Nonlinear Dynamics and Chaos

Physics 413 and Youtube course by Strogatz. MAE 5790

MAE 5790 Lec 1 & Phy413 Lec 1

Logical structure of dynamics.

Differential equations $\dot{x} = f(x)$

$x \in \mathbb{R}^n$ → Phase/State space.

$x = (x_1, \ldots, x_n)$

→ In components, $\dot{x}_1 = f_1(x_1, \ldots, x_n)$

$\vdots$

$\dot{x}_n = f_n(x_1, \ldots, x_n)$

$f_1, \ldots, f_n$ are given functions.

→ The system is linear if all x_i on RHS appear to first power only (no products, powers, or functions of the x_i).

→ Autonomous systems ⇒ RHS does not have time dependance ⇒ (no external forces)

Eg: Simple harmonic oscillator.

$$m\ddot{x} + kx = 0$$

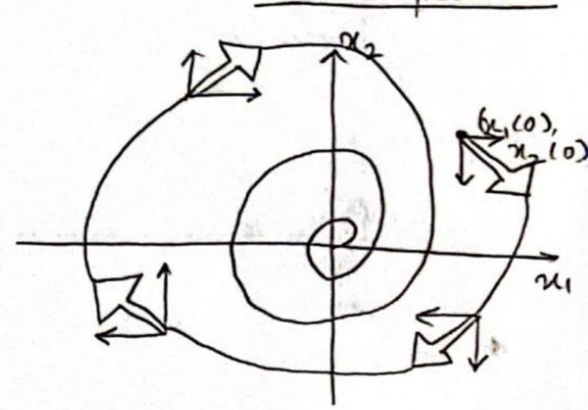
Let $\left.\begin{array}{l} x_1 = x \\ x_2 = \dot{x} \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m} x_1 \end{array}}$ } Linear second order system.

Eg: Damped harmonic oscillator.

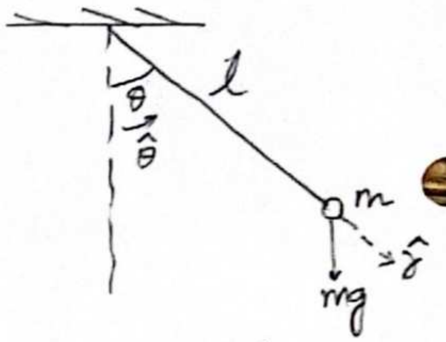
$$a = -\frac{k}{m} x - \frac{b}{m} \dot{x}$$

$\left.\begin{array}{l} x_1 = x \\ x_2 = \dot{x} \end{array}\right\} \quad \boxed{\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m} x_1 - \frac{b}{m} x_2 \end{array}}$

Phase portrait.



Eg:- Pendulum



Angular momentum

$= h = r \times mv = l\hat{r} \times ml\dot{\theta}.\hat{\theta}$

$= ml^2\dot{\theta}\hat{k}$

$\vec{v} = l\dot{\theta}\hat{\theta}$

linear velocity

angular velocity

Torque $= -mgl\sin\theta\ \hat{k}$

$\Rightarrow \quad ml^2\ddot{\theta} = -mgl\sin\theta$

$$\left.\begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} \end{aligned}\right\} \quad \left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 \end{aligned}\right\} \xrightarrow{\text{Small angles}} \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}x_1 \end{aligned}$$

↳ nonlinear, 2^{nd} order.

<u>Including time</u> $\Rightarrow$ add new state $\dot{x} = 1 \Rightarrow x = t$.

→ An external time dependent force is added.

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 + F\cos(\omega x_3) \\ \dot{x}_3 &= 1 \end{aligned}\right\}$$

Nonlinear Autonomous ← (was made into)

<u>Discrete case of dynamical systems.</u>

$$x_1(i+1) = f_1(x_1(i), \ldots, x_n(i))$$
$$\vdots$$
$$x_n(i+1) = f_n(x_1(i), \ldots, x_n(i)).$$

→ Phase portraits are useful in visualizing the behaviour without finding analytical solutions to the given ODEs.

n = no of equations/states.

	$n=1$	$n=2$	$n=3$	$n \gtrsim 3$	$n \gg 1$	"Continuum" $n=\infty$
Linear	RC circuit	simple harmonic oscillator				Wave equation, Maxwell's equation, Schrodinger's eqn
This course → Nonlinear	logistic growth, skydiver dynamics	pendulum	iterated maps. Lorenz (Chaos) fractals.		complex	Systems. Turbulence, general relitivity, fibrillation in heart

$n=\infty \Rightarrow$ need a continuum of information to know how the next state would look.

$\downarrow$ All PDEs.

Maxwell's $\Rightarrow$ Need to know the fields at every point to know the fields at the next instance.

<u>Chapter 2</u> $\dot{x} = f(x), \quad x \in \mathbb{R}$

<u>1D systems:</u>

<u>Eg:</u> $\dot{x} = \sin x \longrightarrow$ Traditional sol: $\displaystyle\int \frac{dx}{\sin x} = \int dt = t + c.$

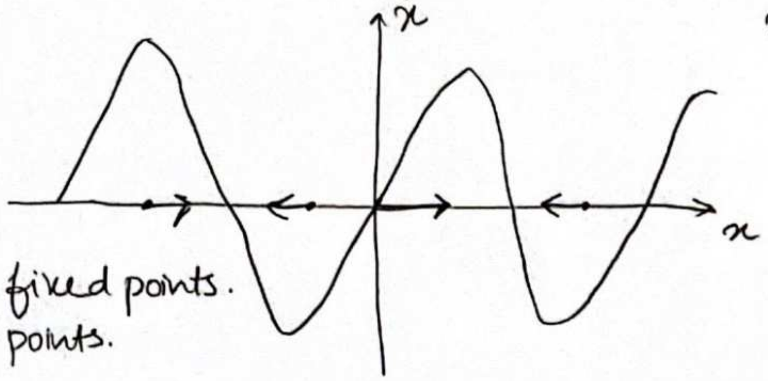
$$\int \operatorname{cosec} x \, dx = -\ln\left|\csc x + \cot x\right|$$

Say $x = x_0$ at $t=0 \Rightarrow t = \ln\left|\dfrac{\operatorname{cosec} x_0 + \cot x_0}{\operatorname{cosec} x + \cot x}\right|$

$\hookrightarrow$ inverting this is painful.

$\rightarrow$ If $x_0 = \frac{\pi}{4}$, what is $\lim\limits_{t\to\infty} x(t)$?

<u>Instead use Phase Portrait.</u>



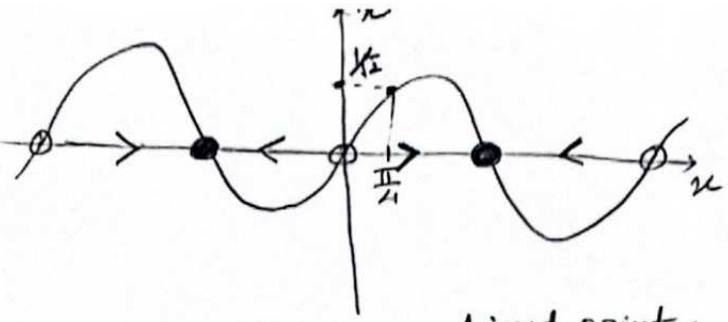
x = position of an imaginary particle

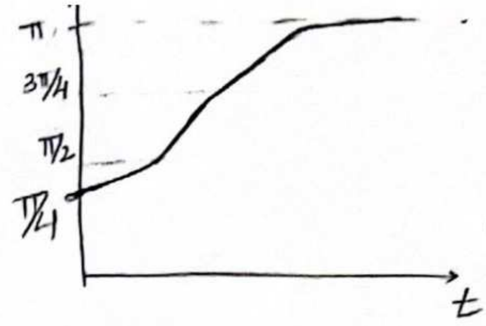
$\dot{x}$ = velocity.

$\Rightarrow \dot{x} = \sin x$ is a velocity vector field on x axis.

When $\dot{x}=0$ at fixed points.

x^* are fixed points.





$\Rightarrow \lim_{t \to \infty} x(t) = \pi$ ↗ fixed point.

→ stable x^* are ● fixed points.

→ unstable x^* are o fixed points.

Eg

Logistic equation in population biology.

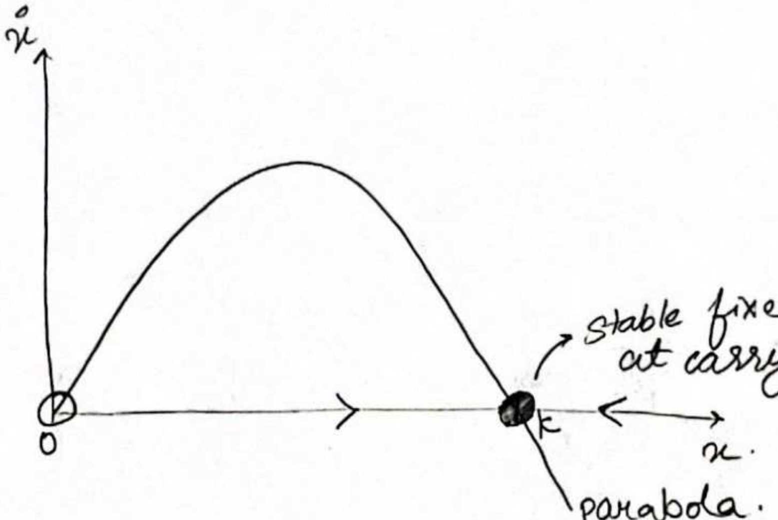
$$\dot{x} = r x \left(1 - \frac{x}{k}\right) \qquad r > 0, \; k > 0$$

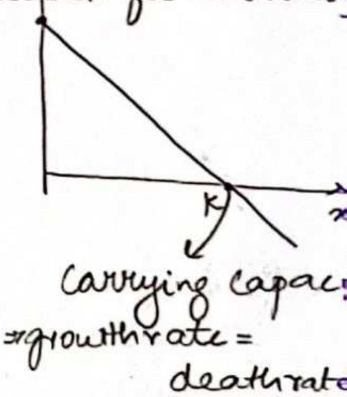
x = population size.

$\dot{x}$ = growth rate

$\frac{\dot{x}}{x}$ = per capita growth rate

logistically, per capita growth rate follows a straight line ⇒ simplest model that accounts for overcrowding





If $x_0 > 0$, $x(t) \longrightarrow k$ as $t \longrightarrow \infty$.

Book sec 2.4 **Linearization.**

- Examine dynamics close to a fixed point x^* quantitatively.

Let $x(t) = x^* + \eta(t)$
↳ Eta, deviation from x^* & $|\eta| \ll 1$.

$$\dot{x} = (\dot{x}^* + \dot{\eta}) = \dot{\eta} = f(x) = f(x^* + \eta)$$
($\dot{x}^* = 0$)

⇓ use Taylor's formula around x^*

$$f(x) = f(x^*+\eta) \approx f(x^*) + \eta \underbrace{f'(x^*)}_{\text{Slope of } f \text{ at the fixed pt.}} + \underbrace{\frac{\eta^2}{2} f''(x^*) + \cdots}_{\text{neglect all these terms.}}$$

($f(x^*) = 0$)

Can only neglect if $f'(x^*)$ is $\neq 0$.

If $f'(x^*) \neq 0$,
$$\Rightarrow \boxed{f(x) = \eta f'(x^*).}$$ → linearized.

- Neglecting terms of order $O(\eta^2)$ yields linearization of system at x^*:

$$\boxed{\dot{\eta} = r\eta} \quad \text{where } r = f'(x^*) \quad \text{since } f(x) = \dot{x}$$

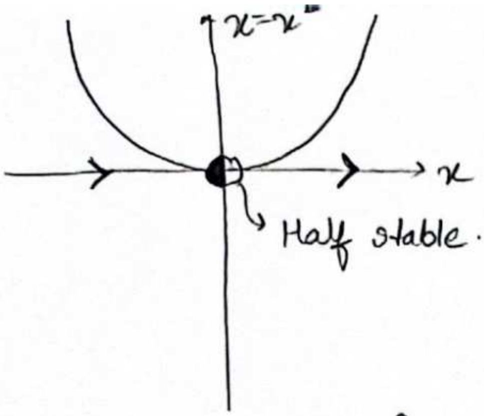
⇒ Exponential growth ⇒ $\eta = \eta_0 e^{rt}$ if $r > 0$.
   " decay if $r < 0$

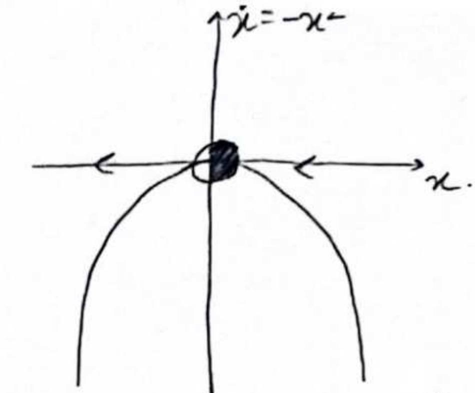
⇒ positive slope at fixed point ⇒ exponential growth!

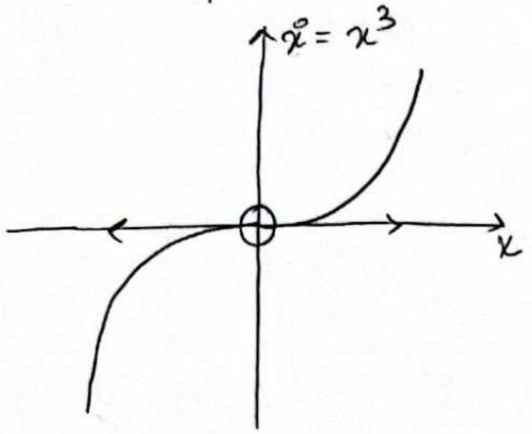
→ If $f'(x^*) = 0$, no information from linearization. ⇒ cannot talk about stability.

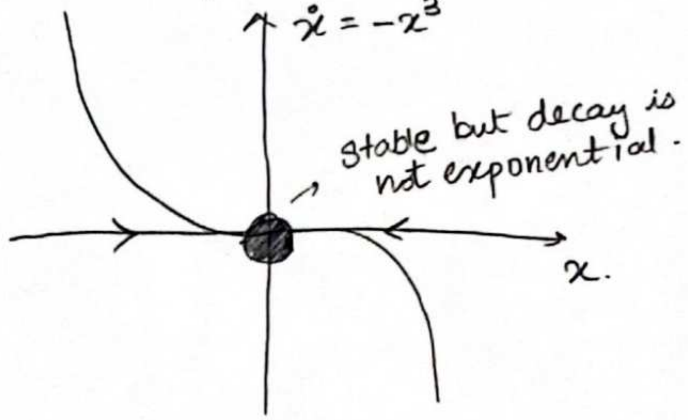
- Eg: $\dot{x} = x^2$, $\dot{x} = -x^2$, $\dot{x} = x^3$, $\dot{x} = -x^3$

$x^* = 0$ is a fixed point for all these. & $f'(x^*) = 0$

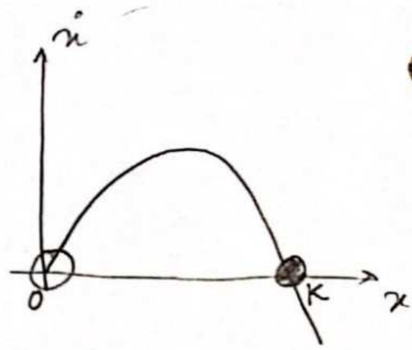








Eg: Logistic equation. using linearization.



$$\dot{x} = rx\left(1 - \frac{x}{K}\right)$$

$\dot{x} = 0$ when $x^* = 0$ or $x^* = K$.

$$f'(x) = r - \frac{2rx}{K}$$

since r is growth rate.

$f'(0) = r > 0 \Rightarrow x^* = 0$ unstable.

$f'(K) = r - 2r = -r < 0 \Rightarrow x^* = K$ stable.

→ The value of $|f'(x^*)|$ gives a quantitative value for magnitude of how fast the value of x changes.

$\Rightarrow \dfrac{1}{|f'(x^*)|}$ is a characteristic time scale.

↳ If this is small, the time required for $x(t)$ is vary signifacantly is small. Like time constant

→ Existance and Uniqueness.

» Solutions to $\dot{x} = f(x)$ do exist and they are unique if $f(x)$ and $f'(x)$ are continuous.

(f is "continuously differentiable").

→ Impossibility of oscillations.

What is the possible behaviour of $x(t)$ as $t \to \infty$, for $\dot{x} = f(x)$?

i) $x(t) \to \pm\infty$ as $t \to \infty$

or ii) $x(t) \to x^*$

⇒ Oscillations are not possible. Not even damped oscillations are possible.

But why?

> All trajectories $x(t)$ increase or decrease monotonically or stay fixed. Since $f(x) = \dot{x}$ is well defined at each point it cannot change value & therefore the trajectory is monotonic.

Ch 3 Bifurcations (in content of 1-D)

> As a parameter changes, qualitative structure of the vector field may change dramatically.

Eg:- A fixed point may be created or destroyed or they may change their stability.

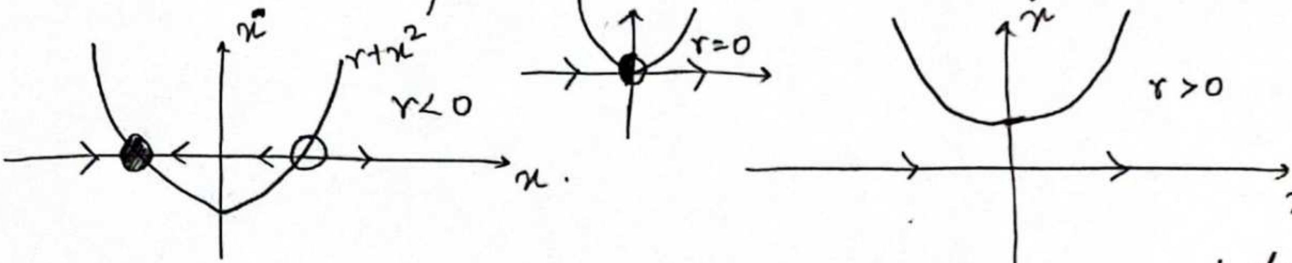
> Bifurcation point or value: Value of the parameter at which the change occurs.

Eg ①

Saddle-node Bifurcation.

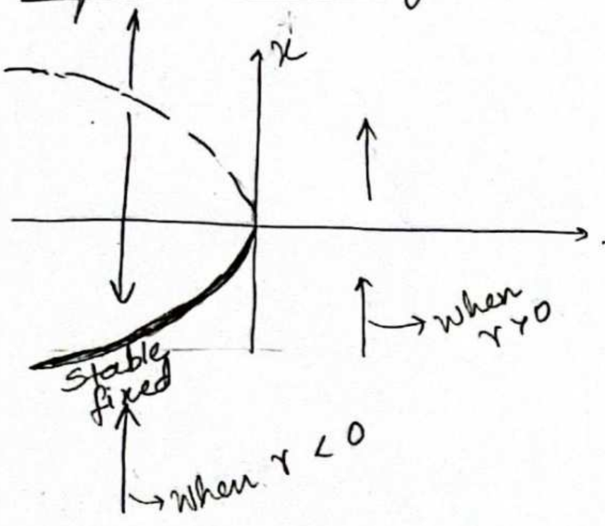
> Basic mechanism for creation/destruction of fixed points.

> Standard example: $\dot{x} = r + x^2$ $r =$ control parameter



→ Half stable fixed points arise at the point of bifurcations.

Bifurcation diagram (x^* vs. r)



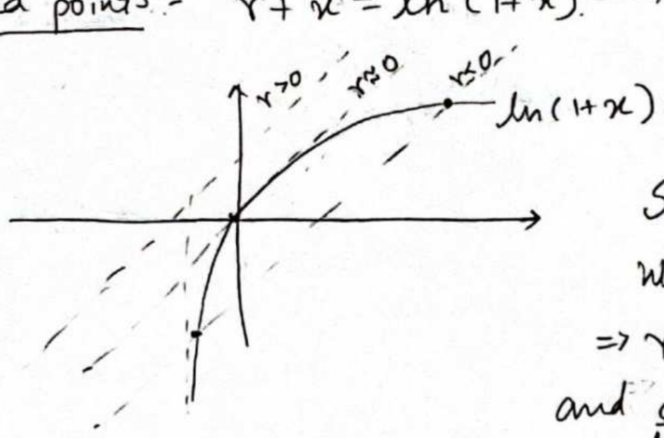
$\dot{x} = 0 \Rightarrow r + (x^*)^2 = 0$

$x^* = \pm\sqrt{-r}$.

↳ AKA fold or turning point bifurcation.

Eg ② : $\dot{x} = r + x - \ln(1+x)$

Fixed points :- $r + x^* = \ln(1+x^*)$ ⟶ can't solve for $x^*(r)$. Use graphical method



⇒ plot $y = r + x$
$y = \ln(1+x)$

Saddle node bifurcation occurs.
when we have a tangential intersection.

$\Rightarrow r + x = \ln(1+x)$

and $\frac{d}{dx}(r+x) = \frac{d}{dx}(\ln(1+x))$

$1 = \frac{1}{1+x} \Rightarrow x^* = 0$ at saddle node bfn.

and then $r + x = \ln(1+x) \Big|_{x=0}$.

$\Rightarrow r = \ln 1 = 0$

$\Rightarrow \boxed{r_c = 0} \longrightarrow$ Critical or bifurcation value.

> Near the bifurcation at $(x, r) = (0, 0)$,

$$\dot{x} = r + x - \ln(1+x)$$
$$\approx r + x - x + \frac{x^2}{2} + \cancel{O(x^3)}^{\,0}$$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots$ when $|x| < 1$

$$\dot{x} \approx \underbrace{r + x^2/2}$$

Since we get $r + x^2$ form again we call this the "normal form".
→ In many cases this form shows up.

Transcritical Bifurcation

Normal form: $\dot{x} = rx - x^2 = x(r - x) \Rightarrow \left.\begin{matrix} x^* = 0 \\ x^* = r \end{matrix}\right\}$ 2 fixed pts.

→ The fixed pt $x^* = 0$ is independent of r and cannot be destroyed. But it's stability can be changed.

Linearization: $f'(x) = \frac{d}{dx}(rx - x^2) = r - 2x$ ↳ slope $-ve \Rightarrow$ stable.

$f'(0) = r$.

$x^* = 0 \begin{cases} \text{stable } r < 0 \\ \text{unstable } r > 0 \end{cases}$

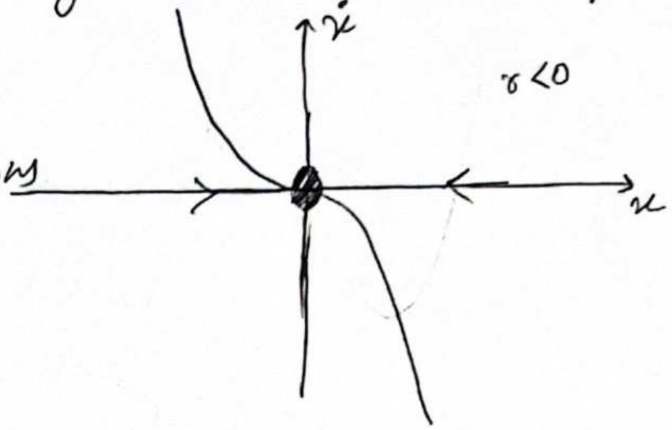
$f'(r) = r - 2r$
$= -r$

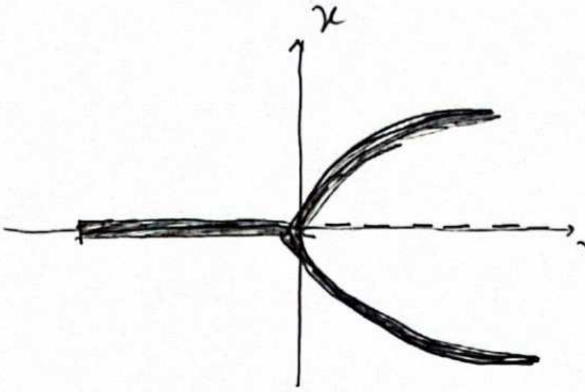
AKA Exchange of stabilities.

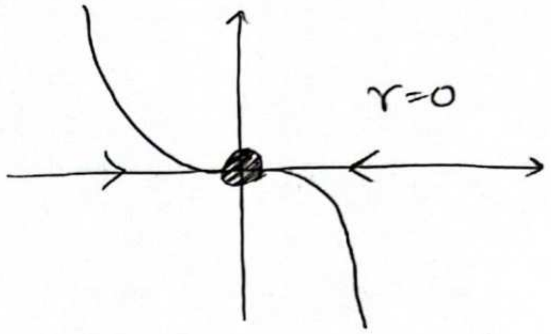
Pitchfork bifurcation – Occurs in systems with symmetry.

$\dot{x} = rx - x^3$ → symmetry b/w x & $-x$ ⇒ mirror symmetry.

AKA supercritical pitchfork.
↳ Bifurcating solutions are stable.

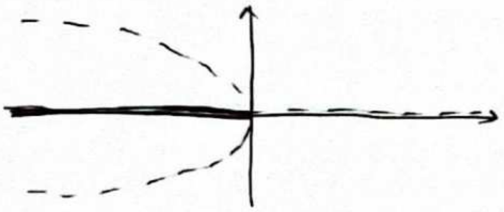


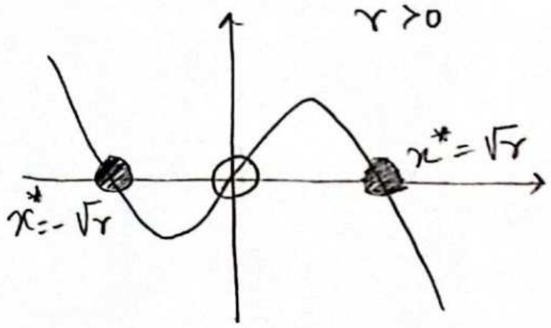




Subcritical

$\dot{x} = rx + x^3$





Normal form of bifurcations.

Saddle node: $\dot{x} = r \pm x^2$

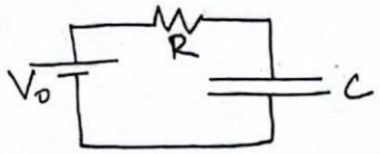
Transcritical: $\dot{x} = rx \pm x^2$ Eg: 3.2.2

Pitchfork: $\dot{x} = rx - x^3$ → supercritical.
$\dot{x} = rx + x^3$ → subcritical.

→ Be careful when $f(x,r)$ is a discontinuous function of r.

Phy 413 Lec 2

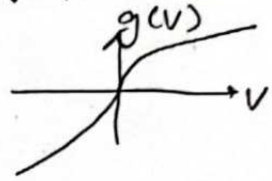
Q



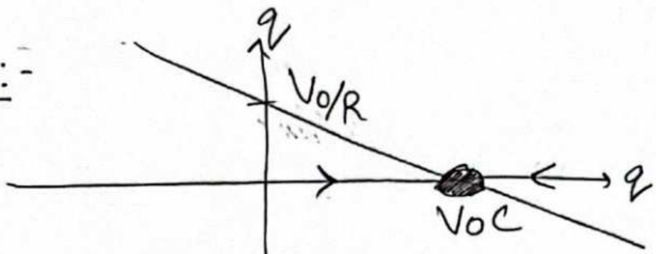
$$\boxed{\dot{q} = \frac{V_0}{R} - \frac{q}{RC}}$$

$V_0 = V_R + V_C$

> Plot $\dot{q}$ as a fn of q
> Fixed points?
> $q(t) = ?$
> Solve ODE formula.
> Replace $V = IR$ with $I = g(V)$
where



Sol:-



Overtime as $t \to \infty$ q reaches V_0C.

$$V_0 = \dot{q}R + q/C$$

$$\frac{dq}{dt} = \frac{V_0}{R} - \frac{q}{RC}$$

$$-R\frac{dq}{dt} = \frac{q}{C} - V_0$$

$$\frac{dq}{q - CV_0} = -\frac{1}{RC}dt.$$

$$\Rightarrow \ln|q - V_0C| - K = \frac{-t}{RC}.$$

$$t = 0 \Rightarrow K = \ln|q_0 - V_0C|$$

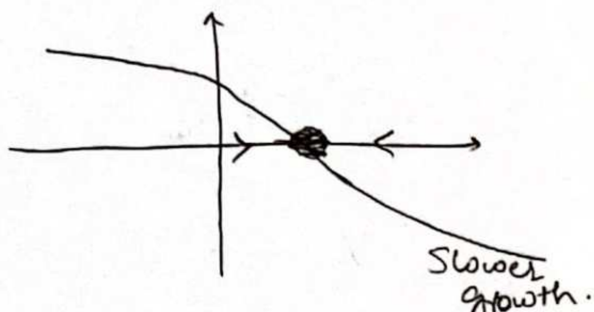
$$\Rightarrow \left|\frac{q - V_0C}{q_0 - V_0C}\right| = e^{-t/RC} \Rightarrow \boxed{q = V_0C + (q_0 - V_0C)e^{-t/RC}}$$

Nonlinear case.

$$I = g(V) \Rightarrow V = g^{-1}(\dot{q})$$

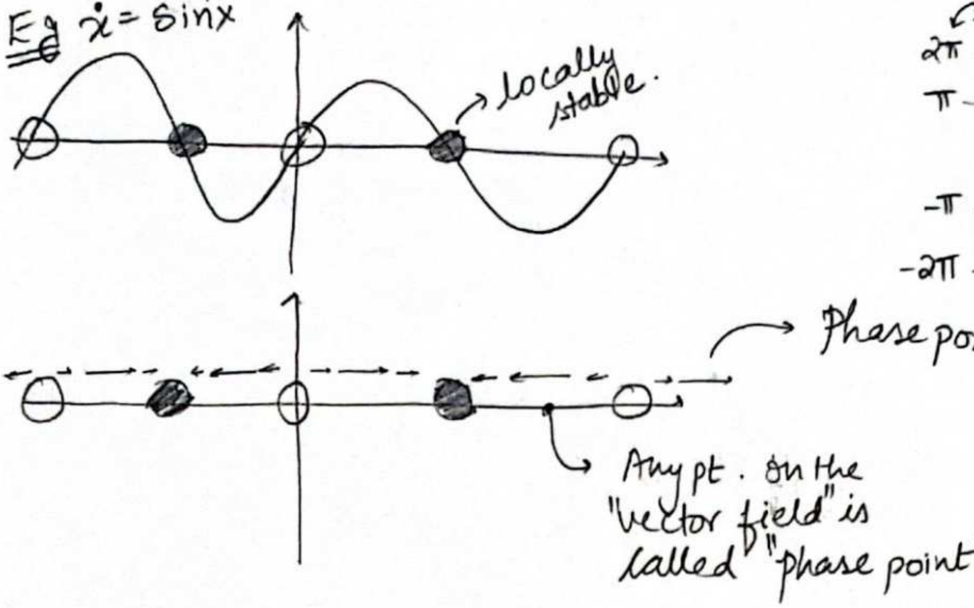
$$\Rightarrow V_0 = g^{-1}(\dot{q}) + \frac{q}{C}$$

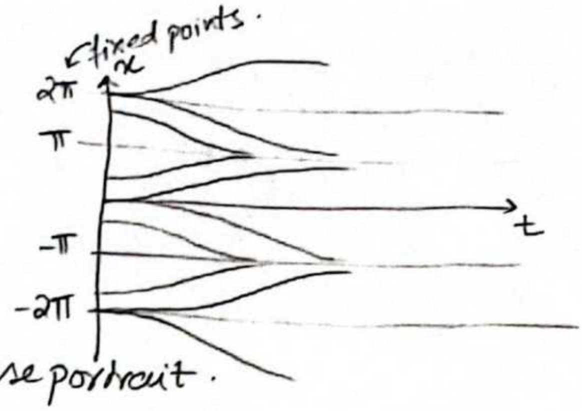
$$\Rightarrow I = \dot{q} = g\left(V_0 - \frac{q}{C}\right).$$



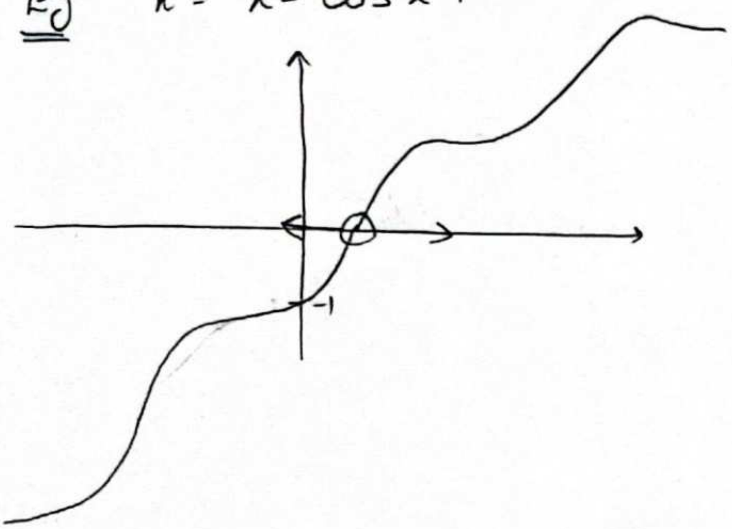
Phase portrait

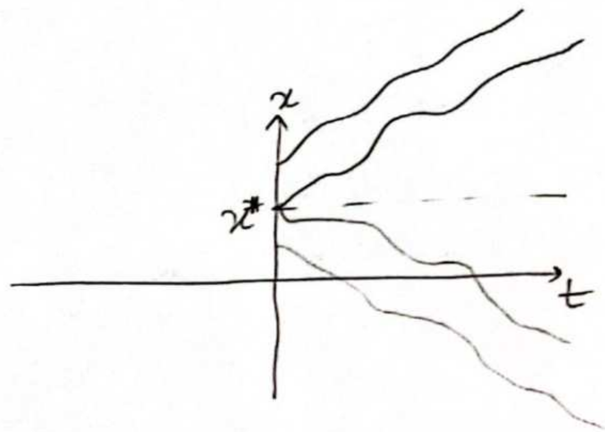
Eg $\dot{x} = \sin x$





Eg $\dot{x} = x - \cos x$.





