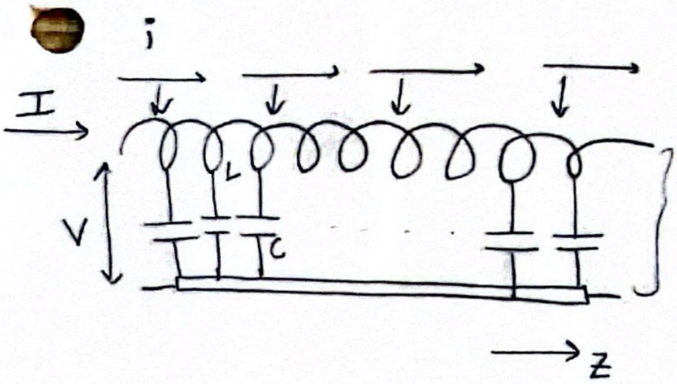
Travelling Wave Tubes - Pierce



$J = -\frac{\partial i}{\partial z}$ (current difference b/w 2 sections flows into the line)

Solutions of the form $\left(e^{j\omega t - \Gamma z}\right)$

$-\Gamma = -j\beta_e + \beta_e C\delta$ → Solutions are of this form.

β_e → prop. constant of electron beam.

C → gain parameter (fn. of circuit & beam impedance)

→ Equation of Γ is of 4^{th} degree ⇒ 4 waves exist. (3 forward, 1 backward)

Analysis

Assumptions: linear (Small signal), constant current across beam, all electrons in a cross section experience the same field. Electrons are displaced by the field only along z. Electron speeds are much slower (non relativistic).

Field caused by impressed current (Part I)

> Disturbance on circuit by bunched electron stream.

> i convection current of electrons flowing close to the line.

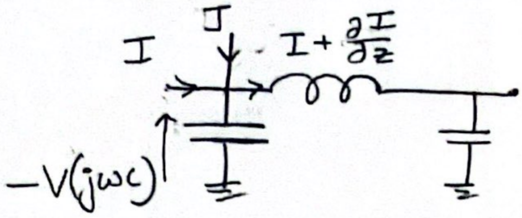
> J displacement current flowing into line [KCL at infinitesimal volume]. (impressed current per unit length)

> Assumption: displacement current along the electron stream is negligible.

$\Rightarrow J = -\frac{\partial i}{\partial z}$.

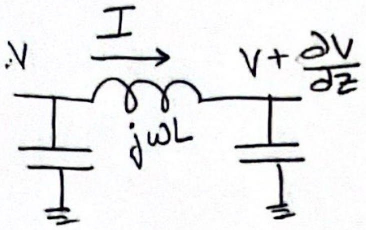
Condition : i & J are sinusoidal.

$\Rightarrow \frac{\partial I}{\partial z} = -jBV + J$



$B = \omega C$
↳ shunt susceptance per unit length.

$\frac{\partial V}{\partial z} = -jXI$



$X = \omega L$
↳ series reactance.

> Phase velocity $\frac{\omega^2}{V_p^2} = BX$ (can be chosen to match the SWS)

> $\frac{X}{B}$ can be chosen such that $-\frac{\partial V}{\partial z}$ ie longitudinal field acting on electrons is equal to the true field for unit power flow

> Time harmonic assumption of the form $e^{-\Gamma z}$ for all quantities gives $\frac{\partial}{\partial z} \rightarrow -\Gamma$

$J = -\frac{\partial i}{\partial z} = \Gamma i$, $-\Gamma I = -jBV + \Gamma i$; $-\Gamma V = -jXI$

$\Rightarrow V(\Gamma^2 + BX) = -j\Gamma X i$ —①

If $i = 0 \Rightarrow$ pure T line, $\boxed{\Gamma_1 = j\sqrt{BX}}$

$\Rightarrow$ Forward wave $\rightarrow e^{-\Gamma_1 z}$ & Backward wave $\rightarrow e^{+\Gamma_1 z}$.

- Characteristic impedance $K = \sqrt{\frac{X}{B}}$ (pure T-Line)

$\Rightarrow X = -jK\Gamma_1$

$$① \Rightarrow \boxed{V = \frac{-j\Gamma X i}{\Gamma^2 + BX} = \frac{-\Gamma\Gamma_1 K i}{\Gamma^2 - \Gamma_1^2}} \text{ — } ①^*$$

Convection current produced by Field. (Part II)

- η - charge to mass ratio of electrons $= 1.759 \times 10^{11}$ C/kg

u_0 - average electron velocity.

V_0 - Voltage by which electrons are accelerated.

$q V_0 = \frac{1}{2} m u_0^2 \Rightarrow u_0 = \sqrt{2\eta V_0}$

I_0 - average electron convection current

ρ_0 - average charge per unit length $\left(-\frac{I_0}{u_0}\right)$

v → ac component of velocity

- ρ → ac component of linear charge density

i → ac component of electron convection current.

$\Big\}$ exp(jωt−Γz)

$$\text{Force on electron} = m\frac{d(u_0+v)}{dt} \overset{\text{defined.}}{\triangleq} \frac{\partial}{\partial z}(qV)$$

Potential energy of electron in a field.

$$\Rightarrow \quad \frac{d}{dt}(u_0+v) = \eta\frac{\partial V}{\partial z}$$

$\Rightarrow$ v is a function of z & t.

$$\Rightarrow \quad \frac{dv}{dt} = \frac{\partial v}{\partial t}\cdot\frac{\partial t}{\partial t} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial t}$$

$$= \frac{\partial v}{\partial t} + \frac{\partial v}{\partial z}\cdot(u_0+v)$$

$$\Rightarrow \quad \boxed{\frac{\partial v}{\partial t} + \frac{\partial v}{\partial z}(u_0+v) = \eta\frac{\partial V}{\partial z}} \quad \text{— (2)}$$

<u>Assumption</u>: $v \ll u_0$ to make the equation linear. Linearization gives wave type soln that have been assumed earlier.

$$\text{Linearization} \Rightarrow \frac{\partial}{\partial t} = j\omega \ \& \ \frac{\partial}{\partial z} = -\Gamma$$

$$(2) \Rightarrow \quad (j\omega - u_0\Gamma)\,v = -\eta\Gamma V$$

Eq. (2a) velocity-Voltage relationship.

$$\Rightarrow \quad v = \frac{-\eta\Gamma V}{j\omega - u_0\Gamma} = \frac{-\eta\Gamma V}{u_0(j\beta_e - \Gamma)}$$

Since $\beta_e \triangleq \frac{\omega}{u_0}$

> Continuity equation gives

$$\nabla \cdot \bar{J} = -\frac{\partial \rho}{\partial t}$$ Here J is current per unit area, ρ is charge per unit volume.

$\iint_S$ $$\Rightarrow \frac{\partial i}{\partial z} = -\frac{\partial \rho}{\partial t}$$ Here i is current & ρ is charge per unit length

$$\Rightarrow -\Gamma i = -j\omega\rho$$

$$\Rightarrow \rho = \frac{-j\Gamma i}{\omega} \quad \text{—— (3)}$$

> $i = \frac{dq}{dt} = \rho\frac{dl}{dt} = \rho v$ since $\rho = \frac{dq}{dl}$

$$\Rightarrow \underbrace{(-I_0 + i)}_{\text{total convection current}} = \underbrace{(u_0 + v)}_{\text{total velocity}}\underbrace{(\rho_0 + \rho)}_{\text{total charge density}}$$

Ignoring product of v & ρ to linearize

$$\Rightarrow i = \rho_0 v + u_0\rho \qquad \text{since } u_0\rho_0 = -I_0 \text{ (page 3)}$$

Sub (3) $$\Rightarrow i = \rho_0 v + u_0\left(\frac{-j\Gamma i}{\omega}\right)$$

$$\Rightarrow \boxed{i = \frac{j\beta_e\rho_0 v}{j\beta_e - \Gamma}}$$

Sub (2a) $$\Rightarrow i = \frac{j\beta_e\rho_0}{(j\beta_e - \Gamma)} \times \frac{-\eta\Gamma V}{u_0(j\beta_e - \Gamma)} = \frac{j\beta_e\Gamma V\overset{I_0}{\overbrace{(-u_0\rho_0)}}(\eta)}{u_0^2(j\beta_e - \Gamma)^2}$$

$$\Rightarrow \boxed{i = \frac{j I_o \beta_e \Gamma V}{2V_o (j\beta_e - \Gamma)^2}} \quad (4)$$

$V \rightarrow$ total voltage (dc+ac)
$V_o \rightarrow$ avg voltage (dc)
$I_o \rightarrow$ avg electron convection current $= -u_o\rho_o$

(4) gives convection current (ac) in terms of applied voltage.
(1)* gives voltage as a function of convection current.
Here voltage is a function of position & is obviously scalar.

Combining (1)* & (4),

$$\Rightarrow \boxed{\frac{j K I_o \beta_e \Gamma^2 \Gamma_1}{2V_o (\Gamma_1^2 - \Gamma^2)(j\beta_e - \Gamma)^2} = 1} \quad (5)$$

Γ satisfying this gives a wave solution.

β_e gives electron propagation. ($\frac{\omega}{u_o}$ prop const. of disturbance in electron beam that propagates at the electron speed).
Γ_1 gives wave propagation. (pure T-line)
Γ gives wave propagation of a mode that is coupled b/w the beam & wave.

> 4 solutions must exist. The 4 boundary conditions to be satisfied are (could be?) voltages at the two ends of the SWS, electron velocity & convection current at the electron feed.

> We are interested in the wave that is about the speed of the electron beam.

> Assume: $-\Gamma_1 = -j\beta_e$ $\Rightarrow$ no loss in SWS & speed matching b/w wave & beam.

> We are looking for $\boxed{-\Gamma = -j\beta_e + \xi}$

$$\Rightarrow -\Gamma = -\Gamma_1 + \xi$$

(5) $\Rightarrow$ $j\beta_e - \Gamma = \xi$; $\Gamma_1^2 = -\beta_e^2$

$$\Rightarrow \Gamma_1^2 - \Gamma^2 = -(\beta_e^2 + \Gamma^2)$$

$$= -(\xi^2 - 2j\beta_e) = 2j\beta_e\xi - \xi^2$$

$$\Gamma^2 = \xi^2 - \beta_e^2 - 2j\beta_e\xi$$

$$\Rightarrow \quad 1 = \frac{-K I_o \beta_e^2 (\xi^2 - \beta_e^2 - 2j\beta_e\xi)}{2V_o(2j\beta_e\xi - \xi^2)(\xi^2)}$$

> Assume: $\xi \ll \beta_e$ $\Rightarrow$ ignore ξ^2 & $\beta_e\xi$ wrt β_e^2 & ξ^2 wrt $2j\beta_e\xi$

$$\Rightarrow \quad \xi^3 = -j\beta_e^3 \frac{K I_o}{4V_o}$$

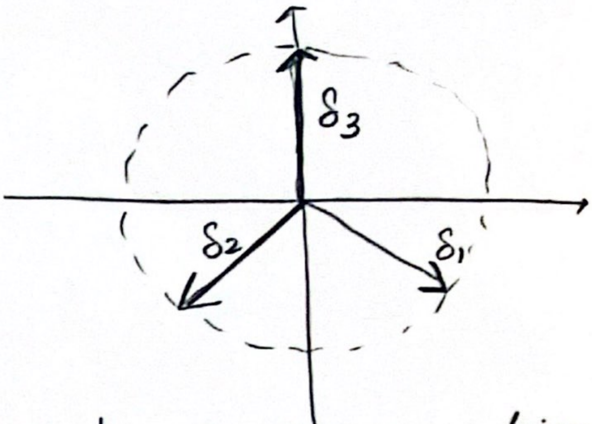
Let $\boxed{\frac{K I_o}{4V_o} = C^3}$ & $\boxed{\xi = \beta_e C \delta}$ & deal with C & δ instead

$\Rightarrow$ ⑤ $\Rightarrow \quad \delta = (-j)^{1/3} = \left(e^{j(2n-\frac{1}{2})\pi}\right)^{1/3}$

$\Rightarrow \quad \delta_1 = e^{\frac{-j\pi}{6}} = \frac{\sqrt{3}}{2} - \frac{j}{2}$

$\delta_2 = e^{-j\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} - \frac{j}{2}$

$\delta_3 = e^{j\frac{\pi}{2}} = j$



> The fourth wave is eliminated by the assumptions.

