \, Re\left\{ \frac{1}{Z_0^*} \right\}$$

$$\boxed{\frac{P(z)\,|_{z=0}}{P(z)\,|_{z=1m}} = \exp(2\alpha)}$$

<u>Loss per meter!</u>

$$\rightarrow \underset{\text{permetre}}{\text{Loss}} = \frac{P(0)}{P(1)} = \exp(2\alpha)$$

$$\Rightarrow \underset{\text{per mtr}}{\text{Loss}} \text{ in dB} = 10 \log_{10}(\exp(2\alpha))$$

$$= \frac{10\,(2\alpha)}{\log_e 10}$$

$$\boxed{\text{Loss per mtr in dB} = 8.7\,\alpha}$$

dB/m

Nepers/m

Low loss transmission line

Low loss Approximation

$$\gamma = \sqrt{(R + j2\pi f L)(G + j2\pi f C)}$$

$$= j2\pi f\sqrt{LC}\sqrt{\left(1 + \frac{R}{j2\pi f L}\right)\left(1 + \frac{G}{j2\pi f C}\right)}$$

When $R \ll 2\pi f L$, $G \ll 2\pi f C$

$$= j2\pi f\sqrt{LC}\left[1 + \frac{R}{j2\pi f L}\right]\left[1 + \frac{G}{j2\pi f C}\right]$$

$$\gamma = j2\pi f\sqrt{LC}\left[1 + \frac{1}{j2\pi f}\left[R/L + G/C\right]\right] = \alpha + j\beta$$

$\beta \cong 2\pi f\sqrt{LC}$ rad/m (like lossless)

$Z_0 \cong \sqrt{\frac{L}{C}}$ Ω (like lossless)

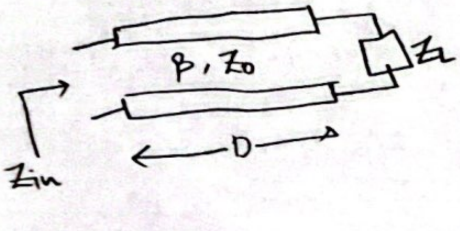
$\alpha \cong \underbrace{\frac{R}{2}\sqrt{\frac{C}{L}}}_{\alpha_c} + \underbrace{\frac{G}{2}\sqrt{\frac{L}{C}}}_{\alpha_d}$ (only new term)

frequency dependent due to skin effect.

α_c - conductor loss $= \frac{R}{2Z_0}$

α_d - dielectric loss $= \frac{G Z_0}{2}$

General Impedance transformation for lossy line



$$\tilde{Z}_{in} = Z_0\left[\frac{\tilde{Z}_L + Z_0 \tanh(\gamma D)}{Z_0 + \tilde{Z}_L \tanh(\gamma D)}\right]$$

$$\tan x = \frac{\exp(jx) - \exp(-jx)}{j[\exp(jx) + \exp(-jx)]}$$

remove j $\Rightarrow$ $\tanh x = \frac{\exp(x) - \exp(-x)}{\exp(x) + \exp(-x)}$

$\tanh jx = j\tan x$

Skin depth

$$\delta = \frac{1}{\sqrt{\pi f \sigma \mu}}$$

frequency (f), conductivity (σ), permeability (μ)

Area of conductor for AC = $\delta \times$ perimeter of cross section.

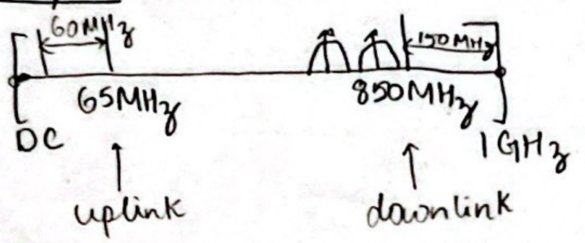
$$\underset{\text{per m}}{R} = \frac{1}{\sigma A} \quad \text{so} \quad R \propto \frac{1}{\sqrt{\sigma}}$$

For coaxial cable

$$\rightarrow \alpha_c = \frac{\sqrt{f}}{4 Z_0}\sqrt{\frac{\mu}{\pi\sigma}}\left[\underset{\substack{\uparrow \\ \text{dominant}}}{\frac{1}{a}} + \frac{1}{b}\right]$$

Coaxial

Example : Cable Modem ISP



> Uplink has low loss and we want this to keep modem cost low

Coaxial line

$a = 0.5\text{mm}$ $\quad$ $Cu - \sigma = 5.96 \times 10^7 \ \Omega^{-1} m^{-1}$

$b = 5\text{mm}$ $\quad$ $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m

$Z_0 = 75\Omega$

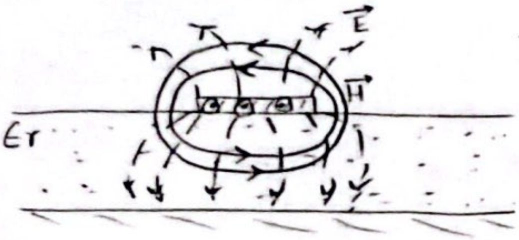
$$\alpha(f) \cong \alpha_c = \frac{\sqrt{f}}{4(75\Omega)} \cdot \sqrt{\frac{\mu}{\sigma\pi}}\left[\frac{1}{0.0005} + \frac{1}{0.005}\right]$$

$$= 6 \times 10^{-7}\sqrt{f} \ \text{Np/m}$$

$\alpha(5\text{MHz}) = 1.3 \times 10^{-3}$ Np/m $= 0.012$ dB/m.

$\alpha(1\text{GHz}) = 0.019$ Np/m $= 0.165$ dB/m. $\rightarrow$ 40 times lower after 100m!

Electrostatics



→ H field
- - → - E field

Need to know EM to understand Microwaves & T Lines

Coulomb's law in Cartesian Space

From Physics

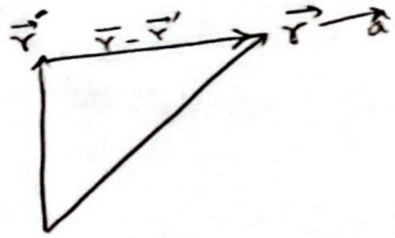
ϵ - permittivity (F/m)

$\vec{F}' \leftarrow Q'$ at $\vec{r}'$ $\qquad$ $Q \rightarrow \vec{F}$, $\vec{a}$ $\qquad$ $\vec{r} = x\hat{x} + y\hat{y} + z \cdot \hat{z}$

$\vec{r}' = x'\hat{x} + y' \cdot \hat{y} + z' \cdot \hat{z}$

units of meters

$$\boxed{\vec{F} = -\vec{F}' = \frac{Q' \cdot Q}{4\pi\epsilon R^2} \cdot \hat{a}}$$



$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$= \| \vec{r} - \vec{r}' \| = \sqrt{(\vec{r}-\vec{r}') \cdot (\vec{r}-\vec{r}')}$$

Pythgorian distance

$$\hat{a} = \frac{\vec{r} - \vec{r}'}{\| \vec{r} - \vec{r}' \|}$$

E-field at $\vec{r}$

$$\vec{E}(\vec{r}) = \frac{\vec{F}_Q}{Q} \boxed{= \frac{Q' (\vec{r} - \vec{r}')}{4\pi\epsilon \| \vec{r} - \vec{r}' \|^3}} \quad \text{V/m}$$

Super position

$\cdot Q_1$ $\vec{r}_1$ $\quad \cdot Q_2$ $\vec{r}_2$ $\quad \cdot Q_N$ $\vec{r}_N$ $\quad \cdot Q_3$ $\vec{r}_3$

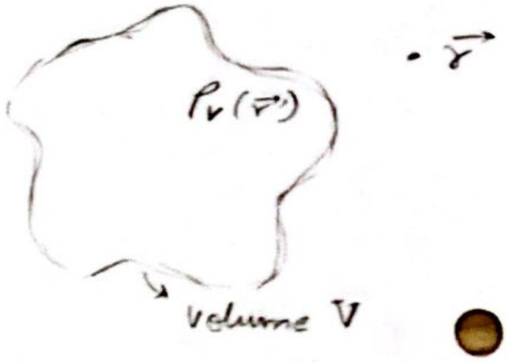
Point of observation

$$\underset{\text{E field}}{\vec{E}(\vec{r})} = \sum_{n=1}^{N} \frac{Q_n (\vec{r} - \vec{r}_n)}{4\pi\epsilon \, \|\vec{r} - \vec{r}_n\|^3}$$

Charge distribution based modelling

(I) Volume Charge Density

$\rho_v(\vec{r}') \rightarrow C/m^3$



Total enclosed charge

$$Q = \int_V \rho_v(\vec{r}') \underbrace{dV'}_{dx'dy'dz'}$$

→ What is the electric field at point $\vec{r}$ (pt. of observation)

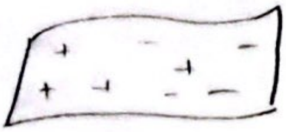
→ Need to find effect of each infinitesimal charge in the volume.

$$\boxed{\vec{E}(\vec{r}) = \int_V \frac{\rho_v(\vec{r}')(\vec{r} - \vec{r}')}{4\pi\epsilon \, \|\vec{r} - \vec{r}'\|^3} \, dv'}$$

$$\vec{E}(x, y, z) = \int dx' \int dy' \int dz' \frac{\rho_v(x', y', z') \left[(x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z} \right]}{4\pi\epsilon \left[(x - x')^2 + (y - y')^2 + (z - z')^2 \right]^{3/2}}$$

(II) Surface Charge Density

S



$\rho_s(\vec{r}')$

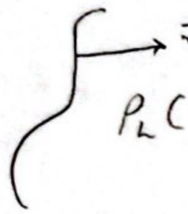
$$\underbrace{\text{Total charge}}_{\text{scalar}} = \int_S \overbrace{\rho_s(\vec{r}')}^{\text{scalar fn of a vector.}} ds'$$

$$= \iint \rho_s(\vec{r}')\, dx'\, dy'$$

E-field

$$\boxed{\underbrace{\vec{E}(\vec{r})}_{\text{vector}} = \int_S \frac{\overbrace{\rho_s(\vec{r}')}^{\text{scalar fn}} (\vec{r} - \vec{r}')\, ds'}{4\pi\epsilon \, \|\vec{r} - \vec{r}'\|^3}}$$

(III) Line charge Density



$\vec{r}$

$\rho_L(\vec{r}')$ C/m

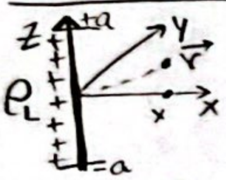
L

Total charge on line

$$= \int_L \rho_L(\vec{r}')\, dl'$$

$$\boxed{\vec{E}(\vec{r}) = \int_L \frac{\rho_L(\vec{r}')\, (\vec{r} - \vec{r}')\, dl'}{4\pi\epsilon \, \|\vec{r} - \vec{r}'\|^3}}$$

Example (1) : Line Charge



→ What is the electric field on the xy plane? Notice symmetry (so we can solve for x)

$$\rightarrow \vec{E}(x,0,0) = \int_{-a}^{+a} \frac{dz'\, \rho_L (x\hat{x} - \cancel{z'\hat{z}}^{\,0 \text{ due to symmetry}})}{4\pi\epsilon\, [x^2 + z'^2]^{3/2}}$$

$\vec{r} = x\hat{x} + 0 + 0$

$\vec{r}' = 0 + 0 + z'\hat{z}$

↳ Variable of integration

$$\Rightarrow \vec{E}(x,0,0) = \frac{\rho_L x \hat{x}}{4\pi\epsilon_0} \underbrace{\int_{-a}^{a} \frac{dz'}{(x^2 - z'^2)^{3/2}}}_{\frac{1}{x^2}\frac{z'}{\sqrt{x^2 + z'^2}}} = \frac{\rho_L a\, \hat{x}}{2\pi\epsilon_0 x \sqrt{x^2 + a^2}}$$

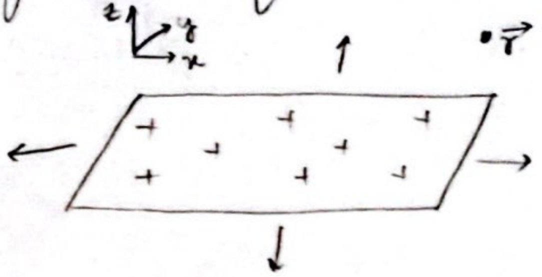
$\Rightarrow \vec{r} - \vec{r}' = x\hat{x} - z'\hat{z}$

$\|\vec{r} - \vec{r}'\| = \sqrt{x^2 + z'^2}$

$$\vec{E}(x,0,0) = \frac{2a\rho_L}{4\pi\epsilon_0 x\sqrt{x^2+a^2}}\hat{x} \xrightarrow{x \gg a} \frac{Q_{total}\,\hat{x}}{4\pi\epsilon_0 x^2}$$

<u>Example</u> :- <u>Surface charge</u>

Infinite uniform sheet of charge



$$\vec{E}(\vec{r}) = \vec{E}(x,y,z)$$

$$= \int_S \frac{\rho_s(\vec{r}-\vec{r}')\,ds'}{4\pi\epsilon_0\|\vec{r}-\vec{r}'\|^3}$$

Integrating along x' and y'

$ds' = dx'\,dy'$; $\vec{r}' = x'\hat{x} + y'\hat{y} + 0\hat{z}$; $\vec{r} = 0\hat{x} + 0\hat{y} + z\cdot\hat{z}$

element of surface integration; variable of integration; due to translational symmetry

$$\|\vec{r}-\vec{r}'\| = \sqrt{z^2+x'^2+y'^2}$$

0 due to symmetry aka odd fn

$$\vec{E}(0,0,z) = \frac{\rho_s}{4\pi\epsilon_0}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dy'\,\frac{z\hat{z} - \cancel{x'\hat{x}} - \cancel{y'\hat{y}}}{[z^2+x'^2+y'^2]^{3/2}}$$