Lec 3 - Phy 413

Existence and Uniqueness Theorem.

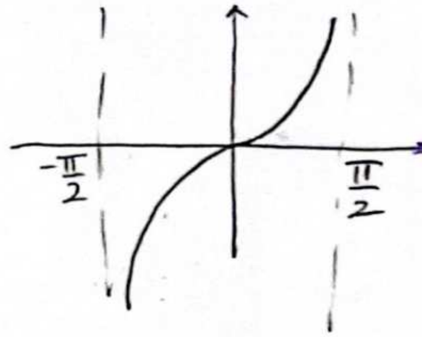
If $f(x)$ and $f'(x)$ are continuous on an interval R and $x_0 \in R$ and we have $\dot{x} = f(x)$ and $x(0) = x_0$. A solution exists and is unique for t in $(-\tau, \tau)$ for $\tau > 0$.

Eg: $\dot{x} = 1+x^2$, $f(x) = 1+x^2$, $f'(x) = 2x \Rightarrow$ continuous.

$R = (-\infty, \infty)$

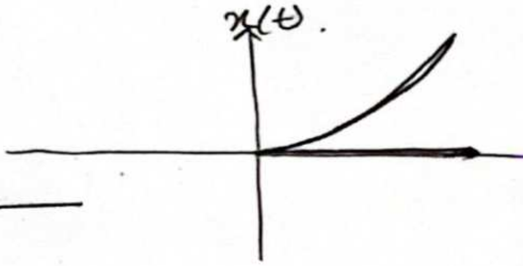
$\int \frac{dx}{1+x^2} = \int dt \Rightarrow x = \tan t$ if $x_0 = 0$.

$\Rightarrow \tau = \frac{\pi}{2}$

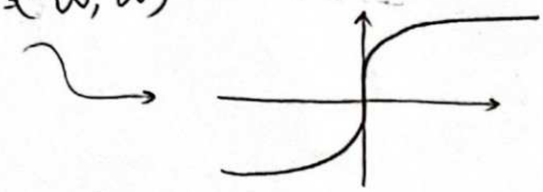


Note ★ → R is defined on x axis. τ on t axis.

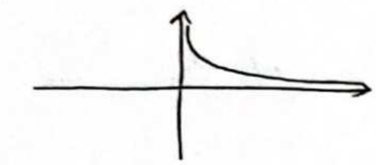
Eg:- $\dot{x} = x^{1/3} \Rightarrow x = \left[\frac{2}{3}t\right]^{3/2}$ but notice $x(t) = 0$ is also a solution.



Why? $f(x) = x^{1/3}$ is continuous for $R = (-\infty, \infty)$



$f'(x) = \frac{1}{3} \cdot x^{-2/3} = \frac{1}{3x^{2/3}}$



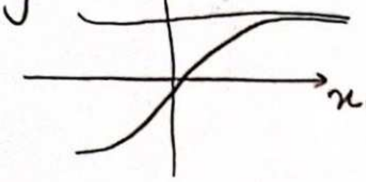
Discontinuous at $x(t) = 0$

Therefore, R cannot include 0. $\Rightarrow x(t) = 0$ is not a solution when $R \in (0, \infty)$.

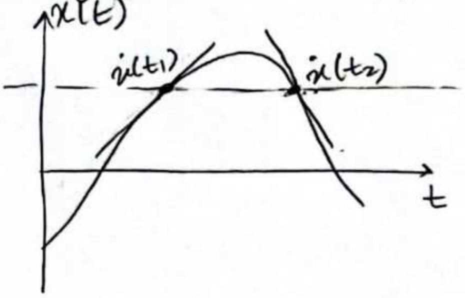
Corollary

① Trajectories cannot cross. Since the phase pt. will not have a unique path as $t \to \infty$

② Trajectories cannot merge in time and space since that would need an ∞ slope $\Rightarrow$ discontinuous. They can only reach the fixed points asymptotically.



Proof of monotonicity.



Assume not monotonic,

$f(x(t_1))$ is +ve.

$f(x(t_2))$ is -ve.

But $x(t_1) = x(t_2)$. $\Rightarrow f(x_1) = f(x_2) = 0$

$\dot{x}(t_1) > 0$ $\Rightarrow$ This can only occur at fixed points.

$\dot{x}(t_2) < 0$

This works because of uniqueness, since $f(x)$ can only have one value for each x.

→ *Linearization.*

$$f(x) = f(x^*) + \frac{f'(x^*)}{1!}(x - x^*) + O((x-x^*)^2)$$

If $|f'(x^*)| > 0$ then as $x \to x^*$ $O((x-x^*)) \to 0$

$\Rightarrow f'(x^*) > 0 \Rightarrow$ exp. growth.

$f'(x^*) < 0 \Rightarrow$ exp. decay.

$\frac{1}{|f'(x^*)|}$ → Characteristic time scale $\Rightarrow$ ↑ or ↓ by e after on timescale.

$$\text{Half life} = \lambda = \frac{\ln 2}{|f'(x^*)|}$$

Potentials.

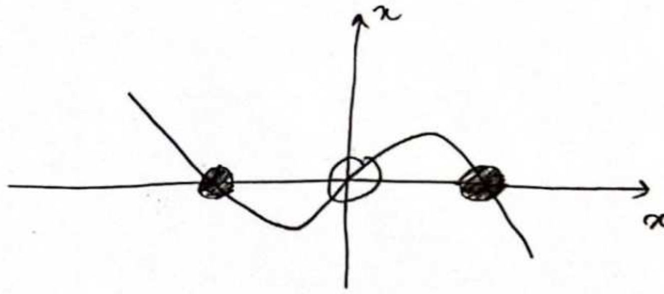
In math $f(x) = -\frac{dV}{dx}$. (force)

$\Rightarrow f(x) = \dot{x} = -\frac{dV}{dx}$. $\rightarrow$ solve for a V, that gives a visual representation of fixed points.

Maxima in $V \Rightarrow$ Unstable x^*

Minima in $V \Rightarrow$ Stable x^*

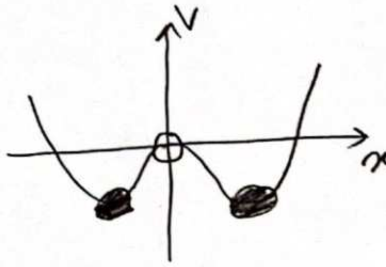
Eg:- $\dot{x} = x - x^3$



$$\frac{dV}{dx} = x^3 - x.$$

$$V = \int x^3 dx - \int x\, dx.$$

$$= \frac{x^4}{4} - \frac{x^2}{2} + c \quad \text{let } c = 0$$



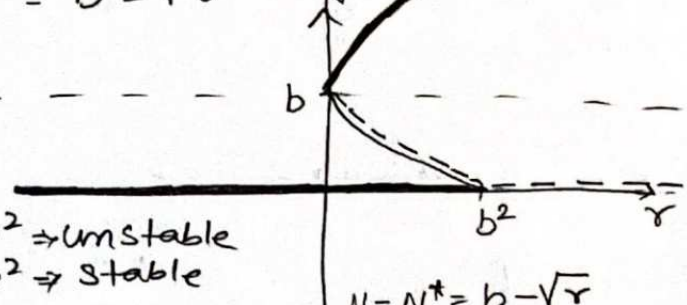
Phy 413 Lec 4 $\rightarrow$ Bifurcations.

Population growth example for bifurcations.

$\frac{\dot{N}}{N} = r - (N-b)^2 \Rightarrow \dot{N} = N(r - (N-b)^2) \Rightarrow N^*(r - (N^*-b)^2) = 0$

$\Rightarrow N^* = 0$ and $(N^*-b)^2 = r \Rightarrow N^* = b \pm \sqrt{r}$

$\frac{d}{dN}\dot{N} = r - b^2 + 4Nb - 3N^2$



for $N = N^* = 0$ $\quad \frac{d}{dN}\dot{N} = r - b^2 \Rightarrow r > b^2 \Rightarrow$ unstable
$r < b^2 \Rightarrow$ stable

for $N = N^* = b + \sqrt{r} \Rightarrow \frac{d}{dN}\dot{N} = -2r - 2b\sqrt{r} \Rightarrow r > 0 \Rightarrow$ stable.

$N = N^* = b - \sqrt{r}$

$\Rightarrow \frac{d}{dN}\dot{N} = -2r + 2b\sqrt{r}$

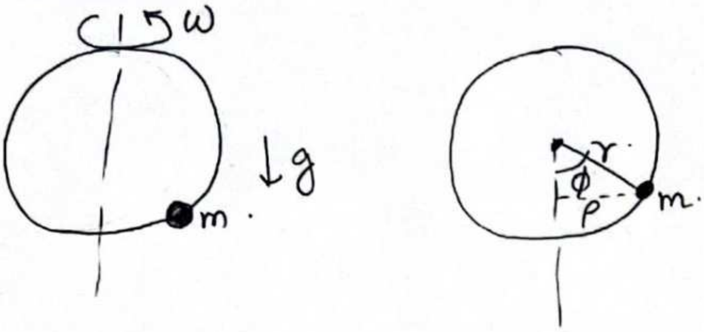
MAE 5790 - Lec 3 <u>Overdamped bead on a rotating hoop.</u>

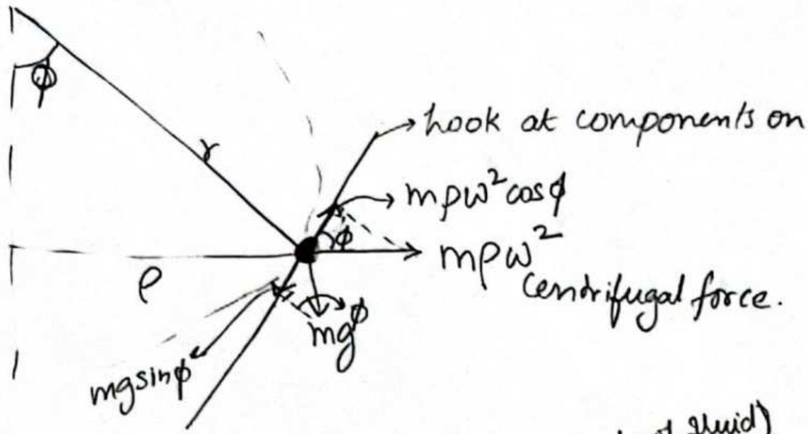
<u>Sec 3.5</u>

<u>Newton's Law.</u>

$F = ma$

do the problem in a frame co-rotating with the hoop.



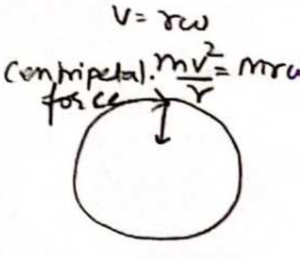


- To model friction imagine the bead is rotating in honey at same ω. So it cannot slide up & down easily.

$\rho = r\sin\phi$

$$\Rightarrow \underbrace{mr\ddot{\phi}}_{ma} = \underbrace{-br\dot{\phi}}_{\text{viscous force}} - mg\sin\phi + mr\omega^2\cos\phi\sin\phi$$

constant {property of fluid & bead only.} ; $r\dot{\phi}$: rotational velocity / Translational velocity



- Coriolis force is along r so it does not have any effect

$$mr\ddot{\phi} = -br\dot{\phi} - mg\sin\phi + mr\omega^2\cos\phi\sin\phi.$$

- Start by ignoring $mr\ddot{\phi}$ term. (Discussed later)
- The dynamics would be governed by $br\dot{\phi} = mg\sin\phi\left[\frac{r\omega^2}{g}\cos\phi - \right.$

$$\Rightarrow \quad br\dot{\phi} = mg\sin\phi\left[\frac{r\omega^2}{g}\cos\phi - 1\right]$$

Fixed points $\Rightarrow \sin\phi = 0$ or when $\cos\phi = \frac{g}{r\omega^2}$

$\sin\phi^* = 0 \Rightarrow \phi = 0$ or $\pi \rightarrow$ top
$\hookrightarrow$ bottom.

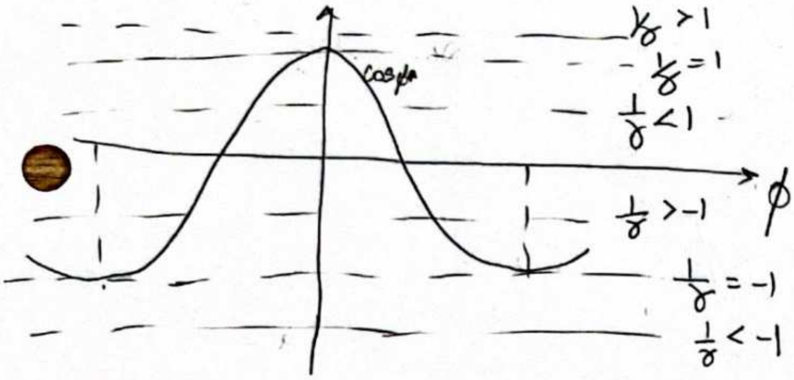
<u>Check:</u> $\phi^* = \pi \rightarrow$ unstable for all $m, g, r, \omega, b > 0$

$\phi^* = 0$ may or maynot be stable $\rightarrow$ Try it.

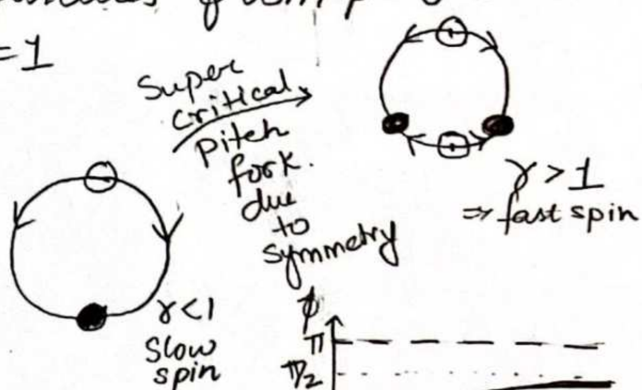
$\cos\phi^* = \dfrac{g}{r\omega^2}$ has solutions iff $g \leq r\omega^2 \Rightarrow$ hoop needs to be fast enough for hoop to leave the bottom.

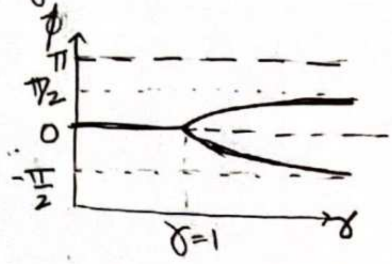
Let $\gamma = \dfrac{r\omega^2}{g}$ (dimensionless - no units).

Then $\cos\phi^* = \dfrac{1}{\gamma}$ must be solved.



A pair of fixed points $\pm\phi^*$ bifurcates from $\phi = 0$ when $\gamma = 1$





$\Rightarrow$ At high γ or ω, the 2 fixed points go to $\frac{\pi}{2}, -\frac{\pi}{2}$

> Supercritical pitchfork bifurcation at $\gamma = 1$.

$\rightarrow$ Going back to the second derivative term $mr\ddot{\phi}$?
When is $mr\ddot{\phi}$ negligible?

> $m \rightarrow 0$ is too crude. Other terms also go to 0.

> <u>Dimensional analysis</u> $\rightarrow$ To reduce the no. of parameters.

> Suppose T = timescale s.t. $\dot{\phi}$ is of the order $\frac{1}{T}$ and $\ddot{\phi} \approx \frac{1}{T^2}$. T is chosen later.

Better way:

Let $\tau =$ dimensionless time

$$\tau = \frac{t}{T}$$

Then $\frac{d\phi}{dt} = \frac{d\phi}{d\tau} \cdot \frac{d\tau}{dt} = \frac{1}{T}\frac{d\phi}{d\tau}$ → notice $\dot{\phi} \approx \frac{1}{T}$

→ Find T s.t. $\frac{d\phi}{d\tau} \xrightarrow[\text{order of 1}]{\text{on the}} 1$ as a parameter goes to 0 to $\pm\infty$.

→ $\frac{d^2\phi}{dt^2} = \frac{1}{T^2}\frac{d^2\phi}{d\tau^2}$. Let $\phi' = \frac{d\phi}{d\tau} \Rightarrow \phi'' = \frac{d^2\phi}{d\tau^2}$

$$\Rightarrow \quad \ddot{\phi} = \frac{1}{T^2}\phi''$$

⇒ Governing equation becomes:

$$\frac{mr}{T^2}\phi'' = -\frac{br}{T}\phi' + mg\sin\phi\,(\gamma\cos\phi - 1)$$

Make the equation dimensionless by dividing by mg.

Why dimensionless? We know when something is small if we compare it to 1.

$$\Rightarrow \underbrace{\left[\frac{r}{gT^2}\right]}_{\text{We want to neglect this term.}}\phi'' = \left[-\frac{br}{mgT}\right]\phi' + f(\phi)$$

$f(\phi)$ ⟶ $\sin\phi\,(\gamma\cos\phi - 1)$

→ ϕ'' & ϕ' are order $O(1)$. Even $f(\phi)$ is order $O(1)$ since γ does not change by changing the parameters (mass, viscocity).

> We want to choose T s.t. $\frac{b\gamma}{mgT} \sim O(1)$ but $\frac{\gamma}{gT^2} \simeq 1$

> Choose $\boxed{T = \frac{b\gamma}{mg}}$

> $\frac{\gamma}{gT^2} = \frac{\gamma}{g\left(\frac{b\gamma}{mg}\right)^2} = \frac{\cancel{\gamma}\, m^2 g^{\cancel{2}}}{\cancel{g}\, b^2 \gamma^{\cancel{2}}} = \boxed{\frac{m^2 g}{b^2\gamma} \ll 1}$

$\Rightarrow$ $\boxed{\boxed{m^2 \ll \frac{b^2\gamma}{g}.}}$ $\longrightarrow$ Only now we can ignore m.

Looking back: $\epsilon\phi'' = -\phi' + f(\phi)$.

$0 \Rightarrow \phi' = f(\phi) \Rightarrow \phi'$ is order 1 as was assumed.

→ What if ϕ'' is very large? Then $\epsilon \to 0$ does not mean $\epsilon\phi'' \to 0$

→ <u>A singular limit:</u> As $\epsilon = \frac{m^2 g}{b^2\gamma} \longrightarrow 0$ we lose the highest order derivative ($\epsilon\phi''$). Then we can't satify both the initial velocity and position. Even when $\epsilon \ll 1$, our approach is only valid <u>after a rapid initial transient.</u>

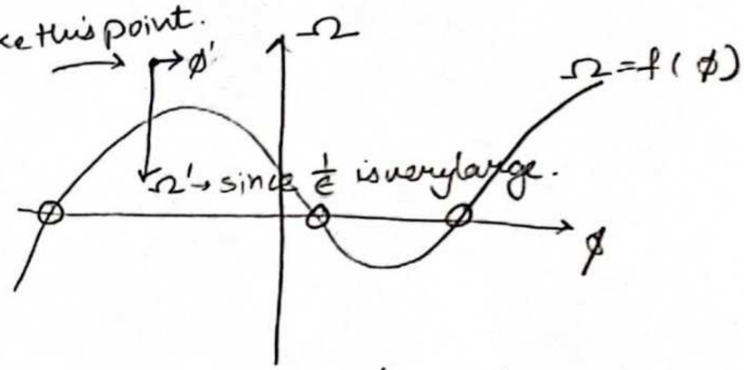
→ That transient time would require a different timescale. So our scaling (T) fails during the transient but correct after that.

→ $\epsilon\phi'' + \phi' = f(\phi)$

→ Let $\Omega = \phi'$; $\Omega' = \phi'' = \frac{1}{\epsilon}\cdot\left[f(\phi) - \Omega\right]$

$\phi' = \Omega$

$\Omega' = \frac{1}{\epsilon}\left[f(\phi) - \Omega\right]$



→ ∴ If the behaviour quickly relaxes to the curve $\Omega = f(\phi)$, we would be OK.

$\Omega = f(\phi) \Rightarrow \phi' = f(\phi)$

↳ this is the one dimensional curve we have been analysing.

→ Ω' is very large when $f(\phi)$ is very different from Ω. This is not true when $f(\phi)$ is close to Ω. When this happens the point stays very close to $f(\phi)$ line & moves along to the fixed points.

→ Therefore there is an initial "jump" at a fast time scale & then a "normal" behaviour which we have analysed.

MAE

Lec 4: Model of an insect outbreak Sec: 3.7

→ A bifurcation problem, involving 2 parameters.
→ Discontinuous jumps are observed.
→ Ludwig et al. (1978) Journal of Animal Ecology. 47, 315

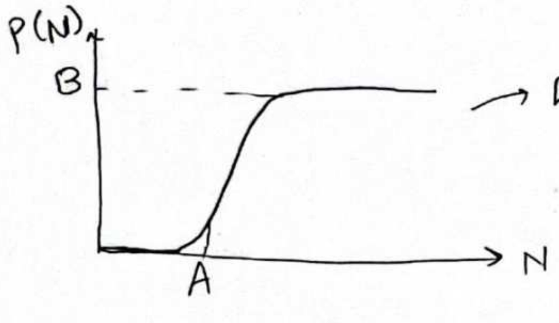
→ Spruce budworms pest. - Canadian timber problems and also in Maine (US).
Can destroy an entire forest in 4 years.

→ $N(t)$ = population of budworms.

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) \longrightarrow \text{logistic model.}$$

→ Including predation by birds.

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - p(N).$$



→ Empirical model of $P(N)$

$$P(N) = \frac{BN^2}{A^2 + N^2}$$

→ Four parameters: A, B, R, K all have units.
→ What are the dynamics of this system for various parameters?
→ Helps to non-dimensionalize ⇒ reduces from 4 to 2 dimensions.
→ Both A & K have the same dimensions as N. What should we use for nondimensionalizing?

Choose scale of N to make the nonlinear $p(N)$ term have no parameters (since it is more complex).

→ Let $x = \frac{N}{A}$. $N = Ax$.
$\dot{N} = A\dot{x}$

$$\therefore A\frac{dx}{dt} = RAx\left(1 - \frac{Ax}{K}\right) - \frac{B\cancel{A^2}x^2}{\cancel{A^2} + \cancel{A^2}x^2}$$

$$\Rightarrow \frac{A}{B}\frac{dx}{dt} = \frac{RA}{B}x\left(1 - \frac{Ax}{K}\right) - \frac{x^2}{1+x^2}$$

→ We want to introduce a dimensionless time since x is dimensionless.

Let $\tau = \frac{Bt}{A}$. This choice makes LHS dimensionless.

$\Rightarrow dt = \frac{A}{B}d\tau$ Let $x' = \frac{d}{d\tau}$

→ Notice $\frac{A}{K}$ is dimensionless. Let $\frac{A}{K} = \frac{1}{k}$

$\frac{RA}{B} = r$. These choices make it look like the logistic model.

$$\Rightarrow \boxed{x' = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}}$$ ⇒ We see now we have only 2 parameters.

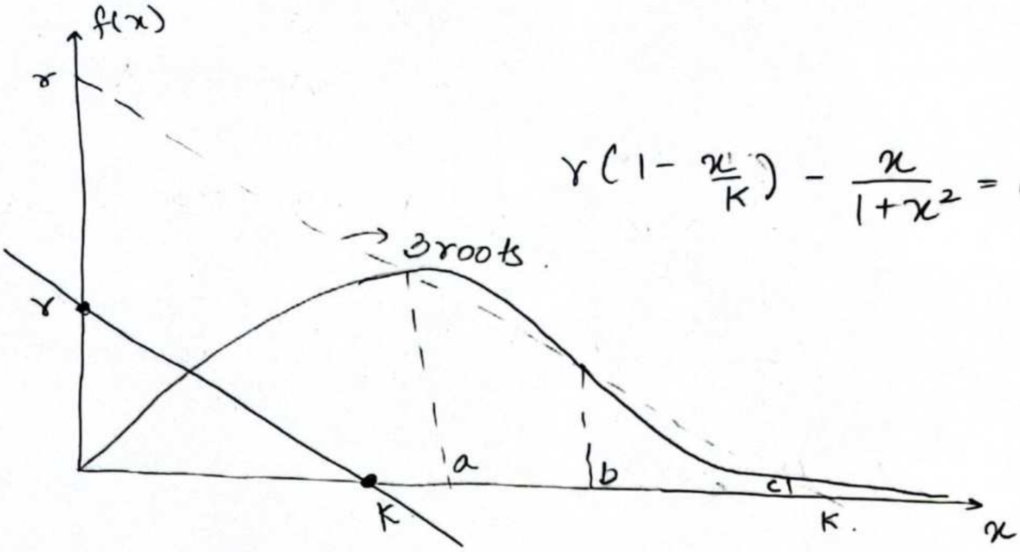
→ Scale models preserve all the dynamics of the system.
The behaviour of the birds has been included into the r & K.

<u>Fixed points.</u>

$x' = 0 \Rightarrow x^* = 0$; No budworms ⇒ no growth rate.

Other x^*'s satisfy $r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}$

Now you see why the parameters were lumped into the simpler equation on the left.

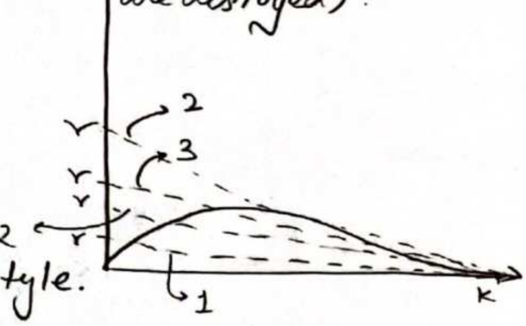


$$r\left(1-\frac{x}{K}\right) - \frac{x}{1+x^2} = 0$$

Steep line → one root
shallow line ⇒ 3 roots.

> We get one intersection for small K, for any r.