The assumptions are not valid for backward waves (since we assume Γ & $j\beta_e$ have same sign)

The fourth wave is given by $-\Gamma = j\beta_e\left(1 - \frac{c^3}{4}\right)$

$C \simeq 0.02 \Rightarrow \quad -\Gamma \cong j\beta_e \Rightarrow$ Backward wave is same as that for a pure SWS!

This is because the BW is not interacting strongly with the electrons (opposite directions).

> $e^{-\Gamma z} = e^{-j\beta_e z} e^{\delta C \beta_e z}$

wave 1 $\Rightarrow$ amplified & slower than electron beam.

wave 2 $\Rightarrow$ attenuated & slower than electron beam.

wave 3 $\Rightarrow$ same amplitude & faster than electron beam.

Recall, $-\Gamma V = -jXI$ was one of the Time Harmonic wave eqns.

$\Rightarrow$ Characteristic wave impedance $K_n = \frac{V}{I} = \frac{jX}{\Gamma_n}$

$$\Rightarrow K_n = \frac{jX}{\Gamma_n} = \frac{jX}{\Gamma_1 - \beta_e C \delta_n} = \frac{jX}{\Gamma_1 + j\Gamma_1 C\delta_n} = \frac{jX}{\Gamma_1(1+jC\delta_n)}$$

$$= \frac{K(1-jC\delta_n)}{1+C^2\delta_n^2}$$

$$\boxed{K_n \simeq K(1-jC\delta_n)}$$

$\hookrightarrow$ A small reactive component is added w.r.t SWS impedance.

> <u>Increasing wave</u>

$$\Gamma = j\beta_e - \xi = j\beta_e - \beta_e C\delta = j\beta_e - \beta_e C\left(-\frac{\sqrt{3}}{2} - \frac{j}{2}\right)$$

$$\Rightarrow Re(\Gamma) = \frac{\sqrt{3}}{2}(\beta_e)(C).$$

After N wavelengths $\beta_e z = \frac{2\pi}{\lambda} \times N\lambda = 2\pi N$

$$\Rightarrow \text{Gain in dB} = 20\log_{10}\left(\exp\left(\frac{\sqrt{3}}{2} \cdot C \cdot 2\pi N\right)\right) = \underline{\underline{47.3 \cdot C \cdot N \text{ dB}}}$$

In reality there is a loss term which will be derived later & the

gain $= -9.54 + 47.3\, CN$ dB

$E_z = \frac{d}{dz}(V) = -\Gamma_1 V \Rightarrow E = \Gamma_1 V$ Here E is the mag. of the field along z.

$$P = \frac{|V|^2}{2K}$$

$$\Rightarrow \frac{E^2}{\beta^2 P} = 2K \quad \text{assuming } \Gamma_1^2 = \beta^2 \Rightarrow \text{low loss circuit.}$$

$$\boxed{C^3 = 2K\left(\frac{I_0}{8V_0}\right) = \left(\frac{E^2}{\beta^2 P}\right)\left(\frac{I_0}{8V_0}\right)}$$

Call $\frac{V_0}{I_0}$ the beam impedance & K the $\underbrace{\text{circuit}}$ impedance.
(Pierce impedance)

Allowable range of phase velocity $\Delta v \approx C u_0$.

$\Rightarrow$ Larger Pierce impedance allows more range in Δv.
(Also larger beam current)

Chapter 7

> From field theory in Chapter 6

forced E_z → unforced E_z

$$E_z = \frac{(\Gamma^2+\beta_0^2)\,\Gamma_1^3\,(E^2/\beta^2 P)}{2(\Gamma_1^2+\beta_0^2)(\Gamma_1^2-\Gamma^2)}\, J e^{-\Gamma z} \qquad (7.1)$$

For slow waves $\beta_0^2 \ll |\Gamma_1^2|$ & $\beta_0^2 \ll |\Gamma^2|$

Recall, from T-Line model we had

$$V = \frac{\Gamma\Gamma_1 k i}{\Gamma_1^2-\Gamma^2} \Rightarrow E_z = \underset{E_z=-\frac{\partial V}{\partial z}}{\Gamma V} = \frac{\Gamma^2\Gamma_1 \overset{\frac{E^2}{2\beta^2 P}}{k}}{\Gamma_1^2-\Gamma^2};$$

which is the same as E_z if we replace i with J. There field theory & circuit theory approaches are consistent. The $e^{-\Gamma z}$ is not explicitly written out but it is there.

> A summation over n modes gives

$$E_z = \left(\tfrac{1}{2}\right)(\Gamma^2+\beta_0^2)\, i \sum_n \frac{(E^2/\beta^2 P)_n\, \Gamma_n^3}{(\Gamma_n^2+\beta_0^2)(\Gamma_n^2-\Gamma^2)}$$

There are several poles at $\Gamma = \Gamma_n$. We are interested in solutions where $\Gamma \simeq \Gamma_n$ for some n. Call this $n=1$. Therefore for $n=1$ the RHS changes rapidly with Γ (near a pole). Other modes are either passive or poorly synchronized with the electron beam.

$$\therefore \underset{E_z}{E} \simeq \left[\overbrace{\frac{\Gamma^2 \Gamma_1 (E^2/\beta^2 P)}{2(\Gamma_1^2 - \Gamma^2)}}^{\text{active mode close to } \Gamma_1} - \frac{j\,\Gamma^2}{\omega C_1} \right] i \quad \text{— (7.2)}$$

All passive & asynchronous modes.

- Here all passive modes & asynchronous have $\frac{E^2}{\beta^2 P}$ almost purely imaginary & the C_1 term will be given a physical interpretation later.
- Assumption: $\beta_0^2 << |\Gamma^2| \;\&\; |\Gamma_1^2|$.
- The second term represents the field due to a local charge density & not from the propagating/synchronized mode.

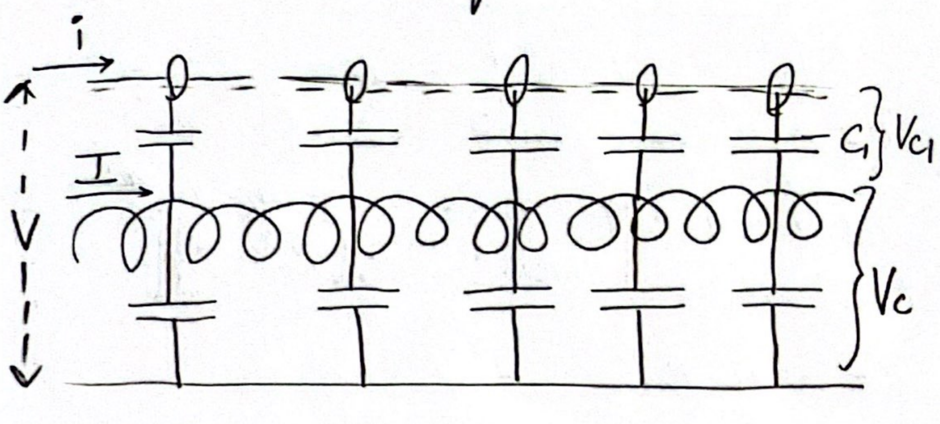
$$\Rightarrow \quad V = \frac{E}{\Gamma} \;\;\&\;\; i = \frac{j\omega}{\Gamma}\rho \quad \left(\overset{\text{from continuity}}{\nabla \cdot J = -\frac{\partial \rho}{\partial t}} \Rightarrow \Gamma i = j\omega\rho \right)$$

(7.2) $\Rightarrow$

$$V = \left[\frac{j\omega \Gamma_1 (E^2/\beta^2 P)}{2(\Gamma_1^2 - \Gamma^2)} + \frac{1}{C_1} \right] \rho$$

(Therefore C_1 is a capacitance per unit length) $\left\{ \frac{q}{l} = \rho = \frac{CV}{l} = C_1 V \right\}$

The model is therefore as follows.



Here V is now the line voltage + voltage across the capacitor.

Consider loss less waves in the line,

$\Gamma_1 = j\beta_1$ & $\Gamma = j\beta$.

$$\Rightarrow V = \left[\underbrace{\frac{\omega\beta_1 (E^2/\beta^2 P)}{2(\beta_1^2 - \beta^2)}}_{\text{Impedance due to line}} + \underbrace{\frac{1}{C_1}}_{\text{Impedance due to } C_1 \text{ ie. impedance due to asynchronous modes.}} \right] P$$

If $\overset{\omega/v}{\beta} < \beta_1$ (wave faster than pure line mode) $\Rightarrow$ the two terms are of same sign & impedance of the circuit (ie. line part (not C_1)) is capacitive

If $\beta > \beta_1$ (wave slower than pure line mode) $\Rightarrow$ Circuit part is inductive

<u>Intuitive explanation:</u>

When β is small, λ is large so current is injected at widely seperated points & the large series inductance is dominated by the shunt capacitance (ie current flows into caps). fast waves $\Rightarrow$ capacitive

When β is large, the currents flow into neighboring junctions & the small inductance dominates (current flows into inductors) slow waves $\Rightarrow$ inductive

From 7.2

$$V = \left[\frac{\Gamma\Gamma_1 (E^2/\beta^2 P)}{2(\Gamma_1^2 - \Gamma^2)} - \frac{j\Gamma}{\omega C_1} \right]; \quad — (7.5a)$$

Recall from Pg 6,

$$i = \frac{j I_0 \beta_e \Gamma V}{2V_0 (j\beta_e - \Gamma)^2}$$

Combining the two gives,

$$1 = \frac{j I_0 \beta_e \Gamma}{2V_0 (j\beta_e - \Gamma)^2} \left[\frac{\Gamma \Gamma_1 (E^2/\beta^2 P)}{2(\Gamma_1^2 - \Gamma^2)} - \frac{j\Gamma}{\omega C_1} \right] \quad \text{— (7.3)}$$

Recall $C^3 = \frac{E^2}{\beta^2 P} \cdot \frac{I_0}{8V_0}$

$$\Rightarrow \quad 1 = \frac{j \beta_e \Gamma}{2(j\beta_e - \Gamma)^2} \cdot \frac{8C^3}{(E^2/\beta^2 P)} \left[\frac{\Gamma \Gamma_1 (E^2/\beta^2 P)}{2(\Gamma_1^2 - \Gamma^2)} - \frac{j\Gamma}{\omega C_1} \right]$$

$$\Rightarrow \boxed{(j\beta_e - \Gamma)^2 = \frac{j 2\beta_e \Gamma^2 \Gamma_1 C^3}{(\Gamma_1^2 - \Gamma^2)} + \frac{4\beta_e \Gamma^2 C^3}{\omega C_1 (E^2/\beta^2 P)}} \quad \text{— (7.4)}$$

We are interested in Γ & Γ_1 that differ from β_e by a small amount,

$$\Rightarrow \boxed{\begin{aligned} -\Gamma &= -j\beta_e + \beta_e C \delta \\ -\Gamma_1 &= -j\beta_e - j\beta_e C b - \beta_e C d. \end{aligned}} \quad \text{— (7.5)}$$

$Re(\delta) > 0 \Rightarrow$ increasing wave

$Im(\delta) > 0 \Rightarrow$ wave faster than electrons.