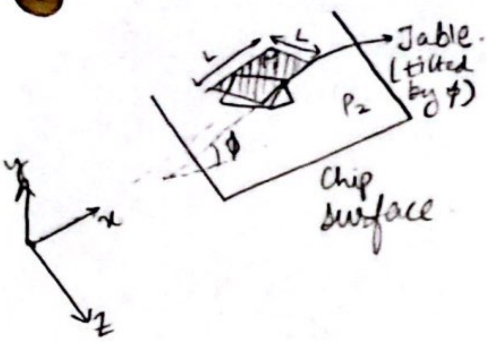
$$\vec{E}(x,y,z) = \frac{\rho_s z\hat{z}}{4\pi\epsilon_0}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dy'\,\frac{1}{[z^2+x'^2+y'^2]^{3/2}}$$

$$= \frac{\rho_s z\hat{z}}{4\pi\epsilon_0}\int_{-\infty}^{\infty}dy'\left[\frac{x'}{\sqrt{z^2+x'^2+y'^2}}\right]\cdot\frac{1}{z^2+y'^2}\Bigg|_{-\infty}^{\infty}$$

$$= \frac{\rho_s z\hat{z}}{2\pi\epsilon_0}\,\frac{1}{z}\tan^{-1}\frac{y}{z}\Bigg|_{-\infty}^{\infty} = \boxed{\frac{\rho_s\hat{z}}{2\epsilon_0}}$$ for $z>0$ Independant of any

Coulomb's Law for Advanced Coordinate Systems

Dynamic Light Projection Chip



→ By charging the surface of the table we can deflect it. [TI Dynamic projector (MEMS)]

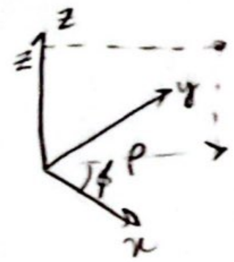
→ Field at a point in space

$$\vec{E}(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})$$

= complicated in cartesian so use cylindrical $\quad -\frac{\rho_2}{2\epsilon_0}\hat{y}$

Cylindrical coordinates

- ρ, ϕ, z $\quad$ ρ - radius
  ϕ - azimuthal angle
  z - z

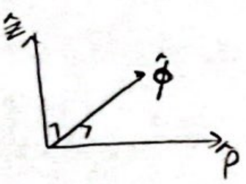


$$\rho^2 = x^2 + y^2$$

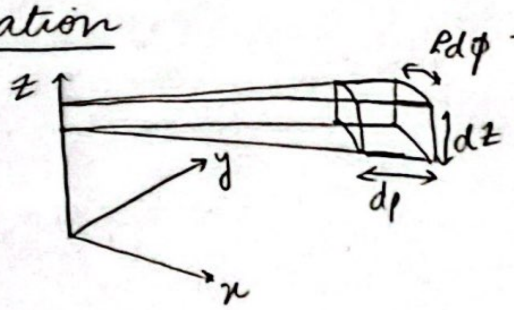
$$\phi = \begin{cases} \tan^{-1}\frac{y}{x} & \text{if } x \geq 0 \\ \tan^{-1}\frac{y}{x} + \pi & \text{if } x < 0 \end{cases}$$

since $\tan^{-1}$ lies from -90 to 90. So we need the π!

Advantages:- $\hat{\rho}$ & $\hat{\phi}$ change based on point of observation.



Integration



$dv = \rho \, d\phi \, dz \, d\rho$ → This what we need.

$ds = dz \, d\rho$ (vertical plane)

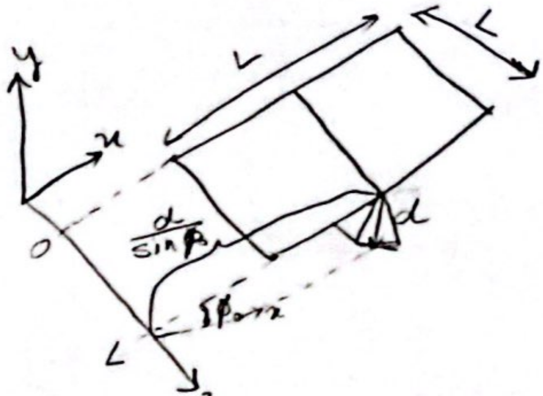
$ds = \rho \, d\rho \, d\phi$ (disk)

$ds = \rho \, d\phi \, dz$ (cylinder)

Steps

1) Identify integral type, argument, Kernel (green's function)
2) Define the region of integration.
3) Write integral in terms of $\vec{r}, \vec{r}'$.
4) Insert scalar position vectors
5) simplify

Evaluate



$$\vec{E}_1(x,y,z) = \int_{S_1} \frac{\rho_1(\vec{r}-\vec{r}')\,ds'}{4\pi\epsilon_0\|\vec{r}-\vec{r}'\|^3}$$

$$= \int_0^L dz' \int_{\frac{d}{\sin\phi_0}-\frac{L}{2}}^{\frac{d}{\sin\phi_0}+\frac{L}{2}} d\rho' \left(\ldots \right)$$

Define z axis such that for any ϕ the extension of L touches z axis. This makes limits of ρ much easier.

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$\vec{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z}$

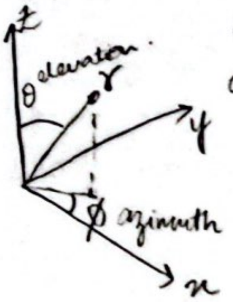
$$\left.\begin{aligned} z' &= z' \\ x' &= \rho' \cos\phi' \\ y' &= \rho' \sin\phi' \end{aligned}\right\} \Rightarrow \vec{r}' = \rho' \cos\phi_0\,\hat{x} + \rho' \sin\phi_0\,\hat{y} + z'\hat{z}$$

ϕ is constant

$$\vec{E}(x,y,z) = \frac{\rho_1}{4\pi\epsilon_0} \int_0^L dz' \int_{\frac{d}{\sin\phi_0}-\frac{L}{2}}^{\frac{d}{\sin\phi_0}+\frac{L}{2}} d\rho' \frac{(x-\rho'\cos\phi_0)\hat{x} + (y-\rho'\sin\phi_0)\hat{y} + (z-z')\hat{z}}{\left[(x-\rho'\cos\phi_0)^2 + (y-\rho'\sin\phi_0)^2 + (z-z_0)^2\right]^{3/2}}$$

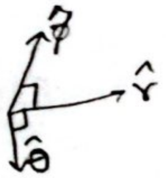
Spherical Coordinate Systems

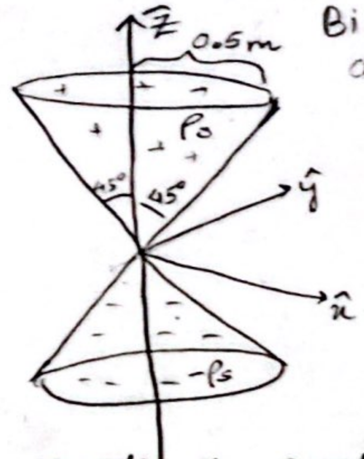
A biconical surface charge is defined by the regions $\theta = 45°, 135°$ and $r < 0.5m$. The upper cone has $+\rho_s$, lower cone has $-\rho_s$. What is the E field on the xy plane?



$0 < \theta < 180$

$0 < \phi < 360$





Biconical antenna!

→ Convert points to and from spherical coordinate systems.

$x = r\sin\theta\cos\phi$

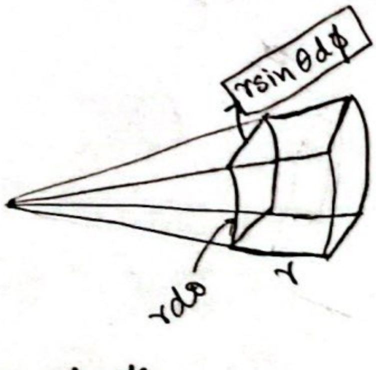
$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$r = \sqrt{x^2+y^2+z^2}$

$\theta = \cos^{-1}\left(\frac{z}{r}\right)$ or $\tan^{-1}\left(\frac{z}{\sqrt{x^2+y^2}}\right)$

$$\phi = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \end{cases}$$



$$\vec{E}(\vec{r}) = E_r(\vec{r})\hat{r} + E_\phi(\vec{r})\hat{\phi} + E_\theta(\vec{r})\hat{\theta}$$

$$E_x(\vec{r}) = \overbrace{\hat{x}\cdot\vec{E}(\vec{r})}^{\text{Projection of } E(\vec{r}) \text{ on } \hat{x}}$$

$= E_r(\vec{r})\hat{r}\cdot\hat{x} + E_r(\vec{r})\hat{\phi}\cdot\hat{x} + E_\theta(\vec{r})\hat{\theta}\cdot\hat{x}$

$E_x = E_r\sin\theta\cos\phi - E_\phi\sin\phi + E_\theta\cos\theta\cos\phi$

$E_y \ldots$

$E_z \ldots$

Dot products

	$\hat{r}$	$\hat{\theta}$	$\hat{\phi}$
$\hat{x}$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\hat{y}$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\hat{z}$	$\cos\theta$	$-\sin\theta$	0

$\vec{r} = x\hat{x} + y\hat{y} + 0\hat{z}$ → no field in z direction

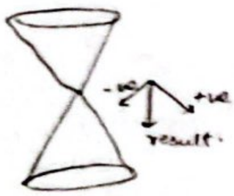
Sweeping $\vec{r}'$ on the two cones.

$$\vec{E}(\vec{r}) = \int_S \frac{\rho_s(\vec{r}')(\vec{r}-\vec{r}')ds'}{4\pi\epsilon\|\vec{r}-\vec{r}'\|^3} = \frac{\rho_s}{4\pi\epsilon}\int_{\text{Top cone}} \frac{(\overbrace{x\hat{x}+y\hat{y}-x'\hat{x}-y'\hat{y}}^{=0 \text{ cancel with bottom cone. look at Note!}}-z'\hat{z})ds'}{\|x\hat{x}+y\hat{y}-x'\hat{x}-y'\hat{y}-z'\hat{z}\|^3}$$

$$+ \frac{-\rho_s}{4\pi\epsilon}\int_{\text{Bottom cone}} \frac{(\overbrace{x\hat{x}+y\hat{y}-x'\hat{x}-y'\hat{y}}^{=0 \to \text{cancel with top cone}}-z'\hat{z})ds'}{\|x\hat{x}+y\hat{y}-x'\hat{x}-y'\hat{y}-z'\hat{z}\|^3}$$

Note!

→ Final E field on xy plane points only downward.



→ Also $z' = r'\cos\theta'$
$x' = r'\sin\theta'\cos\phi'$
$y' = r'\sin\theta'\sin\phi'$

θ' is $45°$ for top cone
θ' is $135°$ for bottom cone.

⇒ $\underbrace{z' = r'/\sqrt{2} \quad x' = r'/\sqrt{2}\cos\phi' \quad y' = r'/\sqrt{2}\sin\phi'}_{\text{top cone}}$

$\underbrace{z' = -r'/\sqrt{2} \quad x' = +r'/\sqrt{2}\cos\phi' \quad y' = +r'/\sqrt{2}\sin\phi'}_{\text{bottom cone}}$

z' → doesn't go to 0 due to -ve in $\vec{E}$

x', y' } go to zero.

$$ds' = dr' \cdot r'\sin\theta' d\phi' = \frac{dr' r' d\phi'}{\sqrt{2}}$$

$$\vec{E}(x,y,0) = \frac{\rho_s}{4\pi\epsilon}\int_0^{0.5m} dr' \int_0^{2\pi} d\phi' \frac{\overset{\text{2 components of } \hat{z}}{2}\, r' \overbrace{(-r'/\sqrt{2}\cdot\hat{z})}^{\text{constant}}}{\sqrt{2}\left\{\left[x-\frac{r'}{\sqrt{2}}\cos\phi'\right]^2 + \left[y-\frac{r'}{\sqrt{2}}\sin\phi'\right]^2 + \frac{r'^2}{2}\right\}^{3/2}}$$

$$= -\frac{\rho_s\hat{z}}{4\pi\epsilon}\int_0^{0.5} dr' \int_0^{2\pi} d\phi' \frac{r'^2}{\left[(x-\frac{r'}{\sqrt{2}}\cos\phi')^2 + (y-\frac{r'}{\sqrt{2}}\sin\phi')^2 + \frac{r'^2}{2}\right]^{3/2}}$$

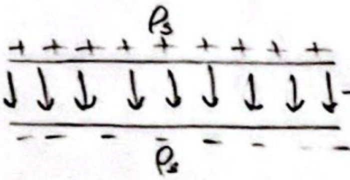
Gauss's Law in Integral form

Coulomb's law → Find field given charge

Gauss's law → Find charge given field.

Electric flux density: $\vec{D} = \underset{\text{Permittivity}}{\epsilon} \vec{E}$ (C/m^2)

→ Measure of flux or how much charge is contained by an area.



$\vec{D}$ field is constant regardless of any material!

$\vec{E}$ field changes.

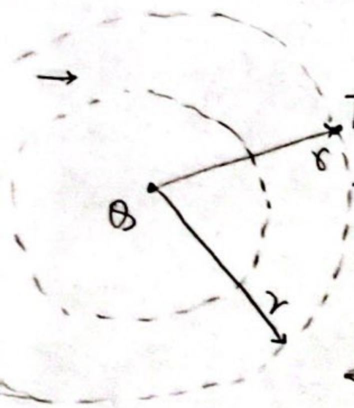
Impressed field: Field between the plates when empty space in b/w

→ When a material is placed inside, the molecules form countervailing dipoles and we get a "polarization field."