> When K is very large and r is small only one intersection.
> Now as r increases we see a bifurcation (saddle node)(b & c are created)
> Again as r increases we see another saddle node (b & a are destroyed).
> For very large r we see only c,

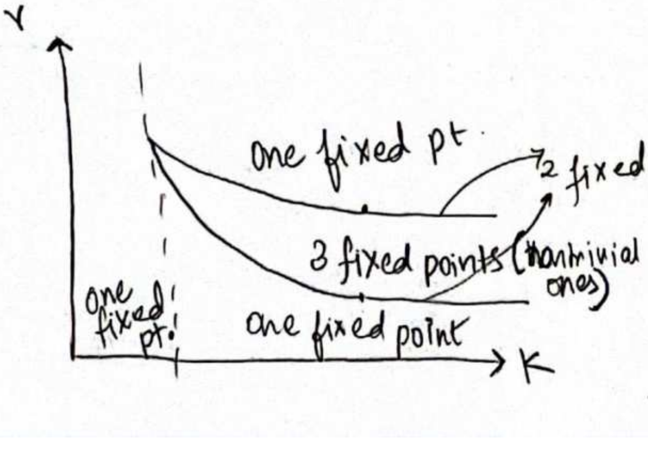


>

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 2 \longrightarrow 1$$

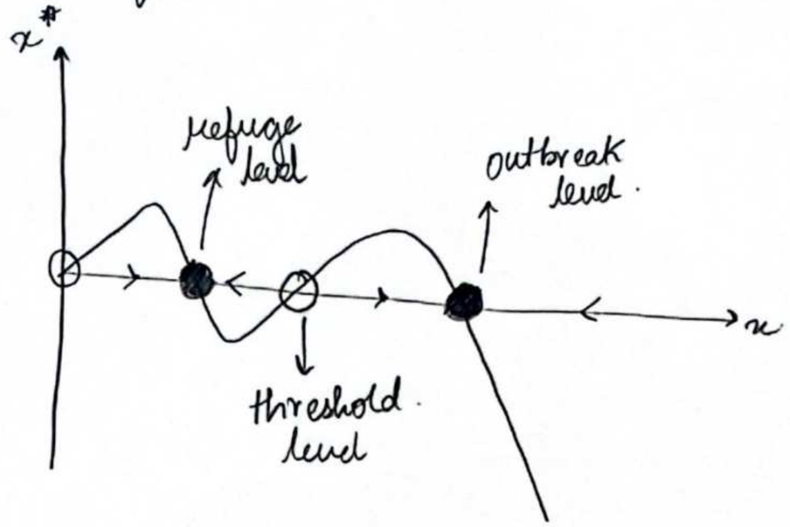
a	a bc	a b c	ab c	c

> b is colliding with a or c in saddle node style.



Bifurcations occur at certain (r, K) pairs.

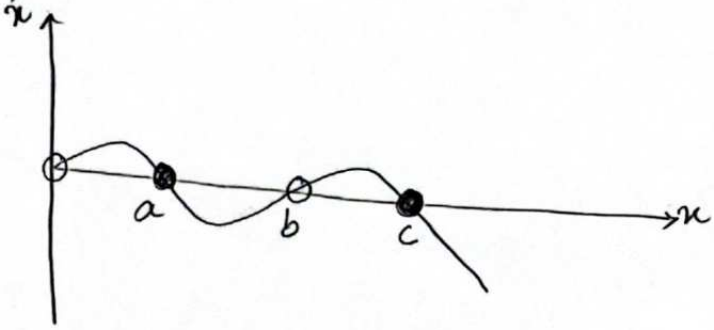
> Suppose 3 fixed points besides $x^* = 0$.



> When x is small
$x' \simeq rx$

> Assuming stability of other 3 fixed pts. alternate.
↳ Topologically this is always true.

> Can get jump phenomenon after a saddle node. Suppose x is at "a" and the parameters start to drift.

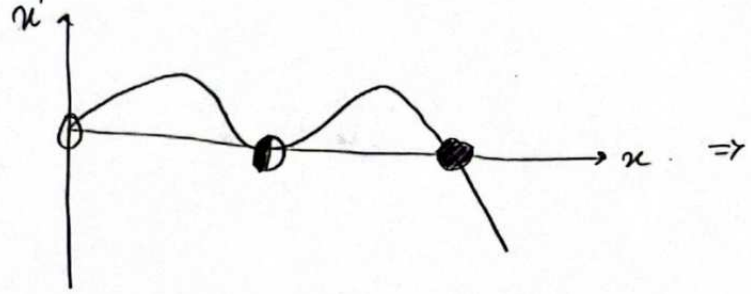


> As the forest ages, r increases slowly.

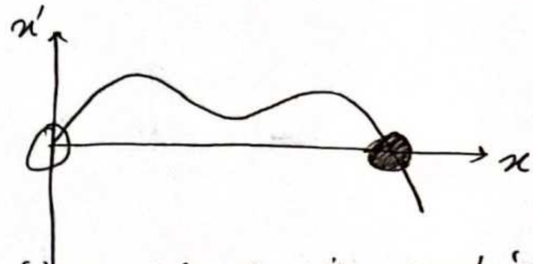
$r = \frac{RA}{B}$

$A \propto$ surface area of foliage
$= A'S \rightarrow$ critical density of budworms per leaf.
$S \rightarrow$ # of leaves.

> As r grows slows, a & b approach each other & annihilate.



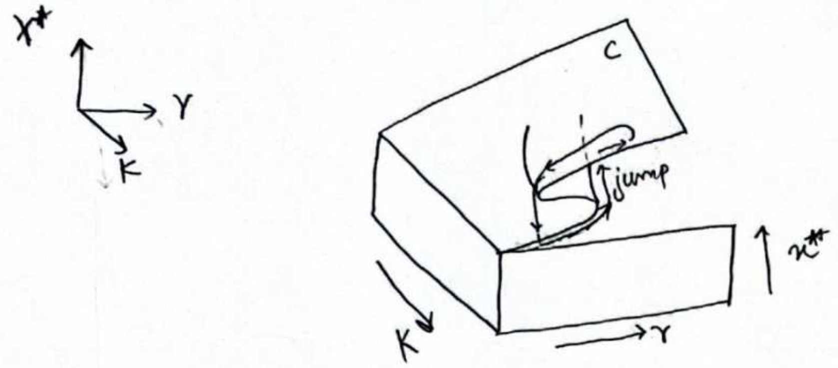
$\Rightarrow$



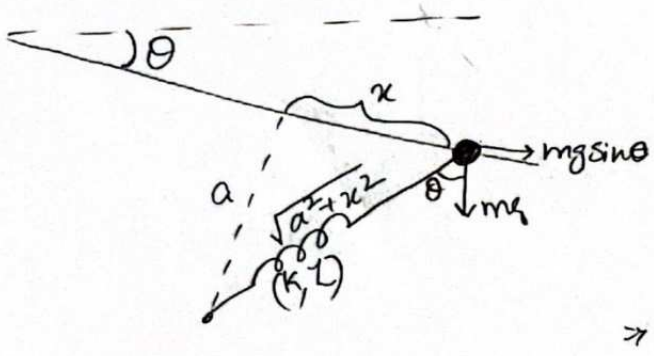
> As r increases all the population sitting at 'a' suddenly jumps to 'c', when the fps a' & b annihilate.

→ Now even if the system goes back to 3 fixed points the population that was at c does not go back to 'a'. Hysteresis. Unless b & c annihilate which would be quite hard.

→ $\boxed{\text{Hysteresis}}$ → Can visualize this as a cusp catastrophe



Lect - Phy 413



Force balance

$$mg\sin\theta = -K\left(L-\sqrt{a^2+x^2}\right)\cdot\frac{x}{\sqrt{x^2+a^2}}$$

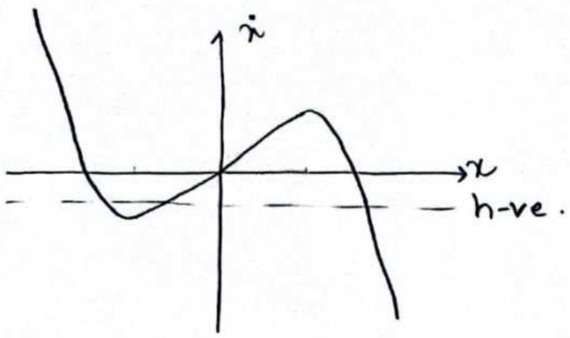
Let $u = \frac{x}{a}$ to non dimensionalize.

$$\Rightarrow mg\sin\theta = -Kau\left(\frac{L}{a}\frac{1}{\sqrt{1+u^2}}-1\right)$$

$$\Rightarrow \underbrace{\frac{mg\sin\theta}{Ka}}\cdot\frac{1}{u} = 1-\underbrace{\frac{L}{a}}\left(\frac{1}{\sqrt{1+u^2}}\right)$$

→ 2 Nondimensional groups ⇒ only 2 parameters control the dynamics.

Eg: $\dot{x} = h + rx - x^3 \rightarrow$ 2 parameters



What value of h gives half stable fp?

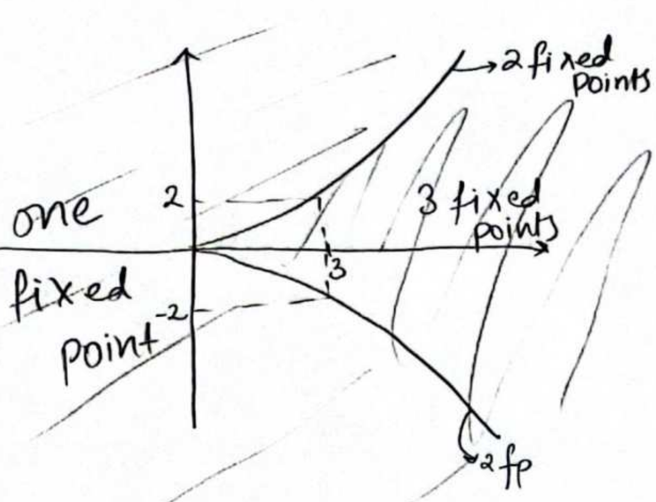
$\dot{x} = rx - x^3$

$f'(x) = r - 3x^2 = 0$

$\Rightarrow x = \sqrt{\frac{r}{3}}$

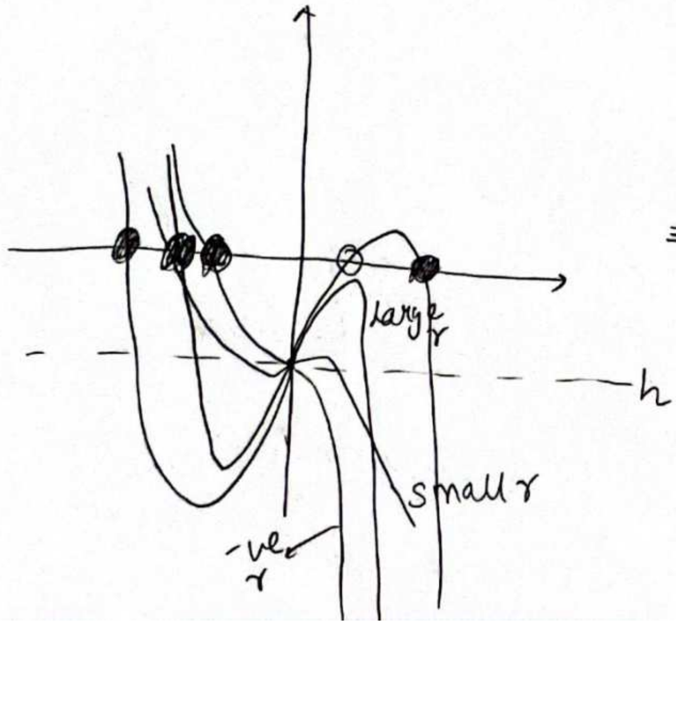
$\Rightarrow \dot{x} = -\frac{2}{3}\sqrt{\frac{r^3}{3}}$ are the y axis values at which slope is 0. These are the values of h that give half stable points.

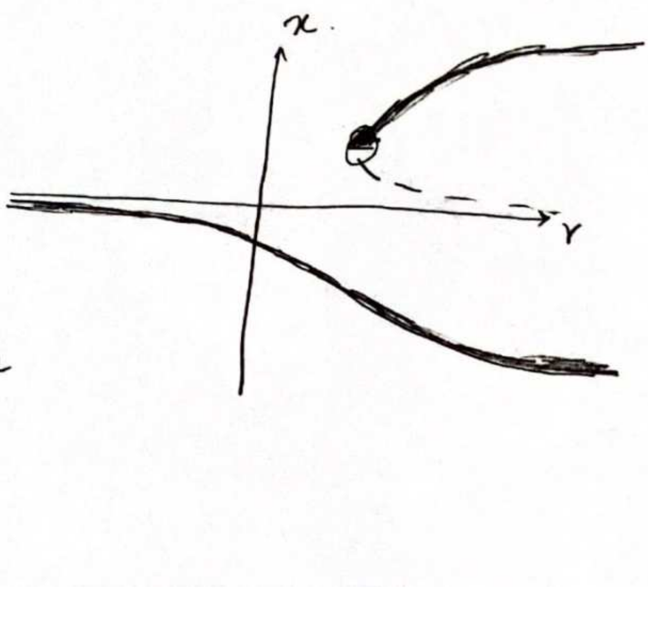
$\Rightarrow h_c = \frac{2}{3}\sqrt{\frac{r^3}{3}}$



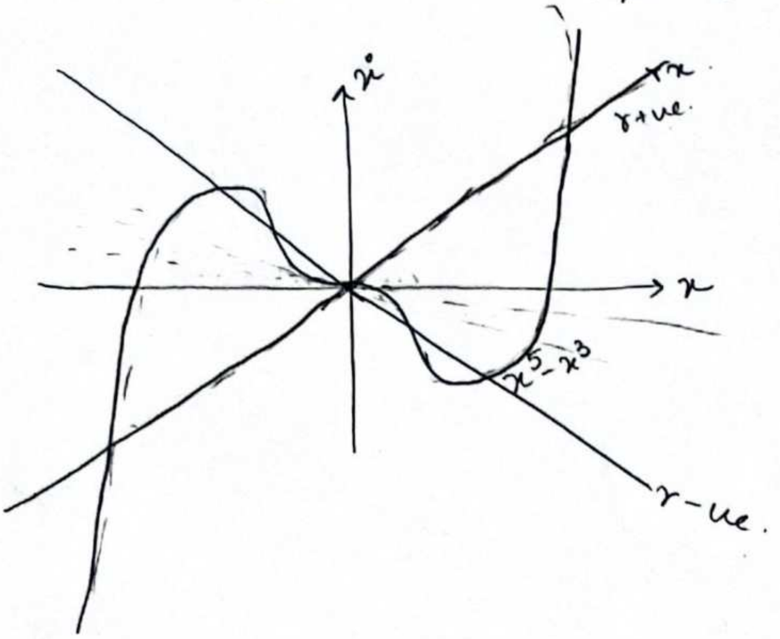
> When h=0 and r=0 we get pitchfork (supercritical).

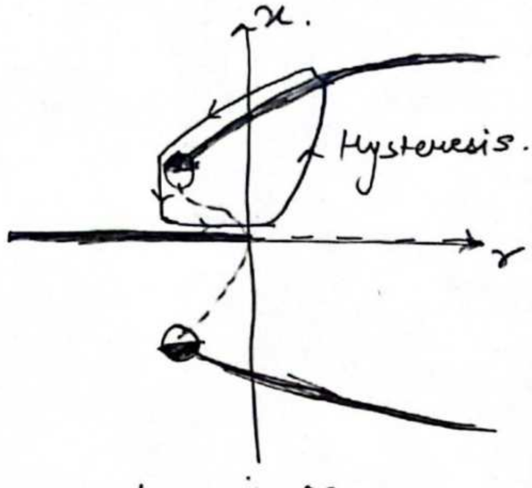
> $h \neq 0, r \neq 0 \Rightarrow$ let h is positive $\Rightarrow \dot{x} = h + rx - x^3$





Eg:- $\dot{x} = rx + x^3 - x^5 \quad = rx - (x^5 - x^3)$





Hysteresis or
Bistable switching.

Eg: Going back to the spring bead example

$$\frac{mg\sin\theta}{Ka} \cdot \frac{1}{u} = 1 - \frac{L}{a}\left(\frac{1}{\sqrt{1+u^2}}\right)$$

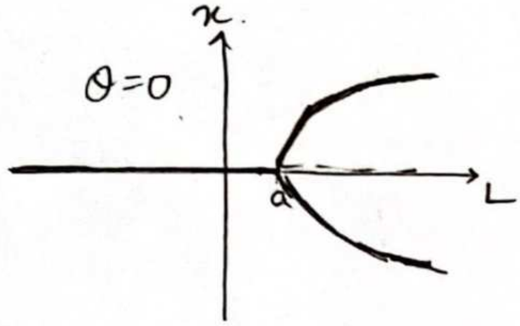
Let $\frac{mg\sin\theta}{Ka} = $ "θ"

$\frac{L}{a} = $ "L"

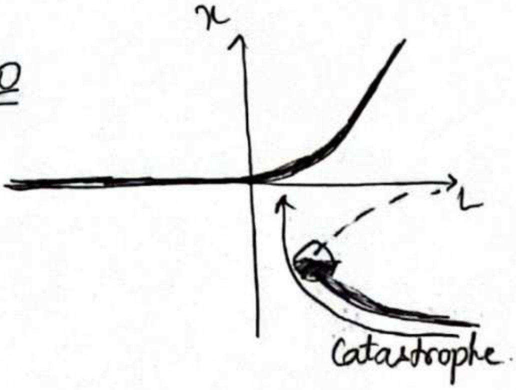
$\underline{\theta = 0}$ $\quad \theta \cdot \frac{1}{u} = 1 - L\left(\frac{1}{\sqrt{1+u^2}}\right)$

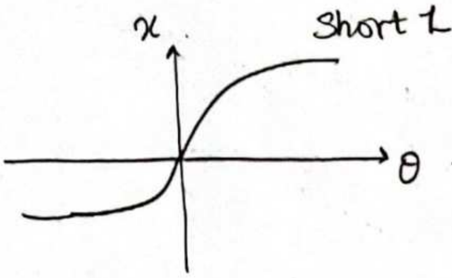
$\frac{L}{a} < 1 \Rightarrow$ only one stable equilibrium

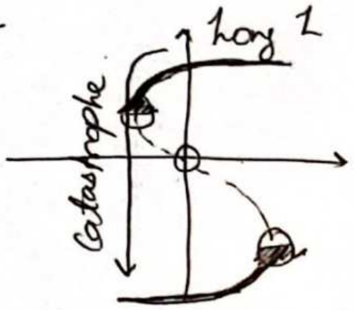
$\frac{L}{a} > 1 \Rightarrow$ spring pushing $\Rightarrow$ 3 equilibrium.



$\underline{\theta \neq 0}$



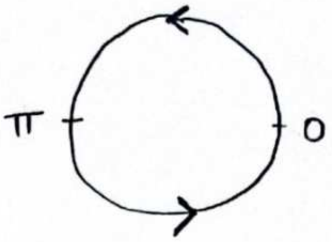




Lec 8 Phy415

Chapter 4 Flows on a Circle

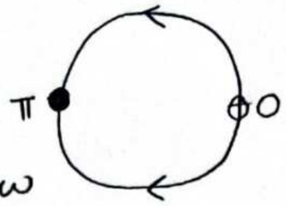
$\dot{\theta} = f(\theta)$

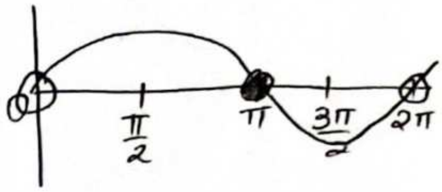


Systems like this can oscillate.

Eg

$\dot{\theta} = \sin\theta$





$\dot{\theta} > 0 \Rightarrow$ counterclockwise flow

$\dot{\theta} < 0 \Rightarrow$ clockwise flow.

→ $f(\theta)$ must be a real valued, 2π periodic function.

Eg:- Uniform oscillator.

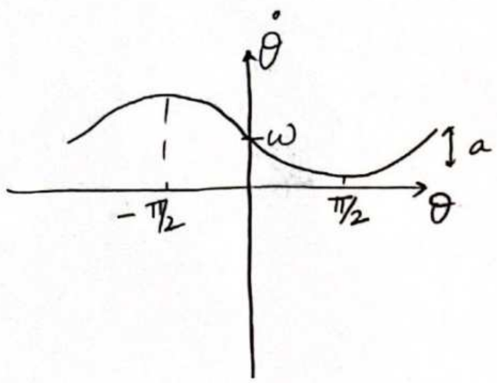
$\dot{\theta} = \omega \Rightarrow \theta(t) = \omega t + \theta_0$. ⇒ uniform motion @angular frequ

$T = \frac{2\pi}{\omega}$

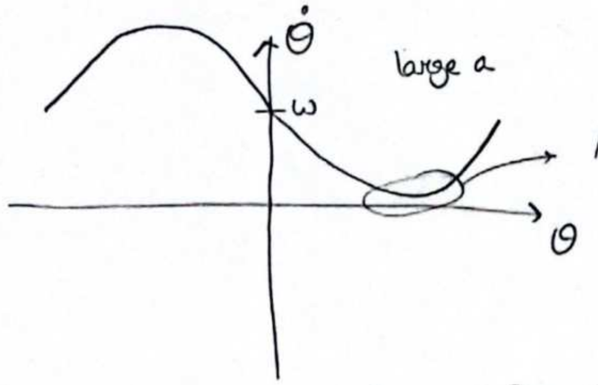
$$T_{beat} = \frac{2\pi}{\omega_1 - \omega_2} = \left[\frac{1}{T_1} - \frac{1}{T_2}\right]^{-1} = \frac{T_1 T_2}{T_2 - T_1}$$

Eg:- Non uniform oscillator.

$\dot{\theta} = \omega - a\sin\theta$.

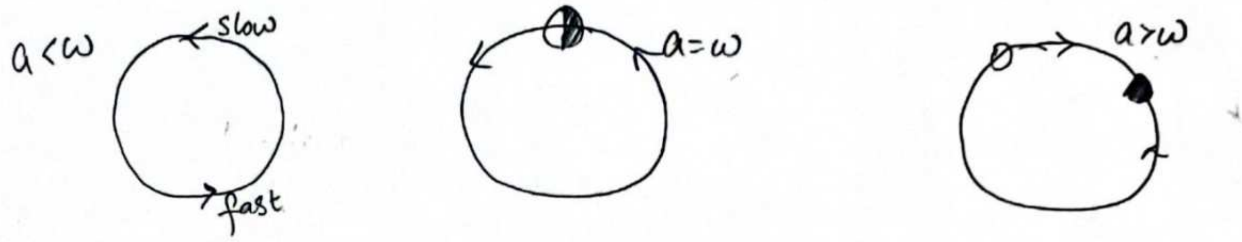


→



Bottleneck where the phase points spends a long time.

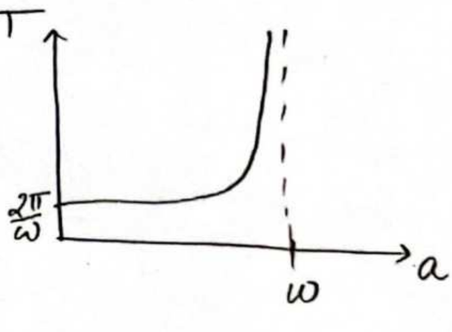
→ $a = \omega \Rightarrow$ No oscillation $\Rightarrow$ Saddle node bifurcation at $\pi/2$.



→ $a<\omega \rightarrow$ Find oscillation period.

$$T = \int dt = \int_0^{2\pi} \frac{dt}{d\theta}\cdot d\theta = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta}$$ → Evaluated using $u = \tan\frac{\theta}{2}$.

$$\Rightarrow \quad T = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$



Since $\sqrt{\omega^2 - a^2} = \sqrt{(\omega+a)(\omega-a)}$

$$\simeq \sqrt{(2\omega).(\omega-a)}$$

As $a \rightarrow \omega$ $\quad T \approx \frac{\pi\sqrt{2}}{\sqrt{\omega}} \cdot \frac{1}{\sqrt{\omega - a}}$

$\Rightarrow$ T blows up at $(a_c - a)^{-1/2}$. Called the square root scaling law.

→ This square root scaling law is very general to saddle node bifurcations.

> ## Ghosts

Just after fps collide there is a 'ghost' of the saddle node when the passage of dynamics are very slow.

> $\dot{\theta} = \omega - a\sin\theta$

Let $\phi = \theta - \frac{\pi}{2}$

$$\Rightarrow \dot{\phi} = \omega - a\cos\phi$$

$$= \omega - a\left(1 - \frac{\phi^2}{2}\right)$$

$$= \omega - a + \frac{a}{2}\phi^2$$

Let $x = \sqrt{\frac{a}{2}}\,\phi^2 \Rightarrow \dot{x}\sqrt{\frac{2}{a}} = r + x^2$ and $\omega - a = r$.

Recall $T = \int_{-\infty}^{\infty} \frac{dx}{r+x^2} \approx \frac{\pi}{r} = \frac{\sqrt{2}\,\pi}{\sqrt{\omega}\sqrt{\omega - a}}$ which takes the same form as what we saw earlier. Therefore, the trajectory spends most of its' time in the ghost.

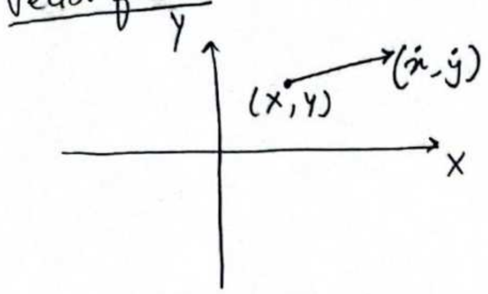
Two Dimensional Flows - Linear - MAE5790

Phase plane analysis

for: $\dot{x} = f(x,y)$ $\quad (x,y) \in \mathbb{R}$

$\dot{y} = g(x,y)$ $\quad (\dot{x},\dot{y})$ are velocity vectors (not actually velocity).

Vector field.



⇒ gives a vector field on the phase plane

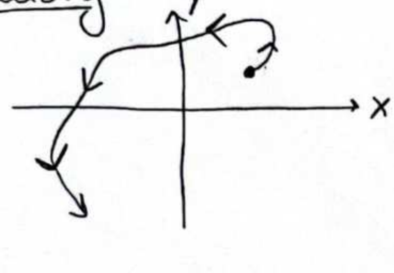
Vector form

$$\dot{\vec{x}} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\dot{\vec{x}} = \vec{f}(\vec{x}) \quad \text{where } x \in \mathbb{R}^2$$

→ If f is continuously differentiable, then solutions $\vec{x}(t)$ exist and are unique, for any initial conditions.

Trajectory



Implication of uniqueness:

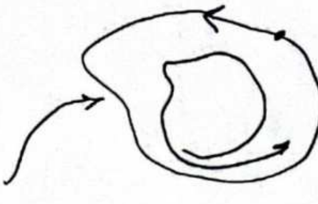
> Trajectories cannot cross!

> But trajectories can approach the same fixed point.

> Fixed point: A point where both $\dot{x}=0$ and $\dot{y}=0$.

$$\Rightarrow \vec{f}(x^*) = 0$$

> Strong topological consequences of noncrossing trajectories in $\mathbb{R}^2$



→ A closed orbit must be a periodic behaviour

⇒ Trajectories outside can't get inside.

⇒ Trajectories inside are like playing "Snake"

Goal: Given $\dot{\vec{x}} = \vec{f}(\vec{x})$, deduce phase portrait (picture of all qualitatively different trajectories), extract qualitative info. from them (existence, stability of fixed points and closed orbits).

Chapter 5

$\dot{\vec{x}} = A\vec{x}$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ → real.

→ Only choosing homogeneity.

$\vec{x}^* = \vec{0}$ is always a fixed point.

Phase portrait: determined by eigenvalues and eigenvectors of A.

> Seek straight line solutions.

$\vec{x}(t) = \vec{V} e^{\lambda t}$ → assume this is a straight line solution.

$$\left.\begin{aligned} \dot{\vec{x}}(t) &= \vec{V}\lambda e^{\lambda t} \\ A\vec{x} &= A(\vec{V}e^{\lambda t}) = e^{\lambda t} A\vec{V} \end{aligned}\right\} \text{Equating the two}$$

$$\dot{\vec{x}}(t) = A\vec{x} \Rightarrow (\vec{V}\lambda)\cancel{e^{\lambda t}} = (A\vec{V})\cancel{e^{\lambda t}}$$

⇒ Solutions of the type $\vec{V}e^{\lambda t}$ exist if $A\vec{V} = \lambda\vec{V}$.

Therefore a linear differential equation system $\dot{\vec{x}} = A\vec{x}$ has the general solution $\vec{x}(t) = \vec{V}e^{\lambda t}$ if $A\vec{V} = \lambda\vec{V}$.

λ is given by $\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$.

$\Rightarrow \lambda^2 - \tau\lambda + \Delta$ where τ is the trace $A = a+d$.
Δ is the det $A = ad - bc$.

$\therefore \boxed{\lambda^2 - \tau\lambda + \Delta = 0}$ is the characteristic equation.

$$\Rightarrow \boxed{\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}}$$

Properties

$$\left.\begin{aligned} \tau &= \lambda_1 + \lambda_2 \\ \Delta &= \lambda_1 \lambda_2 \end{aligned}\right\} \ \tau \,\&\, \Delta \text{ give a lot of information}$$

Classification of fixed points for $\dot{\vec{x}} = A\vec{x}$, $\vec{x} \in \mathbb{R}^2$.

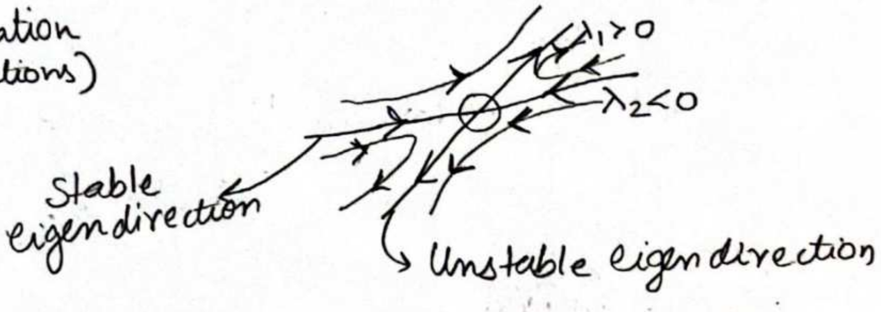
Cases: Case 1) Saddle points $\rightarrow \boxed{\Delta < 0}$

Then $\lambda_1 > 0$ and $\lambda_2 < 0 \Rightarrow \lambda_1 \neq \lambda_2 \Rightarrow$ distinct.

> Distinct eigenvalues $\Rightarrow$ we get 2 eigenvectors that are linearly independant, and can be orthogonalized using Gram Schmidt.

General solution: $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

= (linear combination of the eigen solutions)



Case 2) Attracting (and Repelling) Fixed points.