$b > 0 \Rightarrow$ electrons faster than unforced wave } from SWS alone

$d > 0 \Rightarrow$ attenuated unforced wave

Substituting 7.5 in 7.4 we have.

$$(\beta_e c\delta)^2 = \frac{2j\,\beta_e\,(j\beta_e - \beta_e c\delta)^2 (j\beta_e + j\beta_e cb + \beta_e cd)\,c^3}{(j\beta_e + j\beta_e cb + \beta_e cd)^2 - (j\beta_e - \beta_e c\delta)^2}$$

$$+ \frac{4\beta_e\,(j\beta_e - \beta_e c\delta)^2\,c^3}{\omega C_1\,(E^2/\beta^2 P)}$$

$$\Rightarrow \delta^2 = \frac{2j\,(j - c\delta)^2 (j + jcb + cd)\,c}{(j + jcb + cd)^2 - (j - c\delta)^2} + \frac{4\beta_e\,(j - c\delta)^2\,c}{\omega C_1\,(E^2/\beta^2 P)}$$

$$-(j - c\delta)^2 = -(-1 + c^2\delta^2 - 2jc\delta) = 1 + 2jc\delta - c^2\delta^2 = \left[1 + c(2j\delta - c\delta^2)\right]$$

$$-j(j + jcb + cd) = \left[1 + c(b - jd)\right]$$

denominator 1 $= \cancel{-1} - c^2b^2 + c^2d^2 - 2cb + 2jc^2bd + 2jcd \cancel{+1} + 2jc\delta - c^2\delta^2$

$$\div 2c \Rightarrow -b + jd + j\delta + c\left(jbd - \frac{b^2}{2} + \frac{d^2}{2} - \frac{\delta^2}{2}\right)$$

$$\Rightarrow \delta^2 = \frac{\left[1 + c(2j\delta - c\delta^2)\right]\left[1 + c(b - jd)\right]}{\left[-b + jd + j\delta + c\left(jbd - b^2/2 + d^2/2 + \delta^2/2\right)\right]} - \frac{4\beta_e\left[(1 + c(2j\delta - c\delta^2)\right]c}{\omega C_1\,(E^2/\beta^2 P)}$$

Big assumptions: $|\delta|$ is on the order of 1.

$C << 1$

$|b|$ & $|d|$ are between 0 & 1 or a little larger.

$\Rightarrow$ Ignore all $1+C$ terms, replace with 1.

$$\Rightarrow \boxed{\delta^2 = \frac{1}{(-b+jd+j\delta)} - 4QC} \quad (7.6)$$

where $Q = \dfrac{\beta_e}{\omega C_1 (E^2/\beta^2 P)}$

Q is dimensionless - "Space charge parameter".

Call $\dfrac{\beta_e}{\omega C_1}$ the impedance of asynchronous modes.

$$\Rightarrow Q = \frac{\text{Asynchronous mode impedance}}{\text{Pierce impedance.}}$$

$$\text{Recall } V = V_c + V_{c1} = \Bigg[\underbrace{\frac{\Gamma\Gamma_1 (E^2/\beta^2 P)}{2(\Gamma_1^2 - \Gamma^2)}}_{\frac{V_c}{i}} - \underbrace{\frac{j\Gamma}{\omega C_1}}_{\frac{V_{c1}}{i}} \Bigg];$$

$$\Rightarrow V = \left[1 - \frac{j\Gamma}{\omega C_1} \cdot \frac{2(\Gamma_1^2 - \Gamma^2)}{\Gamma\Gamma_1 (E^2/\beta^2 P)} \right] V_c$$

$$\Rightarrow \boxed{V_c = \left[1 - \frac{j2(\Gamma_1^2 - \Gamma^2)}{\omega C_1 \Gamma_1 (E^2/\beta^2 P)} \right]^{-1} V} \quad (7.7)$$

From 7.6,

$$\left[\delta^2(-b+jd+j\delta)\right]^{-1} = \left[1-4QC(-b+jd+j\delta)\right]^{-1}$$

The two terms in 7.6 are from the two terms in 7.5a,

therefore, (this is not so obvious to me)

$$V_c = \left[1-4QC(-b+jd+j\delta)\right]^{-1} V.$$

Recall,

electron velocity (pg. 4) : $v = \dfrac{-\eta\Gamma V}{u_0(j\beta_e-\Gamma)}$ &

electron convection current (pg. 6) : $i = \dfrac{jI_0\beta_e\Gamma V}{2V_0(j\beta_e-\Gamma)^2}$

Using 7.5,

$$v = \frac{+\eta V(+j\beta_e+\beta_e C\delta)}{u_0(\beta_e C\delta)} = \frac{\eta V(\cancel{C\delta}^{\,0}-j)}{u_0 C\delta}$$

$$\Rightarrow \boxed{\left(\frac{ju_0C}{\eta}\right)v = \frac{V}{\delta}}$$

$$i = \frac{jI_0\beta_e(+j\beta_e-\beta_e C\delta)V}{2V_0(\beta_e C\delta)^2} = \frac{jI_0(+j-\cancel{C\delta}^{\,0})V}{2V_0C^2\delta^2} = \frac{-I_0V}{2V_0C^2\delta^2}$$

$$\Rightarrow \boxed{\left(-\frac{2V_0 C^2}{I_0}\right) i = \frac{V}{\delta^2}}$$

All quantities vary as $e^{-j\beta_e z} e^{\beta_e C \delta z}$.

C - gain parameter

b - relative electron velocity parameter

d - circuit attenuation parameter

Q - space charge parameter.

Chapter 8 :

b - velocity parameter

d - attenuation parameter

QC - space charge parameter

bC → fraction by which electron velocity is greater than the phase velocity of the unforced circuit.

dC → 54.6 dC dB/λ is the circuit attenuation.

Q → depends on circuit impedance, geometry and beam diameter.

Consider eqn 7.6 if $d=0$ (no loss), $Q=0$ (ignoring space charge), we have

$$\delta^2(\delta+jb) = -j$$

Recall, $\beta_e = \omega/u_0$; $-\Gamma_1 = -j\beta_e(1+cb) = -j\frac{\omega}{u_0}(1+cb) = \frac{-j\omega}{v_1}$

where v_1 = phase velocity of unforced circuit.

Therefore $\frac{u_0}{v_1} = 1+cb$ ie. electron velocity/unforced wave velocity.

$b>0 \Rightarrow$ electrons faster than unforced wave.

$b=0 \Rightarrow$ electron speed = unforced wave speed.

If $b=0$ we have $\delta^3 = -j$ (same as in chapter 2)

For $b \neq 0$ assume $\boxed{\delta = x + jy}$

Recall that quantities vary as $e^{-j\beta_e z} e^{\beta_e C \delta z}$ in the forced solution.

$$\Rightarrow e^{-j\beta_e z} e^{\beta_e C \delta z} = e^{-j\beta_e (1 + jC\delta) z}$$

$$= e^{-j\beta_e (1 + jCx - Cy) z}$$

$$= \underline{\underline{e^{-j\beta_e (1 - Cy) z} e^{\beta_e C x z}}}$$

$$\Rightarrow \frac{\omega}{V} = \frac{\omega}{u_0}(1 - Cy)$$

If $Cy \ll 1$ from Taylor expansion of $\frac{1}{1-Cy}$ we have

$$V = (1 + Cy) u_0$$

$\Rightarrow y > 0 \Rightarrow$ fast wave

$y < 0 \Rightarrow$ slow wave.

$x > 0 \Rightarrow$ gain wave

$x < 0 \Rightarrow$ attenuated wave.

Recall in chapter 2 gain was expressed as

BCN dB. where N is number of wavelengths.

$$20 \log_{10} e^{\beta_e C x z} = 20\left(\frac{2\pi}{\lambda}\right) . C . x . z . \log_{10} e$$

Let $B = 20(2\pi)x . \log_{10} e = 54.5x$

$N = \frac{z}{\lambda}$

$\Rightarrow$ $\boxed{B = 54.5x \text{ dB.}}$

Also,

$s^2(s+jb) = -j \Rightarrow (x^2-y^2+2jxy)(x+jy+jb)+j = 0$

$\Rightarrow x^3 + jx^2y + jx^2b + j - y^2x - jy^3 - jby^2 + 2jx^2y - 2xy^2 - 2bxy = 0$

$\Rightarrow \left[jy(x^2-y^2) + jb(x^2-y^2) + 2jx^2y + j\right] + \left[x(x^2-3y^3-2yb)\right] = 0$

$$\Rightarrow \boxed{\begin{aligned} &(x^2-y^2)(y+jb) + 2x^2y + 1 = 0 \quad (8.10) \\ &x(x^2-3y^3-2yb) = 0 \quad (8.11) \end{aligned}}$$

8.11 $\Rightarrow$ $x = 0 \Rightarrow$ unattenuated waves.

$x^2 = 3y^2 + 2yb$

If $x=0$, from 8.10

$$y^2(y+b)=1$$

$$\Rightarrow b=-y+\frac{1}{y^2}.$$

If $x\neq 0$ substitute $x^2=3y^2+2yb$ into 8.10

$$\Rightarrow 2yb^2+8y^2b+8y^3+1=0$$

For a range of b solve for y & x & plot.

Refer fig 8.1

Slow waves have gain. Even if electrons travel slower than unforced wave, the forced wave must be slower than the electrons to see gain.

Effect of attenuation

Assume the undisturbed wave propagates with $e^{-\beta_e c d}$ of attenuation.