→ Total field = Impressed field + Polarization field. (lower them)

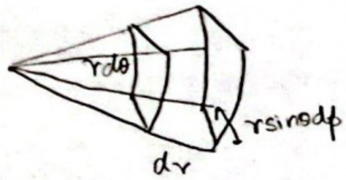
→ ϵ is higher for material that can be polarized easily.

→



$\vec{E}(r, \phi, \theta) = \frac{Q \hat{r}}{4\pi\epsilon_0 r^2}$

$\vec{D}(r, \phi, \theta) = \frac{Q}{4\pi r^2} \cdot \hat{r}$



Surface area of sphere × Magnitude of $\vec{D}$ = Enclosed charge

$$\oint_S \vec{D} \cdot d\hat{n}' = Q$$

$d\hat{n}'$ → surface normal unit vector, $r^2 \sin\theta' d\theta' d\phi' \cdot \hat{r}$

$$= \int_0^{\pi} \int_0^{2\pi} \frac{Q}{4\pi \cancel{r^2}} \cdot \cancel{r^2} \sin\theta' d\theta' d\phi' \, \cancel{\hat{r} \cdot \hat{r}}$$

$$= Q \cdot \frac{\cancel{2\pi}}{\cancel{4\pi}_2} \int_0^{\pi} \sin\theta' d\theta'$$

$$= \frac{Q}{\cancel{2}} \cdot \cancel{2}$$

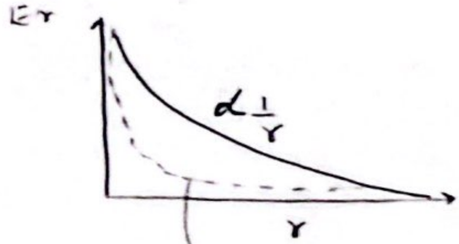
$$= \underline{\underline{Q}}$$

Gauss's law

$$\boxed{\oint_S \vec{D} \cdot d\hat{n}' = \oint_V \rho_v(\vec{r}) \, d\vec{r}'}$$

Example

$\rho_v(r, \phi, \theta) = \frac{\rho_0}{r^2}$ C/m²



$\propto \frac{1}{r^2}$ for pointcharge case (falls off faster)

r' - variable of integration
r - pt. of observation

→ What is the $E_r(r, \phi, \theta)$?

$$\vec{E}(\vec{r}) = E_r(r)\,\hat{r}$$

→ $$\oint_S \vec{D} \cdot d\hat{n}' = \int_V \rho_v \, dv'$$

$$\oint_S \epsilon \cdot E_r(r)\,\hat{r} \cdot r^2 \sin\theta \, d\phi \, d\theta \, \hat{r} = \epsilon E_r(r) \cdot 4\pi r^2$$

$$\int_V \rho_v \, dv' = \int_0^{\pi}\int_0^{2\pi}\int_0^{r} \frac{\rho_0}{\cancel{r'^2}} \, dr' \, \cancel{r'}\sin\theta' d\phi' \cdot \cancel{r'} d\theta'$$

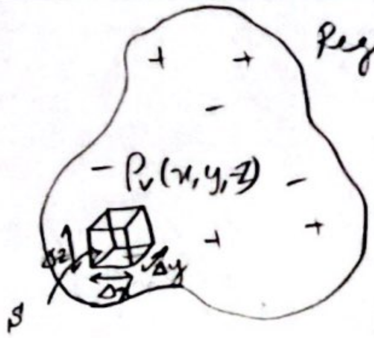
$$= \rho_0 \int_0^{\pi}\int_0^{2\pi}\int_0^{r} dr' \sin\theta' d\phi' d\theta'$$

$$= 4\pi r \rho_0$$

$$\Rightarrow \epsilon E_r(r) \cdot \cancel{4\pi} r^2 = \cancel{4\pi} r \rho_0$$

$$\boxed{E_r(r) = \frac{\rho_0}{\epsilon r}}$$

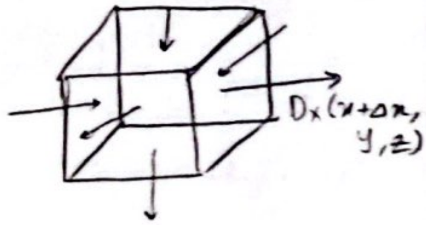
Gauss's Law in Differential Form



Region of charge

$$\vec{D}(x,y,z) = D_x(x,y,z)\hat{x} + D_y(x,y,z)\hat{y} + D_z(x,y,z)\hat{z}$$

$$\oint_{S\text{-cube}} \vec{D} \cdot d\hat{n}' = \int_{V\text{-cube}} \rho_v \cdot dv'$$



net flux through x direction

$$D_x(x+\Delta x, y, z) \cdot \underbrace{\Delta y \cdot \Delta z}_{\text{area}}$$
$$- D_x(x, y, z)\, \Delta y \cdot \Delta z$$
$$+ D_y(x, y+\Delta y, z)\, \Delta x\, \Delta z$$
$$- D_y(x, y, z)\, \Delta x\, \Delta z$$
$$+ D_z(x, y, z+\Delta z)\, \Delta x\, \Delta y$$
$$- D_z(x, y, z)\, \Delta x\, \Delta y \quad = \Delta x \Delta y \Delta z \cdot \rho_v(x,y,z)$$

Divide by $\Delta x\, \Delta y\, \Delta z$;

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \frac{D_x(x+\Delta x, y, z) - D_x(x,y,z)}{\Delta x} + \frac{D_y(x, y+\Delta y, z) - D_y(x,y,z)}{\Delta y} + \frac{D_z(x,y,z+\Delta z) - D_z(x,y,z)}{\Delta z} = \rho_v(x,y,z)$$

$$\Rightarrow \frac{\partial D_x(x,y,z)}{\partial x} + \frac{\partial D_y(x,y,z)}{\partial y} + \frac{\partial D_z(x,y,z)}{\partial z} = \rho_v$$

$$\boxed{\nabla \cdot \vec{D} = \rho_v}$$ Gauss's Law in Differential form

"Point" form

nabla

$$\nabla = \frac{\partial}{\partial x}\cdot\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$ is a vector operator

$$\vec{D} = D_x\hat{x} + D_y\hat{y} + D_z\hat{z}$$

$\nabla.$ (Divergence) = "Sourciness"

$$\oint_S \vec{D} \cdot d\hat{n}' = \int_V \rho_v \, dv' = \int_V \nabla \cdot \vec{D} \, dv'$$

$$\Rightarrow \boxed{\oint_S \vec{D} \cdot \underbrace{d\hat{n}'}_{\text{unit vector.}} = \int_V \nabla \cdot \vec{D} \, \underbrace{dv'}_{\text{scalar.}}}$$ Divergence Theorem

Flux through a closed surface = divergence of flux through the volume.

Cylindrical

$$\vec{D} = D_\rho \cdot \hat{\rho} + D_\phi \hat{\phi} + D_z \hat{z}$$

$$\nabla . \vec{D} = \frac{\partial}{\partial \rho} \cdot D_\rho + \frac{\partial}{\partial \phi} D_\phi + \frac{\partial}{\partial z} D_z$$ X Wrong, because $\hat{\phi}$ varies with ϕ, ρ

$$\boxed{\nabla . \vec{D} = \frac{1}{\rho} \cdot \frac{\partial}{\partial \rho} (\rho D_\rho) + \frac{1}{\rho} \frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}}$$

Spherical

$$\boxed{\nabla . \vec{D} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 D_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (D_\theta \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial D_\phi}{\partial \phi}}$$

Scalar Electric Potential

Have...	Want...	Used...	Eqn
ρ_v	$\vec{E}(\vec{r})$	Coulomb's Law	$\vec{E}(\vec{r}) = \int_V \frac{\rho_v(\vec{r}')(\vec{r}-\vec{r}')dv'}{4\pi\epsilon\lVert\vec{r}-\vec{r}'\rVert^3}$
$\vec{E}(\vec{r})$	$\vec{D}(\vec{r})$	Material relationship	$\vec{D} = \epsilon\vec{E}$
$\vec{D}(\vec{r})$	ρ_v	Gauss's Law	$\nabla\cdot\vec{D} = \rho_v$
$V(\vec{r})$	$\vec{E}(\vec{r})$	Scalar Potential Function	$\vec{E} = -\nabla V$
$\vec{E}(\vec{r})$	V	Line Integral	$V = \int_L \vec{E}\cdot d\vec{l}$
ρ_v	$V(\vec{r})$		

Recall

$$\underset{(\text{work})}{W} = -\int_A^B \underbrace{\vec{F}}_{Q.\vec{E}}\cdot d\hat{l} = -Q\int_A^B \vec{E}\cdot d\hat{l}$$

{-ve sign since E field points against the voltage gradient}

$$\frac{W}{Q} = -\int_A^B \vec{E}\cdot d\hat{l} = \underset{\sim}{V_{BA}} \text{ (voltage difference)}$$

→ Voltage at a single point of observation [Zero reference at infinity]

$$V(\vec{r}) = -\int_\infty^{\vec{r}} E\cdot dl \quad \text{(path doesn't matter)} \Rightarrow \underline{\text{Conservative field}}$$

↳ Like going up and down in altitude. Same as going by shortest path.

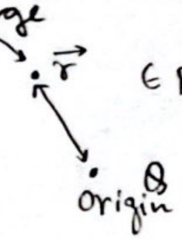
$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$\boxed{\vec{E}(\vec{r}) = -\nabla V(\vec{r})}$ Electric field is -ve of voltage gradient

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

$$\Rightarrow \nabla V(\vec{r}) = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}$$ → only valid for cartesian

<u>Example</u>

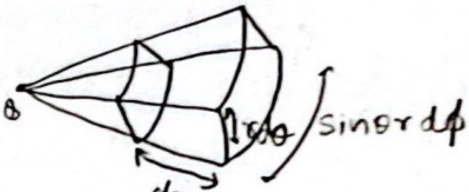


ϵ permittivity

$$\vec{E}(\vec{r}) = \frac{Q\,\hat{r}}{4\pi\epsilon r^2}$$

$$V(r,\phi,\theta) = V(r)$$

$$= -\int_{\infty}^{(r,0,0)} \vec{E} \cdot \underbrace{d\hat{l}}_{\hat{r}}$$



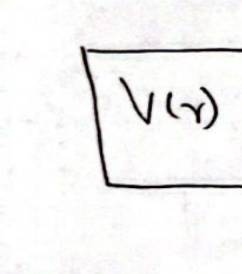
$$V(r) = -\int_{\infty}^{r} \frac{Q\,dr}{4\pi\epsilon r^2}$$

$$= -\frac{Q}{4\pi\epsilon}\int_{\infty}^{r} \frac{dr}{r^2}$$

$$\boxed{V(r) = \frac{Q}{4\pi\epsilon r}}$$

If Qtest is positive & Q is positive work done which is V(r). Qtest is positive!

$$V_{BA} = V(\vec{r}_B) - V(\vec{r}_A)$$



→ Voltage is also linear ⇒ can use superposition

$$V(\vec{r}) = \sum_{n=1}^{N} \frac{Q_n}{4\pi\epsilon_0 \|\vec{r} - \vec{r}_n\|}$$

pt of observation ↗ ↖ location of n^{th} charge

$$V(\vec{r}) = \int_V \frac{\rho_v(\vec{r}')\, dv'}{4\pi\epsilon \|\vec{r}-\vec{r}'\|}$$ (Notice there is no vector here)

Voltage - Charge

$\vec{E} = -\nabla V \qquad V = \dfrac{Q}{4\pi\epsilon r}$

When $\nabla \times \vec{E} = 0 \Rightarrow$ conservative field.

Spherical Coordinates

$$\nabla V(r,\phi,\theta) = \frac{\partial V}{\partial r}\hat{r} + \underbrace{\frac{1}{r}\frac{\partial V}{\partial \theta}}_{0}\hat{\theta} + \underbrace{\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}}_{0}\hat{\phi}$$

Cylindrical Coordinates

$$\nabla V(\rho,\phi,z) = \frac{\partial V}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{z}$$

$\Rightarrow \vec{E} = \dfrac{Q}{4\pi\epsilon r^2}\cdot\hat{r}$

Conductive material $\Rightarrow \vec{E} = 0$; $\vec{D} = 0$

Dielectric material $\Rightarrow \vec{D} = \epsilon_0\epsilon_r\vec{E}$

$\nabla\cdot(\epsilon\vec{E}) = \rho_v$

$-\nabla\cdot(\epsilon\nabla V) = \rho_v$ } poisson's equation

If ϵ is a scalar constant

$\nabla\cdot(\nabla V) = -\dfrac{\rho_v}{\epsilon}$ $\rightarrow$ Divergence of gradient

$\rightarrow$ $\boxed{\nabla^2 V = -\dfrac{\rho_v}{\epsilon}}$ Laplacian operator. (scalar)

→ In a region with no charge density

$\boxed{\nabla^2 V = 0}$ Laplace's Equation

→ <u>Cylindrical coordinates</u>

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

→ <u>Spherical coordinates</u>

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

<u>Example</u>

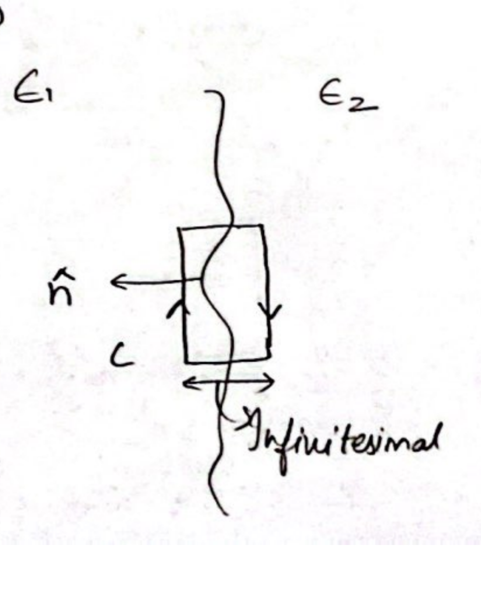
$V(x,y,z) = K - V_0(x,y,z)$

Does this satisfy Laplace's eqn?

$$\nabla^2 V = \frac{\partial^2}{\partial x^2}(K - V_0(x,y,z)) + \frac{\partial^2}{\partial y^2}(K - V_0(x,y,z)) + \frac{\partial^2}{\partial z^2}(K - V_0(x,y,z))$$

$= 0$

→



Two materials ... what happens to the fields?