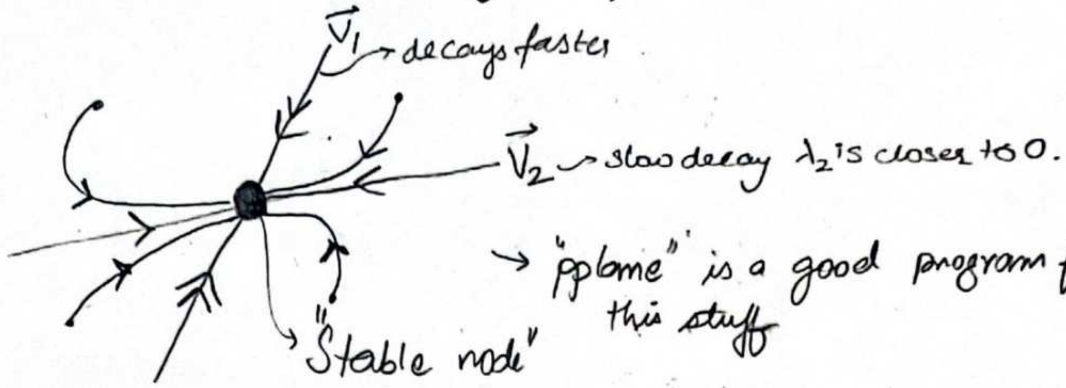
Focus on Attracting $\Rightarrow \boxed{\Delta > 0, \tau < 0}$ Repelling $\boxed{\Delta > 0, \tau > 0}$

2a) Nodes: $\tau^2 - 4\Delta > 0 \Rightarrow$ Real λ, same sign.

Suppose both λ_1 & $\lambda_2 < 0$ for attracting case

and $\lambda_1 < \lambda_2 < 0$

Again $\overline{V}_1$ & $\overline{V}_2$ are linearly independant.



→ "pplane" is a good program for graphing this stuff

> As $t \to \infty$ (typical) trajectories approach $\vec{x}^*$ tangent to the slow direction. In backwards time $t \to -\infty \Rightarrow \vec{x}^*$ is ||del to fast direction

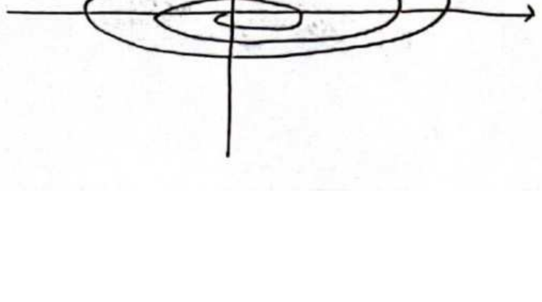
> <u>2b) Spirals.</u>: $\Delta > 0$ $\tau < 0$, $\tau^2 - 4\Delta < 0 \Rightarrow$ complex λs.

No real eigenvectors. $\lambda_1 \neq \lambda_2$ let $\lambda = \mu \pm i\omega$.

Say $\mu < 0$ ($\Rightarrow$ attracting) controls the decay rate, ω controls the rotation rate.

From Linear Algebra → $x(t)$ → each component of $\vec{x}(t)$ is a linear combination of $e^{\mu t}\cos\omega t$ and $e^{\mu t}\sin\omega t$.

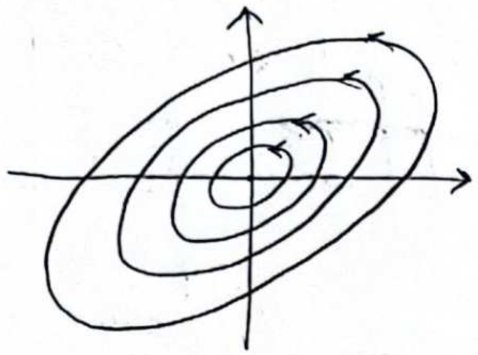
→ Direction of the spiral is not given by τ & Δ, we should just calculate it at one point and see.



→ usually they are elliptical.

Case 3) Center. $\Delta > 0$, $\tau = 0$, $\lambda = \pm i\omega$

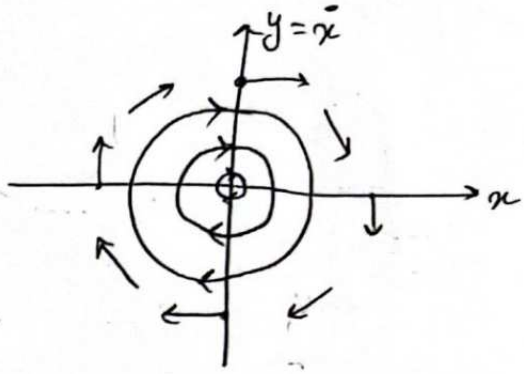
$\Rightarrow$ Every trajectory is closed.



Eg: Undamped oscillator.

$\ddot{x} + x = 0$ could be written as. $\dot{x} = y$.

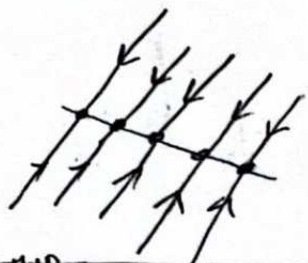
$\dot{y} = -x$.



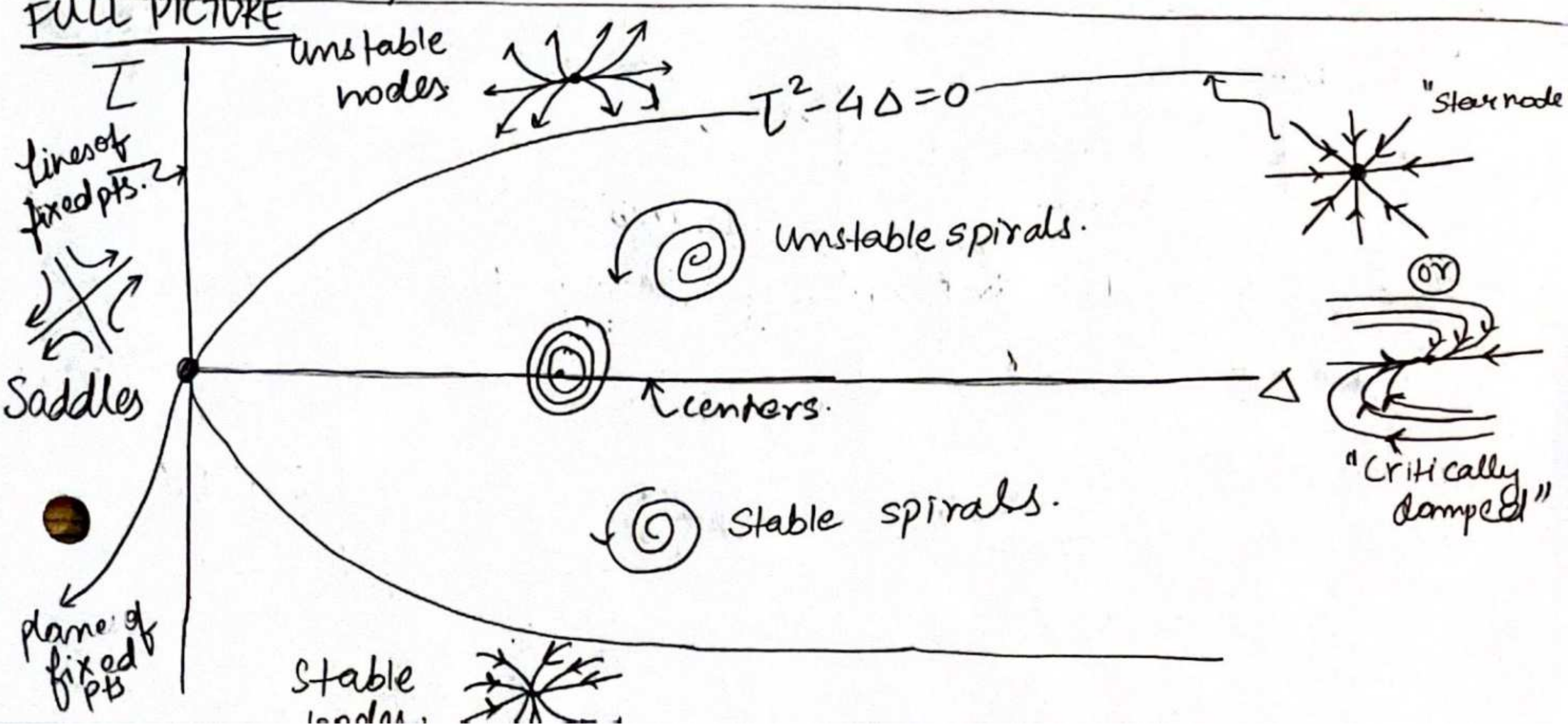
Case 4) $\Delta = 0$

$\Rightarrow$ $Ax = 0$ doesn't have a unique solution.

$\Rightarrow$ line or plane of fixed.

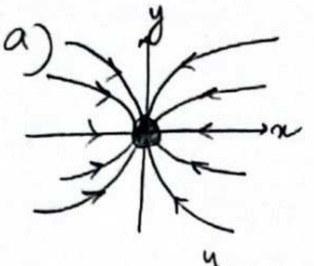


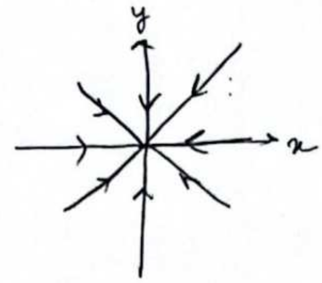
FULL PICTURE

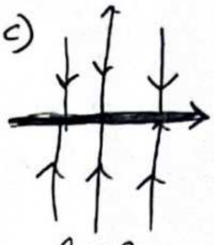


Phy413 Two dimensional linear systems.

Eg: $\dot{x} = ax$, $\dot{y} = dy$ $\rightarrow \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}$

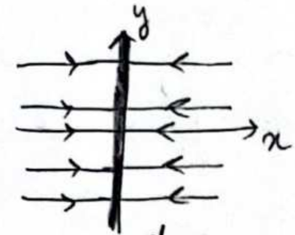
a) 

b) 

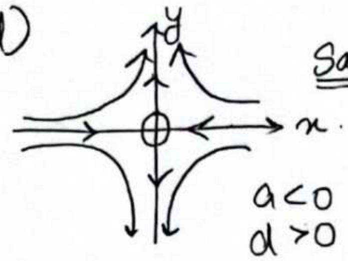
c) 

$a=0$

d \ a	<0	$=0$	>0
<0	$a \neq$ (a) / $=$ (b)	(c)	(d)
$=0$	(c)	(e)	(f)
>0	(d)	(f)	(g) $\neq$ / $=$ (h)

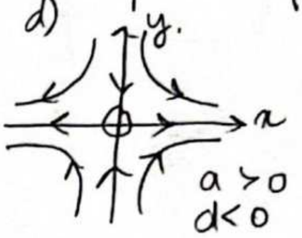
c) 

$d=0$

d) 

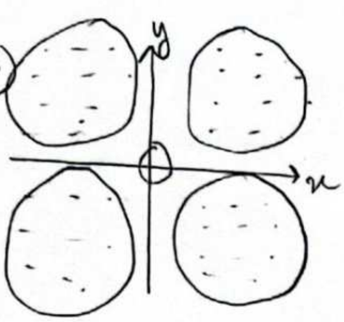
Saddle.

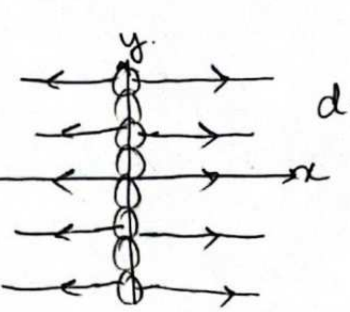
$a<0$
$d>0$

d) 

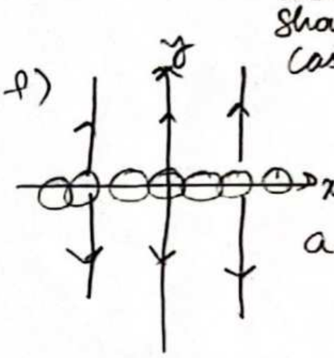
$a>0$
$d<0$

Asymptotes approach the slower side. here we are showing $|a|=|d|$ case so it's not clear.

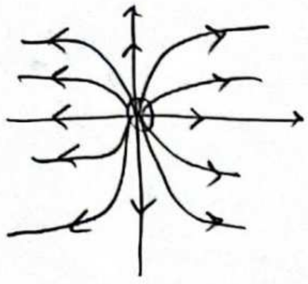
e) 

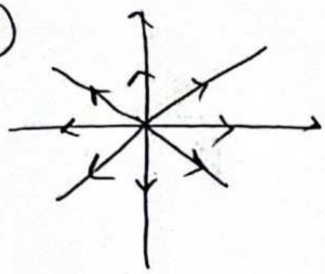
f) 

$d=0$

f) 

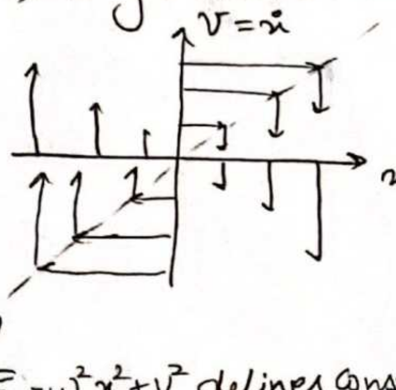
$a=0$

g) 

h) 

$\rightarrow$ If $\dot{x}, \dot{y}$ are coupled, and cannot be decoupled, they would oscillate.

Eg:- $m\ddot{x} = -kx \Rightarrow \dot{x} = v$, $\dot{v} = \frac{-k}{m}x \Rightarrow \dot{v} = -\omega^2 x$



If $\omega=1$ trajectories are circles because they are $\perp$lar to line through origin.

$\frac{d}{dt}(\omega^2x^2+v^2) = 2\omega^2 x\dot{x} + 2v\dot{v} = 2\omega^2 xv - 2v\omega^2 x = 0$

$\Rightarrow \omega^2x^2+v^2 =$ constant, $\Rightarrow$ Energy $= \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \Rightarrow \frac{2E}{m} = \omega^2x^2+v^2$ defines constant

MAE - Lec 6 Two dimensional Nonlinear Systems.

$\dot{x} = f(x)$

Suppose (x^*, y^*) fixed pts. of $\dot{x} = f(x, y)$
$\dot{y} = g(x, y)$

To classify it, consider small deviations

$$u(t) = x(t) - x^*$$

$$v(t) = y(t) - y^*$$

$$\dot{u} = \dot{x} = f(x, y) = f(x^* + u,\ y^* + v)$$

$$= \cancelto{0}{f(x^*, y^*)} + u \frac{\partial f}{\partial x}\Big|_{\vec{x}^*} + v \frac{\partial f}{\partial y}\Big|_{\vec{x}^*} + \text{H.O.T}$$

$$\dot{v} = u \frac{\partial g}{\partial x}\Big|_{\vec{x}^*} + v \frac{\partial g}{\partial y}\Big|_{\vec{x}^*} + \text{H.O.T}$$

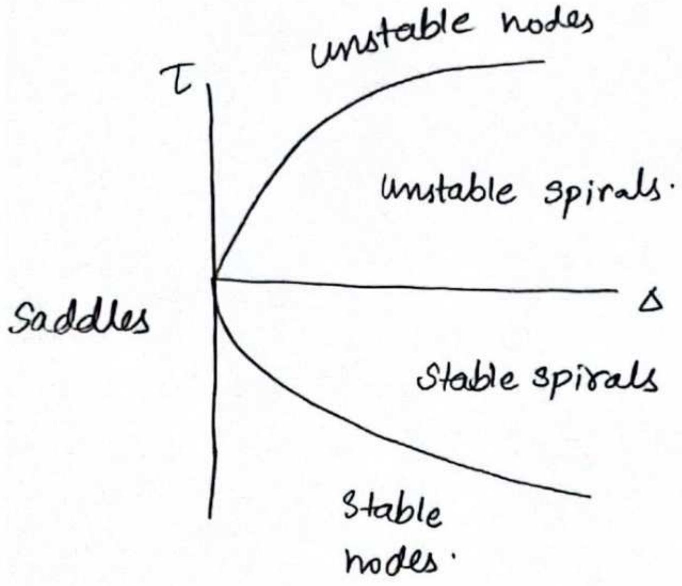
$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)} \begin{pmatrix} u \\ v \end{pmatrix} + \text{H.O.T}$$

Linearizing $\Rightarrow$ Ignore H.O.T

$\Rightarrow$ $\boxed{\dot{\vec{u}} = A \vec{u}}$ A → Jacobian. ⇝ gives the linearization around the fixed point.

> If $\vec{x}^*$ is a saddle, node or spiral then the linearization works.

> But boderline cases (degenerate node, star, center, nonisolated fixed pt) can be altered.



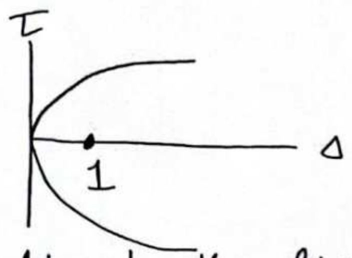
- All these fixed points that live in open spaces can be linearized.
- The "border" line cases lie on the borders! They cannot tolerate small perturbations from H.O.T

Eg:

$\dot{x} = -y + ax(x^2+y^2) \qquad \dot{y} = x + ay(x^2+y^2) \quad , (x^*, y^*)_1 = (0,0)$

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ → This is obvious since if we ignore the H.O.T we just get $\dot{u} = -v$, $\dot{v} = u$ when the fixed point is $(0,0)$.

$\tau = 0, \quad \Delta = 1$



∴ $(0,0)$ is a center, according to the linearization for any 'a'. But this is wrong! It only works when $a=0$. But linearization ignores a.

Let $x = r\cos\theta$, $y = r\sin\theta$
$r = r(t)$, $\theta = \theta(t)$.

$x^2+y^2 = r^2 \Rightarrow 2x\dot{x} + 2y\dot{y} = 2r\dot{r}$

$\Rightarrow \boxed{r\dot{r} = x\dot{x} + y\dot{y}}$

$\dot{x} = -y + axr^2 \; ; \; \dot{y} = x + ayr^2 \; ; \; r\dot{r} = x(-y + axr^2) + y(x + ayr^2)$

$\Rightarrow r\dot{r} = ar^2(x^2+y^2) = ar^4 \qquad \Rightarrow \boxed{\dot{r} = ar^3}$

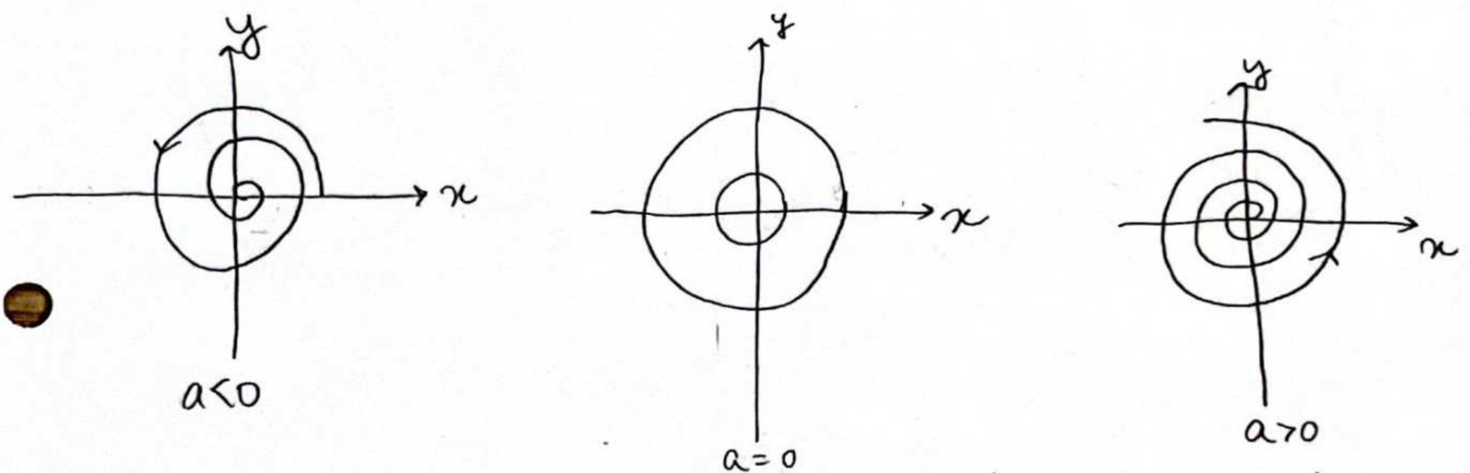
$\theta = \tan^{-1} \frac{y}{x}$

Solving,

$\Rightarrow \boxed{\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}} = \frac{x(x + ayr^2) - y(-y + axr^2)}{r^2}$

$$\boxed{\dot{\theta} = 1}$$

$\Rightarrow \left.\begin{array}{l} \dot{r} = ar^3 \\ \dot{\theta} = 1 \end{array}\right\}$ 2 uncoupled one dimensional systems.



→ The H.O.T are pushing the centre away from it's border!

§6.4 Rabbits Vs. Sheep. Lotka-Volterra model of Competitions.

x = popn. of rabbits.
y = popn. of sheep
$x, y \geq 0$

$\dot{x} = x(3 - x - 2y)$ If $y = 0$ $\dot{x} = x(3 - x)$

Sheep are mean $\Rightarrow -2xy$
Rabbits reproduce fast $\Rightarrow$ 3 is the carrying capacity.

Logistic model!

Carrying capacity.

Fixed points : $(0,0), (3,0), (0,2), (1,1)$.

$$A = \begin{bmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{bmatrix}$$

Classifying fixed points.

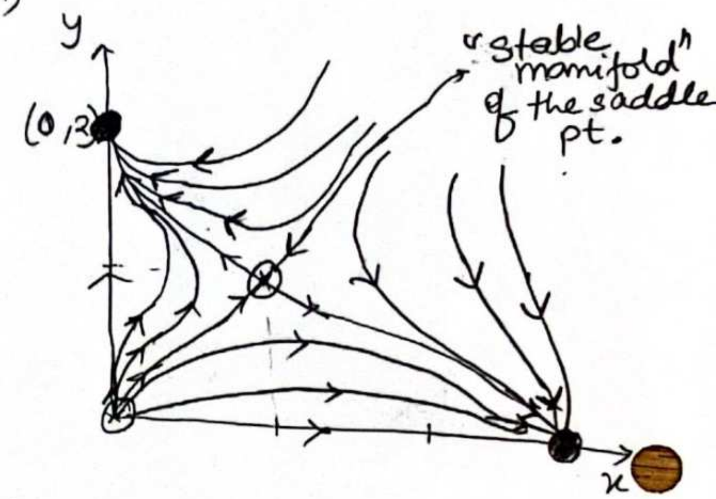
$(0,0) \Rightarrow A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow$ $\lambda_1 = 3$, $\lambda_2 = 2$ — unstable node. $\vec{v}_1 = (1,0)$, $\vec{v}_2 = (0,1)$

Trajectories leave along slow direction $\Rightarrow \vec{v}_2$

$(0,2) \Rightarrow A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \Rightarrow$ $\lambda_1 = -1$, $\lambda_2 = -2$ — Stable node. $\vec{v}_1 = (1,-2)$, $\vec{v}_2 = (0,1)$

$(3,0) \Rightarrow A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$ $\lambda = -3, -1$ — stable node.

$(1,1) \Rightarrow A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$ $\Delta = -1$ $\Rightarrow$ saddle pt.



$\dot{x} = (3 - x - 2y)x$

$\dot{y} = y(2 - x - y)$

If $x = 0 \Rightarrow \dot{x} = 0 \Rightarrow x$ axis & similarly y axis is invariant.

→ Below stable manifold ⇒ rabbits win, Above stable manifold ⇒ sheep win.

What happens if the competition is reduced?

$\dot{x} = x(3 - x - y)$ $\quad \dot{y} = y(2 - y - \frac{1}{2}x)$

⇒ Fixed points at $(0,2)$, $(3,0)$, $(0,0)$, $(2,1)$

But now $(2,1)$ is a stable node and the 2 species can coexist

PHY413

→ **Stability definitions.**

Attracting. $\vec{x}(t) \longrightarrow \vec{x}^*$ as $t \to \infty$

" $\exists\ \delta > 0$ s.t $\|\vec{x}_0 - \vec{x}^*\| < \delta \Rightarrow \vec{x}(t) = \vec{x}^*$ as $t \to \infty$

Lyapunov.

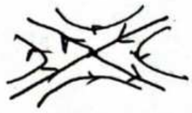
" $\forall\ \epsilon > 0\ \exists\ \delta > 0$ s.t $\|\vec{x}(t) - \vec{x}^*(t)\| < \epsilon$ if $\|\vec{x}_0 - \vec{x}^*\| < \delta$.
at all times".

Fixed points.

Purely real $[\lambda_1 > 0 \quad \lambda_2 > 0] \Rightarrow$ Unstable node (repeller) 

Purely real $[\lambda_1 < 0 \quad \lambda_2 < 0] \Rightarrow$ Stable node (attractor) 

$\lambda_1 = \lambda_2 \Rightarrow$ Stars 

Purely real. $[\lambda_1 > 0 \quad \lambda_2 < 0] \Rightarrow$ Saddle 

λ_1, λ_2 are purely imaginary $\Rightarrow$ Centers 

- λ_1, λ_2 are complex $\Rightarrow$ spirals. (stable $\Rightarrow \tau < 0$, Unstable $\Rightarrow \tau > 0$).
  attractors repellers.

→ **Hartman Grobman Theorem** (When does linearization work?)

> "If fixed point is hyperbolic, the local phase portrait is topologically equivalent to the linearization".

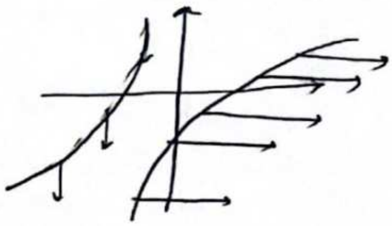
1) F.P is "hyperbolic" if $\forall\ \lambda_i$, $Re(\lambda_i) \neq 0$

Topologically equivalent $\Rightarrow$ There exists a homeomorphism from the linearization to the nonlinear system close to the fixed pt.

→ Stars $\not\Leftrightarrow$ Nodes $\Leftrightarrow$ Spirals.

Nullclines.

Lines along which $\dot{x}=0$ or $\dot{y}=0$



In summary: 2D LSA works if $Re(\lambda_{1,2}) \neq 0$.

MAE Lec 7 - Conservative Systems.

- Consider mechanical system, 1 degree of freedom.

$$m\ddot{x} = \overset{\text{Force}}{F(x)} = -\frac{dV}{dx} \text{ potential energy} \Rightarrow m\ddot{x} + \frac{dV}{dx} = 0 \quad \text{— (1)}$$

- F independant of $\dot{x}$ and t ⇒ no damping + external drive.

⇒ The energy $E = \frac{1}{2}m\dot{x}^2 + V(x)$ is conserved.

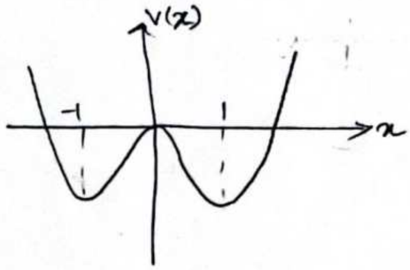
Proof: From (1) $m\ddot{x}\dot{x} + \frac{dV}{dx}\dot{x} = 0 \Rightarrow \frac{d}{dt}\Big(\underbrace{m\frac{\dot{x}^2}{2} + V(x(t))}_{\text{constant}}\Big) = 0$

⇒ E is conserved on trajectories.

- More generally, $\dot{\vec{x}} = \vec{f}(x)$ is conservative if it has a "conserved quantity" $E(\vec{x})$.
  - $E(\vec{x})$ is a continuous real valued fn that is constant on trajectories.
  - Also, this requires $E(\vec{x}) \not\equiv$ constant on any open set. Otherwise something like $E(\vec{x}) = 17$ would be a conserved quantity for every system.

Eg: Particle in a double well potential.

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$$



Suppose $m=1$ for simplicity.

$$\ddot{x} = -\frac{dV}{dx} = x - x^3 \quad , \quad \text{Let } \dot{x} = y, \; \dot{y} = x - x^3$$

$$A = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{bmatrix}$$

F.P: $x^*, y^* = 0, 0$

$x^*, y^* = 1, 0$

$x^* y^* = -1, 0$

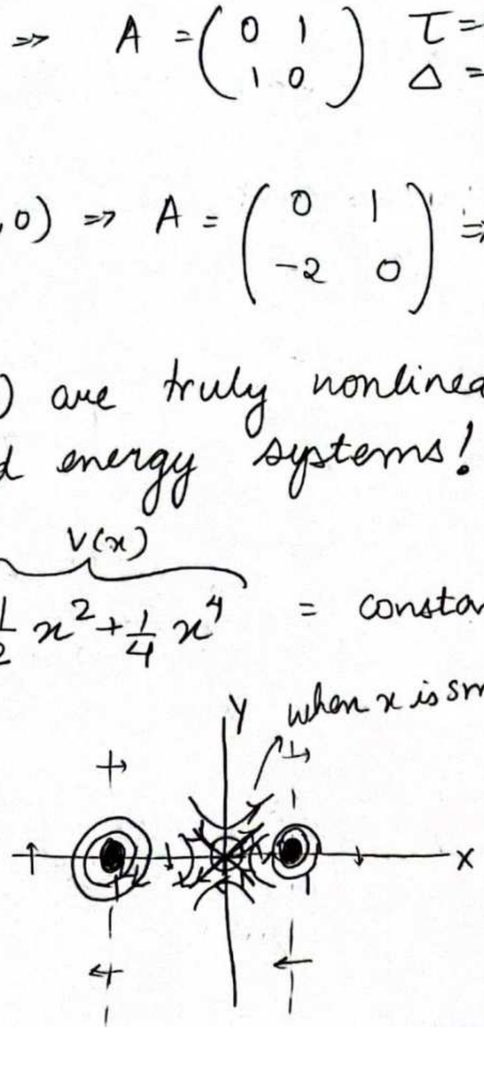
- $x^*, y^* = (0,0) \Rightarrow A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\begin{matrix} \tau = 0 \\ \Delta = -1 \end{matrix}$ $\Rightarrow$ Saddle.

$x^*, y^* = (\pm 1, 0) \Rightarrow A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \Rightarrow \begin{matrix} \Delta = 2 \\ \tau = 0 \end{matrix} \Rightarrow$ Linearization gives a center.

> In fact $(\pm 1, 0)$ are truly nonlinear centres in the case of conserved energy systems!

> $E = \underbrace{\frac{1}{2}y^2}_{\frac{1}{2}m\dot{x}^2} \overbrace{- \frac{1}{2}x^2 + \frac{1}{4}x^4}^{V(x)} = \text{constant}$. These represent closed curves.

when x is small $\frac{1}{2}y^2 - \frac{1}{2}x^2 = \text{const.}$ → hyperbolas.