$\Rightarrow$ Loss L in dB/λ is

$$L = 54.5Cd \text{ dB}/\lambda. \Rightarrow d = 0.01836\,(L/C)$$

$d\neq 0 \Rightarrow$
$$\boxed{\begin{aligned}(x^2-y^2)(y+b)+2xy(x+d)+1=0\\(x^2-y^2)(x+d)-2xy(y+b)=0\end{aligned}}$$

Appendix ① Gas Force Equation

> Zeroth order moment of Boltzmann Equation
> (current continuity equation)

$$\frac{\partial n}{\partial t} + \nabla.(n\vec{v}) = 0 \qquad n \rightarrow \text{no. of charges per unit volume.}$$

> First order moment of Boltzmann Equation
> (momentum conservation equation).

$$\frac{\partial (np_i)}{\partial t} + \nabla.(np_i\vec{v}) = qn(\vec{E}+\vec{v}\times\vec{B})_i - \nabla(nk_BT)_i - \frac{np_i}{\tau_m}$$

$i = x, y, z.$

Let $\vec{B} = 0$; $p_i = mv_i$; $\vec{J} = qn\vec{v}$, $\rho = qn$

$$\Rightarrow np_i = nmv_i = \frac{mJ_i}{q}$$

$$\Rightarrow \frac{m}{q}\frac{\partial J_i}{\partial t} + \frac{m}{q}\nabla.(J_i\vec{v}) = qn(\vec{E})_i - k_BT\,\nabla(n)_i - \frac{m}{q}\frac{J_i}{\tau_m}$$

$$\Rightarrow \frac{\partial \vec{J}}{\partial t} + (\vec{v}.\nabla)\vec{J} + \vec{J}(\nabla.\vec{v}) + \frac{\vec{J}}{\tau_m} + \frac{q}{m}(k_BT)\nabla n = \frac{q^2}{m}n\vec{E}$$

Since $\nabla.(J_i\vec{v}) = J_i(\nabla.\vec{v}) + \underbrace{(\nabla J_i).\vec{v}}_{(\vec{v}.\nabla)J}$

$(\vec{v}.\nabla) \rightarrow$ operator.

For semiconductors,

$$\frac{\partial \bar{J}}{\partial t} + (\bar{v}\cdot\nabla)\bar{J} + \bar{J}\,\nabla\cdot\bar{v} + \frac{\vec{J}}{\tau_m} + \frac{q}{m^*}\overset{\text{Thermal Kinetic Energy.}}{T_{th}}\nabla n = \frac{q^2}{m^*} n\vec{E} \quad \text{①}$$

$$T_{th} = \frac{1}{2} m^* v_{th}^2 \quad \text{where} \quad v_{th} = \sqrt{\frac{3k_BT}{m^*}}$$

Seperate into DC & AC terms.

$\vec{E} = \vec{E_0} + \vec{E_1}$; $\bar{J} = \bar{J_0} + \bar{J_1}$; $\vec{v} = \vec{v_0} + \vec{v_1}$; $n = n_0 + n_1$; $\rho = \rho_0 + \rho_1$.

$$\bar{J} = \rho\bar{v} = q(n_0+n_1)(\vec{v_0}+\vec{v_1}) = \vec{J_0} + \bar{J_1}$$

$$\vec{J_0} = \rho_0\vec{v_0} = q n_0 \vec{v_0} \;;\; \underbrace{\bar{J_1} \cong \rho_0\bar{v_1} + \rho_1\bar{v_0}}_{\text{linearizing.}} = q(n_0\bar{v_1} + n_1\bar{v_0})$$

$\Rightarrow$ Linearizing ① & equating dc & ac terms

$$\Rightarrow \quad \frac{\vec{J_0}}{\tau_m} = \frac{q^2}{m^*} n_0 \bar{E_0}$$

$$\& \quad \frac{\partial \vec{J_1}}{\partial t} + (\bar{v_0}\cdot\nabla)\bar{J_1} + \vec{J_0}\,\nabla\cdot\bar{v_1} + \frac{\vec{J_1}}{\tau_m} + \frac{q}{m^*}(k_BT)\nabla n_1$$

$$= \frac{q^2}{m^*} n_0\bar{E_1} + \frac{q^2}{m} n_1 \bar{E_0}$$

$$\Rightarrow \frac{\partial \bar{J_1}}{\partial t} + (\bar{v_0}\cdot\nabla)\bar{J_1} + \vec{J_0}\,\nabla\cdot\bar{v_1} + \frac{\bar{J_1}}{\tau_m} + \frac{(k_BT)\nabla\rho_1}{m}$$

$$= \frac{q}{m}\rho_0\bar{E_1} + \frac{q}{m}\rho_1\bar{E_0}$$

Albrecht's model eliminates $\vec{v}_1$ to be left with just $\bar{J}_1, \rho_1$ as the unknowns since the 3 are related.

It can be shown that

$$J_0 \nabla\cdot\bar{v}_1 = \bar{V}_0 \nabla\cdot\bar{J}_1 - \bar{V}_0 (\bar{V}_0\cdot\nabla)\rho_1 = \bar{V}_0 \nabla\cdot\bar{J}_1 - \bar{V}_0 (\bar{V}_0\cdot\nabla) q n_1$$

$$\Rightarrow \frac{\partial \bar{J}_1}{\partial t} + (\bar{V}_0\cdot\nabla)\bar{J}_1 + \vec{V}_0 \nabla\cdot\bar{J}_1 - \bar{V}_0(\bar{V}_0\cdot\nabla)(q n_1) + \frac{\bar{J}_1}{\tau_m} + \frac{q}{m}(k_B T)\nabla n_1 = \frac{q^2}{m} n_0 \bar{E}_1 + \frac{q^2}{m} n_1 \bar{E}_0$$

Let $\nu_m = \dfrac{1}{\tau_m}$

$$\Rightarrow \boxed{\frac{\partial \bar{J}_1}{\partial t} + (\bar{V}_0\cdot\nabla)\bar{J}_1 + \bar{V}_0 \nabla\cdot\bar{J}_1 - \bar{V}_0(\bar{V}_0\cdot\nabla)\rho_1 + \nu_m \bar{J}_1 + \frac{T_{th}\nabla\rho_1}{m} = \frac{q}{m}\rho_0\bar{E}_1 + \frac{q}{m}\rho_1 \bar{E}_0}$$

Gas Force Equation.

Albrecht - Pierce model.

$$\bar{J} = \overset{DC}{\bar{J}_0} + \overset{AC}{\bar{J}_1}$$ current density

$$\rho = \rho_0 + \rho_1$$ charge density

$$\bar{V} = \bar{V}_0 + \bar{V}_1$$ velocity. (drift + ac velocity)

$$\bar{J}_0 = \rho_0 \bar{V}_0$$

$$\bar{J}_1 = \rho_0 \bar{V}_1 + \rho_1 \bar{V}_0$$ (ignore nonlinear term $V_1 \rho_1$)

$\nu \to$ frequency

Continuity

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot \bar{J}_1 = 0$$

Gas force equation (See Appendix 1)

$$\frac{\partial \bar{J}_1}{\partial t} + \nu \bar{J}_1 + \bar{V}_0 \nabla \cdot \bar{J}_1 + \underbrace{(\bar{V}_0 \cdot \nabla)}_{\text{operator (not } \nabla \cdot \bar{V}_0)} \bar{J}_1 - \bar{V}_0 (\bar{V}_0 \cdot \nabla) \rho_1$$

(thermal kinetic energy)

$$+ \left(\frac{T}{m}\right) \nabla \rho_1 = \left(\frac{q}{m}\right) \rho_0 \bar{E}_1 + \left(\frac{q}{m}\right) \rho_1 \bar{E}_0 \qquad (2)$$

Maxwell's Equation

$$\nabla \times \left(\frac{1}{\mu} (\nabla \times \bar{E}) \right) = -j\omega \left(\bar{J}_1 + \omega^2 e \bar{E}_1 \right)$$

$$\hookrightarrow \frac{\partial}{\partial t} = j\omega$$

The unknowns are : $\rho_1, \bar{J}_1, \bar{E}_1$

We need : $\rho_0, \bar{V}_0, \bar{E}_0$.

Gas force equation for 2DEG

$$\bar{V}_0 = V_{beam}\,\hat{z} \qquad\qquad \frac{\partial}{\partial t} \rightarrow -i\omega$$

$$\bar{J}_1 = J_{z1}\hat{z} \qquad\qquad \nabla \rightarrow i\bar{K}$$

$$\bar{K} = K_z\,\hat{z}$$

$$T = \frac{1}{2} m V_{th}^2$$

(2)

$$\Rightarrow -i\omega J_{z1} + \nu J_{z1} + V_{beam}(i K_z J_{z1}) + i K_z V_{beam} J_{z1}$$

$$- V_{beam}(i K_z V_{beam})\rho_1 + \frac{T}{m} i k_z \rho_1 = \frac{q}{m}\rho_0 E_{z1} + \frac{q}{m}\rho_1 E_{z0}$$

We know $q E_0 = m V_{beam} \nu$ → collision frequency. [From Cold plasma model at 0C See Apdx 1]

& $\rho_1 = \dfrac{K_z J_{z1}}{\omega}$ from continuity.

multiply both sides by $i\omega$ & substitute these two equations,

$$\Rightarrow \left(\omega^2 + i\nu\omega - 2K_z\omega V_{beam} + K_z^2 V_{beam}^2 - \frac{1}{2} V_{th}^2 k_z^2\right) J_z = i\omega\left(\frac{q}{m}\right)\rho_0 E_{z1}$$

$$\frac{+ i\phi \nu V_{beam} k_z J_z}{\phi}$$

$$\Rightarrow J_{z1}\left(\omega^2 + k_z^2 V_{beam}^2 - \frac{\nu^2}{4} - 2k_z\omega V_{beam} + i\nu\omega - i\nu V_{beam} k_z\right.$$

$$\left. + \frac{\nu^2}{4} - \frac{1}{2}k_z^2 V_{th}^2\right) = i\omega\left(\frac{q}{m}\right)\rho_0 E_{z1}.$$

$$\Rightarrow \boxed{\left[\left(\omega + \frac{i\nu}{2} - k_z V_{beam}\right)^2 + \frac{\nu^2}{4} - \frac{1}{2}k_z^2 V_{th}^2\right] J_{z1} = i\omega\left(\frac{q}{m}\right)\rho_0 E_z}$$

J_{z1} is surface current density along z. $\left(\frac{A}{m}\right)$ of the 2DEG

To be consistent with Albrecht's paper, we use the following substitutions in notation.

$$\left.\begin{array}{l}\rho_1 \to \sigma_1 \\ \rho_0 \to \sigma_0\end{array}\right\} \text{surface charge densities. } (C/m^2)$$

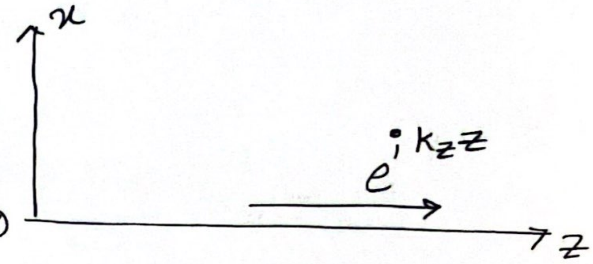
$$J_{z1} \to K_{z1} \;\} \text{ surface current density } (A/m)$$

$$\Rightarrow \boxed{\left[\left(\omega + \frac{i\nu}{2} - k_z V_{beam}\right)^2 + \frac{\nu^2}{4} - \frac{1}{2}k_z^2 V_{th}^2\right] K_{z1} = i\omega\left(\frac{q}{m}\right)\sigma_0 E_{z1}}$$

This gives one equation between E_{z1} & K_{z1}.

Observe that the moving gas of charges J_{z1} produces E_{z1} which must also satisfy Maxwell's Equations. This gives the second equation relating the two quantities.

Case i : 2DEG in yz plane at $x=0$, top & bottom dielectrics with ϵ.



> Assume an oscillating sheet of charges $\sigma_1(y,z) \propto e^{ik_z z}$. Therefore the fields have phase variation $\propto e^{ik_z z}$ where k_z is a complex number.