We know, $\oint \vec{E} \cdot d\vec{l} = 0$

$$\oiint_S \vec{D} \cdot d\vec{s} = Q_{enclosed}$$

Let's take a contour C across the boundary.

$$\oint_C \vec{E}\cdot d\vec{l} \simeq \int_{C_1} \vec{E_1}\cdot d\vec{l} + \int_{C_2} \vec{E_2}\cdot d\vec{l}$$

$$\simeq \bar{E_1}^{tan} \Delta l \cdot \hat{\imath} + \bar{E_2}^{tan} \Delta l \cdot (-\hat{\imath})$$

$$= (\bar{E_1}^{tan} \Delta l - \bar{E_2}^{tan} \Delta l)\cdot \hat{\imath} = 0$$

$\bar{E}^{tan}$ is the tangential component at the interface.

$$\Rightarrow \boxed{\vec{E_1}^{tan} = \vec{E_2}^{tan}} \quad (1)$$

Tangential components must be equal. From $\int E\cdot dl = 0$

2 special cases

1) Region 2 is a conductor: $\Rightarrow \bar{E_2} = 0 \Rightarrow \bar{E_2}^{tan} = 0$

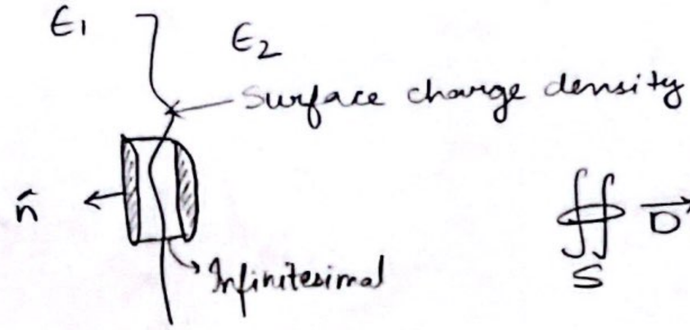
$$\Rightarrow \boxed{\bar{E_1}^{tan} = 0}$$ at the interface.

- Tangential field near the surface of a conductor must go to '0'. $\boxed{\hat{n}\times(\bar{E_1}-\bar{E_2})\Big|_{interface} = 0}$ Another way of writing Eq (1)

2) Region 2 is a dielectric : Nothing special $\bar{E_1}^{tan} = \bar{E_2}^{tan}$.

$\rightarrow \quad \vec{D_1} = \epsilon_1 \vec{E_1}$ & $\vec{D_2} = \epsilon_2 \vec{E_2}$

$\Rightarrow D_1^{tan} \neq D_2^{tan} \Rightarrow$ Tangential components of flux is discontinuous.



$$\oiint_S \vec{D}\cdot\vec{ds} \simeq \iint_{face1} \vec{D}\cdot\vec{ds} + \iint_{face2} \vec{D}\cdot\vec{ds}$$

$$\Rightarrow \oiint_S \vec{D}\cdot\vec{ds} = \bar{D}_1\cdot\hat{n}\,\Delta A - \bar{D}_2\cdot\hat{n}\,\Delta A \simeq \overbrace{\rho_s\,\Delta A}^{Q_{enclosed}}$$

$$\Rightarrow \hat{n}\cdot(\vec{D}_1 - \vec{D}_2) = \rho_s$$

$$\Rightarrow \boxed{D_1^{norm} - D_2^{norm} = \rho_s}$$

D^{norm} is a scalar

2 special cases

1) Region 2 is a conductor $\Rightarrow D_2 = 0 = D_2^{norm}$

$$\Rightarrow \hat{n}\cdot\vec{D}_1\Big|_{interface} = \rho_s$$

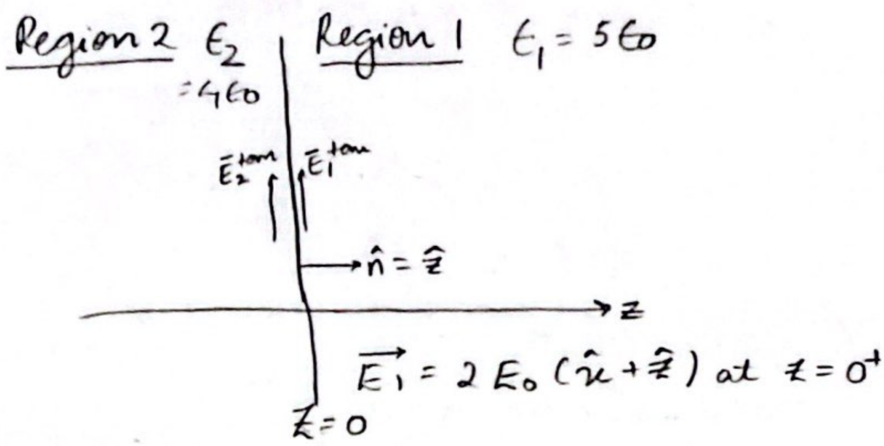
$$\Rightarrow \hat{n}\cdot\epsilon_1\vec{E}_1\Big|_{interface} = \rho_s$$

2) If both regions are dielectrics $\Rightarrow$ no surface charge ρ_s

$$\boxed{D_1^{norm} = D_2^{norm}}$$

$$\boxed{\epsilon_1 E_1^{norm} = \epsilon_2 E_2^{norm}}$$

Example



What is $\vec{E}_2$ at $z = 0^-$?

→

$$\vec{E}_1 = 2E_0(\hat{x}+\hat{z})$$

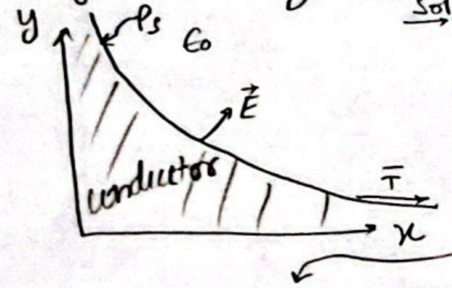
$$\vec{E}_1^{tan} = 2E_0\hat{x} \qquad \vec{E}_1^{norm} = 2E_0\hat{z}$$

$$\Rightarrow \vec{E}_2^{tan} = 2E_0\hat{x}$$

$$\Rightarrow \vec{E}_2^{norm} = \frac{\epsilon_1}{\epsilon_2}\vec{E}_1^{norm}$$

$$\vec{E}_2 = \vec{E}_2^{tan} + \vec{E}_2^{norm} = 2E_0\left(\hat{x} + \frac{5}{4}\hat{z}\right)$$

Example: The region $xy<1$ be a conductor with a surface charge density $\rho_s = 4\epsilon_0$



Sol: $\hat{n} = \dfrac{x\hat{x}+y\hat{y}}{\sqrt{x^2+y^2}}$; $\vec{E}^{norm} = \dfrac{\rho_s}{\epsilon_0}\hat{n} = \dfrac{4\epsilon_0}{\epsilon_0}\dfrac{x\hat{x}+y\hat{y}}{\sqrt{x^2+y^2}}$

$xy=1$: find tangent vector

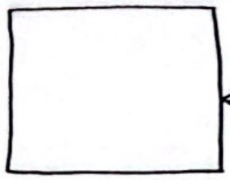
$d\bar{l} = dx\,\hat{x} + dy\,\hat{y}$

We know $x\,dy + y\,dx = 0 \Rightarrow dy = -\frac{y}{x}dx$

$\Rightarrow d\bar{l} = dx\,\hat{x} - \frac{y}{x}dx\,\hat{y} \Rightarrow \bar{T} = \hat{x} - \frac{y}{x}\hat{y} \Rightarrow \vec{N} = \hat{z}\times\bar{T} = \hat{y} + \frac{y}{x}\hat{x} \Rightarrow \hat{n} = \frac{\vec{N}}{|\vec{N}|} = \frac{x\hat{x}+y\hat{y}}{\sqrt{x^2+y^2}}$

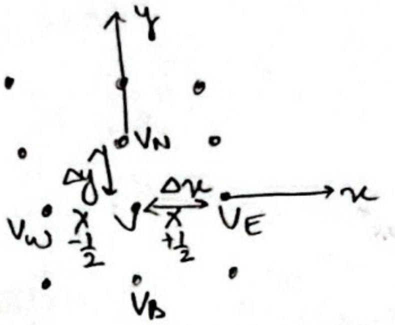
→ Often we solve Laplace's Eqn. because it is scalar. Use techniques like seperation of variables or use approximate ways of doing it.
↳ Method of finite differences.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \text{— (1)}$$



V given on boundary should be enough to solve Eq - (1)

(or) → Use discrete spaces



$$\left.\frac{dV}{dx}\right|_{1/2} = \frac{V_E - V}{\Delta x}$$ as $\lim_{\Delta x \to 0}$ we get the answer

$$\left.\frac{dV}{dx}\right|_{-1/2} = \frac{V - V_W}{\Delta x}$$

$$\left.\frac{d^2V}{dx^2}\right|_0 \cong \frac{V_E - 2V + V_W}{(\Delta x)^2}$$

$$\left.\frac{d^2V}{dy^2}\right|_0 \cong \frac{V_N - 2V + V_S}{(\Delta y)^2}$$

From Laplace's Equation

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{V_E - 2V + V_W}{(\Delta x)^2} + \frac{V_N - 2V + V_S}{(\Delta y)^2} = 0$$

→ When we are close to a boundary we would have some of these values.

If $\Delta x = \Delta y \Rightarrow$ $V_E + V_W + V_N + V_S - 4V = 0$ ⇒ Voltage in the centre is the average of the 4 voltages around.

Example

100	100	100	100	100
0	a	b	c	0
0	d	e	f	0
0	g	h	i	0
0	0	0	0	0

→ Start by assuming a - i as 0s.
→ Start averaging & fill in a → i
→ keep iterating till the values converge.
→ Initially 1st iteration

25	25	25
0	0	0
0	0	0

→ AKA 5 point finite difference operator.

→ Eventually at some iteration...

35.94	42.69	35.94
9.63	12.5	9.63
1.56	1.56	1.56

→ http://emclab.mst.edu/csoft.html have a whole bunch of numerical techniques to solve EM problems

What we have so far

Charge: $Q, \rho_s, \rho_L, \rho_v$

E-field: $\vec{E}$ (V/m)

D-field: $\vec{D}$ (C/m^2)

Volts: V (V)

ϵ → could be nonlinear → $f_n(\vec{E})$
→ it could also be a matrix depending on x, y, z directions (anisotropic)

Want	Have	Use
$\vec{E}$	charge	Coulomb's law $\vec{E}(\vec{r}) = \int_{vol} \frac{\rho_v(\vec{r}')(\vec{r}-\vec{r}')dv'}{4\pi\epsilon \lVert\vec{r}-\vec{r}'\rVert^3}$
V	Charge	Voltage integral $\vec{V}(\vec{r}) = \int_V \frac{\rho_v(\vec{r}')dv'}{4\pi\epsilon \lVert\vec{r}-\vec{r}'\rVert}$
$\vec{D}$	$\vec{E}$	$\vec{D} = \epsilon\vec{E}$ ← scalar constant (not always)
ρ_v	$\vec{D}$	Gauss's Law $\rho_v = \nabla \cdot \vec{D}$
$\vec{E}$	V	$-\nabla V = \vec{E}$

$$-\nabla V = \vec{E} = \frac{1}{\epsilon}\vec{D}$$

Where, $\nabla^2 = \nabla\cdot\nabla$ (Laplacian)
divergence of gradient

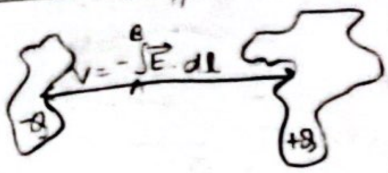
$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$-\nabla\cdot\nabla V = \frac{1}{\epsilon}\nabla\cdot\vec{D} = \frac{\rho_v}{\epsilon}$$

$\Rightarrow$ $\boxed{\nabla^2 V = -\frac{\rho_v}{\epsilon}}$ Poisson's Equation

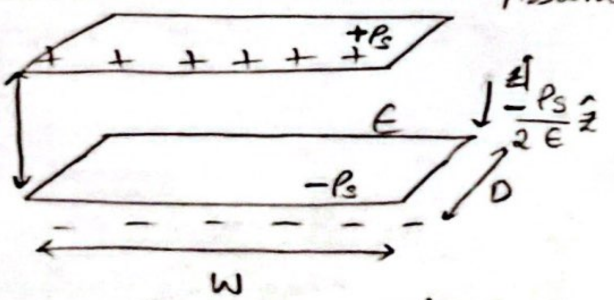
Laplace's Equation $\boxed{\nabla^2 V = 0}$ In a charge free region.