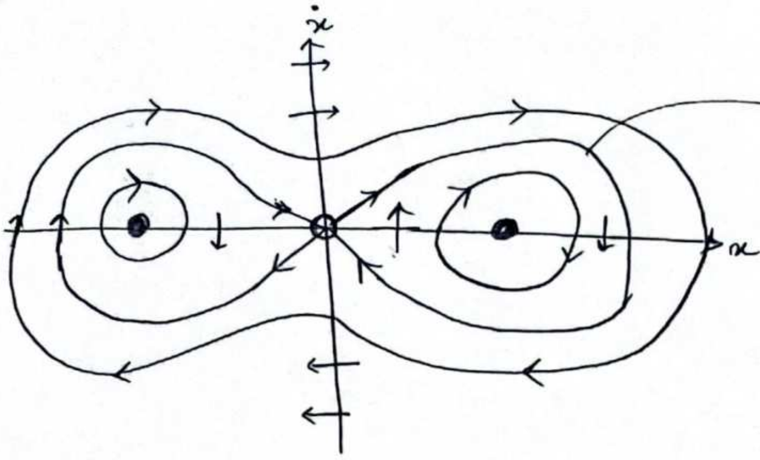


$\dot{x} = y$

$\dot{y} = x - x^3$

$\Rightarrow$ nullclines $\dot{y} = 0$ are y axis & $x = 1, -1$.

nullclines $\dot{x} = 0 \Rightarrow$ x axis

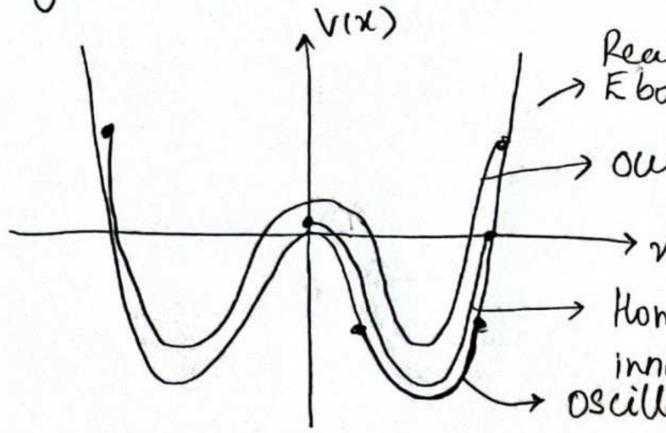


"Homoclinic orbit"
↳ same ↳ slope (incline)

> inclined to go back to same place.

⇒ A trajectory that kind of Starts & ends at the same fixed point

> Homoclinic orbits are not periodic or $T = \infty$. All other trajectories here are periodic.



Because here we are ignoring the y axis!

Really these trajectories move on constant E boundaries - This picture is just for intuition.

→ outer oscillatory orbits (sample both wells)

→ Homoclinic trajectory.

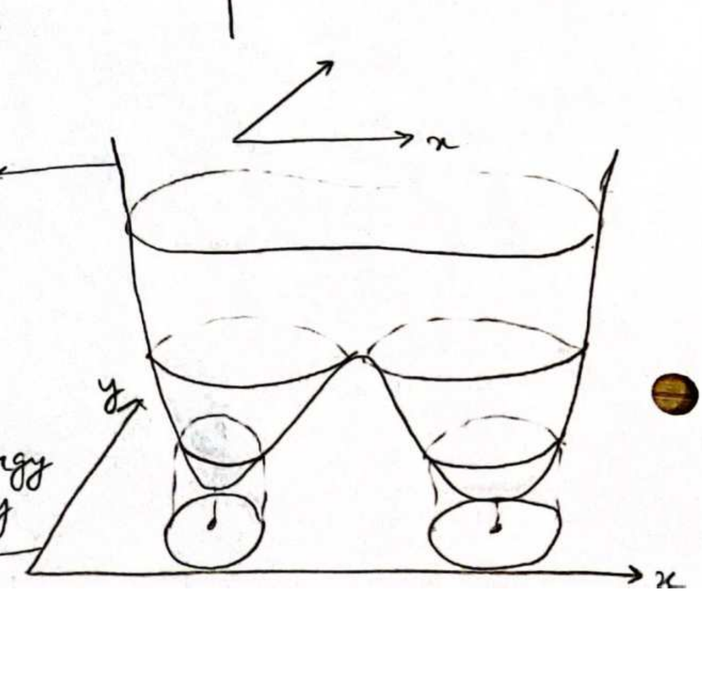
→ inner oscillatory trajectories.

Energy Surface.

$$E(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$$

→ On the energy surface E is conserved so particle move around with a constant height.

Energy Surface



Projection of Energy Surface onto x, y plane gives the phase portrait.

Openset: A set which doesn't include it's boundary. Similar to open interval but in 2 dimensions.

*→ In conservative systems, we cannot have any attracting fixed points!

Theorem: (Nonlinear centres in 2D conservative systems)

Suppose $\dot{\vec{x}} = \vec{f}(\vec{x})$ = conservative. f is continuously differentiable

$\vec{x} \in \mathbb{R}^2$, $E(x)$ is a conserved quantity.

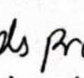
> $\vec{x}^*$ is an isolated fixed point ⇒ No other fixed points near it.

or

> If that fixed point is a local min or local max of $E(x)$, then $\vec{x}^*$ is a centre (all trajectories close to $\vec{x}^*$ are closed).

Idea of proof: > E is constant on trajectories ⇒ Trajectories lie in contours of E.

It is "in" the contour & not the whole contour since for homoclinic case it occupies only half the contour.

> Contours are closed curves near a minimum or a maximum. (needs proof) but intuitively cutting near a max or a min gives contours. 

> They need to be closed trajectories since there are no fixed points near by.

> Saddle points for example are fixed pts. without being a min. or max.

Eg: Pendulum (dimensionless)

$\ddot{\theta} + \sin\theta = 0$, Let $V = \dot{\theta}$ = angular velocity.

$$\left.\begin{aligned} \dot{V} &= -\sin\theta \\ \dot{\theta} &= V \end{aligned}\right\} \text{No small angle approximation.}$$

$$A = \begin{bmatrix} \frac{d\dot{\theta}}{d\theta} & \frac{d\dot{\theta}}{dv} \\ \frac{d\dot{v}}{d\theta} & \frac{d\dot{v}}{dv} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\cos\theta & 0 \end{bmatrix} \qquad (0,0) \Rightarrow A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$\Rightarrow \tau = 0, \Delta = 1 \Rightarrow$ Linear centre.

→ This is a conservative system $E = \frac{1}{2}V^2 - \cos\theta$ = constant, has a local min at (0, 0)

$$E = \frac{1}{2}V^2 - \left[1 - \frac{\theta^2}{2} + \dots\right]$$

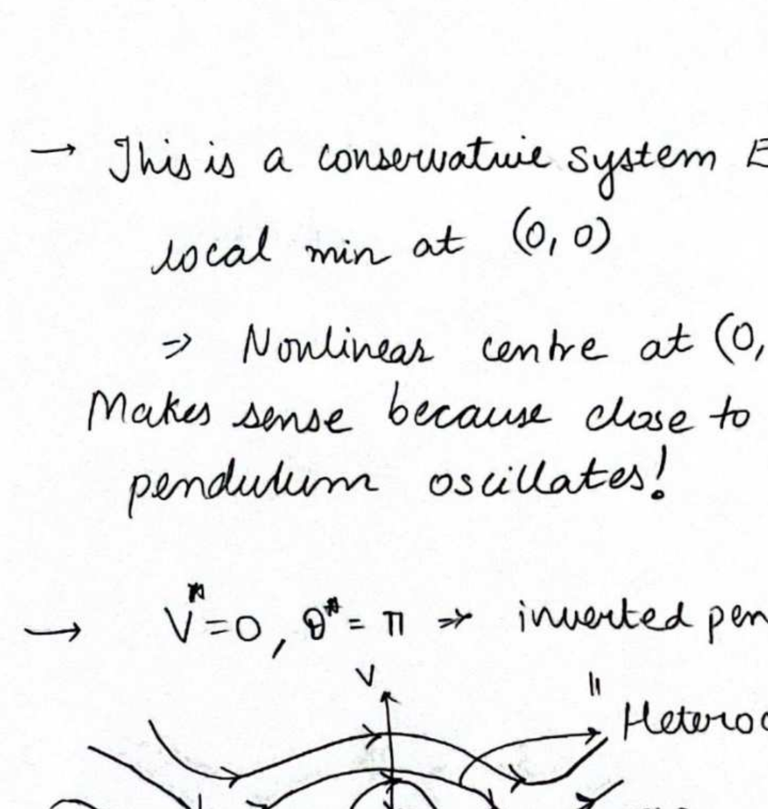
$$= \frac{1}{2}(V^2 + \theta^2) + \text{constant}$$

↳ paraboloid with min at (0,0) & contours are circles.

⇒ Nonlinear centre at (0,0)

Makes sense because close to 0,0 the pendulum oscillates!

→ $V^* = 0$, $\theta^* = \pi$ ⇒ inverted pendulum. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ⇒ saddle.



"Heteroclinic orbit" → Connects 2 saddles.

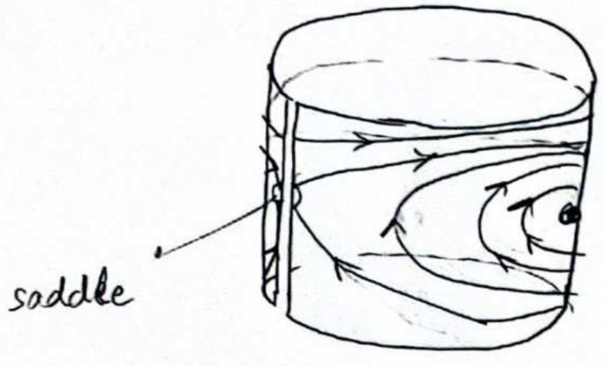
⇒ Pendulum leaves the top, makes a full circle & stops at the top.

Pendulum goes around in circles.

> If we regard all θ mod 2π ⇒ The plane becomes a cylinder.

<u>Cylindrical Phase Space</u>

>



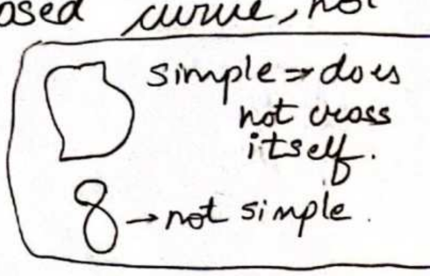
> Only one centre and one saddle.

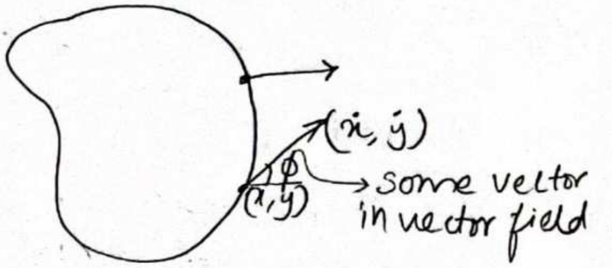
> On the homoclinic orbit the trajectory takes infinite time to go from the start to the stop. However all other orbits go around their trajectories in finite time. The closer the particle gets to the homoclinic orbit's fixed point the slower it gets.

<u>MAE – Lec 8</u> <u>Index Theory</u>

> Provides "global" info. about phase portraits.

> <u>Index of a closed curve C</u>: C = simple closed curve, not necessarily a closed trajectory, should not pass through a fixed point.





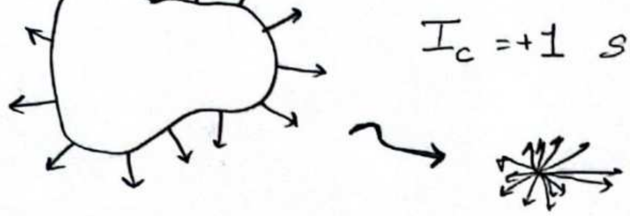
$$\phi = \tan^{-1} \frac{\dot{y}}{\dot{x}}$$

As $\bar{x} = (x, y)$ goes around C once counterclockwise, then ϕ changes continuously (based on the vector field at those points) if $\dot{x}, \dot{y}$ are continuous functions of $\bar{x}$.

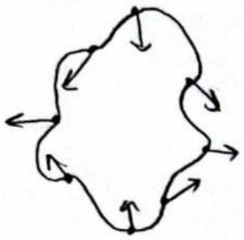
> Let $[\phi]_c$ = net change in ϕ when we go around c.
Divide by 2π => $I_c = \frac{1}{2\pi}[\phi]_c$ is the index of c w.r.t the vector field $(\dot{x}, \dot{y})$

> It is basically the no. of times a chalk along the vector rotates as it is slid around the contour.

Eg:



$I_c = +1$ since chalk goes around counterclockwise



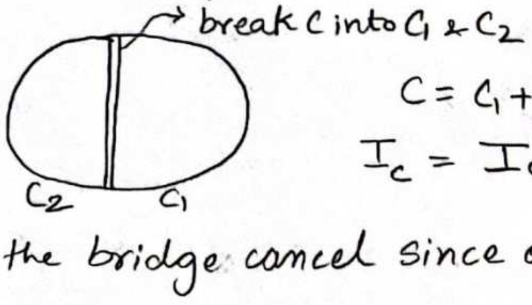
=> $I_c = -1$

Index is always an integer? Yes!!

Properties.

i) Index of a closed trajectory. => The vectors would be tangent to the trajectory/curve => $\boxed{I_c = +1}$ "Cannot be -1".

ii) Index is additive if we subdivide the curve 'C'.



$C = C_1 + C_2$

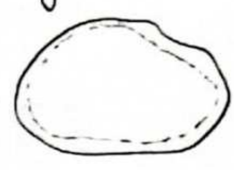
$I_c = I_{c_1} + I_{c_2}$

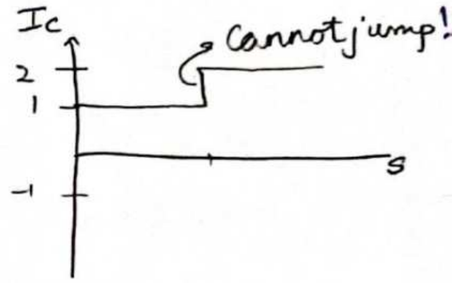
The angles on the bridge cancel since one goes ↓ & other goes ↑.

iii) If C is deformed continuously into C' without passing through a fixed pt., then $I_C = I_{C'}$. This is like Gauss' Law, Flux through a surface doesn't change if charge doesn't change.

Proof: I_c depends continuously on C. Eg: shrink it a little
$\Rightarrow$ vectors change a little & ϕ also changes very little.

But I_c is an integer $\Rightarrow$ if we have a continuous function that is integer valued.
$\Rightarrow$ it must be a constant.





> Fixed pts. cause problems because the vector there has no ϕ so it is undefined.

iv) If C does not enclose a fixed point $\Rightarrow I_c = 0$.
On a small closed curve vector field is constant $\Rightarrow I_c = 0$. Any curve can be shrunk down to a small curve if there are no fixed points inside.

v) If $t \to -t$ all the arrows rotate by $180°$ $\Rightarrow$ $\vec{x}$ now is $-\vec{x}$ $\Rightarrow$ ϕ at each point goes to $\phi + \pi$ $\Rightarrow$ $[\phi]_c$ unchanged. Index is invariant to reversing the time. It doesn't say anything about stability.

<u>Index of a point P</u> = I_c, for any C that encloses P and no other fixed points. (By iii))

Index of points.

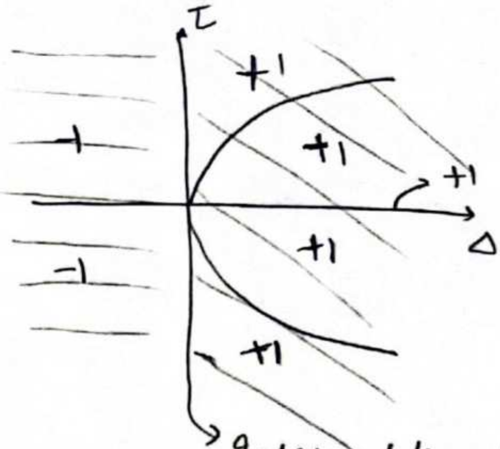
Index of node $I_{node} = +1$ for both stable & unstable.

$I_{saddle} = -1$

$I_{spiral} = +1$

$I_{center} = +1$

$I_{ordinary\ pt.} = 0$



Index undefined since FPs are non isolated

Eg:- $\dot{z} = z^2$ & take $z = x+iy$.

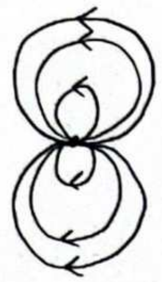
$\Rightarrow \dot{x}+i\dot{y} = x^2-y^2+i(2xy)$

$\Rightarrow \left.\begin{array}{l}\dot{x} = x^2-y^2 \\ \dot{y} = 2xy\end{array}\right\}$ linearization gives $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$ useless!

There is a single point with Index = 2 or -2 (not sure)

In general $\dot{z} = z^n$ gives more exotic fixed points. Other case is $\dot{z} = \bar{z}^n$

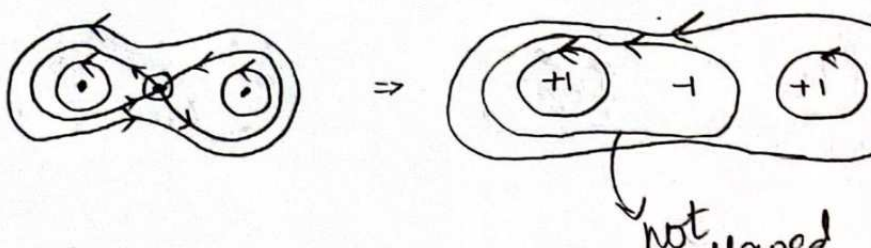
Eg:-



$I = 2$

Theorem: Any closed trajectory on $\mathbb{R}^2$ must enclose fixed points whose indices obey $\sum_{k=1}^{N} I_k = +1$. Fixed points must be isolated

Eg:-



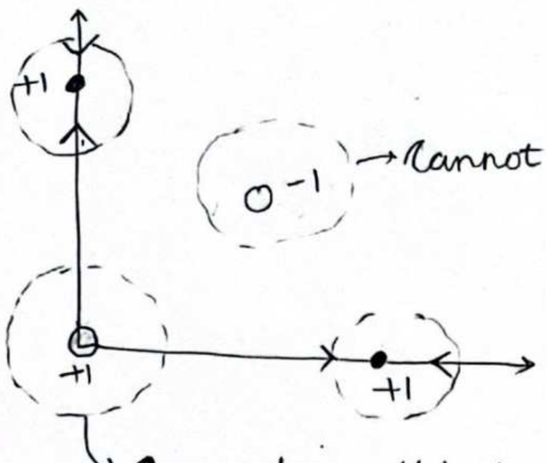
not allowed

Proof: $I_c = +1$ for closed trajectory & rule iii)

Eg: Can use index theory sometimes to rule out closed trajectories.

Rabbits vs. Sheep.

$\dot{x} = x(3-x-2y) \qquad \dot{y} = y(2-x-y)$



Cannot have this since it crosses a trajectory & same for any other closed curves.

→ L. Glass, Science, 1977, vol. 198, 321.

> If you cut off salamander hands they grow back. If you cut off the right hand & attach it to the left side (after cutting the hand off on the left side), the salamander grows 2 left hands to preserve the "index" of the left arm.

→ Hairy Ball Theorem

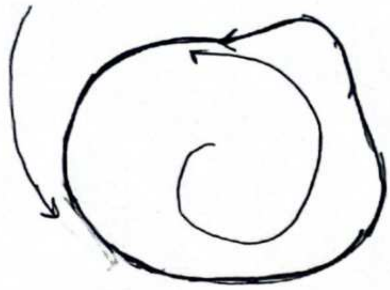
→ The index of a spherical surface is always 2.

→ ⇒ If the vectors were hair, they could not be combed down without zeros ⇒ no vectors there. In a taurus (dough nut) shape you can.

→ In general the index of a surface $I = 2 - 2g$. Where g is the genus. g = # handles. Sphere ⇒ no handles. Taurus ⇒ $g = 1$

→ Poincare Hopf index theorem: $\Sigma I_k = 2 - 2g$.

Chapter 7 Limit Cycles - Isolated closed trajectories!



→ Could have unstable & half stable cycles.

Eg: Heartbeat, biological rythms (bodytemperature, hormones), feedback control systems, aeroelastic flutter, mechanical vibrations, chemical oscillations.

* Linear systems DO NOT have limit cycles.
↳ $\dot{\vec{x}} = A\vec{x}$ ⇒ periodic solutions are not isolated.

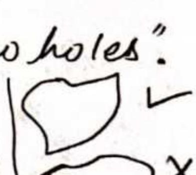
Since if $\vec{x}(t)$ is periodic, so is $c\vec{x}(t)$ for any c.
Eg:- Simple Harmonic Oscillator (remembers variations forever).

Lec 9 - MAE: Testing for closed orbits.

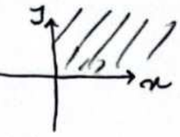
Ruling out closed orbits.

a) Index theory:- A closed orbit in $\mathbb{R}^2$ must encircle fixed points whose indices add up to +1.
Less powerful.

b) Dulac's criterion:- Let $\dot{\vec{x}} = \vec{f}(\vec{x})$ smooth, $\vec{x} \in \mathbb{R}^2$, R = "simply connected" region in $\mathbb{R}^2$. Simply connected ⇒ "no holes".
If $\exists$, a smooth function $g(\vec{x})$ such that $\nabla \cdot (g\dot{\vec{x}})$ has one sign in R, then $\nexists$ a closed orbit within R.



⇒ $\nabla \cdot (g\dot{\vec{x}})$ is strictly +ve or strictly -ve in R. Cannot be 0.

<u>Ex:</u> Show that, $\dot{x} = x(2-x-y)$
$\dot{y} = y(4x-x^2-3)$ has no closed orbits in the region $x>0, y>0$.  The problem is coming up with a function g!

<u>Solution</u>: Pick $g = \frac{1}{xy}$ (usually through trial & error).

$$\nabla\cdot(g\vec{\dot{x}}) = \frac{\partial}{\partial x}(g\cdot\dot{x}) + \frac{\partial}{\partial y}(g\cdot\dot{y}) = \frac{\partial}{\partial x}\left(\frac{2-x-y}{y}\right) + \cancelto{0}{\frac{\partial}{\partial y}\left(\frac{4x-x^2-3}{x}\right)}$$

$\Rightarrow \nabla\cdot(g\vec{\dot{x}}) = -\frac{1}{y} < 0$ in R $\Rightarrow$ no closed orbits.

<u>Achilles heel</u>: Hard to guess g. Try $g=1$, $g=\frac{1}{xy}$, $g=\frac{1}{x^a y^b}$, e^{kx}, $e^{\ldots}$

<u>Proof of Dulac</u>: Proof by contradiction.
Let T = closed orbit in the region R, A = region inside T.
$\vec{\dot{x}}$ is tangential to T

<u>Green's Theorem</u>:
<u>Div Thm</u>

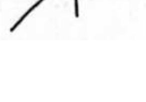
$$\iint_A \nabla\cdot\vec{F}\,dA = \oint_C \vec{F}\cdot\vec{n}\,dl.$$

$$= \underbrace{\iint_A \underbrace{\nabla\cdot(g\vec{\dot{x}})}_{\text{has one sign}}\,dA}_{\text{has same sign}} = \oint_C \underbrace{g\vec{\dot{x}}\cdot\vec{n}\,dl}_{\text{always 0 since } \vec{\dot{x}} \,\&\, \vec{n} \text{ are } \perp^{\text{lar}}}.$$

$\Rightarrow$ There cannot be such a closed trajectory since sign = 0 is a contradiction.

<u>Ex</u>: $\dot{x} = y$; $\dot{y} = -x-y+x^2+y^2$, let $g = e^{-2x}$ (magic!)

$$\frac{\partial}{\partial x} y e^{-2x} + \frac{\partial}{\partial y}\left[(\cancel{-x}-y+\cancel{x^2}+y^2)e^{-2x}\right] = -2ye^{-2x} + e^{-2x}(+2y-1)$$

$= -e^{-2x}$  always $-$ve $\Rightarrow$ this system has no closed orbits anywhere!

Section 7.3 **How to prove the existence of a closed orbit.**

Poincaré - Bendixson Theorem. (valid in 2-D).

I) > Suppose we have a closed bounded region in $\mathbb{R}^2$.

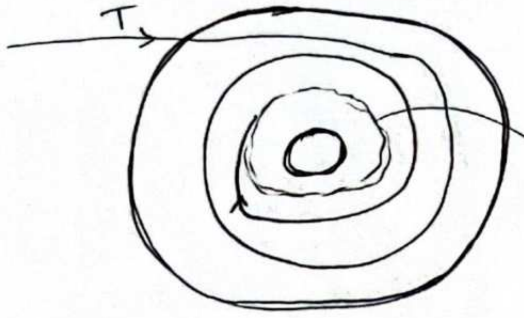
II) > $\dot{\vec{x}} = \vec{f}(\vec{x})$ is smooth.

III) > There are no fixed points in the region R.

IV) $\exists$ a trapped trajectory T, i.e. $T = (x(t), y(t))$ lies in R for $t = t_0$ and stays in R for all $t > t_0$.

Then: T is either a closed trajectory, or T spirals toward a closed trajectory. as time goes to ∞. Either way there is a closed trajectory in the region.

> The region R must be "washer" (annulus) shaped to contain trapped trajectories without containing fixed points.



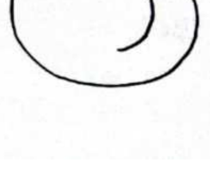
→ closed orbit.

Standard trick : Find an annulus such that the vector field points into the annulus on its boundary. In this case all trajectories are stuck inside.

Ex: Consider $\dot{r} = r(1-r^2) + \mu r\cos\theta$ and $\dot{\theta} = 1$, when

$\mu = 0$; $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1 \Rightarrow$ decoupled.

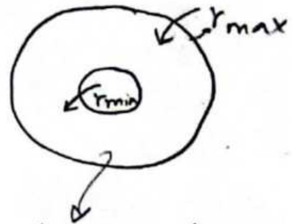
$\Rightarrow$



$\Rightarrow$ stable limit cycle at $r = 1$

→ Show that the closed orbit still exists if μ is small but not zero.

<u>Sol'n</u>:- Find 2 circles with inward flow towards R.



Also called trapping region.

Choose r_{max} s.t $\dot{r}<0$ when $r=r_{max}$

$$\dot{r} = r\left((1-r^2)+\mu\cos\theta\right)$$

$r_{max} > \sqrt{1+\mu} \Rightarrow \dot{r}<0$ since $\max(\cos\theta)=1$

$r_{min} < \sqrt{1-\mu} \Rightarrow \dot{r}>0$ & μ is small $\Rightarrow \mu<1$.

$\Rightarrow$ There is a closed orbit between r_{max} & r_{min}.

<u>Ex</u>: <u>Glycolysis</u>

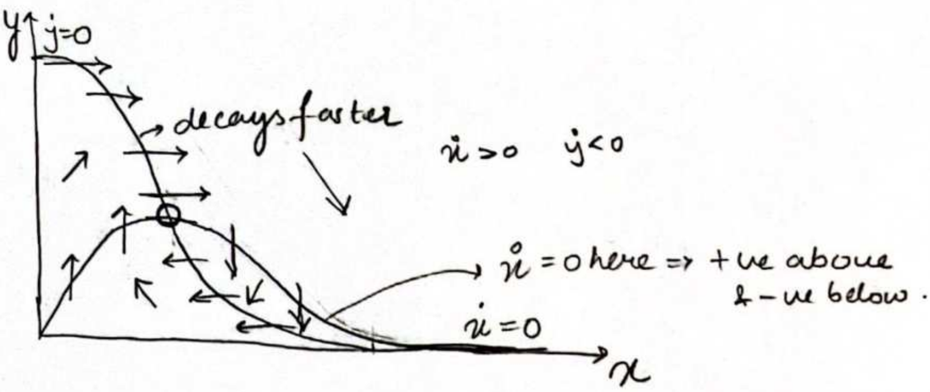
$$\dot{x} = x + ay + x^2y$$
$$\dot{y} = b - ay - x^2y$$

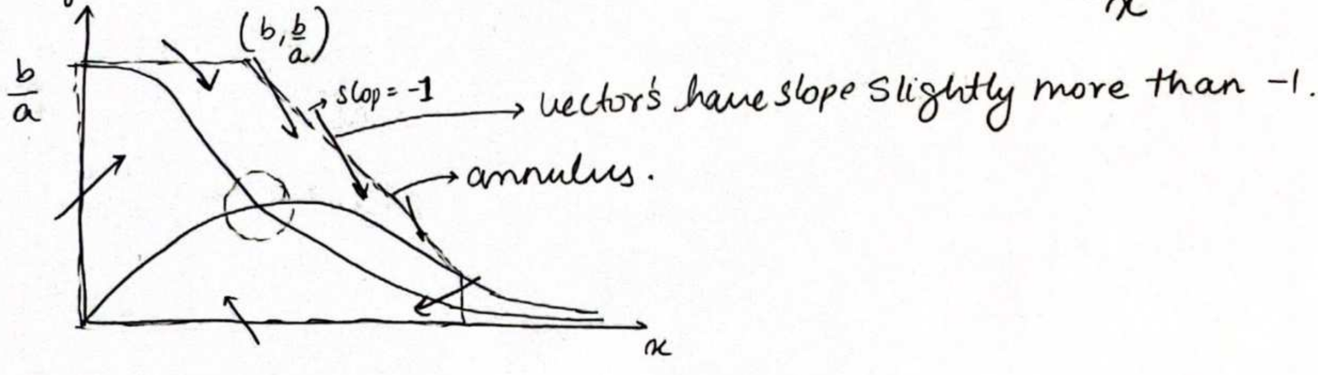
$x = [ADP] \rightarrow$ adinosene di phosphate.
$y = [F6P] \rightarrow$ fructose 6 phosphate
$a, b > 0$ Kinetic parameters.
$x, y > 0$ ($\because$ conc.)

① Construct a trapping region, using <u>nullclines</u>.