> Symmetry in y & z $\Rightarrow$ potential function varies only with x. $\psi_l(x)$ where $l=1,2$ correspond to $x>0$, $x<0$ respectively.

> This potential function must satisfy the scalar wave equation in order to find propagating modes along z.

$$\Rightarrow \quad \nabla_t^2 \psi + k_{cl}^2 \psi = 0$$

Where $k_{cl}^2 = k_l^2 - k_z^2$. Since a slow wave solution is required we can assume $k_z >> k_l$ $\Rightarrow$ $k_{cl}^2 = -k_z^2$

$$\Rightarrow \boxed{k_{cl} = \pm j\, k_z}$$

$$\nabla_t^2 \psi_\ell + k_{c\ell}^2 \psi_\ell = 0$$

$$\Rightarrow \quad \frac{d^2 \psi_\ell}{dx^2} + k_{c\ell}^2 \psi_\ell = 0$$

The solutions are of the form

$$\psi_\ell = A_\ell e^{jk_{c\ell} x}$$

$$\Rightarrow \quad \psi_1 = A_1 e^{jk_{c1} x} \quad ; \quad \psi_2 = A_2 e^{jk_{c2} x}.$$

$$\Rightarrow \quad \psi_1 = A_1 e^{\pm k_z x} \quad ; \quad \psi_2 = A_2 e^{\pm k_z x}.$$

k_z is a complex number. $x > 0$ & $x < 0$ are both semi-infinite & radiation boundary condition implies only decaying waves exist. $\Rightarrow$ the real part of $\pm k_z x$ must always be negative. Note that we must allow $Re\{k_z\}$ to be both + & -ve to solve for amplifying & decaying modes. Therefore,

let $\sigma_\pm = Re\{k_z\} \Rightarrow Re\{-\sigma_\pm k_z x\} < 0$ if $x > 0$ &

$Re\{\sigma_\pm k_z x\} < 0$ if $x < 0$

$$\Rightarrow \quad \psi_1 = A_1 e^{-\sigma_\pm k_z x} \quad \& \quad \psi_2 = A_2 e^{\sigma_\pm k_z x}.$$ are the

desired potential functions.

Fields

$$\bar{E} = \left[\pm i\overset{k_z}{\beta}\, \nabla_t \psi + k_c^2 \psi \,\hat{z}\right] e^{\pm i\overset{k_z}{\beta} z}$$

$$\bar{H} = -i\omega\epsilon \left(\nabla_t \psi \times \hat{z}\right) e^{\pm i\overset{k_z}{\beta} z}.$$

We only want $+ik_z z$

$$\Rightarrow \bar{E}_1 = \left[ik_z \nabla_t\left(A_1 e^{-\sigma_\pm k_z x}\right) - k_z^2 \left(A_1 e^{-\sigma_\pm k_z x}\right)\hat{z}\right] e^{ik_z z}$$

$$= \left[ik_z A_1 (-\sigma_\pm k_z) e^{-\sigma_\pm k_z x}\,\hat{x} - k_z^2 A_1 e^{-\sigma_\pm k_z x}\,\hat{z}\right] e^{ik_z z}$$

$$\bar{E}_1 = -A_1 k_z^2 \left[i\sigma_\pm \hat{x} + \hat{z}\right] e^{-\sigma_\pm k_z x} e^{ik_z z}.$$

Similarly,

$$\bar{E}_2 = A_2 k_z^2 \left[i\sigma_\pm \hat{x} - \hat{z}\right] e^{\sigma_\pm k_z x} e^{ik_z z}.$$

$$\bar{H}_1 = -i\omega\epsilon \left(-A_1 \sigma_\pm k_z\right) e^{-\sigma_\pm k_z x} (-\hat{y})\, e^{ik_z z}$$

$$= -i\omega\epsilon A_1 \sigma_\pm k_z\, e^{-\sigma_\pm k_z x} e^{ik_z z}\,\hat{y} \,//$$

Similarly,

$$\bar{H}_2 = -i\omega\epsilon \left(A_2 \sigma_\pm k_z\right) e^{+\sigma_\pm k_z x} (-\hat{y})\, e^{ik_z z}$$

$$= i\omega\epsilon A_2 \sigma_\pm k_z\, e^{\sigma_\pm k_z x} e^{ik_z z}\,\hat{y} \,//$$

Applying boundary conditions to find A_1, A_2.

$$H_{1y} - H_{2y} = J_{1z} \qquad (\because \hat{n}(\bar{H}_1 - \bar{H}_2) = \bar{J}) \text{ at } x=0.$$

Recall, $J_{1z} = \dfrac{\omega \rho_1}{k_z}$ from continuity of time harmonic currents.

$$\Rightarrow J_{1z} = \frac{\omega \sigma_1(y,z)}{k_z}$$

At $x=0$ & $z=0$

$$\Rightarrow -i\omega\epsilon\sigma_{\pm} k_z (A_1 + A_2) = \frac{\omega\sigma_1(y,z)}{k_z}$$

$$\Rightarrow A_1 + A_2 = \frac{i\sigma_1(y,z)}{\epsilon\sigma_{\pm} k_z^2}$$

$E_{1z} = E_{2z}$ at $x=0$ & $z=0$

$$\Rightarrow -A_1 k_z^2 = -A_2 k_z^2 \Rightarrow A_1 = A_2$$ (which should be obvious from symmetry)

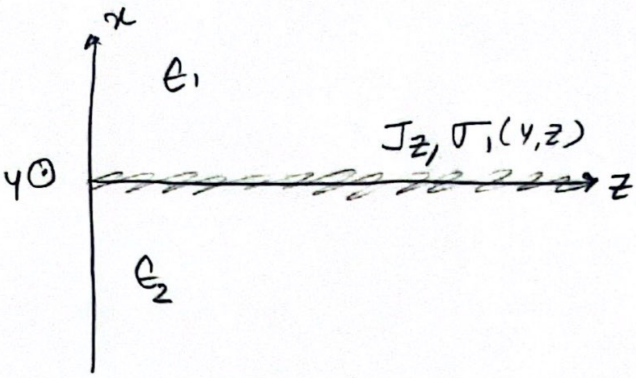
$$\Rightarrow A_1 = A_2 = \frac{1}{2} \frac{i\sigma_1(y,z)}{\epsilon\sigma_{\pm} k_z^2}$$

$$\Rightarrow \bar{E}_1 = -\frac{1}{2} \frac{i\sigma_1(y,z)}{\epsilon\sigma_{\pm} \cancel{k_z^2}} \cancel{k_z^2} (i\sigma_{\pm}\hat{x} + \hat{z}) e^{-\sigma_{\pm} k_z x} e^{ik_z z}$$

$$= \frac{1}{2}\epsilon^{-1}\sigma_1(y,z)(\hat{x} - i\sigma_{\pm}\hat{z}) e^{-\sigma_{\pm} k_z x} e^{ik_z z}$$

$$\bar{E}_2 = \frac{1}{2}\epsilon^{-1}\sigma_1(y,z)(-\hat{x} - i\sigma_{\pm}\hat{z}) e^{\sigma_{\pm} k_z x} e^{ik_z z}.$$

Case ii :



Almost same solution,

$$k_{c\ell} = \pm j\, k_z$$

$$\psi_\ell = A_\ell\, e^{j k_{c\ell} x} \Rightarrow \psi_1 = A_1 e^{\pm k_z x} \; ; \; \psi_2 = A_2\, e^{\pm k_z x}.$$

$$\psi_1 = A_1\, e^{-\sigma_\pm k_z x} \; ; \; \psi_2 = A_2\, e^{\sigma_\pm k_z x}.$$

Fields

$$\bar{E}_1 = -A_1 k_z^2 \left[i\sigma_\pm \hat{x} + \hat{z} \right] e^{-\sigma_\pm k_z x} e^{i k_z z}$$

$$\bar{E}_2 = A_2 k_z^2 \left[i\sigma_\pm \hat{x} - \hat{z} \right] e^{\sigma_\pm k_z x} e^{i k_z z}$$

$$\bar{H}_1 = -i\omega \epsilon_1 A_1 \sigma_\pm k_z\, e^{-\sigma_\pm k_z x} e^{i k_z z} \hat{y}$$

$$\bar{H}_2 = i\omega \epsilon_2 A_2 \sigma_\pm k_z\, e^{\sigma_\pm k_z x} e^{j k_z z} \hat{y}$$

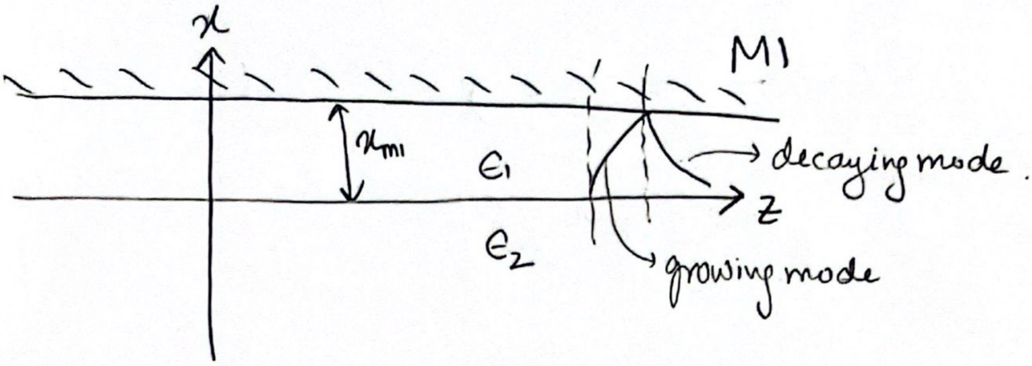
BC of H,E $\Rightarrow$ $\epsilon_1 A_1 + \epsilon_2 A_2 = \dfrac{i\sigma_1(y,z)}{\sigma_\pm k_z^2}$ & $A_1 = A_2$

$$\Rightarrow \quad A_1 = A_2 = \frac{i\,\sigma_1(y,z)}{(\epsilon_1+\epsilon_2)\,\sigma_\pm k_z^2}$$

$$\Rightarrow \bar{E}_1 = \frac{1}{(\epsilon_1+\epsilon_2)} \sigma_1(y,z)\,(\hat{x} - i\,\sigma_{\pm}\,\hat{z})\, e^{-\sigma_{\pm} k_z x}\, e^{i k_z z}$$

$$E_2 = \frac{1}{(\epsilon_1+\epsilon_2)} \sigma_1(y,z)\,(-\hat{x} - i\,\sigma_{\pm}\,\hat{z})\, e^{\sigma_{\pm} k_z x}\, e^{i k_z z} \; //.$$

Case iii : Introduce metal layer at some distance $x > 0$.



$k_{ce} = \pm j k_z$. & solutions are of the form $\psi_l = A_l e^{i k_{cl} x}$.

Note: In medium 1, radiation b.c no longer applies, so we must allow for growing modes. The growing modes come from the reflection on M1.

$$\Rightarrow \psi_1 = A_1 e^{-\sigma_{\pm} k_z x} + B_1 e^{\sigma_{\pm} k_z x}$$

$$\psi_2 = A_2 e^{\sigma_{\pm} k_z x}$$

Potential functions for this case.