Capacitance

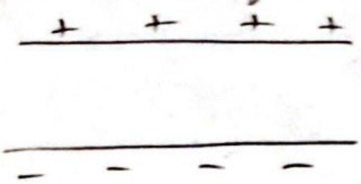


Capacitance $C = \frac{Q}{V} = \frac{\text{charge seperation}}{\text{Voltage.}}$

Parallel Plate Capacitor



Assuming plates were infinite.



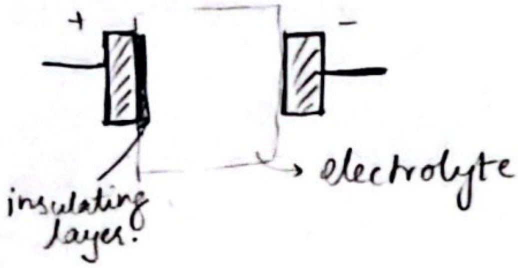
$\downarrow \frac{-\rho_s}{2\epsilon}\hat{z}$	$\uparrow \frac{+\rho_s}{2\epsilon}\hat{z}$
$\downarrow \frac{-\rho_s}{2\epsilon}\hat{z}$	$\downarrow \frac{-\rho_s}{2\epsilon}\hat{z}$
$\uparrow \frac{+\rho_s}{2\epsilon}\hat{z}$	$\downarrow \frac{-\rho_s}{2\epsilon}\hat{z}$
due to bottom plate	due to top plate

$$V = \int_0^d \frac{-\rho_s}{\epsilon}\hat{z}\cdot dz'\hat{z} = -\int_0^d \frac{\rho_s}{\epsilon}\,dz'$$

$$\boxed{V = \frac{\rho_s d}{\epsilon}} \quad \boxed{Q = \text{Area}\cdot\rho_s} \quad \boxed{\Rightarrow C = \frac{\text{Area}\,\epsilon}{d}}$$

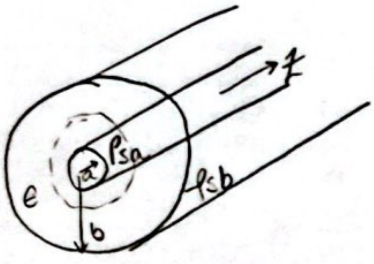
Doesn't depend on ρ_s

Electrolytic Caps → To reduce 'd'

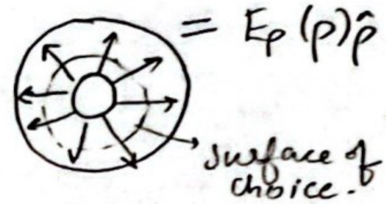


- Hit it with a dc potential
- You form a thin insulating layer (oxide maybe) on the anode
- The electrolyte is ionized and acts as an extension of the cathode.
- This is why you cannot reverse the polarity. Otherwise the insulating layer will melt. ⇒ Short!

→ Going back to T-Lines - Coaxial Cable. What is C per unit length?



$\vec{E}(\rho, \phi, z) = \vec{E}(\rho)$ φ symmetry & z symmetry



$= E_\rho(\rho)\hat{\rho}$ → only ρ component & only depends on ρ

$\vec{D} = \epsilon E_\rho(\rho)\hat{\rho}$

$\oiint_S \vec{D} \cdot d\hat{n}$ = Enclosed charge

↳ 3 components on surface of choice ⇒ $\cancel{\text{end}_1}$ + $\cancel{\text{end}_2}$ + curved side

(0 due to $d\hat{z}$)

$$\Rightarrow \underbrace{2\pi\rho D \epsilon E_\rho(\rho)}_{Q} = 2\pi a D \rho_{sa} \Rightarrow \boxed{E_\rho(\rho) = \frac{a\,\rho_{sa}}{\rho\,\epsilon}}$$

$$V = -\int_a^b \frac{a\rho_{sa}}{\rho\epsilon}\hat{\rho}\cdot\hat{\rho}\,d\rho$$

$$\boxed{V = +\frac{a\rho_{sa}}{\epsilon}\left[\ln\frac{b}{a}\right]}$$

$$\boxed{Q = 2\pi a \rho_{sa} D}$$

$$C = \frac{Q}{V} = \frac{2\pi D\epsilon}{\ln b/a}$$

$$\Rightarrow \boxed{\frac{C}{D} = \frac{2\pi\epsilon}{\ln b/a}}$$

Magnetostatics

Current + Current Density

Point charge

Current $I = \frac{dQ}{dt}$ C/s (Amp) → current flowing through a line

Surface charge

Current density $\vec{J}(\vec{r})$ A/m² → current flowing through a volume → This is a vector!

$I_{total} = \int_S \vec{J} \cdot d\hat{n}$ → not an enclosed integral → The normal component of J going through a surface

Line charge

Current sheet density $\vec{K}(\vec{r})$ (A/m) → current flowing through a surface

→ $\oint_S \vec{J} \cdot d\hat{n} = \begin{cases} = 0 & \text{if there is no charge storage} \\ \neq 0 & \text{change in charge storage} \end{cases}$ → magnetostatics!

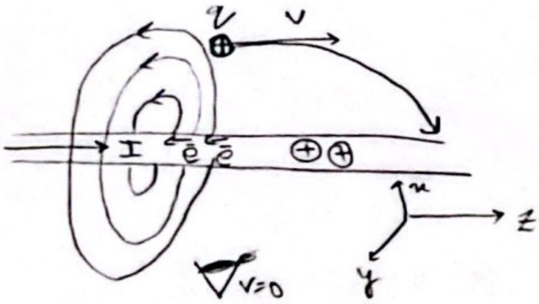
→ From divergence theorem

$$\oint_S \vec{J} \cdot d\hat{n} = \int_V (\nabla \cdot \vec{J})\, dv'$$

→ Magnetostatic condition : $\nabla \cdot \vec{J} = 0$

Electromagnetic force Vs. Relativity

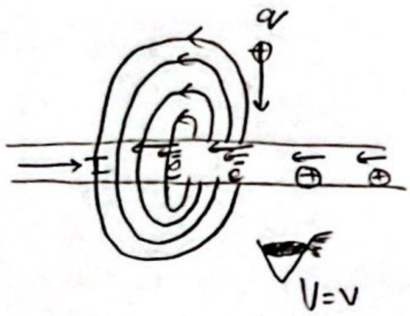
Case ①



> A wire is carrying a current I :=> electrons are moving in -ve z direction and protons are standing still

> Charge q moving with velocity v experiences a Force towards the wire. No surprise. Force is magnetic $F = q \cdot (\vec{v} \times \vec{B})$

Case ②



> Same wire and charge, now observer moves at velocity v along with q. The current remains same because now electrons are moving faster in -z direction but protons are also moving in -ve z direction

> However since electrons are moving faster they experience a greater Lorentz contraction. So the wire appears to be negatively charged and applies an electrostatic force on q and sure enough it starts accelerating towards the wire as expected

> Therefore a magnetic force and electric force are the same thing in two different reference frames.

$\vec{B} = \mu \cdot \vec{H} \rightarrow (A/m)$

(Wb/m^2) ↳ permeability (H/m)

$\vec{H} \rightarrow$ Magnetic field

$\vec{B} \rightarrow$ Magnetic flux density.

Ohm's Law

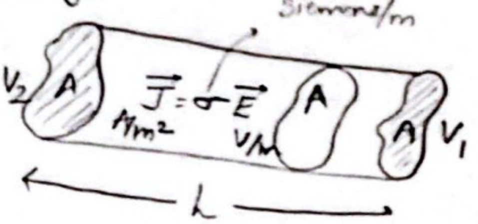
$\vec{J} = \sigma \vec{E}$

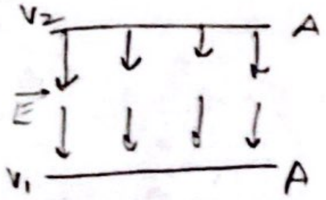
$$\underbrace{\int_A \vec{J} \cdot d\hat{n}}_{I} = \underbrace{\int_A \sigma \vec{E} \cdot d\hat{n}}_{\frac{\sigma \cdot V}{L} \int_A \hat{n} \cdot d\hat{n}}$$

$$\Rightarrow I = V \cdot \frac{\sigma A}{L}$$

$$\Rightarrow \boxed{V = \frac{L}{\sigma A} \cdot I} \qquad \boxed{R = \frac{L}{\sigma A}}$$

Irregular prism

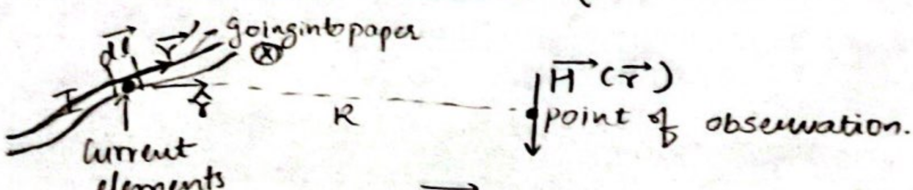




$\vec{E} \simeq \frac{V_2 - V_1}{L}(-\hat{z})$

Constant

Biot - Savart Law (Current distribution ↔ Magnetic field)



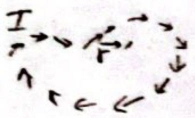
$$d\vec{H} = \frac{I\, d\vec{l} \times \hat{r}}{4\pi R^2}$$ "point" current relationship

Integral forms

$$\vec{H}(\vec{r}) = \oint_{path} \frac{I\, d\vec{l}' \times \hat{r}}{4\pi R^2}$$

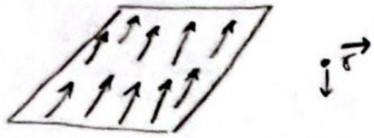
$R = \| \vec{r} - \vec{r}' \|$

$\hat{r} = \frac{(\vec{r} - \vec{r}')}{\| \vec{r} - \vec{r}' \|}$



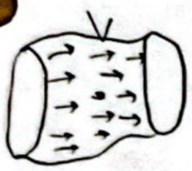
$$\boxed{\vec{H}(\vec{r}) = \oint_{path} \frac{I\,d\vec{l}' \times (\vec{r} - \vec{r}')}{4\pi \|\vec{r} - \vec{r}'\|^3}}$$

Surface current density ($\vec{K}$) (A/m)



$$\vec{H}(\vec{r}) = \int_S \frac{\vec{K}(\vec{r}') \times (\vec{r} - \vec{r}')\,ds'}{4\pi \|\vec{r} - \vec{r}'\|^3}$$

Volume current density ($\vec{J}$) (A/m^2)

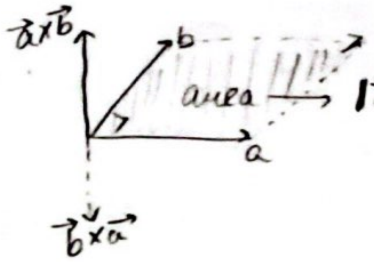


$$\vec{H}(\vec{r}) \int_V \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')\,dv'}{4\pi \|\vec{r} - \vec{r}'\|^3}$$

Review of cross products

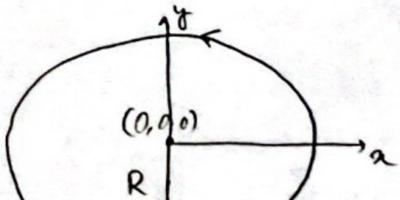
$$\hat{a} \times \hat{b} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= (a_y b_z - a_z b_y)\hat{x} + (a_z b_x - a_x b_z)\hat{y} + (a_x b_y - a_y b_x)\hat{z}$$



Magnetic field due to a loop current

→ Find $\vec{H}$ field at the centre of a counterclockwise loop carrying current I at a radius R, lying in the xy plane, centred at origin.



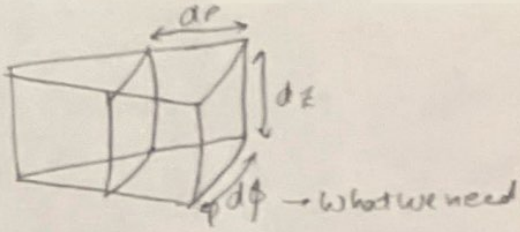
$$\vec{H} = \oint \frac{I\,d\vec{l}' \times (\vec{r} - \vec{r}')}{4\pi \|\vec{r} - \vec{r}'\|^3}$$

$\vec{r} = 0\hat{x} + 0\hat{y} + 0\hat{z}$ → (origin)

$\vec{r}' = \rho\cos\phi'\hat{x} + \rho\sin\phi'\hat{y} + 0\hat{z}$

$\vec{r}' = R\cos\phi'\hat{x} + R\sin\phi'\hat{y}$

$$\vec{r} - \vec{r}\,' = -R\cos\phi'\,\hat{x} - R\sin\phi'\,\hat{y}$$



$$d\vec{l} = R\,d\phi'\,\hat{\phi}$$ → direction of current.

$$\hat{\phi} = -\sin\phi'\,\hat{x} - \cos\phi'\,\hat{y}$$

$$d\vec{l} \times (\vec{r} - \vec{r}\,') = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -R\sin\phi' d\phi' & +R\cos\phi' d\phi' & 0 \\ -R\cos\phi' & -R\sin\phi' & 0 \end{vmatrix}$$

$$= \left(R^2 \sin^2\phi'\,d\phi' + R^2\cos^2\phi'\,d\phi'\right)\hat{z}$$

$$d\vec{l} \times (\vec{r} - \vec{r}\,') = R^2\,d\phi'\,\hat{z}$$

$$\|\vec{r} - \vec{r}\,'\| = R$$

$$\vec{H} = \frac{I R^2 \hat{z}}{4\pi R^3} \int_0^{2\pi} d\phi = \frac{I}{2R}\hat{z}$$

$$\Rightarrow \boxed{\vec{H} = \frac{I}{2R}\hat{z}}$$

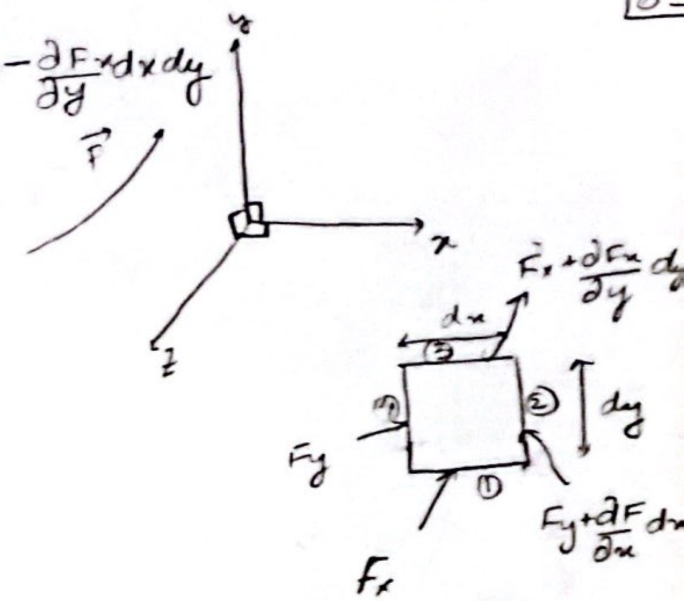
<u>Gauss's Law</u> (If $\vec{F}$ is a vector field)

$$\int_V \vec{\nabla} \cdot \vec{F}\, dv = \oint_S \vec{F} \cdot \hat{n}\, dA$$

<u>Stokes Law</u>

$$\oint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, dA = \oint \vec{F} \cdot d\vec{l}$$

$$\oint \vec{F}\cdot d\vec{l} = \underset{①}{F_x dx} + \underset{②}{F_y dy + \frac{\partial F_y}{\partial x} dx\,dy} - \underset{③}{F_x dy - \frac{\partial F_x}{\partial y} dx\,dy} - \underset{④}{F_y dy}$$



$$= \left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right] dx\,dy$$

No idea what this is.