$\dot{x} = 0 \Rightarrow y = \dfrac{x}{a+x^2}$

$\dot{y} = 0 \Rightarrow y = \dfrac{b}{a+x^2}$





> Intuition on how to arrive at the annulus/trapping region

$x, y \gg 1 \Rightarrow \dot{x} \simeq x^2 y \quad \dot{y} = -x^2 y$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} \cong -1 \Rightarrow \dfrac{dy}{dx} \cong -1$ when $x, y \gg 1$

Compare $\dot{x}$ and $-\dot{y}$: $\dot{x} - (-\dot{y}) = \dot{x} + \dot{y} = b - x$

$\Rightarrow -\dot{y}$ is $>$ $\dot{x}$ if $x > b$. $\Rightarrow$ when $x > b \Rightarrow$ vectors have a slope steeper than -1.

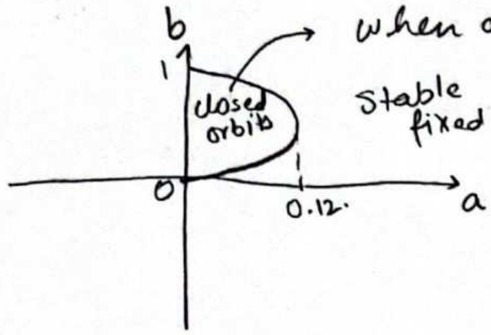
> What is happening around the hole? We want the fixed pt. to be a repeller.

$$A_{\text{Jacobian}} = \begin{bmatrix} -1 + 2xy & a + x^2 \\ -2xy & -a + x^2 \end{bmatrix} \qquad x^* = b, \quad y^* = \frac{b}{a+b^2}$$

$\Rightarrow \Delta = (a+b)^2$, $\tau = -\dfrac{(b^4 + (2a-1) b^2 + (a + a^2))}{a + b^2}$, we want $\tau > 0$

$\Delta > 0$

since we want a repeller. $\Rightarrow b^2 = \frac{1}{2}(1 - 2a \pm \sqrt{1 - 8a})$. is when $\tau = 0$



when a & b live here $\tau > 0 \Rightarrow$ hole is repelling $\Rightarrow$ we get a closed orbit.

Lec-10 Van der Pol oscillator

> $\ddot{x} + \mu \dot{x}(x^2-1) + x = 0$

> Nonlinear damping, x is large ⇒ damping is +ve & oscillations decay. x is small ⇒ negative damping ⇒ pumping!

> Using Poincaré-Bendixson, we can prove $\exists!$ stable limit cycle for all $\mu > 0$.
uniquely exists

> Here let us look at limiting cases for μ to make it simpler.
⇒ Very nonlinear and weakly nonlinear cases.

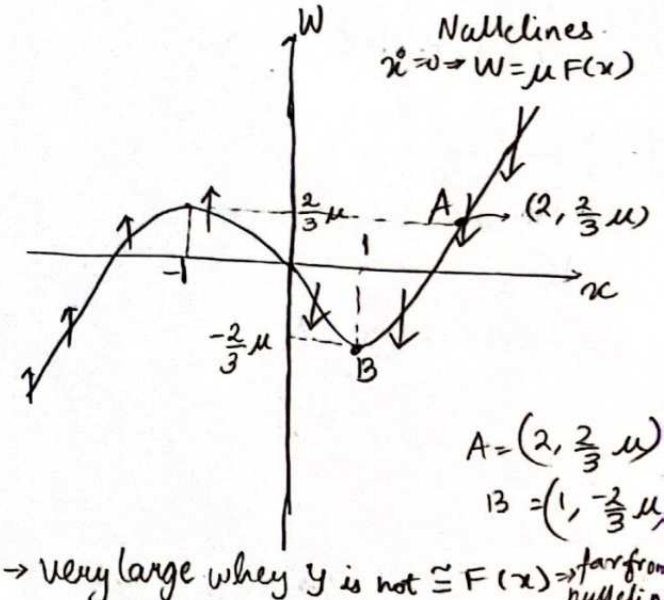
Very nonlinear, μ large ⇒ relaxation oscillators § 7.5

> Use the Liénard Transformation. Causes the limit cycle to approach a constant shape as $\mu \to \infty$. (Very useful!)

> Note: $\ddot{x} + \mu\dot{x}(x^2-1) = \frac{d}{dt}\left(\dot{x} + \mu\frac{1}{3}x^3 - \mu x\right)$. Let $w = \dot{x} + \mu F(x)$

⇒ $\dot{w} = -x$ from VdP

⇒ $\boxed{\begin{aligned}\dot{x} &= w - \mu F(x)\\ \dot{w} &= -x\end{aligned}}$

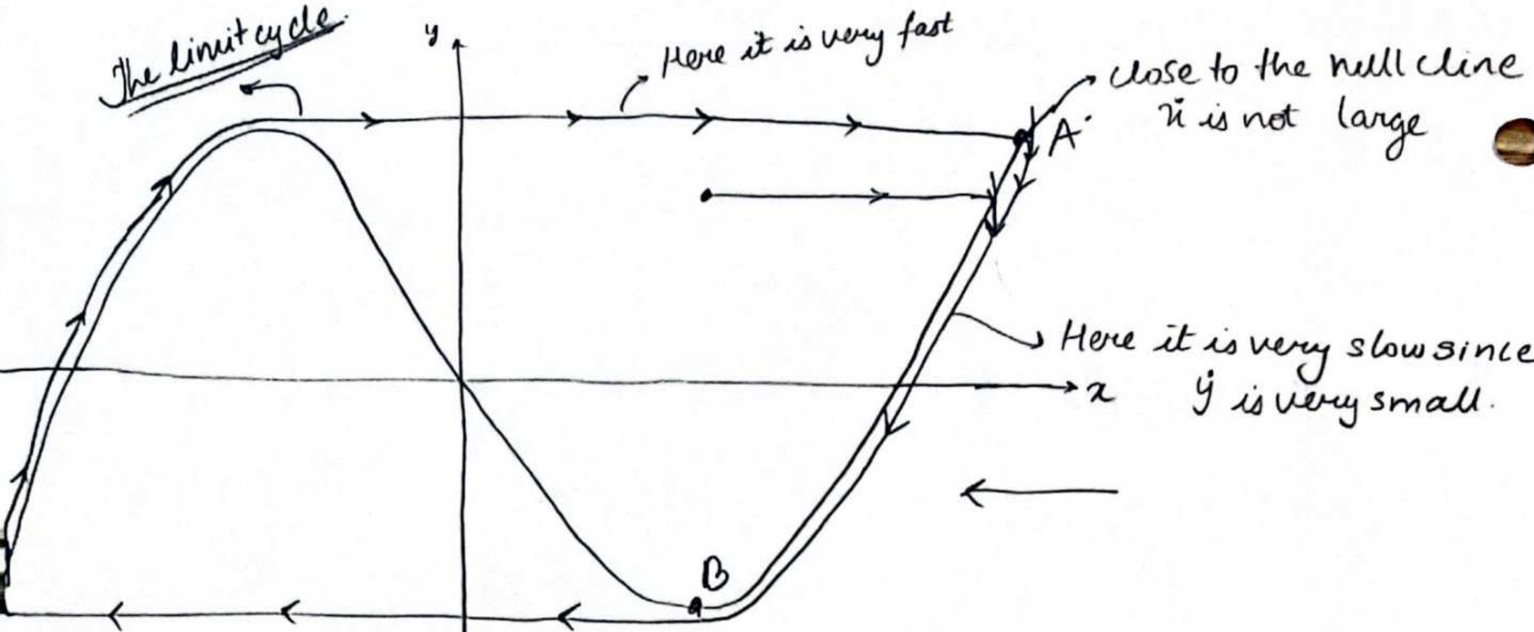


$A = (2, \frac{2}{3}\mu)$
$B = (1, -\frac{2}{3}\mu)$

→ Let $\boxed{y = \frac{w}{\mu}}$ to rescale, since A, B have μ in their y coordinate & $\mu \to \infty$.
$y \sim O(1)$ for large μ

$\dot{x} = \mu(y - F(x))$
$\dot{y} = -\frac{1}{\mu}x$

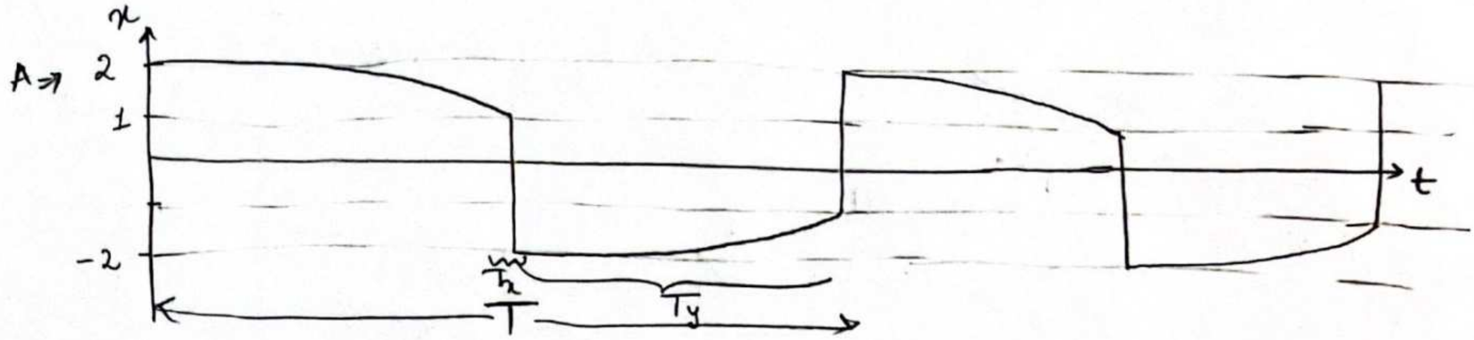
⇒ $\dot{x} \sim O(\mu) \to$ very large when y is not $\cong F(x)$ ⇒ far from nullcline
$\dot{y} \sim O(\frac{1}{\mu}) \to$ very small. also $\dot{y}$ is −ve when x is +ve
⇒ all vectors basically point ⟶



- Since there is a slow build up & fast release we see relaxation oscillation.

$\dot{x} = \mu(y - F(x)) \Rightarrow \dot{x} < 0$ iff $y < F(x)$. $\Rightarrow$ x moves to the left only when $y < F(x)$ therefore the trajectory hugs the curve on the outer side.

- Waveform!



- y has velocity $\sim O(\frac{1}{\mu})$ & distance $\sim O(1) \Rightarrow \underset{T_y}{\text{Time}} \sim O(\mu)$,
& $T_x \sim O(\frac{1}{\mu}) \Rightarrow$ Period of oscillation is $\sim O(\mu)$.

→ Estimate period $T(\mu)$: $T \sim 2$(time to go from A to B).

Use fact that $w \simeq \mu f(x) \Rightarrow$ we are essentially on the nullcline.

$$T = 2\int_{t_A}^{t_B} dt \approx -2\int_1^2 \frac{dt}{dw}\frac{dw}{dx}\, dx$$

$\dot{w} = -x$; $\frac{1}{\dot{w}} = \frac{-1}{x}$; $\frac{dw}{dx} \cong \frac{d}{dx}(\mu F(x)) = \mu F'(x) = \mu(x^2-1)$

$$\Rightarrow T \simeq -2\int_1^2 \frac{-1}{x}\,\mu(x^2-1)\,dx$$

$$= 2\mu\left(\tfrac{1}{2}x^2 - \ln x\right)\Big|_1^2$$

$$T = \boxed{2\mu\left(\tfrac{3}{2} - \ln 2\right)} = O(\mu)$$

Weakly Nonlinear Van der Pol ($\S$ 7.6) $\mu \ll 1$ More mathy

$$\ddot{x} + x + \epsilon\dot{x}(x^2-1) = 0 \qquad 0 \le \epsilon \ll 1$$

- When $\epsilon = 0$, we have harmonic oscillator. Here freq is 1.
  & all orbits are circular, period $= 2\pi$.

$x(t) = A\cos t$, $A =$ constant & depends on initial condition.

- For small ϵ, except all orbits to be nearly circular & since centers are so sensitive, they are spirals that repeat every 2π.
- How to find the limit cycle & what's the amplitude.

Method 1 (crude): Look at the change in energy over 1 cycle. ΔE.
On cycle, we have $\Delta E = 0$. Other trajectories have $\Delta E > 0$, $\Delta E < 0$.

- $E = \frac{1}{2}x^2 + \frac{1}{2}\dot{x}^2$; $\frac{dE}{dt} = x\dot{x} + \dot{x}\ddot{x} = \dot{x}(x+\ddot{x}) = \dot{x}(-\epsilon\dot{x}(x^2-1))$

- Now suppose $x = A\cos t + O(\epsilon)$, here A to be determined.
  $\Rightarrow \dot{x} = -A\sin t + O(\epsilon)$

The change in energy over 1 cycle; $\Delta E = \int^{T} \frac{dE}{dt}\,dt = -\epsilon\int^{2\pi + O(\epsilon)} A^2\sin^2 t\,(A^2\cos^2 t - 1)$

Due to $O(\epsilon^2)$ we can ignore $O(\epsilon)$ in the limit of integration.

$$\Rightarrow \quad \Delta E = -\epsilon \left[A^4 \underbrace{\langle \sin^2 t \cos^2 t \rangle}_{\frac{1}{8}} 2\pi - A^2 \underbrace{\langle \sin^2 t \rangle}_{\frac{1}{2}} 2\pi \right]$$

$$= -2\pi \epsilon A^2 \left[\frac{A^2}{8} - \frac{1}{2} \right]$$

$\Rightarrow \quad A^2 = 4 \Rightarrow \boxed{\boxed{A=2}}$ gives $\Delta E = 0 \Rightarrow$ Energy is conserved $\Rightarrow$ limit cycle.

$\Rightarrow$ VdP for weakly nonlinear case has an amplitude 2.

$$\Rightarrow \boxed{x(t) = 2\cos t + O(\epsilon).}$$

Lec 11 Averaging Theory for weakly Nonlinear Oscillators.

> Consider: $\ddot{x} + x + \epsilon h(x, \dot{x}) = 0$, $\epsilon \ll 1$

Small perturbation to linear SHO.

In the book he presents seperation of timescales which is different.

$\dot{x} = y$

$\dot{y} = -x - \epsilon h(x, y)$

> When $\epsilon = 0$, soln's of the form $\left.\begin{array}{l} x(t) = r\cos(t+\phi) \\ y(t) = -r\sin(t+\phi) \end{array}\right\}$ r, ϕ are constant on trajectories.

> When $\epsilon \neq 0$, we expect slow drift of r, ϕ, but trajectories will stay nearly circular with a "period" $\simeq 2\pi$.

* > Let $\left.\begin{array}{l} x(t) = r(t)\cos(t+\phi(t)) \\ y(t) = -r(t)\sin(t+\phi(t)) \end{array}\right\}$ View this as a definition of $r(t)$ and $\phi(t)$. i.e. $r(t) = \sqrt{x^2(t) + y^2(t)}$

$\tan(t+\phi(t)) = \dfrac{-y(t)}{x(t)}$

> Find eqns for $\dot{r}, \dot{\phi}$: $r^2 = x^2 + y^2 \Rightarrow r\dot{r} = x\dot{x} + y\dot{y}$

$$= xy + y(-x - \epsilon h)$$

$$= -\epsilon y h = -\epsilon h(-r\sin(t+\phi))$$

$$\Rightarrow \boxed{\dot{r} = \epsilon h \sin(t+\phi)}$$

Similarly,

since $t + \phi(t) = \arctan^{-1} \frac{y(t)}{x(t)}$.

$\frac{d}{dt}$ on both sides gives,

$$** \boxed{\dot{\phi} = +\frac{\epsilon h}{r} \cos(t+\phi)} **$$

Both $\dot{r}$ and $\dot{\phi}$ are order $\epsilon \Rightarrow$ evolution is slow..

$\Rightarrow h = h(x,y) = h(r\cos(t+\phi), -r\sin(t+\phi)) \longrightarrow$ it is time dependant

$\Rightarrow$ We have made our system <u>non autonomous</u>! $\Rightarrow$ time dependant

$\Rightarrow$ The arrows in phase plane are no longer frozen.

> To fix this we use seperation of time scales. Since time scale for oscillations are much faster than time scale for $\dot{r}$ & $\dot{\phi}$.

$\rightarrow$ Iron out fast oscillations by averaging over one cycle of length 2π. Given $g(t)$, define average over one cycle about the point t as

$$\bar{g}(t) = \langle g \rangle (t) = \frac{1}{2\pi} \int_{t-\pi}^{t+\pi} g(s)\, ds.$$

$\rightarrow$ Time scale of the faster time

$\rightarrow$ This is a running average. As t moves along we have a window around it that averages things. "Moving avg"

$\rightarrow$ Taking my height as an example. the instantaneous value of my height is equal to the moving average if the time scale is on the order of seconds or higher. Therefore as long as the timescale is approriately chosen we can replace a variable by its average. The error here is within an epsilon order in one cycle.

<u>Observe:</u> $\bar{\dot{g}} = \dot{\bar{g}}$ because $\bar{\dot{g}} = \frac{1}{2\pi}\int_{t-\pi}^{t+\pi} \frac{dg}{ds}\,ds.$

$$= \frac{1}{2\pi}\left[g(t+\pi) - g(t-\pi)\right]$$

> Recall the formula of taking time derivative when limits have 't'. from
> Fundamental theorem of Calculus.

$$\& \ \dot{\bar{g}} = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{t-\pi}^{t+\pi} g(s)\,ds\right]$$

$$= \frac{1}{2\pi}\left[g(t+\pi) - g(t-\pi)\right]$$

> <u>Derive equations for $\bar{r}$, $\bar{\phi}$.</u>

$$\bar{\dot{r}} = \dot{\bar{r}} = \langle \epsilon h \sin(t+\phi)\rangle_{t} \rightarrow \text{avg. w.r.t } t.$$

$$\bar{\dot{\phi}} = \dot{\bar{\phi}} = \left\langle \frac{\epsilon h}{r}\cos(t+\phi)\right\rangle_t$$

So far everything is exact.

> But we want $\dot{\bar{r}}$ & $\dot{\bar{\phi}}$ to depend on $\bar{r}$ and $\bar{\phi}$. This would not be exact.

> Let us now replace r & ϕ by $\bar{r}$ and $\bar{\phi}$ which incurs some error

> Over one cycle, $r = \bar{r} + O(\epsilon)$, $\phi = \bar{\phi} + O(\epsilon)$.

> Replacing r, ϕ by $\bar{r}$ & $\bar{\phi}$ everywhere & the $O(\epsilon)$ error becomes $O(\epsilon^2)$ since it multiplies with ϵ everywhere.

> <u>Beauty of it</u>: We now have autonomous equations we can analyze using phase plane methods.

> We can treat $\bar{r}$ & $\bar{\phi}$ as not only equal to r & ϕ but they are also constant within the integral over one cycle of short or fast time scale. Like the height over one second is constant but over 5 years is not.

Approximation

- Treat $\bar{r}, \bar{\phi}$ as constants when performing these averages $\langle\cdot\rangle_t$.

Eg: Van der Pol.

$$\ddot{x} + x + \epsilon\dot{x}(x^2-1) = 0 \quad \Rightarrow h = \dot{x}(x^2-1) = y(x^2-1)$$

$$h = y(x^2-1)$$

$$= -r\sin(t+\phi)\left[r^2\cos^2(t+\phi)-1\right]$$

$$h = -\bar{r}\sin(t+\bar{\phi})\left[\bar{r}^2\cos^2(t+\bar{\phi})-1\right] + O(\epsilon)$$

↳ Notice it is explicitly time dependent

- $$\dot{\bar{r}} = \langle \epsilon h \sin(t+\bar{\phi})\rangle + O(\epsilon^2)$$

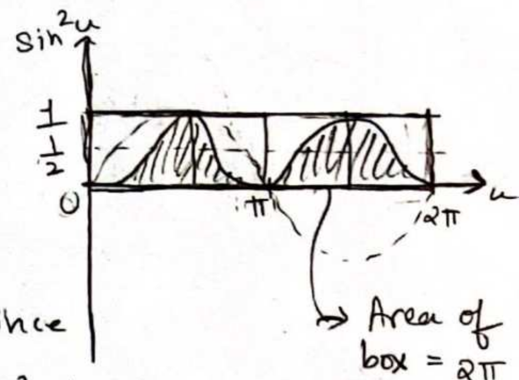
$$= \langle -\epsilon\bar{r}\sin^2(t+\bar{\phi})\left[\bar{r}^2\cos^2[t+\bar{\phi}]-1\right]\rangle + O(\epsilon^2).$$

This also includes $O(\epsilon)$ from h since it multiplies with ϵ

$$= -\epsilon\bar{r}\left(\bar{r}^2\langle\sin^2\cos^2\rangle - \langle\sin^2\rangle\right) + O(\epsilon^2).$$

Average of $\sin^2$

$$\langle\sin^2\rangle = \frac{1}{2\pi}\int_{t-\pi}^{t+\pi}\sin^2 s\, ds = \frac{1}{2}$$



⇒ Area of box = 2π

⇒ Area of integral = π from Symmetry in each box of $\frac{\pi}{2}$ width

⇒ $\langle\sin^2\rangle = \frac{1}{2}$

Another way to see it is $\langle\sin^2\rangle = \langle\cos^2\rangle$ since they are only out of phase. Therefore $2\langle\sin^2\rangle = 1$

$$\Rightarrow \langle\sin^2\rangle = \frac{1}{2}$$

- $$\langle\sin u\cos u\rangle = \frac{1}{2}\langle\sin 2u\rangle$$

$$\Rightarrow \langle\sin^2 u\cos^2 u\rangle = \frac{1}{4}\langle\sin^2 2u\rangle = \frac{1}{4}\left[\frac{1}{2}\right] = \underline{\frac{1}{8}}$$

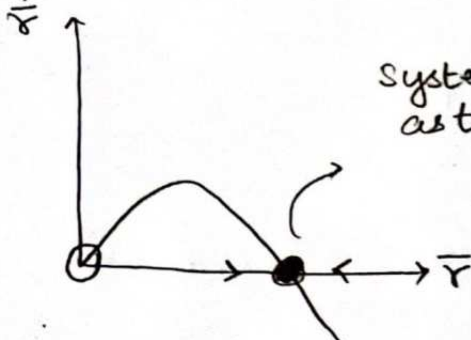
$$\therefore \dot{\bar{r}} = -\epsilon \bar{r} \left(\bar{r}^2 \cdot \frac{1}{8} - \frac{1}{2} \right) + O(\epsilon^2).$$

$$\therefore \dot{\bar{r}} = -\frac{\epsilon}{8} \bar{r}^3 + \frac{\epsilon}{2} \bar{r}$$

<u>Idea Summary</u>

> Since we only want to look at dynamics of $\dot{r}$ & $\dot{\theta}$ which change over a slow time scale (like height over a year) we can replace them with $\bar{\dot{r}}$ & $\bar{\dot{\phi}}$ (= $\dot{\bar{r}}$ & $\dot{\bar{\phi}}$) where the average is taken over the fast time scale (like height replaced by $\overline{\text{height}}$ over one second). This averaging does not really affect r & θ but kills the fast timescale fluctuations (which was the oscillation). Basically by replacing $r \to \underset{\text{or } \bar{r}}{\langle r \rangle}$ and $\phi \to \underset{\text{or } \bar{\phi}}{\langle \phi \rangle}$ they are "protected" since averaging does not affect them. It does however affect stuff that changes fast & irons them out.

$$\boxed{\dot{\bar{r}} = \frac{\epsilon \bar{r}}{8} (4 - \bar{r}^2)}$$



System $\Rightarrow \bar{r} \to 2$ as $t \to \infty$.

$\Rightarrow$ Amplitude of vdP $= 2$.

> We could also explicitly solve for $\bar{r}$ as a fn of t which is very powerful.

> Eg: $x(0) = 1$; $\dot{x}(0) = 0$, then $\bar{r}(0) = 1 \Rightarrow \boxed{\bar{r}(t) = \frac{2}{\sqrt{1 + 3e^{-\epsilon t}}}}$

notice t is order $\frac{1}{\epsilon}$ to show results.

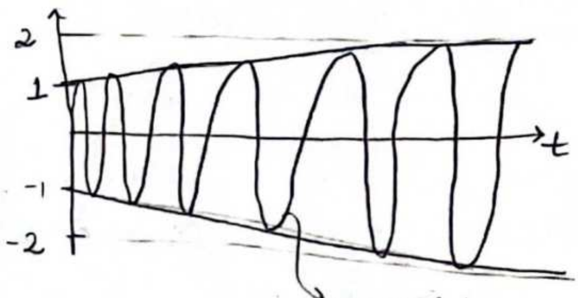
$$\dot{\bar{\phi}} = \langle \epsilon \frac{h}{r} \cos(t+\phi) \rangle = -\epsilon \langle r^2 \langle \cos^3 \sin \rangle - \langle \sin \cos \rangle$$

$$= \int_0^{2\pi} \cos^3 u \sin u \, du = -\cos^4 u \Big|_0^{2\pi} = 0$$

$\Rightarrow \boxed{\bar{\phi} = 0}$ $\Rightarrow \dot{\phi} = O(\epsilon^2) \Rightarrow$ superslow timescale. It is basically constant. Therefore the <u>period of the</u> vdP is almost 2π

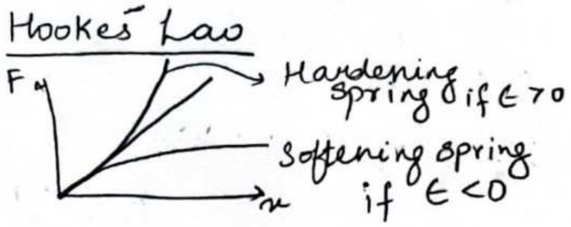
$$x(t) = r(t)\cos(t+\phi(t)).$$

$$\Rightarrow x(t) = \frac{2}{\sqrt{1+3e^{-\epsilon t}}}\cos t + O(\epsilon^2)$$



Eg: Duffing's Equation.

$$\ddot{x} + x + \epsilon x^3 = 0$$



$$h = x^3$$
$$= r^3\cos^3(t+\phi).$$

$$\dot{r} = \langle \epsilon h \sin(t+\phi)\rangle$$

$$= \langle \epsilon \underbrace{\cos^3(t+\phi)\sin(t+\phi)\rangle}_{=0} r^3$$

Physically if $\epsilon>0$
$\Rightarrow$ harder spring decreases timeperiod $\Rightarrow \phi$ changes faster as r is increased

$$\Rightarrow \dot{r} = O(\epsilon^2).$$

$$\dot{\phi} = \langle \epsilon \frac{h}{r}\cos(t+\phi)\rangle = \epsilon r^2\langle\cos^4\rangle + O(\epsilon^4)$$
$$= \frac{3}{8}\epsilon r^2$$

$\theta = \phi + t$ (θ fast, ϕ slow, t fast)
$\dot{\theta} = 1$ since $\dot{\phi} \approx 0$ over fast time scale.

$$\boxed{\dot{\phi} = \frac{3}{8}\epsilon r^2}$$

$$\omega = \frac{d}{dt}[\underbrace{t+\phi}_{\theta}] = 1+\dot{\phi} = 1+\frac{3}{8}\epsilon r^2$$

$\Rightarrow$ Frequency increases by $\frac{3}{8}\epsilon r^2$ due to cubic term. $\Rightarrow T = 2\pi\left(1-\frac{3}{8}\epsilon r^2\right) + O(\epsilon^2)$

→ Averaging irons out the fast oscillations contributions to $\dot{r}$ and $\dot{\phi}$ and keeps the slow variations of r & ϕ.

Recipe: $x(t) = r(t)\cos(t+\phi(t))$ $\quad y(t) = -r(t)\sin(t+\phi(t))$

$$\boxed{\dot{r} = \langle\epsilon h \sin(t+\phi)\rangle \quad \dot{\phi} = \langle\frac{\epsilon h}{r}\cos(t+\phi)\rangle}$$

where h is coefficient of ϵ in $\ddot{x} + x + \epsilon h(x,\dot{x}) = 0$.

We saw $\dot{r} = \epsilon h \sin(\phi + t)$

$\dot{\phi} = \frac{\epsilon h}{r} \cos(\phi + t)$.

$\overline{\dot{r}} = \langle \epsilon h \sin(\phi + t) \rangle$

$\overline{\dot{\phi}} = \langle \frac{\epsilon h}{r} \cos(\phi + t) \rangle$

$\therefore \overline{\dot{r}} = \dot{\bar{r}}$

$\Rightarrow \dot{\bar{r}} = \langle \epsilon h \sin(\phi + t) \rangle$

$\dot{\bar{\phi}} = \langle \frac{\epsilon h}{r} \cos(\phi + t) \rangle$

average over fast time

$T = \epsilon t$

slow

fast

$\therefore \quad r' = \frac{dr}{dT} = \frac{dr}{d\epsilon t} = \frac{1}{\epsilon}\frac{dr}{dt} = \frac{1}{\epsilon}\dot{r} = \frac{1}{\epsilon}\dot{\bar{r}}$

replace instantaneous with avg over fast time T.

$\phi' = \frac{1}{\epsilon}\dot{\bar{\phi}}$

$$\Rightarrow \boxed{\begin{aligned} r' &= \langle h(\theta) \sin(\theta) \rangle \\ \phi' &= \langle \frac{h(\theta)}{r} \cos(\theta) \rangle \end{aligned}}$$

$\langle \; \rangle = \frac{1}{2\pi}\int_0^{2\pi} \cdots \, d\theta$

$\theta = \phi + t$

fast

slow

fast

derivative w.r.t slow time.

$\Rightarrow$ How do r & ϕ change over long periods of time.

Perturbation theory for weakly nonlinear oscillators.

$$\ddot{x} + x + \epsilon h(x, \dot{x}) = 0 \quad \text{——— } ①$$

Seek solutions of ① as a power series in ϵ.

$$x(t,\epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots \quad \text{— } ②$$

$x_k(t)$ are to be determined.

→ Idea is that all the important information is contained in the first two terms. Called perturbation theory.