Fields

$$\bar{E}_1 = \Big[i k_z \nabla_t \left(A_1 e^{-\sigma_\pm k_z x} + B_1 e^{\sigma_\pm k_z x} \right)$$

$$- k_z^2 \left(A_1 e^{-\sigma_\pm k_z x} + B_1 e^{\sigma_\pm k_z x} \right) \hat{z} \Big] e^{i k_z z}.$$

$$= \Big[i k_z \left\{ A_1 (-\sigma_\pm k_z) e^{-\sigma_\pm k_z x} + B_1 (\sigma_\pm k_z) e^{\sigma_\pm k_z x} \right\} \hat{x}$$

$$- k_z^2 \left(A_1 e^{-\sigma_\pm k_z x} + B_1 e^{\sigma_\pm k_z x} \right) \hat{z} \Big] e^{i k_z z}$$

$$\bar{E}_1 = \Big[\left\{ -A_1 k_z^2 (i \sigma_\pm \hat{x} + \hat{z}) e^{-\sigma_\pm k_z x} \right\} + \left\{ B_1 k_z^2 (i \sigma_\pm \hat{x} - \hat{z}) e^{\sigma_\pm k_z x} \right\} \Big] \times e^{i k_z z}.$$

$$\bar{E}_2 = A_2 k_z^2 (i \sigma_\pm \hat{x} - \hat{z}) e^{\sigma_\pm k_z x} e^{i k_z z}$$

$$\bar{H}_1 = + i \omega \epsilon_1 \left\{ A_1 (-\sigma_\pm k_z) e^{-\sigma_\pm k_z x} + B_1 (\sigma_\pm k_z) e^{\sigma_\pm k_z x} \right\} (+\hat{y}) e^{i k_z z}$$

$$\bar{H}_2 = i \omega \epsilon_2 A_2 \sigma_\pm k_z e^{\sigma_\pm k_z x} e^{i k_z z} \hat{y}$$

<u>BC on H</u> $(x=0, z=0)$

$$+ i \omega \epsilon_1 \left\{ A_1 (-\sigma_\pm k_z) + B_1 (\sigma_\pm k_z) \right\} - i \omega \epsilon_2 \left\{ A_2 \sigma_\pm k_z \right\} = \frac{\omega \sigma_1 (y, z)}{k z}$$

$$\Rightarrow \; -i \omega \, \sigma_\pm k_z \left\{ (+A_1 - B_1) \epsilon_1 + A_2 \epsilon_2 \right\} = \frac{\omega \sigma_1 (y, z)}{k z}$$

$$\Rightarrow A_1\epsilon_1 + A_2\epsilon_2 - B_1\epsilon_1 = \frac{i\,\sigma_1(y,z)}{\sigma_\pm\, k_z^2} \quad \text{— ①}$$

<u>BC on E</u> $(x=0,\ z=0)$

$$\bar{E}_{1z}(0,0) = -A_1 k_z^2 + B_1 k_z^2$$

$$\bar{E}_{2z}(0,0) = -A_2\, k_z^2$$

$$\Rightarrow \quad A_1 + B_1 - A_2 = 0 \quad \text{— ②}$$

<u>BC on E at x_{m1}</u>

$$\bar{E}_{1z}(x_{m1}, 0) = -A_1 k_z^2\, e^{-\sigma_\pm k_z x_{m1}} - B_1 k_z^2\, e^{\sigma_\pm k_z x_{m1}} = 0$$

$$\Rightarrow \quad A_1 e^{-\sigma_\pm k_z x_{m1}} + B_1 e^{\sigma_\pm k_z x_{m1}} = 0 \quad \text{— ③}$$

<u>Solving for A_1, B_1, A_2</u> (Assuming $\epsilon_1 = \epsilon_2$).

$$① + ② \Rightarrow A_1 = \frac{1}{2}\frac{i\,\sigma_1(y,z)}{\epsilon\,\sigma_\pm\, k_z^2}$$

From ③

$$B_1 = -A_1 e^{-2\sigma_\pm k_z x_{m1}} = -\frac{1}{2}\frac{i\,\sigma_1(y,z)}{\epsilon\,\sigma_\pm\, k_z^2}\, e^{-2\sigma_\pm k_z x_{m1}}$$

From ②

$$A_2 = \frac{1}{2}\left(\frac{i\sigma_1(y,z)}{\epsilon\,\sigma_\pm\, k_z^2}\right)\left(1 - e^{-2\sigma_\pm k_z x_{m1}}\right)$$

Solving for $A_1, A_2\ B_1$ (when $\epsilon_1 \neq \epsilon_2$)

① + ② × ϵ_2

$$\Rightarrow \left(A_1 \epsilon_1 + \cancel{A_2 \epsilon_2} - B_1 \epsilon_1\right)$$

$$+ \left(A_1 \epsilon_2 - \cancel{A_2 \epsilon_2} + B_1 \epsilon_2\right) = \frac{i\sigma_1(y,z)}{\sigma_\pm K_z^2}$$

$$\Rightarrow A_1(\epsilon_1+\epsilon_2) + B_1(\epsilon_2-\epsilon_1) = \frac{i\sigma_1(y,z)}{\sigma_\pm K_z^2}$$

From ③

$$B_1 = -A_1 e^{-2\sigma_\pm K_z x_{m1}}$$

$$\Rightarrow A_1(\epsilon_1+\epsilon_2) + A_1 e^{-2\sigma_\pm K_z x_{m1}}(\epsilon_1-\epsilon_2) = \frac{i\sigma_1(y,z)}{\sigma_\pm K_z^2}$$

$$\Rightarrow A_1\left((\epsilon_1+\epsilon_2) + (\epsilon_1-\epsilon_2)\, e^{-2\sigma_\pm K_z x_{m1}}\right) = \frac{i\sigma_1(y,z)}{\sigma_\pm K_z^2}$$

$$\Rightarrow \boxed{A_1 = \frac{i\sigma_1(y,z)}{\sigma_\pm K_z^2}\left[(\epsilon_1+\epsilon_2) + (\epsilon_1-\epsilon_2)\, e^{-2\sigma_\pm K_z x_{m1}}\right]^{-1}}$$

$$\boxed{B_1 = \frac{-i\sigma_1(y,z)}{\sigma_\pm\ k_z^2}\left[(\epsilon_1+\epsilon_2)\,e^{2\sigma_\pm k_z x_{m1}} + (\epsilon_1-\epsilon_2)\right]^{-1}}$$

From ② $A_2 = A_1 + B_1$

$$\boxed{A_2 = \frac{i\sigma_1(y,z)}{\sigma_\pm\ k_z^2}\left[\frac{-2\epsilon_2 + (\epsilon_1+\epsilon_2)e^{2\sigma_\pm k_z x_{m1}} + (\epsilon_1-\epsilon_2)e^{-2\sigma_\pm k_z x_{m1}}}{2(\epsilon_1-\epsilon_2)(\epsilon_1+\epsilon_2) + (\epsilon_1-\epsilon_2)^2 e^{-2\sigma_\pm k_z x_{m1}} + (\epsilon_1+\epsilon_2)^2 e^{2\sigma_\pm k_z x_{m1}}}\right]}$$

> Substituting $\epsilon_1 = \epsilon_2$ gives the right result.

> Substituting $x_m \to \infty$ gives the right results. (for both $\epsilon_1 = \epsilon_2$ & $\epsilon_1 \neq \epsilon_2$)

Final Fields

$$\bar{E}_1 = \left[\left\{-A_1 k_z^2(i\sigma_\pm\hat{x} + \hat{z})e^{-\sigma_\pm k_z x}\right\} + \left\{B_1 k_z^2(i\sigma_\pm\hat{x} - \hat{z})e^{\sigma_\pm k_z x}\right\}\right]e^{ik_z z}$$

$$E_2 = A_2 k_z^2(i\sigma_\pm\hat{x} - \hat{z})\,e^{\sigma_\pm k_z x}\,e^{ik_z z}.$$

$$\bar{H}_1 = +i\omega\epsilon_1\left\{A_1(-\sigma_\pm k_z)e^{-\sigma_\pm k_z x} + B_1(\sigma_\pm k_z)e^{\sigma_\pm k_z x}\right\}\hat{y}\,e^{ik_z z}$$

$$\bar{H}_2 = i\omega\,\epsilon_2 A_2\,\sigma_\pm k_z\,e^{\sigma_\pm k_z x}\,e^{ik_z z}\,\hat{y}.$$

> We now have E_z vs J_z/k_z from Maxwell's Equations & the Gas force plasma equation. Both must be satisfied, giving us a dispersion relation.

<u>Case i</u> z component of $\bar{E}$ is

$$E_{z1} = -\frac{1}{2} i \epsilon^{-1} \sigma_{\pm} e^{-\sigma_{\pm} k_z x} e^{i k_z z} \sigma_1(y,z)$$

$$\text{replacing } \sigma_1(y,z) = \frac{J_{1z} k_z}{\omega} = \frac{K_{z1} k_z}{\omega}$$

$$\Rightarrow E_{z1} = -\frac{1}{2} i \epsilon^{-1} \sigma_{\pm} \omega^{-1} k_z e^{-\sigma_{\pm} k_z x} e^{i k_z z} K_{z1}$$

Let $K_{z1} e^{i k_z z} = K_{z1}$ (basically make the wave rotate implicit in K_{z1})

We want E_{z1} at $x=0$ since this is the field that interacts with the plasma.

$$\Rightarrow \boxed{E_{z1} = -\frac{1}{2} i (\epsilon\omega)^{-1} \sigma_{\pm} k_z K_{z1}} \rightarrow E_{z1}$$

TPW (travelling plasma wave)

Gas force relation gives.

$$\boxed{E_{z1} = -i \frac{m}{q\omega\sigma_0}\left[\left(\omega + \frac{i\nu}{2} - k_z V_{beam}\right)^2 + \frac{\nu^2}{4} - \frac{1}{2} k_z^2 v_{th}^2\right] K_{z1}}$$

$\rightarrow E_{z1}$

GF. (Gas Force)

Equating the two $\left(\underset{TPW}{E_{z1}} = \underset{GP}{E_{z1}} \right)$ (ignoring SWS)

$$\frac{1}{2}(\epsilon\phi)^{-1}\sigma_{\pm} k_z = m(q\phi\sigma_0)^{-1}\left[\left(\omega + \frac{i\nu}{2} - k_z V_{beam}\right)^2 + \frac{\nu^2}{4} - \frac{1}{2}k_z^2 V_{th}^2\right]$$

$$\Rightarrow \quad \frac{1}{2\epsilon}\sigma_{\pm} k_{z1} \cdot \frac{q}{m}\sigma_0 = \left(\omega + \frac{i\nu}{2} - k_z V_{beam}\right)^2 + \frac{\nu^2}{4} - \frac{1}{2}k_z^2 V_{th}^2$$

Let $\omega V_{p20} = \frac{1}{2}\left(\frac{q}{m}\right)\left(\frac{\sigma_0}{\epsilon}\right)$

$$\Rightarrow \boxed{\left(\omega + \frac{i\nu}{2} + k_z V_{beam}\right)^2 - \omega\sigma_{\pm} k_z V_{p20} = \frac{1}{2}k_z^2 V_{th}^2 - \frac{\nu^2}{4}}$$

In case of a cold collision less gas ($\nu = 0$; $V_{th} = 0$)

$$\Rightarrow \omega = k_z V_{beam} \pm \left(\omega\sigma_{\pm} k_z V_{p20}\right)^2$$

which is similar to the 3D beam where

$$\omega = k_z V_{beam} \pm \omega_p \qquad \text{where } \omega_p = \sqrt{\frac{q\rho_0}{m\epsilon}}$$

Explained on next page.