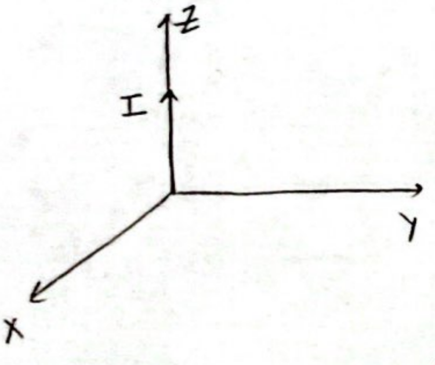
→ $\oint \vec{H}\cdot d\vec{l} = \oiint_S (\nabla \times \vec{H})\cdot \hat{n}\, dA$

→ $\oint \vec{E}\cdot d\vec{l} = 0$ when there are no changes. (iff Conservative)

Ampere's Law

{Notations of coordinate systems may differ)

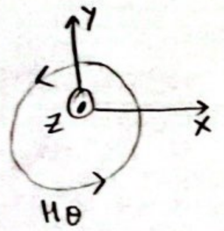
$\vec{H} = f(\vec{r})$ only a fn of $\vec{r}$



Biot Savart

$$\vec{H} = \frac{I}{2\pi R}\hat{\theta}$$

Ampere's Law: $\oint_L \vec{H}\cdot d\hat{l}$ = enclosed current! | $\oint_L \vec{H}\cdot d\vec{L} = \oint_S \vec{J}\cdot d\hat{n}$

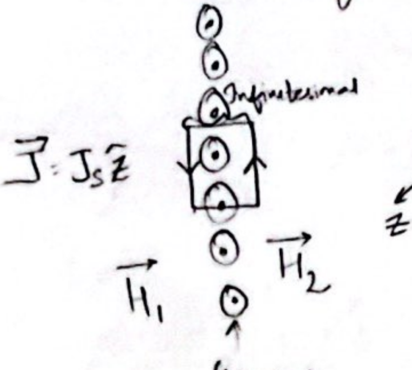


$$\oint \vec{H}\cdot d\hat{l} = \oint (0, H_\theta, 0)\cdot(dr, r\,d\theta, dz)$$

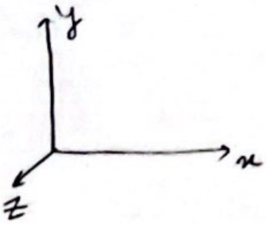
$$= \oint_0^{2\pi} H_\theta\, r\, d\theta = H_\theta r \int_0^{2\pi} d\theta = 2\pi r H_\theta$$

$$\Rightarrow 2\pi r H_\theta = I \Rightarrow \boxed{\vec{H_\theta} = \frac{I}{2\pi r}\hat{\theta}}$$

Sheet of Current



Infinite in y & z

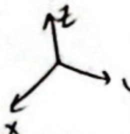


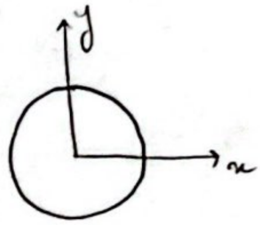
$$\oint \vec{H} \cdot \vec{dl} = H_{y2}\,dy - H_{y1}\,dy = (H_{y2} - H_{y1})\,dy$$

$$H_{y2} - H_{y1} = J_s$$

$$\boxed{H_{T_{an}^2} - H_{T_{an}^1} = J_s} \qquad \vec{H_2} = \frac{J_s}{2}\hat{y} \; ; \quad \vec{H_1} = -\frac{J_s}{2}\hat{y}$$

Cylinder of current





$(r < a)$

$$\oint \vec{H} \cdot d\hat{l} = 0$$

$(r > a)$

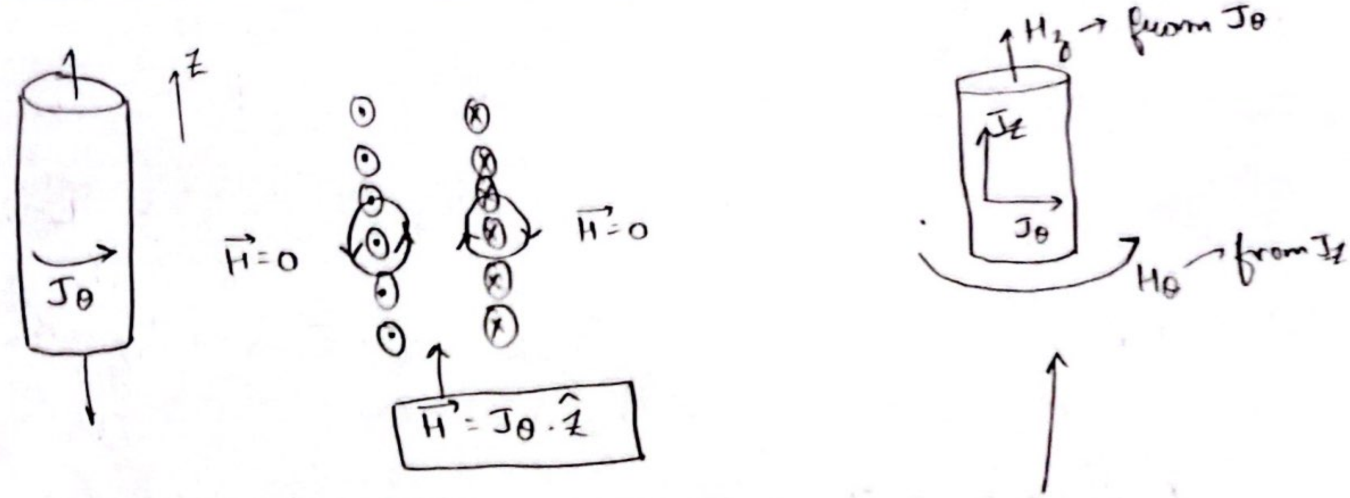
$$\oint H \cdot dl = 2\pi r H_\theta = I = 2\pi a J_s$$

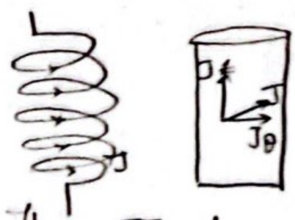
$$\Rightarrow \boxed{\vec{H} = \frac{I}{2\pi r}\hat{\theta}} \qquad \boxed{\vec{H} = \frac{a}{r} J_s \hat{\theta}}$$

$\Rightarrow$ Inside the conductor no fields.

$\Rightarrow$ Outside the conductor circular fields

Cylinder with circular current density



→ If we had an inductor (Solenoid)  It has 2 components, a J_θ and a J_z. The J_θ term has fields inside in $\hat{z}$ direction, whereas J_z term has fields outside in $\hat{\theta}$ direction. Here we are talking about infinite length cylinder. //

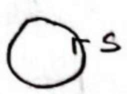
Recap (Analogy)

Ampere's Law 

$$\oint_L \vec{H}\cdot d\vec{l} = \oint_S \vec{J}\cdot d\hat{n} = \int_S (\nabla \times \vec{H})\cdot d\hat{n}$$ ↳ from Stokes

$$\Rightarrow \boxed{\nabla \times \vec{H} = \vec{J}}$$

Closed path integral of $\vec{H}$ = enclosed current

Gauss's Law 

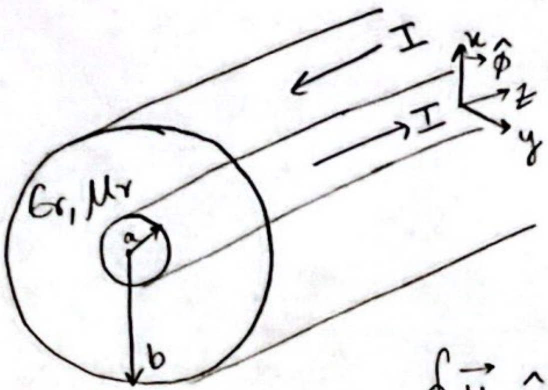
Closed surface integral of $\vec{D}$ = enclosed charge

Inductance

Capacitance $= \dfrac{\text{Electric Flux (Charge)}}{\text{Voltage}}$

Inductance $= \dfrac{\text{Magnetic Flux (Wb)}}{\text{Current}}$

Co-axial inductor



→ Current density is much higher in the internal conductor.

$$\vec{H}(\rho, \phi, z) = \vec{H}(\rho) = \vec{H}_\phi(\rho)\,\hat{\phi}$$

ϕ, z → 0, 0

Since current flows in $\hat{z}$.

$$\oint \vec{H}.\hat{dl} = \int_0^{2\pi} \underbrace{\rho\, d\phi'\,\hat{\phi}}_{d\vec{l}} . H_\phi(\rho)\,\hat{\phi}$$

$$= \int_0^{2\pi} \rho\, d\phi'\, H_\phi(\rho) = 2\pi\rho\, H_\phi(\rho) = I \quad \text{b/w } a \,\&\, b \qquad (1)$$

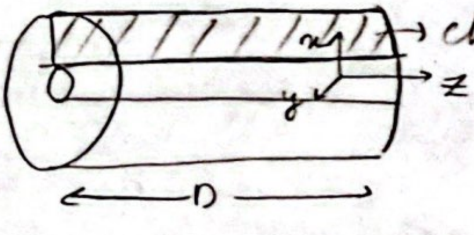
> Outside the cable, current inside is 0 so no magnetic field exists.

> From (1) $H_\phi(\rho) = \dfrac{I}{2\pi\rho}$ $\qquad \vec{B} = \mu \vec{H}$ $\qquad \dfrac{Wb}{m^2}\ \ \dfrac{H}{m}\ \ \dfrac{A}{m}$

> Origins of μ are pretty complex

$$\Rightarrow \vec{B} = \frac{\mu . I}{2\pi\rho}\,\hat{\phi}$$

say we have a length of D



$$\int_S \vec{B}.\underbrace{d\hat{n}}_{\hat{\phi}\, dx' dz} = \int_a^b dx' \int_0^D dz' \frac{\mu I}{2\pi x'}$$

$$\boxed{\vec{\Phi}_M \text{ total flux} = \frac{\mu D I}{2\pi} \ln\frac{b}{a}} \Rightarrow \boxed{L = \frac{\mu D}{2\pi}\ln\frac{b}{a}}\ *$$

Per unit length

$$\boxed{\underset{H/m}{L} = \frac{\mu}{2\pi}\ln\frac{b}{a}}\ *$$

What we covered so far

- $\nabla \cdot \vec{D} = \rho_v$

$\nabla \times \vec{H} = \vec{J} + \ldots$

$\nabla \times \vec{E} = 0$ when field is conservative (electrostatics)

What about $\nabla \cdot \vec{B} = ? = 0$

→ $Q = CV$

$\frac{dQ}{dt} = C \cdot \frac{dV}{dt} \Rightarrow I = C \cdot \frac{dV}{dt}$

→ $\overset{\text{ind. curr}}{L I} = \overset{\text{mag flux}}{\Phi_M} \Rightarrow L \frac{dI}{dt} = \underbrace{\frac{d\Phi_m}{dt} = V}$

- ⇒ Change in magnetic flux is equal to Voltage → Faraday's law

Faraday's law

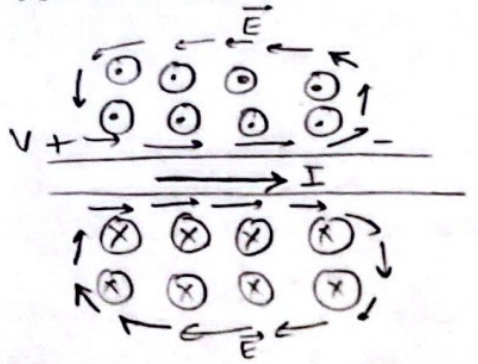
$$\boxed{\underbrace{\oint \vec{E} \cdot d\hat{l}}_{\text{voltage}} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\hat{n}} = -\int_S \dot{\vec{B}} \cdot d\hat{n} \quad \text{if material is not changing}$$

Stoke's ⇒ $\oint \vec{E} \cdot d\hat{l} = \int_S \nabla \times \vec{E} \cdot d\hat{n} = \int_S \frac{-\partial \vec{B}}{\partial t} \cdot d\hat{n}$

$$\Rightarrow \boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

$\vec{E}$ field swirls around a changing magnetic field.