Eg: $\ddot{x} + 2\epsilon\dot{x} + x = 0$ — ③ initial conditions: $x(0) = 0$, $\dot{x}(0) = 1$

The exact sol. is $x(t,\epsilon) = (1-\epsilon^2)^{-1/2} e^{-\epsilon t} \sin\left[(1-\epsilon^2)^{1/2} t\right]$

Using P.T

Substituting ② in ③

$$\frac{d^2}{dt^2}(x_0 + \epsilon x_1 + \dots) + 2\epsilon \frac{d}{dt}(x_0 + \epsilon x_1 + \dots) + (x_0 + \epsilon x_1 + \dots)$$

Grouping in terms of ϵ.

$$[\ddot{x}_0 + x_0] + \epsilon[\ddot{x}_1 + 2\dot{x}_0 + x_1] + O(\epsilon^2) = 0$$

→ Ignore this ← This makes it fail sometimes

Key Idea: Since the equation above is true for all small values of ϵ, each term individually must be 0.

$$O(1): \ddot{x}_0 + x_0 = 0$$

$$O(\epsilon): \ddot{x}_1 + 2\dot{x}_0 + x_1 = 0$$

Applying initial conditions,

$$x_0(0) + \epsilon\, x_1(0) + \dots = 0 \quad \underset{\substack{\text{argument}\\ \text{as key}\\ \text{idea}}}{\overset{\text{same}}{\Rightarrow}} \quad x_0(0)=0,\ x_1(0) = 0 \dots$$

Similarly

$$\dot{x}_0(0) + \epsilon\, \dot{x}_1(0) + \dots = 1 \quad \underset{\text{argument}}{\overset{\text{same}}{\Rightarrow}} \quad \dot{x}_0(0) = 1,\ \dot{x}_1(0) = 0$$

Solve $O(1)$ equation, $\ddot{x}_0 + x_0 = 0$ where $x_0(0)=0$, $\dot{x}_0(0)=1$

$$\Rightarrow x_0(t) = \sin t$$

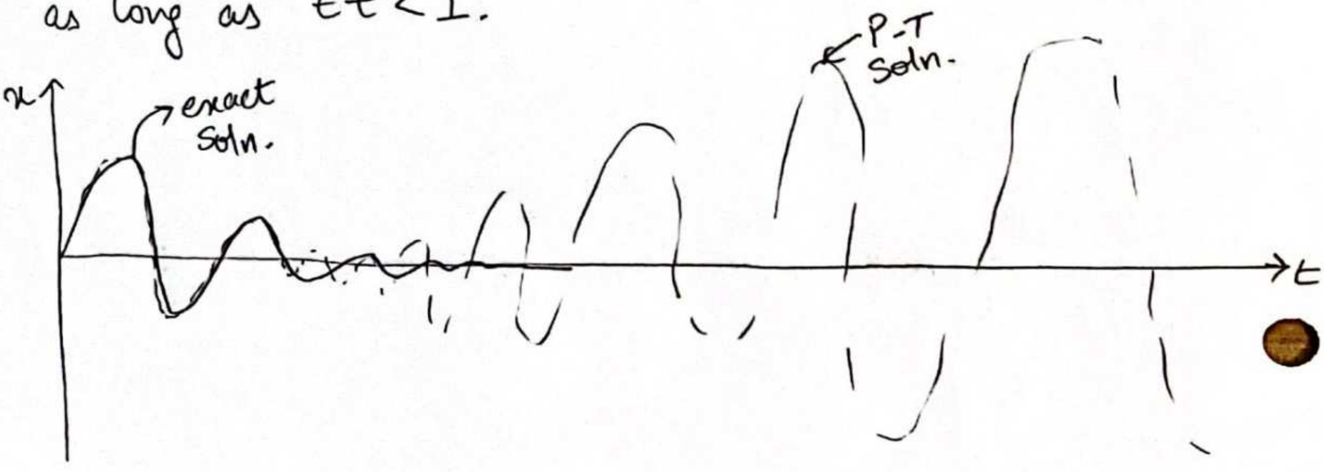
Plug into $O(\epsilon)$ equation $\Rightarrow \ddot{x}_1 + 2\cos t + x_1 = 0$ where $x_1(0)=0$, $\dot{x}_1(0)=0$

$$\Rightarrow x_1(t) = -t\sin t \rightarrow \text{blows up as } t \to \infty$$

$\hookrightarrow$ a.k.a secular term due to blow up.

$$\therefore\ x(t,\epsilon) = \sin t - \epsilon t \sin t + O(\epsilon^2).$$

These are infact the first two terms of the power series expansion of the actual solution which converges. This solution alone diverges and therefore fails. It only works as long as $\epsilon t < 1$.



Problems?

① > P.T Solution does not capture slow time scale ⇒ envelope behaviour. To get accurate results we need infinite terms in the series (impractical).

② > Original exact solution give frequency $\simeq 1 - \frac{1}{2}\epsilon^2$ whereas P.T gives frequency of 1. This is an error on a super slow time scale that builds up as $t \uparrow$.

Solution? Two timing.

Let $\tau = t$ denote fast time of $O(1)$.
Let $T = \epsilon t$ denote slow time of $O(\epsilon)$

Treat them as independant variables.

→ T is considered slow enough that functions in T are constant over τ.

Let $x(t, \epsilon) = x_0(\tau, T) + \epsilon x_1(\tau, T) + O(\epsilon^2)$

Chain rule ⇒ $\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial \tau} + \frac{\partial x}{\partial T}\frac{\partial T}{\partial t} = \frac{\partial x}{\partial \tau} + \epsilon \frac{\partial x}{\partial T}$

notation for $\frac{\partial}{\partial t}$

$\dot{x} = \partial_\tau x + \epsilon \partial_T x$

$\Rightarrow \dot{x} = \partial_\tau x_0 + \epsilon(\partial_T x_0 + \partial_\tau x_1) + O(\epsilon^2)$ —— ⑤

Similarly,

$\rightarrow \ddot{x} = \partial_{\tau\tau} x_0 + \epsilon(\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0) + O(\epsilon^2)$ —— ⑥

Also, $x = x_0 + \epsilon x_1 + O(\epsilon^2)$

<u>Example</u>. $\ddot{x} + 2\epsilon \dot{x} + x = 0$ $\quad x(0) = 0 \quad \dot{x}(0) = 1$

Substituting ⑤ & ⑥ here

$$\partial_{\tau\tau} x_0 + \epsilon(\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0) + 2\epsilon\,\partial_\tau x_0 + x_0 + \epsilon x_1 + O(\epsilon^2) = 0$$

$O(1)$: $\quad \partial_{\tau\tau} x_0 + x_0 = 0$

$O(\epsilon)$: $\quad \partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0 + 2\partial_\tau x_0 + x_1 = 0$

$O(1) \Rightarrow \quad x_0 = A\sin\tau + B\cos\tau.$

<u>Key Idea</u>.

Here the constants A, B are actually functions of T.

Substituting in $O(\epsilon)$

$$\partial_{\tau\tau} x_1 + x_1 = -2(A' + A)\cos\tau + 2(B' + B)\sin\tau. \quad — ⑦$$

where $A' = \dfrac{\partial A}{\partial T}$

RHS of ⑦ produces secular terms, we therefore use a standard procedure in two timing and set the coefficients to 0

$\Rightarrow$ "Set the coefficients of resonant terms to zero".

$\Rightarrow \quad A' + A = 0; \quad B' + B = 0. \quad \Rightarrow \quad A(T) = A(0)e^{-T}$

$\quad\quad B(T) = B(0)e^{-T}$

To find $A(0)$ and $B(0)$ we use initial conditions $x(0) = 0;\ \dot{x}(0) = 1$

$x(0) = x_0(0,0) + \epsilon x_1(0,0) + O(\epsilon^2) = 0.$ For all ϵ this is true implies that $x_0(0,0) = 0.$

Similarly, from 5

$$\dot{x}(0) = \partial_\tau x_0(0,0) + \epsilon(\partial_T x_0(0,0) + \partial_\tau x_1(0,0)) + O(\epsilon^2)$$

$$= 1.$$

$$\Rightarrow \quad \partial_\tau x_0(0,0) = 1.$$

$$x_0 = A\sin\tau + B\cos\tau$$

$$x_0(0,0) = \boxed{B(0) = 0}$$

$$\frac{\partial x_0}{\partial \tau} = +A\cos\tau - B\sin\tau$$ since A & B are constant w.r.t τ.

$$\frac{\partial}{\partial \tau} x_0(0,0) = A(0) = 1 \qquad \Rightarrow \begin{matrix} A(T) = e^{-T} \\ B(T) = 0 \end{matrix}$$

$$\Rightarrow \boxed{x_0 = e^{-T}\sin\tau}$$

$$\Rightarrow \quad x(t) = e^{-T}\sin\tau + O(\epsilon)$$

$$\boxed{x(t) = e^{-\epsilon t}\sin t + O(\epsilon).}$$

→ Surprisingly this approximates the exact solution very well!

> Could further solve for $x_1(t)$ to get $O(\epsilon)$ terms.

Example. Vander Pol. $\ddot{x} + x + \epsilon(x^2-1)\dot{x} = 0$

$$\dot{x} = \partial_\tau x_0 + \epsilon(\partial_T x_0 + \partial_\tau x_1) + O(\epsilon^2).$$

$$\ddot{x} = \partial_{\tau\tau} x_0 + \epsilon(\partial_{\tau\tau} x_1 + \partial_{T\tau} x_0) + O(\epsilon^2).$$

$$\Rightarrow \quad \partial_{\tau\tau} x_0 + \epsilon(\partial_{\tau\tau} x_1 + \partial_{T\tau} x_0) + x_0 + \epsilon x_1 + \epsilon(x^2-1)(\partial_\tau x_0 + \epsilon(\partial_T x_0 + \partial_\tau x_1)) + O(\epsilon^2) = 0.$$

For any ϵ $\Rightarrow$

$$\partial_{\tau\tau} x_0 + x_0 = 0. \quad \text{— (1)}$$

$$\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0 + x_1 + (x_0^2-1)(\partial_\tau x_0) = 0 \quad \text{— (2)}$$

From (1), $x_0 = A\sin\tau + B\cos(\tau)$

or $x_0 = r(T)\cos(\tau + \phi(T))$.

Substituting into (2),

$$\partial_{\tau\tau} x_1 + x_1 = -2\partial_{T\tau}\left[r(T)\cos(\tau+\phi(T))\right] - (x_0^2-1)\,\partial_\tau\left[r(T)\cos(\tau+\phi(T))\right]$$

$$= 2\partial_T r(T)\sin(\tau+\phi(T)) + (x^2-1)\, r(T)\sin(\tau+\phi(T))$$

$$= 2\partial_T r(T)\sin(\tau+\phi(T)) + (r^2\cos^2(\tau+\phi)-1)\, r\sin(\tau+\phi)$$

$$= 2(r'\sin(\tau+\phi) + r\phi'\cos(\tau+\phi))$$
$$+ (r^2\cos^2(\tau+\phi)-1)\, r\sin(\tau+\phi)$$

$$\partial_{\tau\tau} x_1 + x_1 = 2(r' \sin(\tau+\phi) + r\phi' \cos(\tau+\phi)) + r^3 \sin(\tau+\phi)\cos^2(\tau+\phi) - r\sin(\tau+\phi).$$

$$\sin\theta\cos^2\theta = \frac{1}{4}(\sin\theta + \sin 3\theta).$$

$$\Rightarrow \partial_{\tau\tau} x_1 + x_1 = \left(2r' - r + \frac{1}{4}r^3\right)\sin(\tau+\phi) + [2r\phi']\cos(\tau+\phi) + \frac{1}{4}r^3 \sin 3(\tau+\phi).$$

To avoid secular terms we have.

$$2r' - r + \frac{1}{4}r^3 = 0$$
$$2r\phi' = 0$$

$$\Rightarrow \quad r' = \frac{1}{8}r(4-r^2) \quad \Rightarrow \text{ limit cycle at } r^* = 2$$

Also $\phi' = 0 \Rightarrow \phi(T) = \phi_0 \Rightarrow \boxed{x(\tau, T) = 2\cos(\tau + \phi_0)}$.

$$\Rightarrow \quad x(t) \longrightarrow 2\cos(\tau+\phi_0) + O(\epsilon). \text{ as } t \to \infty$$

Let $\theta = \tau + \phi(T) \Rightarrow \omega = \frac{d\theta}{dt} = 1 + \frac{d\phi}{dT}\cdot\frac{dT}{dt} = 1 + \epsilon\phi' = \underline{\underline{1}}$

since $\phi \equiv 0$.

$\therefore \omega = 1 + O(\epsilon^2)$

General recipe for two timing.

$\ddot{x} + x + \epsilon h(x, \dot{x}) = 0.$

$O(1)$: $\partial_{\tau\tau} x_0 + x_0 = 0$

$O(\epsilon)$: $\partial_{\tau\tau} x_1 + x_1 = -2\partial_{\tau T} x_0 - h.$ where $h = h(x_0, \partial_\tau x_0)$

Solution of $O(1) =$ $x_0 = r(T)\cos(\tau + \phi(T))$.

Substitute into $O(\epsilon)$ & make coefficients of $\cos(\tau+\phi)$ & $\sin(\tau+\phi) = 0$.

RHS of $O(\epsilon)$ is $2[r'\sin(\tau+\phi) + r\phi'\cos(t+\phi)] - h$

where $h = h[r\cos(t+\phi), -r\sin(\tau+\phi)]$

We use Fourier analysis to extract terms in h proportional to $\sin(\tau+\phi)$ & $\cos(\tau+\phi)$.

Let $\theta = \tau + \phi$.

$$h(\theta) = \sum_{k=0}^{\infty} a_k \cos k\theta + \sum_{k=0}^{\infty} b_k \sin k\theta.$$

where $$a_0 = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\, d\theta.$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos k\theta\, d\theta, \quad k \geq 1$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\sin k\theta\, d\theta, \quad k \geq 1$$

$$\therefore \quad 2\left[r'\sin\theta + r\phi'\cos\theta\right] - \sum_{k=0}^{\infty} a_k \cos k\theta - \sum_{k=1}^{\infty} b_k \sin k\theta.$$

$\frac{dr}{dT}$

$T = \epsilon t$;

$\tau = t$

$$\boxed{\begin{aligned} r' &= \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\sin\theta\, d\theta = \langle h \sin\theta \rangle \\ r\phi' &= \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\cos\theta\, d\theta = \langle h\cos\theta\rangle \end{aligned}}$$

avg over one period

$\langle \cos \rangle = \langle \sin \rangle = 0$

$\langle \sin\cos \rangle = 0$

$\langle \sin^3 \rangle = \langle \cos^3 \rangle = 0$

$\langle \sin^{2n+1} \rangle = \langle \cos^{2n+1} \rangle = 0$

$\langle \sin^2 \rangle = \langle \cos^2 \rangle = \frac{1}{2}$

$\langle \sin^4 \rangle = \langle \cos^4 \rangle = \frac{3}{8}$

$\langle \sin^2\cos^2 \rangle = \frac{1}{8}$

$\langle \sin^{2n} \rangle = \langle \cos^{2n} \rangle = \frac{1.3.5\cdots(2n-1)}{2.4.6\cdots(2n)}$

Steps:

① Find r' & $r\phi'$.

② $\omega = 1 + \epsilon\phi'$

$r(0) = \sqrt{x(0)^2 + \dot{x}(0)^2}$

$\phi(0) \simeq \tan^{-1}\left(\frac{\dot{x}(0)}{x(0)}\right) - 1.$

③ Solve for $r(T)$ & $\phi(T)$ explicitly using $r(0)$, $\phi(0)$.

④ $x(t,\epsilon) \simeq x_0(\tau, T) + O(\epsilon)$

$\simeq r(T)\cos(t + \phi(T)) + O(\epsilon).$

⑤ f.p of $r(T)$ are limit cycles.

Lec 12

Bifurcations in 2D systems.

> Suppose $\exists$ stable equilibrium or closed orbit.
> > How can it vanish or change stability as we vary a parameter?

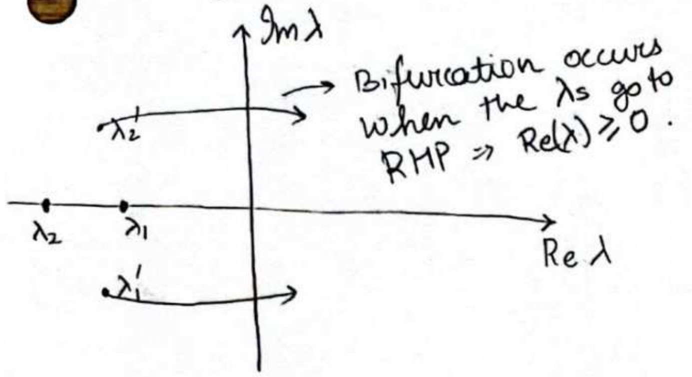
> In this lecture prototypical examples (normal forms) are discussed.

I. Bifurcations of fixed points.

a) $\lambda = 0$ bifurcations (saddle node, transcritical, pitchfork)

b) $\lambda = \pm i\omega$ (Hopf bifurcations) → lead to creation of closed orbits since complex λs become purely imaginary

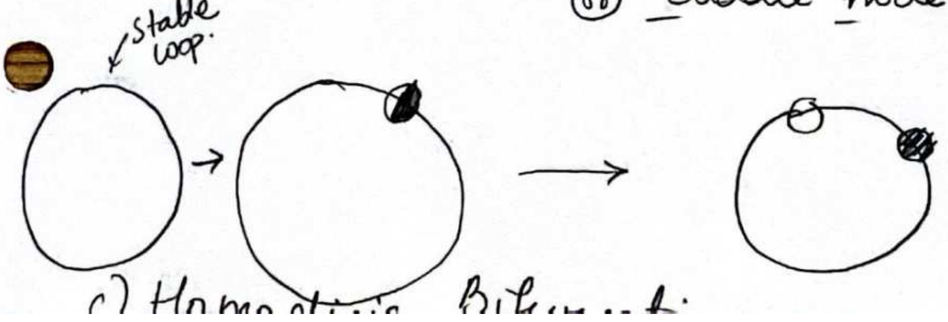
> Importance of eigenvalues: when $Re\ \lambda_1, \lambda_2 < 0$, fixed point is linearly stable.



II. Bifurcations of closed orbits.

a) Coalessence of cycles → Stable & unstable cycles annihilate. aka Saddle node bifurcation of cycles

b) Sniper or Snic – Saddle node infinite period. bif^n (or) Saddle node invariant circle.



> We see a saddle node Bif on a circle.
> At the bifurcation timeperiod goes to ∞.

c) Homoclinic Bifurcation

I.a $\lambda = 0$ Bifurcations.

1) Saddle-node.

canonical form of saddle node $\rightarrow$ $\dot{x} = a - x^2$

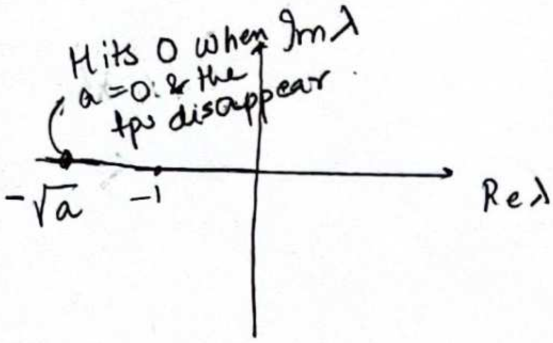
$\dot{y} = -y$

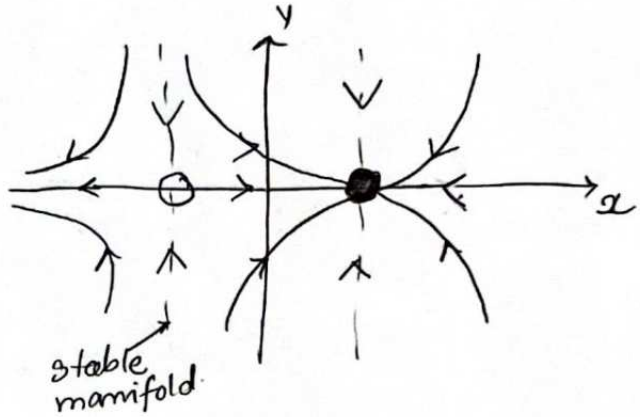
Fixed points: $(\sqrt{a}, 0)$, $(-\sqrt{a}, 0)$ when $a > 0$.

$$A = \begin{bmatrix} -2x^* & 0 \\ 0 & -1 \end{bmatrix} \qquad \lambda_1 = -2x^* \qquad \lambda_2 = -1$$

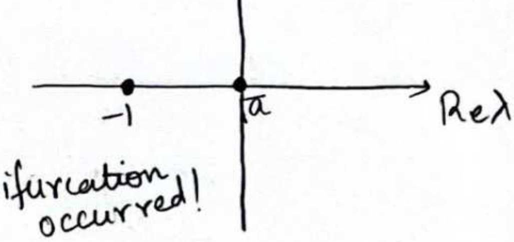
If $\left.\begin{matrix} x^* = \sqrt{a} \\ y^* = 0 \end{matrix}\right\}$ stable node. $\qquad \left.\begin{matrix} x^* = -\sqrt{a} \\ y^* = 0 \end{matrix}\right\}$ saddle.

Therefore a saddle and a node are colliding. AKA saddle node.

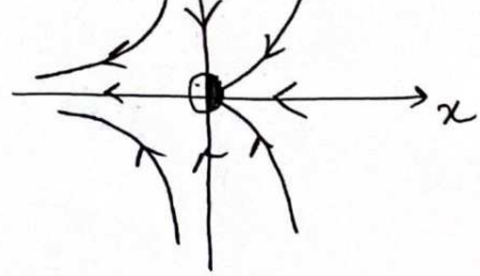




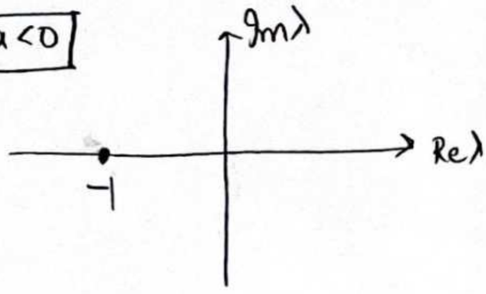
$\boxed{a=0}$

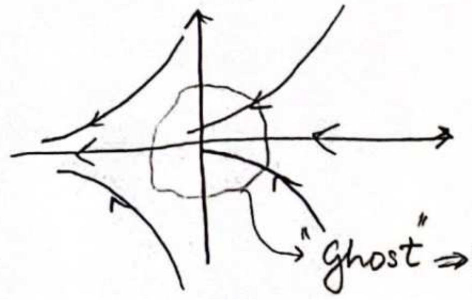


Bifurcation occurred!



$\boxed{a<0}$





"Ghost" $\Rightarrow$ slow! or "Bottleneck.

2) <u>Transcritical</u> (Analyze it yourself)

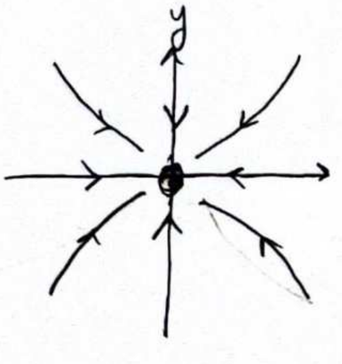
$$\dot{x} = ax - x^2$$
$$\dot{y} = -y$$

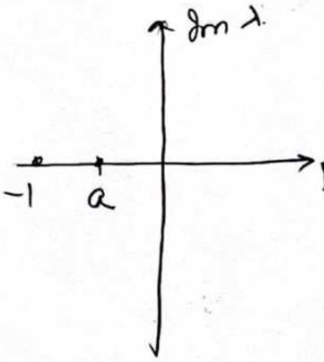
*3) <u>Pitchfork</u>

<u>Supercritical</u> : $\dot{x} = ax - x^3 \quad \dot{y} = -y \qquad \bar{x}^* = (\pm\sqrt{a}, 0)$.

$\bar{x}^* = (0, 0)$.

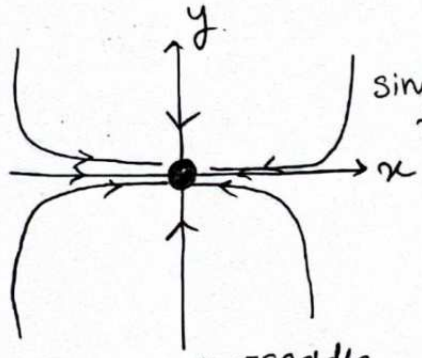
<u>$a < 0$</u>



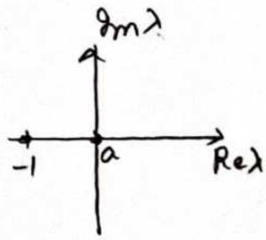


$$A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$$

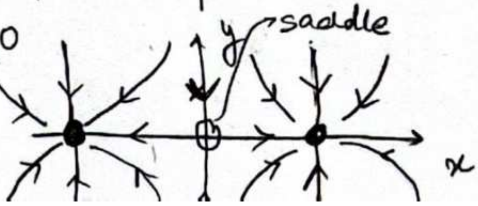
<u>$a = 0$</u>



similar to $\dot{x} = -x^3$ system.
$\Rightarrow$ Called critical slowing down.



<u>$a > 0$</u>



> Supercritical since newly created fp are <u>stable</u>.

Ib Hopf Bifurcation $(\lambda = \pm i\omega)$ → frequency of limit cycle at birth.

Equivalently $\tau = 0$, $\Delta > 0$

Supercritical Hopf bifurcation

→ Stable spiral ⟶ Unstable spiral. + surrounded by a small amplitude limit cycle that is nearly elliptical!

→ Hopf theorem actually guarantees the existance of a limit cycle & not just closed orbits.

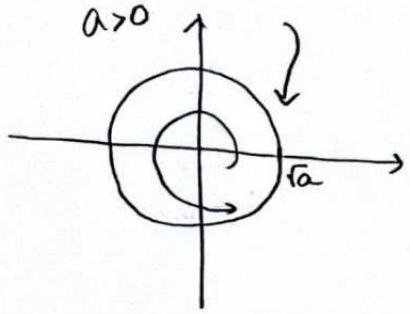
→ At the bifurcation the λs would be $\lambda_1, \lambda_2 = \pm i\omega$.

Ex: $\dot{r} = ar - r^3$, $\dot{\theta} = \omega + br^2$

$r=0$ is a fixed point for the 1D dynamics. $\dot{r} = ar - r^3$.

Stable when $a \leq 0$ } $a=0$ is where bifurcation occurs.
Unstable when $a > 0$ } 1D dynamics ⇒ $r = \sqrt{a}$ is a perfect circle after bifurcation.

→ $a>0$



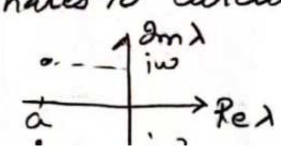
> At birth the limit cycle has 0 amplitude.

→ Parkinson's could actually be a Hopf bifurcation because of the Hopf bifurcation.

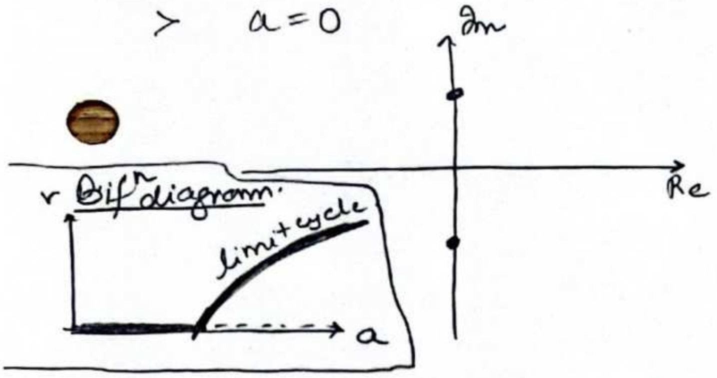
→ As g_m increases inside an electrical oscillator, a Hopf bif^n occurs creating a limit cycle.

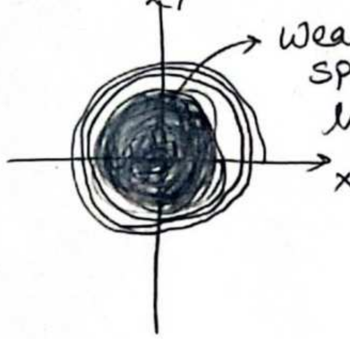
→ $a<0$

→ Convert to cartesian coordinates to calculate Jacobian. $\lambda = a \pm i\omega$



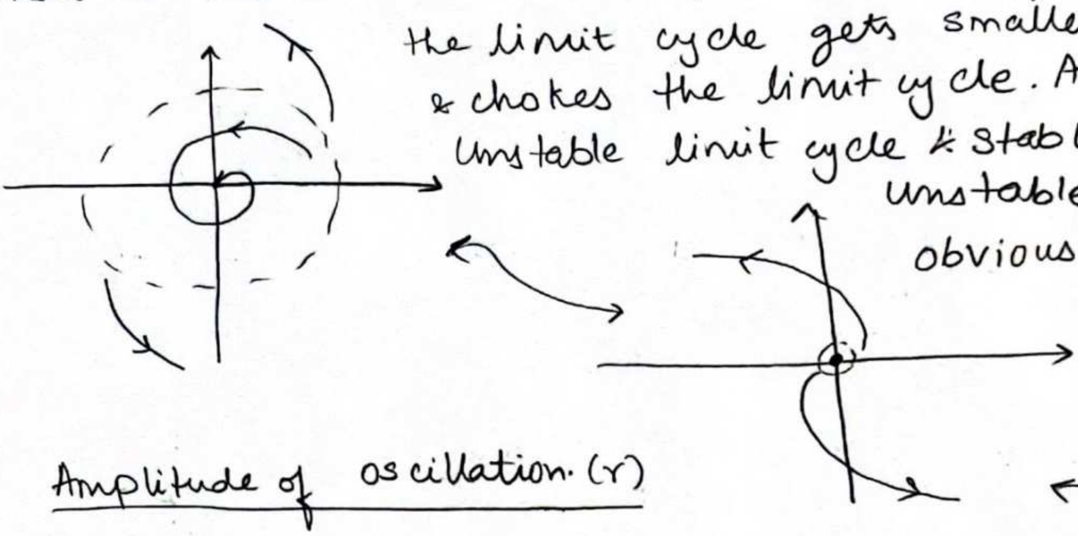
> $a = 0$

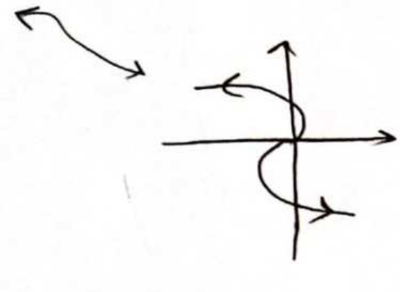




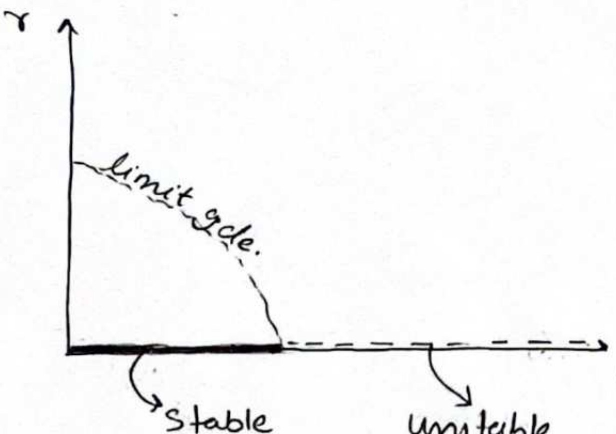
Lec 13 Subcritical Hopf bifurcation → actually very different from supercritical case.