> The dispersion relation above is for the plasma wave with no SWS mode.

Finally we want to include the circuit with the plasma and solve for the combined mode dispersion.

Convention

In the pierce book the convention of forward modes is

$e^{-\Gamma z}$; $e^{-\Gamma_1 z}$ & $e^{-j\beta_e z}$

In Albrecht's paper the convention of forward modes is

$e^{jk_z z}$; $e^{j\frac{\omega}{V_{ph}}}$; $e^{j\frac{\omega}{V_{beam}}}$

$\Rightarrow \Gamma = -jk_z$; $\Gamma_1 = \frac{-j\omega}{V_{ph}}$; $\beta_e = \frac{-\omega}{V_{beam}}$.

Recall, the perturbative solution we look for is

$-\Gamma = -j\beta_e + \xi \quad \Rightarrow \quad jk_z = j\frac{\omega}{V_{beam}} + \xi$ $\longrightarrow$ Page ⑦

$\Rightarrow \boxed{\omega = k_z V_{beam} + j\,\xi V_{beam}.}$ $\Rightarrow \pm\omega_p = j\,\xi\, V_{beam}$ & therefore ω_p is the perturbation term.

Next we express E_{z1} in terms of I_1 for the circuit ignoring space charge effects.

Recall, $E_{z1} = \frac{\Gamma^2 \Gamma_1 \overrightarrow{K}^{Zckt} I_1}{\Gamma^2 - \Gamma_1^2}$ (page ③) $(E_{z1} = +\Gamma V)$

$$\Rightarrow E_{z1} = \frac{(+k_z^2)\left(\frac{-j\omega}{V_{ph}}\right)(Z_{ckt})(I_1)}{(-k_z^2) - \left(-\frac{\omega^2}{V_{ph}^2}\right)}$$

$$\Rightarrow \boxed{E_{z1} = \frac{-j\omega Z_{ckt} k_z^2 V_{ph}}{\omega^2 - k_z^2 V_{ph}^2} I_1}$$

Define,

$\alpha = \frac{k_z V_{ph}}{\omega}$ (normalized dispersion root) ; $\lambda = \frac{V_{beam}}{V_{ph}}$ (tuning parameter) ; $C^3 = \frac{1}{4}\frac{Z_{ckt}}{Z_{beam}}$ (Pierce gain parameter)

$4QC^3 = \frac{\omega_p^2}{\omega^2}$ (Pierce space charge parameter).

The dispersion relation (ignoring space charge) is

$$\frac{jK I_0 \beta_e \Gamma^2 \Gamma_1}{2V_0(\Gamma_1^2 - \Gamma^2)(j\beta_e - \Gamma)^2} = 1$$

$$\Rightarrow \frac{j Z_{ckt}}{2 Z_{beam}} \cdot \frac{\left(\frac{-\omega}{V_{beam}}\right)(-k_z^2)\left(\frac{-j\omega}{V_{ph}}\right)}{\left(-\frac{\omega^2}{V_{ph}^2} + k_z^2\right)\left(\frac{-j\omega}{V_{beam}} + jk_z\right)^2} = 1$$

$\frac{V_0}{I_0}$

$$\Rightarrow \quad \frac{2c^3(\omega^2 k_z^2) V_{ph} V_{beam}}{(\omega^2 - k_z^2 V_{ph}^2)(-\omega + k_z V_{beam})^2} = 1$$

$$\Rightarrow \quad \frac{2c^3(k_z^2)(V_{ph} V_{beam})}{\omega^2\left(1 - \frac{k_z^2 V_{ph}^2}{\omega^2}\right)\left(-1 + \frac{k_z V_{beam}}{\omega}\right)^2} = 1$$

$$\Rightarrow \quad \frac{2c^3 x^2 \lambda}{(1-x^2)(1-\lambda x)^2} = 1$$

$$\Rightarrow \quad \boxed{(1-\lambda x)^2(1-x^2) - 2\lambda c^3 x^2 = 1}$$

Next we look at the case when space charge is included,

The dispersion relation from Pierce is given by (Page 14)

$$1 = \frac{j I_0 \beta_e \Gamma}{2V_0 (j\beta_e - \Gamma)^2}\left[\frac{\Gamma\Gamma_1 (E^2/\beta^2 P)}{2(\Gamma_1^2 - \Gamma^2)} - \frac{j\Gamma}{\omega C_1}\right]$$

$$\Rightarrow 1 = \frac{j}{2} \cdot \frac{1}{Z_{beam}} \cdot \left(\frac{-\omega}{V_{beam}}\right)(-jk_z)\left(\frac{1}{\frac{-j\omega}{V_{beam}} + jk_z}\right)^2$$

$$\times \left[\frac{(-jk_z)\left(\frac{-j\omega}{V_{ph}}\right)(Z_{ckt})}{\left(\left(\frac{-j\omega}{V_{ph}}\right)^2 - (-jk_z)^2\right)} - \frac{j(-jk_z)}{\omega C_1}\right]$$

$$\Rightarrow 1 = 2c^3 \left(\frac{\omega k_z V_{beam}}{(k_z V_{beam} - \omega)^2} \right) \left[\frac{-k_z \omega V_{ph}}{k_z^2 V_{ph}^2 - \omega^2} - \frac{k_z}{\omega Z_{ckt} C_1} \right]$$

$$\Rightarrow 1 = 2c^3 \left(\frac{x\lambda}{(x\lambda - 1)^2} \right) \left[\frac{-x}{x^2 - 1} - \frac{k_z}{\omega Z_{ckt} C_1} \right]$$

Recall that on page 16 we define $Q = \frac{\beta_e}{\omega C_1 (E^2/\beta^2 P)}$

However here we ignore the c terms from the last eq. on Page 15

If we do not ignore them we have

$$Q = \frac{\beta_e \left[1 + c(2j\delta - c\delta^2) \right]}{2\omega C_1 Z_{ckt}} \quad \text{where } \delta = \frac{-\Gamma + j\beta_e}{\beta_e c} \text{ (page 1}$$

$$\Rightarrow \delta = \frac{jk_z - j\frac{\omega}{V_{beam}}}{\beta_e c} = j \frac{\left(\frac{k_z V_{beam}}{\omega} - 1 \right)}{\frac{\beta_e V_{beam}}{\omega} c} = \frac{j}{c}(x\lambda - 1)$$

$$\Rightarrow 1 + c(2j\delta - c\delta^2) = 1 + c\left(2j\left(\frac{j}{c}(x\lambda - 1)\right) - c\left(-\frac{1}{c^2}(x\lambda - 1)^2\right)\right)$$

$$= 1 - 2(x\lambda - 1) + (x\lambda - 1)^2 = x^2\lambda^2$$

$$\Rightarrow Q = \frac{\beta_e x^2 \lambda^2}{2\omega C_1 Z_{ckt}} \Rightarrow \frac{k_z}{\omega Z_{ckt} C_1} = \frac{2Q k_z}{\beta_e x^2 \lambda^2} = \frac{-2Q k_z V_{beam}}{\omega x^2 \lambda^2}$$

$$= \frac{-2Q}{x\lambda}$$

$$\Rightarrow 1 = \frac{2c^3x^2\lambda}{(1-\lambda x)^2}\left[\frac{-1}{x^2-1} - \frac{k_z}{Z_{ext}\,\omega\, G\, x}\right]$$

$$\Rightarrow 1 = \frac{2c^3x^2\lambda}{(1-\lambda x)^2}\left[\frac{-1}{x^2-1} + \frac{2\Theta}{x^2\lambda}\right]$$

$$\Rightarrow (1-\lambda x)^2 = \frac{-2c^3x^2\lambda}{x^2-1} + 4\Theta c^3$$

$$\Rightarrow \boxed{\left[(1-\lambda x)^2 - 4\Theta c^3\right](1-x^2) - 2\lambda c^3 x^2 = 0}$$

Finally we want to find the dispersion relation when the Travelling plasma wave & slow wave interact (instead of slow wave & vaccum $\bar{e}$-beam).

$$\Rightarrow \underset{GF}{E_{z1}} = \underset{TPW}{E_{z1}} + \underset{SWS}{E_{z1}}$$

Case i <u>Same ϵ on both sides of 2DEG.</u>

Recall,

$$\underset{TPW}{E_{z1}} = -\frac{1}{2} i \, (\epsilon\omega)^{-1} \sigma_{\pm} \, k_z \, k_{z1} \qquad (\text{Page } 7 \ \& \ 14)$$

$$\underset{GF}{E_{z1}} = -\frac{im}{q\omega\sigma_0}\left[\left(\omega + \frac{i\nu}{2} - k_z V_{beam}\right)^2 + \frac{\nu^2}{4} - \frac{1}{2} k_z^2 v_{th}^2\right] k_{z1} \qquad (\text{Page } 3)$$

$$\underset{SWS}{E_{z1}} = \frac{\Gamma^2 \Gamma_1 (Z_{ckt})}{(\Gamma_1^2 - \Gamma^2)} \overbrace{k_{z1} W_{beam}}^{I_1} \qquad (\text{Page } 12 \text{ of Pierce notes})$$

(ignoring higher order nonsynchronous modes)

$$\Rightarrow -\frac{im}{q\omega\sigma_0}\left[\left(\omega + \frac{i\nu}{2} - k_z V_{beam}\right)^2 + \frac{\nu^2}{4} - \frac{1}{2} k_z^2 v_{th}^2\right] = -\frac{1}{2}\frac{i\sigma_{\pm} k_z}{\epsilon\omega} + \frac{\Gamma^2\Gamma_1 (Z_{ckt}) W_{beam}}{(\Gamma_1^2 - \Gamma^2)}$$

$$\Rightarrow \frac{m\omega}{q\sigma_0}\left[(1 + i\eta - x\lambda)^2 + \eta^2 - \zeta^2 x^2\right] = \frac{1}{2}\frac{\sigma_{\pm} k_z}{\epsilon\omega} + \frac{i\Gamma^2\Gamma_1 (Z_{ckt}) W_{bea}}{(\Gamma_1^2 - \Gamma^2)}$$

where $\eta = \frac{1}{2}\frac{\nu}{\omega}$ & $\zeta^2 = \frac{1}{2}\frac{V_{th}^2}{V_{bi}^2}$

$$\Rightarrow \left[(1+j\eta - x\lambda)^2 + \eta^2 - \zeta^2 x^2\right] = \frac{1}{2}\frac{\sigma_\pm k_z}{\epsilon\omega}\frac{q\sigma_0}{m\omega}$$

$$+ \frac{q\sigma_0}{m\omega} \times j \frac{(+k_z^2)\left(\frac{j\omega}{V_{ph}}\right)(Z_{ckt})\, W_{beam}}{\left(-\frac{\omega^2}{V_{ph}^2} + k_z^2\right)}$$

$$\Rightarrow \left[(1+j\eta - x\lambda)^2 + \eta^2 - \zeta^2 x^2\right] = \theta\sigma_\pm x - \frac{q\sigma_0}{m\omega}\frac{\left(\frac{k_z^2}{\omega}V_{ph}\right)\left(Z_{ckt}W_{beam}\right)}{(x^2-1)}$$

$$\Rightarrow \left[(1+j\eta - x\lambda)^2 + \eta^2 - \theta\sigma_\pm x - \zeta^2 x^2\right](1-x^2) = 2\left(\frac{1}{2}\frac{q\sigma_0}{m\omega\epsilon V_{ph}}\right)(x^2)\left(Z_{ck}W_{beam}\epsilon\omega\right)$$

$$\boxed{\Rightarrow \left[(1+j\eta - \lambda x)^2 - \zeta^2 x^2 - \theta\sigma_\pm x + \eta^2\right](1-x^2) - 2\theta C_{2DEG}^2 x^2 = 0}$$

Where $\theta = \frac{1}{2}\frac{q\sigma_0}{m\epsilon\omega V_{ph}}$; $C_{2DEG}^2 = \epsilon\omega Z_{ckt} W_{beam}$; $\eta = \frac{1}{2}\frac{\nu}{\omega}$; $\zeta^2 = \frac{1}{2}\frac{V_{th}^2}{V_{ph}^2}$

$\sigma_0 \rightarrow$ dc surface charge density

$m, q \rightarrow$ effective mass & charge of $\bar{e}$.