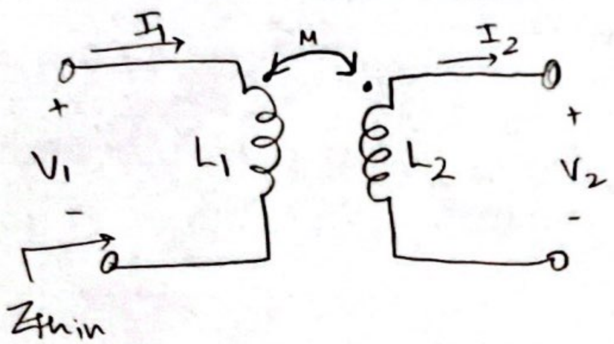
→ What is the inductance?



If we turn off the current I, the $\vec{B}$ field falls and this produces an E field which produces a voltage V'. This voltage drop produces a current I'. Therefore, the current has some "momentum" & energy stored in the magnetic field which is the inductance

→ magnetic field drops off by $\frac{1}{64}$ if length ↑ by 2 for a square wire carrying I. Used for near field comm.

Mutual Inductance



$$L_1 = \frac{\Phi_1}{I_1} \quad \text{(mag. flux)} \qquad L_2 = \frac{\Phi_2}{I_2}$$

$$M = \frac{\Phi_{21}}{I_1} = \frac{\Phi_{12}}{I_2}$$

Coupling constant $k = \frac{M}{\sqrt{L_1 L_2}}$

$0 \leq k \leq 1$ → ideal transformer

$$V_1 = L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt}$$

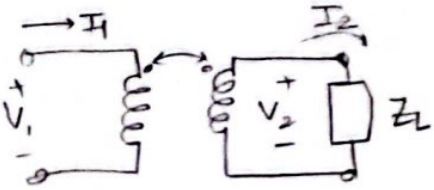
$$V_2 = -L_2 \frac{dI_2}{dt} - M \frac{dI_1}{dt}$$

Quasistatic approximation

$\vec{H}_{AC} \cong \vec{H}_{DC} \cdot \cos 2\pi f t$ when f is small enough such that λ is >> dimensions of coil.

→ We want Z_{thin} for a load at V_2.

Converting to Phasor Domain



$$\tilde{V}_1 = j2\pi f\left[L_1\tilde{I}_1 + M\tilde{I}_2\right]$$

$$\tilde{V}_2 = -j2\pi f\left[M\tilde{I}_1 + L_2\tilde{I}_2\right]$$

$$\begin{bmatrix}\tilde{V}_1\\ \tilde{V}_2\end{bmatrix} = j2\pi f\begin{bmatrix}L_1 & M\\ -M & -L_2\end{bmatrix}\begin{bmatrix}\tilde{I}_1\\ \tilde{I}_2\end{bmatrix}$$

$$\frac{\tilde{V}_2}{\tilde{I}_2} = \tilde{Z}_L \quad\text{———— (1)}$$

$$\frac{\tilde{V}_2}{\tilde{I}_2} = j2\pi f\left[-L_2 - M\frac{\tilde{I}_1}{\tilde{I}_2}\right] \quad\text{— (2)}$$

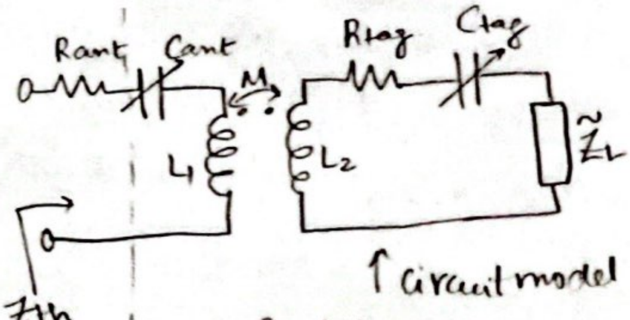
Solving (1) & (2)

$$\frac{\tilde{I}_1}{\tilde{I}_2} = \frac{Z_L + j2\pi f L_2}{-j2\pi f M}$$

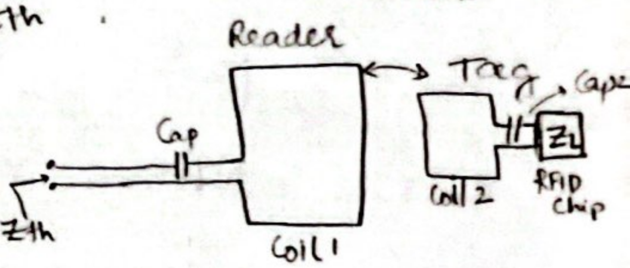
$$\tilde{Z}_{th} = \frac{\tilde{V}_1}{\tilde{I}_1} = j2\pi f\left[L_1 + M\frac{\tilde{I}_2}{\tilde{I}_1}\right]$$

$$\boxed{\tilde{Z}_{th} = j\omega L_1 + \frac{\omega^2 M^2}{\tilde{Z}_L + j\omega L_2}}$$

Assuming no resistances



↑ circuit model



The cap is used to resonate with L_1.

→ Now Z_{th} is purely a function of Z_L & (M, L_2). The RFID chip (Z_L) varies Z_L b/w 2 loads to send a logic signal back to $\tilde{Z}_{th}$. The chip is woken up by a charge pump that is fed by the induced emf in coil 2. Cap 2 is used to cancel L_2.

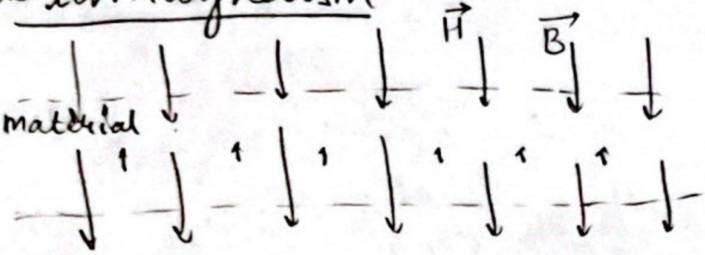
Magnetic Materials

Remember :- $\vec{D} = \epsilon \vec{E}$, $\epsilon > 1$

$\vec{B} = \mu \vec{H}$, $\mu_0 = 4\pi \times 10^{-7}$ H/m | $\mu_r \mu_0 = \mu_0 (1 + \chi_v)$

$\mu \rightarrow$ Has different sources
- $\rightarrow$ Could be less than 1
- $\rightarrow$ Could be VERY large

① Diamagnetism



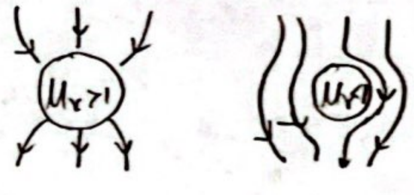
> When the material is placed inside a magnetic flux $\vec{B}$, the response of the electrons in lower orbits close to the nucleus produces diamagnetism. It is a weak phenomenon.

> The electrons produce a (Lenz law) counter field and the $\vec{B}$ field drops below the applied $\vec{H}$ field $\Rightarrow \mu_r < 1$ (Slightly)

$0.999 < \mu_r < 1$

> Bismuth $\chi_v = -1.7 \times 10^{-4}$

> Water $\chi_v = -9 \times 10^{-6}$



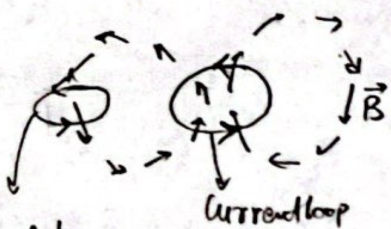
$\mu_r < 1 \rightarrow$ deflects mag. fields

$\Rightarrow$ Floating frogs!

② Paramagnetism

> Atoms or molecules that have a net magnetic moment.

> A classical model:



$\rightarrow$ Iron that gets magnetised under an imposed field.

> $1 < \mu_r < 1.1$

> Relatively weak

forced to flip  This reinforces the original field $\Rightarrow \mu_r > 1$

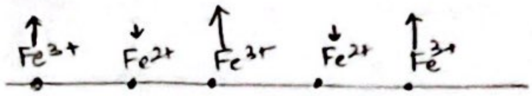
③ Ferrimagnetism (Quantum)

> Magnetite : Fe_3O_4

$$Fe_3O_4 \to Fe^{+8/3},\ O^{-2}$$

> $Fe^{+8/3}$ is composed of Fe^{3+} & Fe^{2+}

> Fe^{3+} & Fe^{2+} have different magnetic moments (Fe^{3+} is higher). The formation process of this material imparts a net magnetic moment.

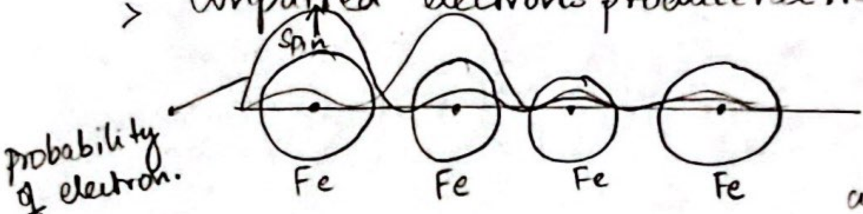
> 

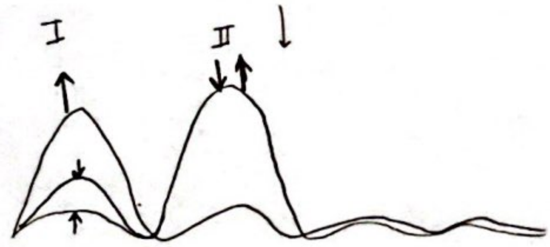
In trying to reduce local magnetic moment, the net magnetic moment is quite high.

> $\mu_r > 1$

④ Ferromagnetism (Quantum)

> Unpaired electrons produce net magnetic moment





atom I : Let's say it is spin ↑

atom II : Could be spin ↑ or ↓. These two spins have probability islands around atom I.

> Spin ↓ has a higher prob. at atom I due to Pauli's Ex. ⇒ it can occupy the same state as the spin ↑ $\bar{e}$ in atom I.

> Spin ↑ for the same reason has lower probability in atom I. But, since the spin ↑ & ↓ states in atom I are of the same sign they have an electrostatic repulsion which "reduces" the probability of spin ↓ in atom I. The difference in electrostatic energy b/w ↑ & ↓ is called the Exchange Energy = Electrostatic energy in the electron — magnetic energy.

→ If the exchange energy is positive (rare) ⇒ The dipole moments of neighbouring atoms line up!

⇒ $\mu_r >>> 1$

→ Heat reduces the exchange energy ⇒ destroys the magnet

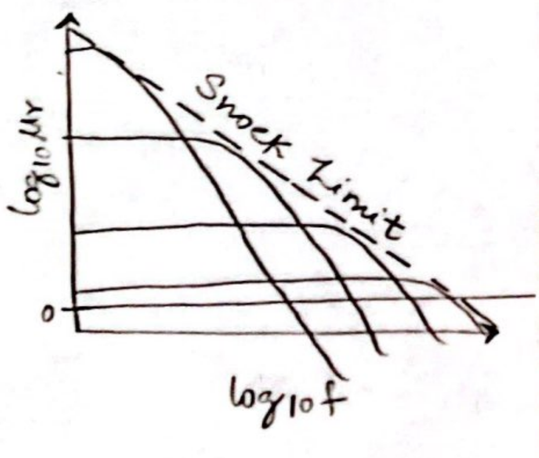
→ $$\text{Work} = \frac{1}{\mu}\iiint \|\vec{B}\|^2 \, dv'$$

↳ $\mu >> 1$ ⇒ can tolerate ↑B and still minimize work.

$\mu << 1$ ⇒ Cannot tolerate ↑B ⇒ flux passes around

→ Higher μ_r ⇒ prevents fields from changing fast ⇒ puts a cap on max. frequency you can achieve.

frequency ↑			↓ Strength
→ THz	$\mu_r < 1$	weak	diamagnetism
→ Microwave	$1 < \mu_r < 1.1$		paramagnetism
→ Lower f	$\mu_r > 1$		ferrimagnetism
→ 10s of kHz	$\mu_r >> 1$		ferromagnetism



MAXWELL'S EQUATIONS

What do we Know so far?

$$\oint_S \vec{D} \cdot d\hat{n}' = \int_V \rho_v \, dv'$$

	Differential form	Word form	Integral form
Gauss Coulomb	$\vec{\nabla} \cdot \vec{D} = \rho_v$	Electric charges spawn electric flux.	$\oint_S \vec{D} \cdot d\hat{n} = \int_V \rho_v \, dv'$
Ampere	$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	Magnetic field circulates around electric current & changing electric flux	$\oint_L \vec{H} \cdot d\hat{l} = \int_S \vec{J} \cdot d\hat{n} + \frac{\partial}{\partial t} \int_S \vec{D} \cdot d\hat{n}'$ Displacement current (Capacitor paradox)
Faraday	$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$	Electric field circulates around changing magnetic flux	$\oint_L \vec{E} \cdot d\hat{l} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\hat{n}$
Gauss Maxwell	$\vec{\nabla} \cdot \vec{B} = 0$	There are no magnetic charges in the universe.	$\oint_S \vec{B} \cdot d\hat{n}' = 0$

$$\rightarrow \cancel{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H})}^{\,0} = \vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \frac{\partial \vec{D}}{\partial t} = \nabla \cdot \vec{J} + \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{D}$$

→ A source of current must deplete charge! ⇒ $\boxed{\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}}$ ⇒ Continuity equation — Charge is conserved!