$\epsilon \rightarrow$ permittivity of medium.

$\omega \rightarrow$ excitation frequency

$V_{ph} \rightarrow$ phase velocity of SWS at ω

$Z_{ckt} \rightarrow$ pierce impedance of SWS

$W_{beam} \rightarrow$ width of 2DEG

$\nu \rightarrow$ collision frequency

$V_{th} \rightarrow$ thermal velocity $\left(V_{th} = \sqrt{\frac{3kT}{m^*}}\right)$

$\lambda = V_{beam}/V_{ph}$

$x = \frac{k_z V_{ph}}{\omega}$

Case ii Different ϵ ie ϵ_1 for $x>0$, ϵ_2 for $x<0$

$$E_{z1} = \frac{-i\sigma_{\pm}\, \sigma_1(y,z)}{\epsilon_1+\epsilon_2}\, e^{-\sigma_{\pm} k_z x}\, e^{ik_z z}$$

Make $e^{ik_z z}$ implicit in $\sigma_1(y,z)$ & take $z=0$.

& note $\sigma_1(y,z) = \frac{k_{z1} k_z}{\omega}$

$$\Rightarrow \underset{TPW}{E_{z1}} = -\frac{i\sigma_{\pm} k_z k_{z1}}{(\epsilon_1+\epsilon_2)\omega}$$

$\underset{SWS}{E_{z1}}$ & $\underset{GF}{E_{z1}}$ are identical so just replace ϵ with $\frac{\epsilon_1+\epsilon_2}{2}$ in the final equation.

$$\Rightarrow \boxed{\left[(1+j\eta-\lambda x)^2 - \zeta^2 x^2 - \theta\sigma_{\pm} x + \eta^2\right](1-x^2) - 2\theta\, C_{2DEG}^2\, x^2 = 0}$$

$\theta = \frac{q\sigma_0}{m\omega V_{ph}(\epsilon_1+\epsilon_2)}$; $C_{2DEG}^2 = \left(\frac{\epsilon_1+\epsilon_2}{2}\right)\omega Z_{ckt} W_{beam}$; $\eta = \frac{1}{2}\frac{\nu}{\omega}$; $\zeta^2 = \frac{1}{2}\frac{V_{th}^2}{V_{ph}^2}$

<u>Case iii</u> (ϵ for all x & M_1 at $x = x_{m_1}$)

From Page (13)

$$\bar{E}_{z1} = -A_1 k_z^2 e^{-\sigma_\pm k_z x} e^{ik_z z} - B_1 k_z^2 e^{\sigma_\pm k_z x} e^{ik_z z}$$

$$\bar{E}_{z2} = -A_2 k_z^2 e^{\sigma_\pm k_z x} e^{ik_z z}$$

<u>At $x=0$</u>

$$\bar{E}_{z1} = (-A_1 k_z^2 - B_1 k_z^2) e^{ik_z z}$$

$$\bar{E}_{z2} = -A_2 k_z^2 e^{ik_z z}$$

we know $A_2 = A_1 + B_1$ so consider A_2 (Assuming $\epsilon_1 = \epsilon_2$)

$$\Rightarrow \bar{E}_{z1} = -\frac{1}{2}\left(\frac{i\sigma_1(y,z)}{\epsilon\sigma_\pm \cancel{k_z^2}}\right)\left(1 - e^{-2\sigma_\pm k_z x_{m_1}}\right)\cancel{k_z^2}\, e^{ik_z z}$$

$$= -\frac{1}{2}\left(\frac{i\sigma_1(y,z)}{\epsilon\sigma_\pm}\right)\left(1 - e^{-2\sigma_\pm k_z x_{m_1}}\right) e^{ik_z z}$$

Let $\sigma_1(y,z) e^{ik_z z} = \frac{k_{z1} k_z}{\omega}$ (make $e^{ik_z z}$ implicit as before)

$$\Rightarrow \bar{E}_{z1} = -\frac{i}{2}\left(\frac{\sigma_\pm k_z}{\omega\epsilon}\right)\left(1 - e^{-2\sigma_\pm k_z x_{m_1}}\right) k_{z1}$$

- It is obvious that the form is almost identical to case i & hence the dispersion relation is:

$$\boxed{\left[(1+j\eta-\lambda x)^2-\xi^2 x^2-\theta\sigma_\pm x\left(1-e^{-2\sigma_\pm k_z x_{m1}}\right)+\eta^2\right](1-x^2)-2\theta\, \zeta_{2DEG}^2\, x^2=0}$$

Case iv (ϵ_1 for $x>0$, ϵ_2 for $x<0$ & M1 at $x=x_{m1}$)

$$E_{z1} = -A_2 k_z^2 e^{ik_z z} \quad \text{at } x=0.$$

-

$$= \frac{-i\sigma_1(y,z)}{\sigma_\pm \cancel{k_z^2}} \cdot \cancel{k_z^2} \cdot e^{ik_z z} \cdot \tilde{\psi}(x_{m1})$$

Where $$\tilde{\psi}(x_{m1}) = \frac{-2\epsilon_2 + (\epsilon_1+\epsilon_2)e^{2\sigma_\pm k_z x_{m1}} + (\epsilon_1-\epsilon_2)e^{-2\sigma_\pm k_z x_{m1}}}{2(\epsilon_1-\epsilon_2)(\epsilon_1+\epsilon_2) + (\epsilon_1-\epsilon_2)^2 e^{-2\sigma_\pm k_z x_{m1}} + (\epsilon_1+\epsilon_2)^2 e^{2\sigma_\pm k_z x_{m1}}}$$

$$\Rightarrow E_{z1} = \frac{-i\, k_z k_{z1}}{\omega} \sigma_\pm\, \tilde{\psi}(x_{m1})$$

-

$$\Rightarrow \quad E_{z1} = \frac{-i}{2} \frac{\sigma_\pm k_z}{\omega} \left(2\tilde{\psi}(x_{m1})\right) k_{z1}$$

The dispersion relation is given by (solve for k_z $\rightarrow$ complex)

$$\boxed{\left[(1+j\eta-\lambda x)^2-\zeta^2x^2-\phi\sigma_{\pm}\,x\,\tilde{\psi}(x_{m1})+\eta^2\right](1-x^2)-2\phi\,\tilde{C}_{2DEG}^2\,x^2=0}$$

where,

$$\phi=\frac{q\sigma_0}{m\omega V_{ph}}\ ;\quad \tilde{C}_{2DEG}^2=\omega Z_{ckt}W_{beam}\ ;\quad \eta=\frac{1}{2}\frac{\nu}{\omega}\ ;\quad \zeta^2=\frac{1}{2}\frac{V_{th}^2}{V_{ph}^2}$$

$$\tilde{\psi}(x_{m1})=\frac{-2\epsilon_2+(\epsilon_1+\epsilon_2)e^{2\sigma_{\pm}k_z x_{m1}}+(\epsilon_1-\epsilon_2)e^{-2\sigma_{\pm}k_z x_{m1}}}{2(\epsilon_1-\epsilon_2)(\epsilon_1+\epsilon_2)+(\epsilon_1-\epsilon_2)^2e^{-2\sigma_{\pm}k_z x_{m1}}+(\epsilon_1+\epsilon_2)^2e^{2\sigma_{\pm}k_z x_{m1}}}$$

$$x=\frac{k_z V_{ph}}{\omega}\qquad \lambda=\frac{V_{beam}}{V_{ph}}$$

$V_{beam}=1e4$ (J/m)
V_0, n_e,

$\sigma_0 \rightarrow$ dc surface charge density (C/m²) $\rightarrow q \times n_e \times t$ (t: 1-7nm)

$m \rightarrow$ effective mass of electrons in Si ; $q \rightarrow \bar{e}$ charge

$\epsilon_1 \rightarrow$ permittivity for $x>0$; $\epsilon_2 \rightarrow$ permittivity for $x<0$

V_{ph} (complex) $\rightarrow$ phase velocity of SWS at ω

$Z_{ckt} \rightarrow$ pierce impedance

$W_{beam} \rightarrow$ width of 2DEG

$\nu \rightarrow$ collision frequency $\rightarrow$ 1.58e12 Hz

$V_{th} \rightarrow$ thermal velocity ($V_{th}=\sqrt{\frac{3kT}{m^*}}$)

$V_{beam} \rightarrow$ drift velocity of $\bar{e}$ in channel. (sweep)

$n_{2Deff} \rightarrow 1.32e17$ (1/m²) $\boxed{\sigma_0=q\,n_{2Deff}}$.

$$\tau_m=\frac{\mu\times m^*}{q}$$

$\tilde{\sigma}_0 = n_e \times q \times \mu$ (conductivity)

$$\nu=\frac{1}{\tau_m}$$

$$\left.\begin{aligned} m^* &= 1.18\,m_0\\ n_e &= 5e^{24}\ m^{-3}\\ \mu &= 0.14\end{aligned}\right\}\ \text{Unknowns}$$

x_{m1} = 550nm. | W = 35μm ·

We know $k_z x_{m1} << 1$ & we use the Taylor expansion of $\psi(x_{m1})$ which is easily shown to be:

$$\tilde{\psi}(x_{m1}) = \frac{1}{2\epsilon_1} - \frac{\epsilon_2}{2\epsilon_1} \cdot \frac{1}{\epsilon_1 + 2\epsilon_2 \sigma \pm k_z x_{m1}}$$

This $\tilde{\psi} \rightarrow \frac{1}{2\epsilon_1}$ as $x_{m1} \rightarrow \infty$ whereas the exact $\psi \rightarrow \frac{1}{\epsilon_1 + \epsilon_2}$ when $x_{m1} \rightarrow \infty$. Which is expected since the Taylor expansion assumes x_{m1} is small & is not valid for $x_{m1} \rightarrow \infty$.

> ν_c, x_{m1}, σ_0 can be found by comparing HFSS & analytic model (when $V_{beam} = 0$). Use equivalent ϵ & σ for a plasma.