→ <u>Units</u>

$\vec{E}$ = Electric field - (V/m)

$\vec{H}$ = Magnetic field - (A/m)

$\vec{B}$ = magnetic flux density - (Wb/m²) aka (Tesla)

$\vec{D}$ = electric flux density - (C/m²)

$\vec{J}$ = current density - (A/m²)

ρ_v = volume charge density (C/m³)

The Scalar Wave Equation

› <u>Differential, Phasor Form</u>

$$\vec{E}(x,y,z,t) = E_x \cos(2\pi f t + \phi_x)\hat{x} + E_y \cos(2\pi f t + \phi_y)\hat{y} + E_z \cos(2\pi f t + \phi_z)\hat{z}$$

Here, $E_x E_y E_z$ & $\phi_x \phi_y \phi_z$ are also functions of x, y, z

<u>In phasor form</u>

→ $\vec{\tilde{E}}(x,y,z) = E_x \exp(j\phi_x)\hat{x} + E_y \exp(j\phi_y)\hat{y} + E_z \exp(j\phi_z)\hat{z}$

→ $\vec{E}(\vec{r},t) = \mathrm{Re}\{\vec{\tilde{E}}(x,y,z) \cdot \exp(j2\pi f t)\}$

→ Let's convert Mx Eq to phasor domain

→ $\nabla \times \vec{\tilde{H}} = \vec{\tilde{J}} + j(2\pi f)\vec{\tilde{D}}$ → $\nabla \cdot \vec{\tilde{B}} = 0$

→ $\nabla \times \vec{\tilde{E}} = -j2\pi f \vec{\tilde{B}}$ → $\nabla \cdot \vec{\tilde{D}} = \tilde{\rho}_v$

→ <u>Simple, source free medium</u>

- Linear
- Homogeneous
- Isotropic (material properties are same in all directions)

$\Rightarrow \left.\begin{array}{l} \vec{D} = \epsilon \vec{E} \\ \vec{B} = \mu \vec{H} \end{array}\right\}$ Simple

- Source free ⇒ no charges

→ Nonlinear ⇒ ϵ, μ are functions of $\vec{E}$ & $\vec{H}$

→ Nonhomogeneous ⇒ ϵ, μ are fns of positions

→ Anisotropic ⇒ ϵ, μ are matrices (w.r.t directions).

→ Now the equations become...

$$\nabla \times \vec{\tilde{H}} = +j2\pi f \epsilon \vec{\tilde{E}} \qquad \nabla \cdot \vec{\tilde{H}} = 0$$

$$\nabla \times \vec{\tilde{E}} = -j2\pi f \mu \vec{\tilde{H}} \qquad \nabla \cdot \vec{\tilde{E}} = 0$$

→ Uncoupling $\vec{E}$ & $\vec{H}$ to observe one of them...

$$\nabla \times \vec{\tilde{E}} = -j2\pi f \mu \vec{\tilde{H}}$$

$$\nabla \times \nabla \times \vec{\tilde{E}} = -j2\pi f \mu \underbrace{\nabla \times \vec{H}}_{j2\pi f \epsilon \vec{\tilde{E}}}$$

$$\nabla \times \nabla \times \vec{\tilde{E}} = \underbrace{(2\pi f)^2 \mu \epsilon}_{K^2} \vec{\tilde{E}}$$

$K = 2\pi f \sqrt{\mu \epsilon}$

$= \text{rad/m}$

= wavenumber.

$$\Rightarrow \boxed{\nabla \times \nabla \times \vec{\tilde{E}} - K^2 \vec{\tilde{E}} = 0}$$

Vector wave equation

> β is used in bounded spaces like waveguides or T-Lines

> K is used in free space

$K = \frac{2\pi}{\lambda}$

→ From math $\nabla \times \nabla \times \vec{A} = \underbrace{\nabla(\nabla \cdot \vec{A})}_{\text{grad(div)}} - \underbrace{\nabla^2 \vec{A}}_{\nabla \cdot \nabla \vec{A}\ \text{div (grad)}}$

$\nabla^2 \vec{A} = \nabla^2 A_x \hat{x} + \nabla^2 A_y \hat{y} + \nabla^2 A_z \hat{z}$

$$\Rightarrow \cancel{\nabla(\nabla \cdot \vec{\tilde{E}})}^{\,0} - \nabla^2 \vec{\tilde{E}} - K^2 \vec{\tilde{E}} = 0$$

$$\Rightarrow \nabla^2 \vec{\tilde{E}} + K^2 \vec{\tilde{E}} = 0$$

Helmholtz Wave Equation

$$\boxed{(\nabla^2 + K^2)\vec{\tilde{E}} = 0}$$

Scalar wave equation, because it really is → 3 uncoupled scalar wave equations.

$$(\nabla^2 + K^2)\begin{pmatrix} \tilde{E}_x(x,y,z) \\ \tilde{E}_y(x,y,z) \\ \tilde{E}_z(x,y,z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In totality,

$$\Rightarrow \boxed{(\nabla^2 + K^2)\begin{pmatrix} \vec{\tilde{E}} \\ \vec{\tilde{H}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$ → 6 seperate scalar equations.

→ What does this mean?

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+K^2\right]\tilde{E}_x=0 \quad —①$$

Recall,

$$\left[\frac{\partial^2}{\partial z^2}+(2\pi f)^2 LC\right]\begin{pmatrix}\tilde{V}\\ \tilde{I}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$ → Sinusoidal Telegrapher's Equation

Travelling wave with constant V in z direction

Eq ①

⇒ Travelling wave in all directions at a constant velocity!

Plane Wave (solution of Helmholtz Wave Equation)

Uniform planewave

$$\tilde{\vec{E}}(\vec{r}) = E_0 \hat{e} \exp(j[\phi_0 - K\hat{k}\cdot\vec{r}]) \text{ V/m}$$

$\hat{e}$: unit vector of polarization

$$\tilde{\vec{H}}(\vec{r}) = \frac{E_0}{\eta}\hat{h}\exp(j[\phi_0 - K\hat{k}\cdot\vec{r}])$$

$\vec{r} = x\hat{x}+y\hat{y}+z\hat{z}$ (m)

E_0 = amplitude of E field wave (V/m)

$\hat{e}$: polarization of E field

k = wave number = $\frac{2\pi}{\lambda}$

ϕ_0 = phase (rad)

$\hat{k}$ = direction of propagation.

$\hat{h}$ = polarization of H field

η - intrinsic impedance of the medium.

→ Uniform since E_0 is constant.

→ Plane wave since changing the phase gives a different plane.

→ $\eta = \sqrt{\frac{\mu}{\epsilon}}$ (Ω)

in cases where we have elliptical polarization ($\Rightarrow$ e & h are imaginary)

→ $\hat{e} \times \hat{h}^* = \hat{k}$ (mutually orthogonal)

Problem

measure: $\vec{\tilde{E}}(\vec{r}) = -1 \times 10^{-5}$ V/m $[4\hat{x} - 2\hat{y} + \hat{z}] \cdot$

$\exp(-j0.01[\hat{x} + 3\hat{y} + 2\hat{z}] \cdot \vec{r})$

What is the $\vec{H}(\vec{r})$?

Solution

$\vec{E}(\vec{r}) \Rightarrow$ normalize $\frac{4\hat{x} - 2\hat{y} + \hat{z}}{\sqrt{16+4+1}} = \hat{e}$

$$= \underbrace{-4.58 \times 10^{-5} \left(\frac{V}{m}\right)}_{E_0} \underbrace{\left[\frac{4}{\sqrt{21}}\hat{x} - \frac{2}{\sqrt{21}}\hat{y} + \frac{1}{\sqrt{21}}\hat{z} \right]}_{\hat{e}}$$

$\Rightarrow$ normalize $\exp\left(-j\sqrt{14}\,0.01 \left[\frac{1}{\sqrt{14}}\hat{x} + \frac{3}{\sqrt{14}}\hat{y} + \frac{2}{\sqrt{14}}\hat{z}\right] \cdot \vec{r}\right)$

$= \exp\left(-j \cdot \frac{2\pi}{168}\left[\frac{1}{\sqrt{14}}\hat{x} + \frac{3}{\sqrt{14}}\hat{y} + \frac{2}{\sqrt{14}}\hat{z}\right] \cdot \vec{r}\right)$

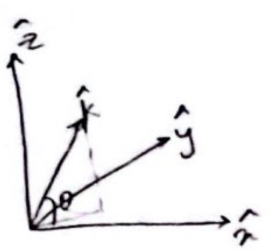
$\eta = \sqrt{\frac{\mu_0}{\epsilon_0}} = \boxed{377\,\Omega}$; $\hat{e} \times \hat{h} = \hat{k} \Rightarrow \hat{k} \times \hat{e} = \hat{h}$

$\vec{\tilde{H}}(\vec{r}) = -1.22 \times 10^{-7}$ A/m $\left[\frac{1}{\sqrt{6}}\hat{x} - \frac{1}{\sqrt{6}}\hat{y} - \frac{2}{\sqrt{6}}\hat{z}\right]$

$\cdot \exp\left(-j \cdot \frac{2\pi}{168}\left[\frac{\hat{x}}{\sqrt{14}} + \frac{3\hat{y}}{\sqrt{14}} + \frac{2}{\sqrt{14}}\hat{z}\right] \cdot \vec{r}\right)$

Direction of propagation aka bearing angle?

$\hat{k} = \frac{1}{\sqrt{14}}\hat{x} + \frac{3}{\sqrt{14}}\hat{y} + \frac{2}{\sqrt{14}}\hat{z}$



Direction of arrival is $-\hat{k}$.

DOA: $\phi = \begin{cases} \tan^{-1}\frac{k_y}{k_x} & \text{if } x<0 \\ \pi + \tan^{-1}\frac{k_y}{k_x} & \text{if } x>0 \end{cases}$

$$\theta = -\tan^{-1}\frac{k_z}{\sqrt{k_x^2+k_y^2}}$$

(wrt horizon)

Here: $\theta = -32°$

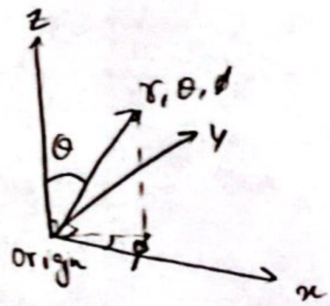
$\phi = 252°$

Spherical Wave Link Budgets

$\boxed{C = B \log_2 (1 + SNR)}$ Shannon capacity theorem.

C → max. capacity; B → B.W; SNR → $\frac{\text{signal power}}{\text{noise power}}$ → determines capacity!

Spherical wave



$$\vec{\tilde{E}}(\vec{r}) = \frac{E_0(\theta,\phi)}{(r/\lambda)} \hat{e} \exp(-jk\hat{r}\cdot\vec{r})$$

$$\vec{\tilde{H}}(\vec{r}) = \frac{E_0(\theta,\phi)}{\eta(r/\lambda)} \hat{h} \exp(-jk\hat{r}\cdot\vec{r})$$

$$\vec{S}_{av} = \frac{1}{2} Re\{\vec{\tilde{E}} \times \vec{\tilde{H}}^*\}$$ → poynting vector → How much energy flows per second per sq-meter.

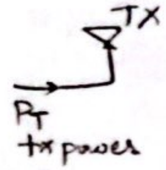
$$= \frac{1}{2} Re\left\{ \frac{E_0}{(r/\lambda)} \hat{e} \exp(-jk\hat{r}\cdot\vec{r}) \times \frac{E_0}{(\frac{r}{\lambda})\eta} \hat{h}^* \exp(+jk\hat{r}\cdot\vec{r}) \right\}$$

$$= \frac{E_o^2(\theta,\phi)}{2\eta(r/\lambda)^2} \operatorname{Re}\{\hat{e} \times \hat{h}^*\}$$

$$\vec{S}_{av} = \frac{E_o^2(\theta,\phi)}{2\eta(r/\lambda)^2} \cdot \hat{r} \quad (\text{Watts}/m^2)$$

→ $E_o(\theta,\phi)$ is a function of antenna. But antenna's usually mention gain.

→ Define, Gain :- Assume all power is isotropic. The poynting vector would be $\vec{S}_{iso} = \frac{P_T}{4\pi r^2}\hat{r}$



$$\Rightarrow G_T(\theta,\phi) = \frac{\|\vec{S}_{av}(\theta,\phi)\|}{\|\vec{S}_{iso}(r)\|} \quad \frac{\text{Actual power}}{\text{Isotropic power}}$$

$$\boxed{G_T(\theta,\phi) = \frac{2\pi E_o^2(\theta,\phi)\lambda^2}{\eta P_T}}$$

usually in dB

→ Another definition of gain based on receiver's perspective!

$$\boxed{G_R = \frac{4\pi}{\lambda^2}\cdot A_{em}}$$

A_{em} – electromagnetic aperture (m^2)

Reciprocity theorem :- $G_T = G_R$ in a linear medium.

→ $\|\vec{S}_{av}\| = \dfrac{P_T G_T}{4\pi r^2}$

→ $P_R = A_{em} \|\vec{S}_{av}\| = \left[\dfrac{\lambda^2 G_R}{4\pi}\right] \dfrac{P_T G_T}{4\pi r^2}$

→ <u>Friis's Free Space Equation</u>

$$\boxed{P_R = P_T \cdot \frac{G_T G_R \lambda^2}{(4\pi r)^2}}$$