→ In subcritical Hopf, there is an unstable limit cycle that surrounds a stable spiral. As a parameter changes the limit cycle gets smaller & smaller & chokes the limit cycle. At the bif^n point the unstable limit cycle & stable spiral become an unstable spiral. This can obviously go the other way.





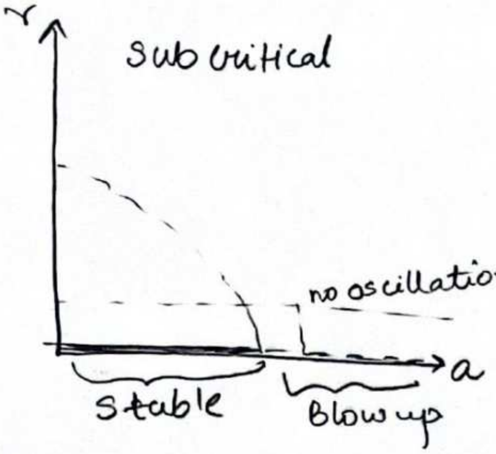
→ Amplitude of oscillation. (r)

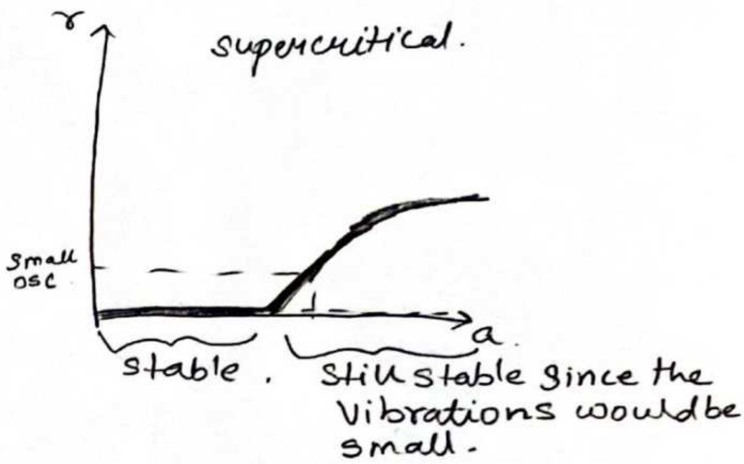


→ Supercritical Hopf aka "soft", "continuous", "safe"

→ Subcritical Hopf aka "hard", "discontinuous", "dangerous".

→ Airplane wings could be stable & suddenly start violently vibrating & break if a subcritical Hopf occurs.

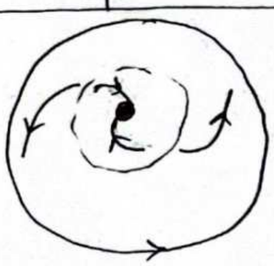




Ex. $\dot{r} = ar + r^3 - r^5$

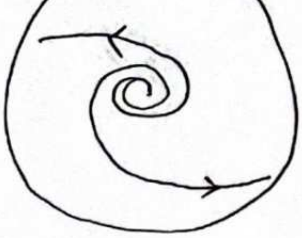
$\dot{\theta} = 1$

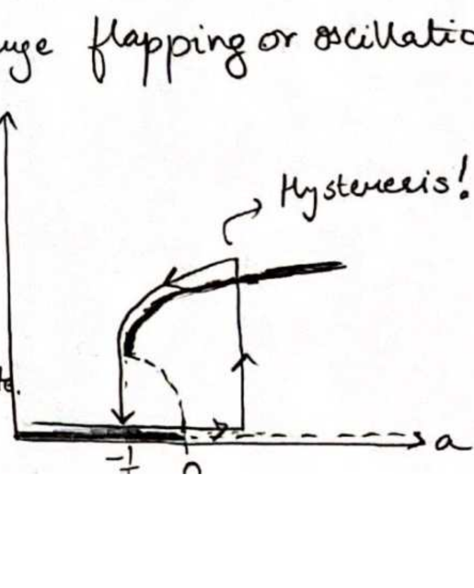
<u>If</u> $-\frac{1}{4} < a < 0$ ⇒ Stable Spiral, surrounded by a circular unstable limit cycle. In this case even outside the limit cycle there is another limit cycle outside that is stable.



<u>If</u> $a > 0$

huge limit cycle ⇒ huge flapping or oscillations.





> Below $-\frac{1}{4}$ the two limit cycles annihilate.

> Hysteresis of limit cycles is observed.

→ Linearization cannot predict the difference b/w sub & super critical Hopf. Notice at $r \cong 0$ they look the same & in the equations r^3 term sign is where the difference exists which is ignored by the linearization.

→ An analytical criterion to tell the difference exists. But it's complicated. See Guckenheimer & Holmes. § 3.4. Or do exercise §8.2.12.

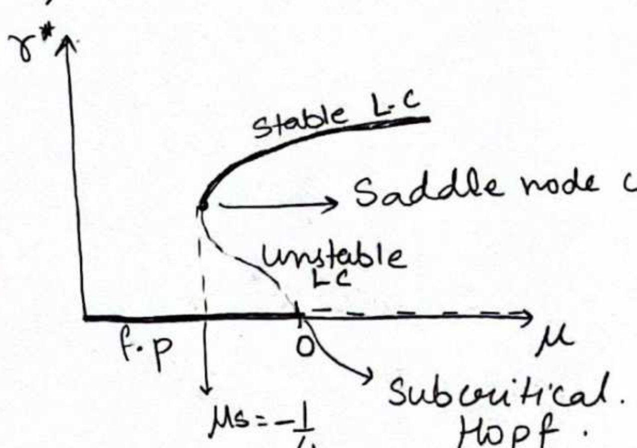
POI

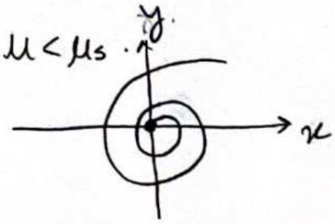
→ This is also subcritical since the bifurcating object is unstable.

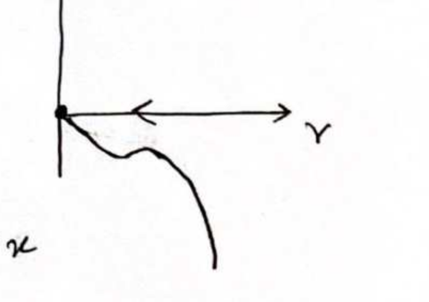
Lec 14 Global Bifurcations in cycles. §8.4.

→ Saddle node coalescence of limit cycles. Illustrated by

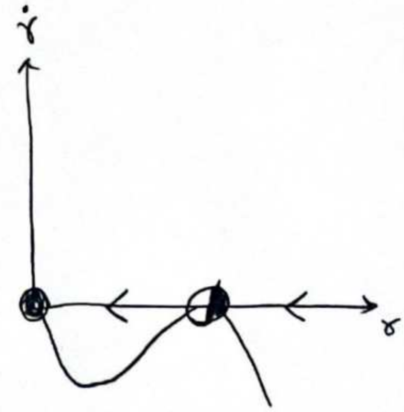
$\dot{r} = \mu r + r^3 - r^5 \qquad \dot{\theta} = 1 + b r^2$

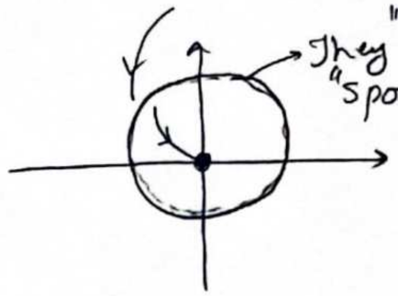






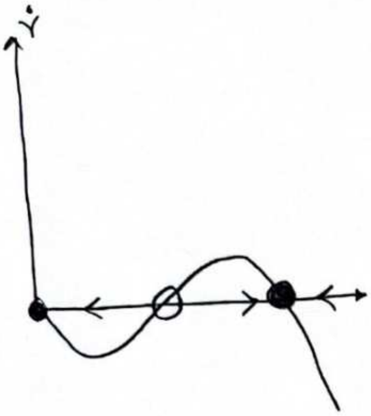
$\mu = \mu_s$

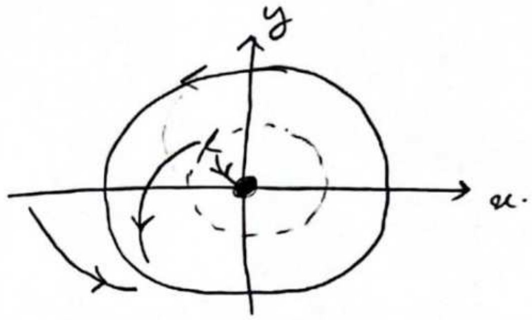




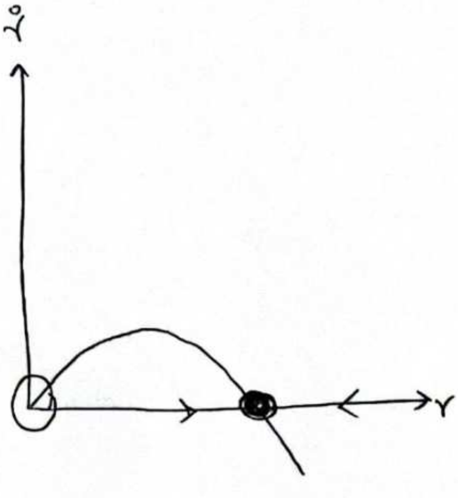
They "collide" or "spontaneously generate" as a half stable limit cycle

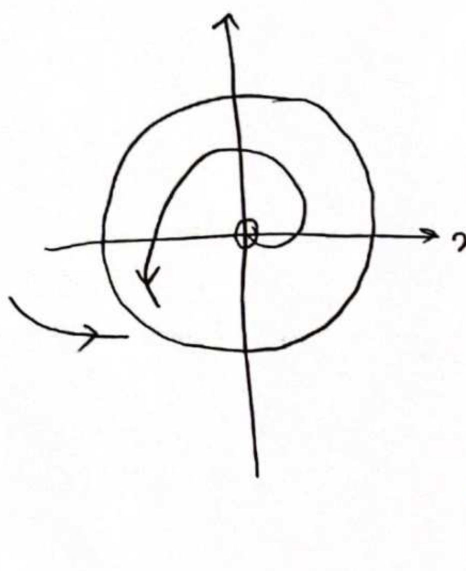
$0 > \mu > \mu_s$





$\mu > 0$





> Cycles born at $\mu = \mu_s$ with an amplitude of $O(1)$.

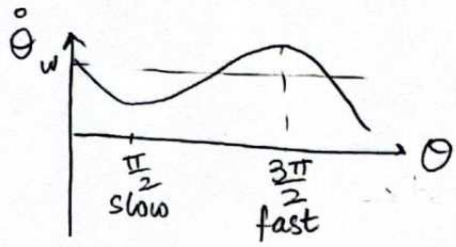
$O(1)$ is with respect to $|\mu - \mu_s|$. The frequency is also of $O(1)$, since $\dot{\theta} = 1 + b r^{*2}$.

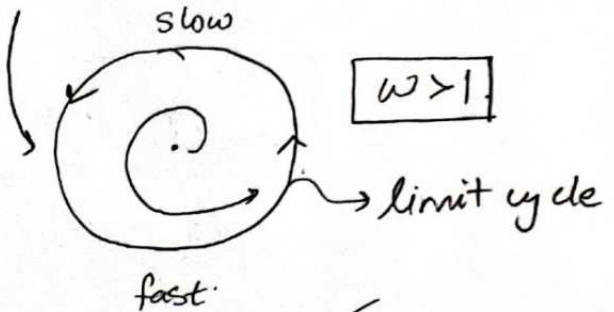
<u>SNIPER.</u> <u>Saddle node infinite period bifurcation</u> occuring on the cycle.

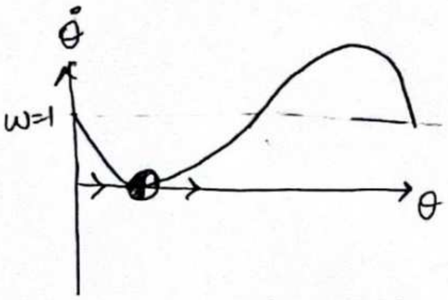
<u>Ex:</u>

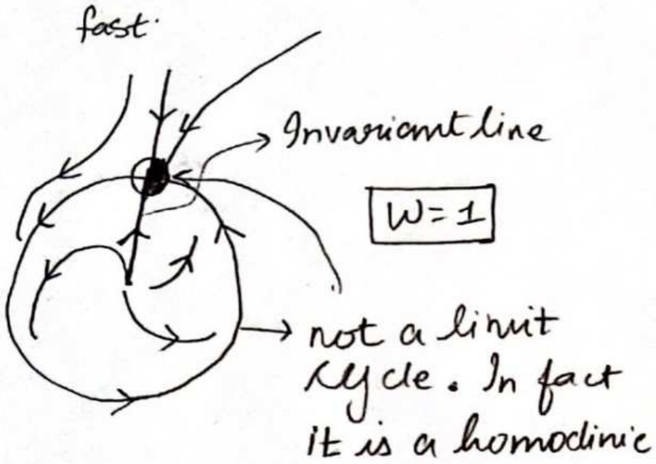
$$\dot{r} = r(1 - r^2)$$

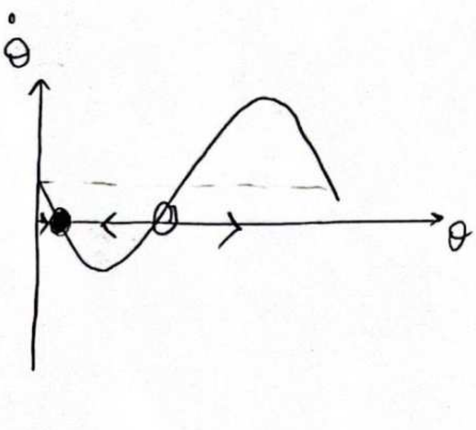
$$\dot{\theta} = \omega - \sin\theta$$

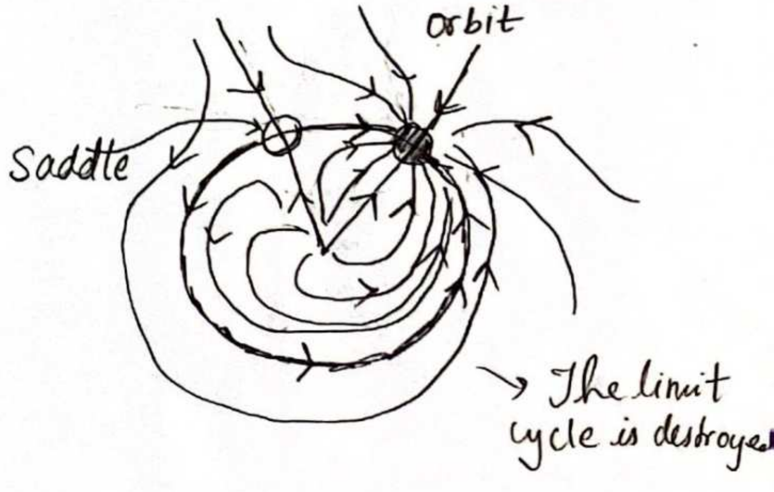


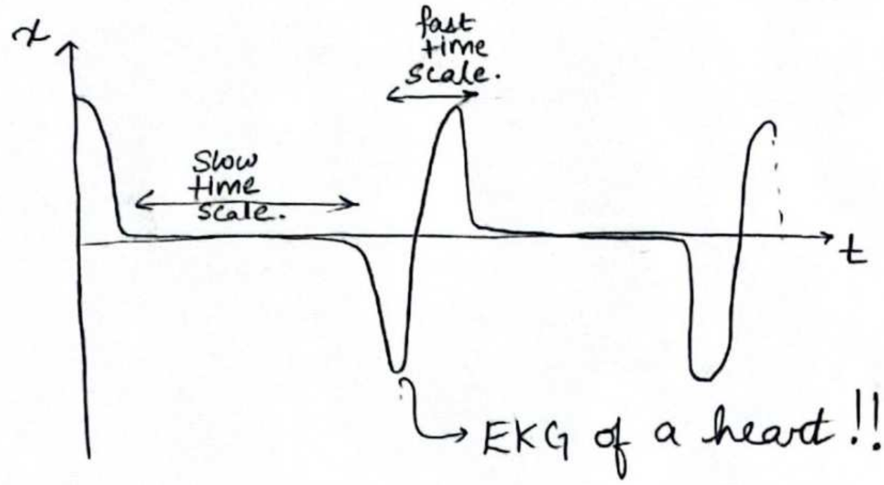


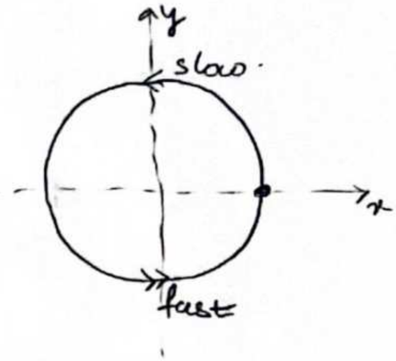












→ This bifurcation occurs in heartbeat firing & nerve cell action potential.

→ <u>Period calculation</u> $T = \int_0^T dt = \int_0^{2\pi} \frac{d\theta}{\omega - \sin\theta}$

Residue Theorem can be used.

Can also be solved using a trig substitution.

<u>Note.</u>

$\int \frac{1}{a - \sin x} dx$ substitute $u = \tan\left(\frac{x}{2}\right)$

$$\Rightarrow du = \frac{1}{2} dx \sec^2\left(\frac{x}{2}\right)$$

$$\cos(x) = \frac{1-u^2}{1+u^2} \quad \& \quad dx = \frac{2du}{u^2+1}$$

$$\sin x = \frac{2u}{u^2+1}$$

$$\Rightarrow \int \frac{2}{(u^2+1)\left(a - \frac{2u}{u^2+1}\right)} du = \frac{-2\tan^{-1}\left(\frac{1 - a\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-1}}\right)}{\sqrt{a^2-1}} + c$$

$$\therefore \boxed{T = \frac{2\pi}{\sqrt{\omega^2 - 1}}} \simeq \frac{2\pi}{\sqrt{2}} \cdot \frac{1}{\sqrt{\omega - 1}} \quad \text{as } \omega \to 1$$

In general $T \sim \frac{1}{\sqrt{\mu}}$, In general μ is the distance from saddle node bif^n. Here $\mu = \omega - 1$.

Universal behaviour near bifurcations of cycles.

Let μ = distance from bif^n ($\mu \ll 1$). ⇒ When thing are just born

	Amplitude of Stable cycle	Period of cycle
Supercritical Hopf	$O(\mu^{1/2})$ small.	$O(1)$.
Saddle node	$O(1)$	$O(1)$.
SNIPER	$O(1)$	$O(\mu^{-1/2})$ large.
Homoclinic ↳ Not yet discussed. (In sec 8.5)	$O(1)$	$O(\lvert\ln\mu\rvert)$ large.

- Useful to tell the kind of bif^n by measurement of the these parameters. Which bif^n killed the cycle?

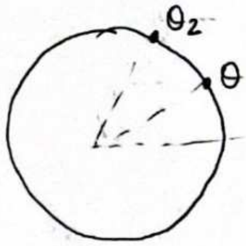
---

§ 8.6 Coupled Oscillators. (and quasiperiodicity).

2D phase spaces: Plane (done a lot), cylinder (done), sphere (like plane), Torus

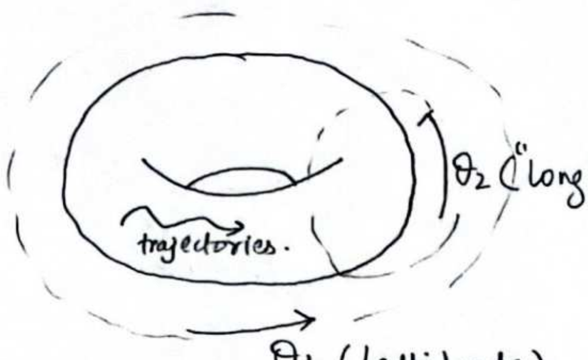
$$\boxed{\begin{aligned}\dot{\theta}_1 &= f_1(\theta_1, \theta_2)\\ \dot{\theta}_2 &= f_2(\theta_1, \theta_2)\end{aligned}}$$

fs are 2π periodic in both arguments. These are studied on the Torus (doughnut).

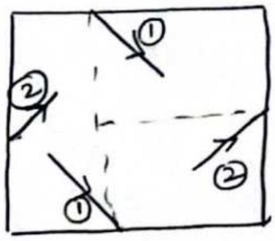


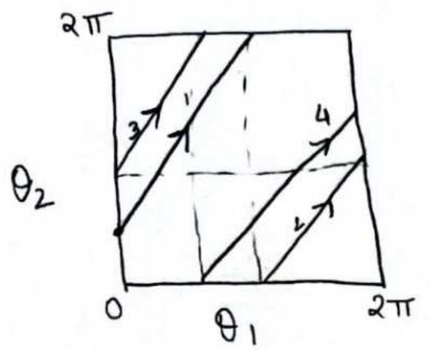
> Here 2 points represent the state of the system.
> However we want one point to represent both states.

Therefore use a Taurus with coordinates θ_1, θ_2.



or use a square with periodic boundary conditions.





Eg

$$\left.\begin{aligned}\dot{\theta}_1 &= \omega_1 \\ \dot{\theta}_2 &= \omega_2\end{aligned}\right\} \text{No coupling.}$$

$$\boxed{\begin{aligned} \underline{\underline{\text{In general}}}\quad \dot{\theta}_1 &= \omega_1 + \underbrace{K\sin(\theta_2 - \theta_1)}_{\text{coupling.}} \\ \dot{\theta}_2 &= \omega_2 + K\sin(\theta_1 - \theta_2) \end{aligned}}$$

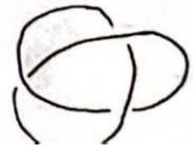
Case 1: $\dfrac{\omega_1}{\omega_2} = \dfrac{p}{q}$ where $p, q \in \mathbb{Z}$ & they are relatively prime. $\Rightarrow$ Slope is rational!

→ This causes a closed curve on the Torus.

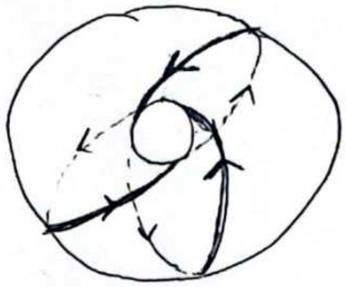
> Basically we go p times around one circle as we go q times around the other circle in the same time.

> Think of them as 2 runners @ 3 laps/time (P) & 2 laps/same time (q).

Eg: $p = 3$, $q = 2$. The trajectory on the Torus would have a knot in it. All trajectories would be closed with knots in them. In Knot theory they are called a 'Trefoil Knot".



→



→ Torus Knots occur when p&q are relatively prime.

→ Trefoil knots are 3:2 Torus Knots.

→ However rational numbers are a tiny ∞ (a.k.a countable).

Case 2 : $\frac{\omega_1}{\omega_2}$ = irrational. - get quasiperiodicity.

→ The lines never touch! Every trajectory on the Torus does not ever close and every trajectory is dense on the Torus.

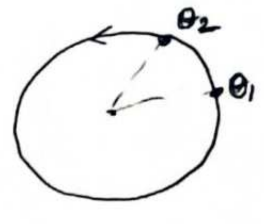
→ Mathematically speaking all points on the Torus are NOT covered by a single trajectory.

→ By picking a point on the Torus. We can show that the trajectory need not hit the point. However, if we pick an ϵ disk around the point where $\epsilon \neq 0$ but ϵ can be arbitrarily small, you can show that the trajectory eventually will penetrate the disk. (Weird!)

→ Is this Chaos? Not truly! In a chaotic system 2 neighbouring points diverge exponentially fast (not forever though). Here they do not.

Coupled oscillators.

Let $\phi = \theta_1 - \theta_2$



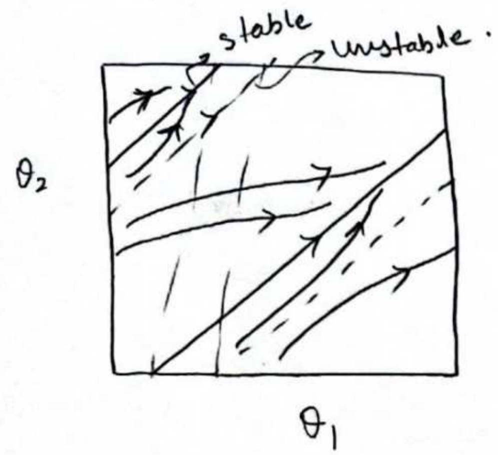
$$\dot{\theta}_1 = \omega_1 + k \sin(\theta_2 - \theta_1)$$

$$\dot{\theta}_2 = \omega_2 + k \sin(\theta_1 - \theta_2)$$

If $\theta_2 > \theta_1$, $\dot{\theta}_1 = \omega_1 +$ (positive term) $\Rightarrow$ θ_1 speeds up (& θ_2 slows down) until $\dot{\theta}_1 = \dot{\theta}_2$ and we have $\theta_1 - \theta_2 = \phi$ which would be constant. They are phase locked.

$$\dot{\phi} = \omega_1 - \omega_2 - 2k \sin\phi$$

→ Coupled oscillators also behave like Adler's equation! Therefore there are 2 fixed points. (stable & unstable).



> ϕ = constant $\Rightarrow$ $\theta_1 - \theta_2$ = Constant.
$\Rightarrow$ diagonal line with a slope of one.

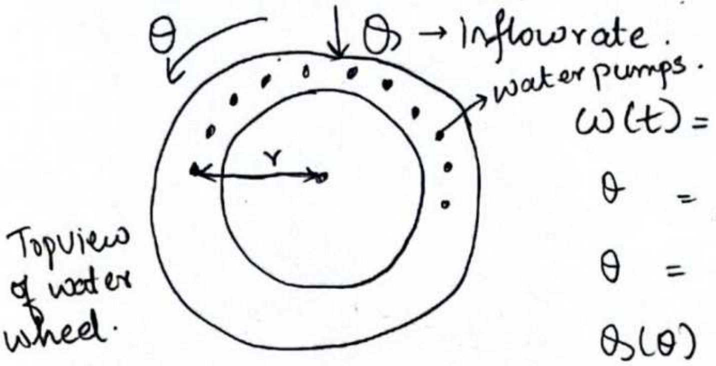
> If $2K > \omega_1 - \omega_2$ $\Rightarrow$ we get a constant phase difference between the 2 oscillators. This is phase locking!

$$\theta_1 - \theta_2 = \phi^*.$$

> $\dot{\theta}_1$ & $\dot{\theta}_2$ are equal to ω_1 & ω_2 (the free running frequencies) when coupling is 0. When coupling is introduced, the phase difference between them undergoes a beat phenoma. If coupling is strong enough they "lock" in phase & have the Same free running frequency.

Lec 15 Chaotic Waterwheel → Exact analog of the Lorenz equations.

Chapter 9 Lorenz Equations.



$\omega(t)$ = angular velocity of wheel.

θ = angle in the lab frame.

$\theta = 0 \to$ 12:00 in the lab frame.

$Q(\theta)$ = rate at which water is pumped in

r = radius.

$m(\theta, t) \to$ mass distribution of water around the rim.

→ Mass between 2 angles is $\int_{\theta_1}^{\theta_2} M(\theta, t)\, d\theta$.

→ Variables: $\omega(t)$, $m(\theta, t)$.

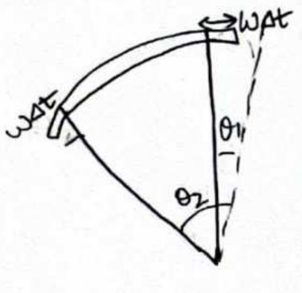
① Conservation of mass:

$$\frac{\partial m}{\partial t} = Q - Km - \omega \frac{\partial m}{\partial \theta}$$

inflow — drainage rate coefficient — Transport term since wheel is moving.

Derivation

Consider any sector $[\theta_1, \theta_2]$



Let $M = \int_{\theta_1}^{\theta_2} m(\theta, t)\, d\theta$ — ①

> In time Δt, $\Delta M \simeq \Delta t \left[\int_{\theta_1}^{\theta_2} Q(\theta)\, d\theta - \int_{\theta_1}^{\theta_2} Km\, d\theta \right]$

mass that came in

$+ m(\theta_1)\, \omega \Delta t \to$ mass transported in.

$- m(\theta_2)\, \omega \Delta t \to$ mass transported out.

$$m(\theta_1) - m(\theta_2) = -\int_{\theta_1}^{\theta_2} \frac{\partial m}{\partial \theta} d\theta$$ to make it all in the form $\int \cdots d\theta$

$$\Delta M \cong \Delta t \left[\int_{\theta_1}^{\theta_2} \left(Q - K m - \omega \frac{\partial m}{\partial \theta} \right) d\theta \right]$$

taking the limit

$$\dot{M} = \int_{\theta_1}^{\theta_2} \left[Q - K m - \omega \frac{\partial m}{\partial \theta} \right] d\theta .$$

$$\int_{\theta_1}^{\theta_2} \left[\frac{\partial m}{\partial t} \right] d\theta$$ from (1)

Since θ_1 & θ_2 are arbitrary

$$\boxed{\frac{\partial m}{\partial t} = Q - K m - \omega \frac{\partial m}{\partial \theta}} \rightarrow \text{(I)}$$

(2) Newton's Law (expressed as torque balance)

Suppose. $I(t)$ = moment of inertia of wheel. → depends on time since the amount of mass changes.

$$(I\omega)^{\cdot} = -\nu\omega + \text{torque due to gravity.}$$

(linear damping due to brake + inertial damping)

Note: Water enters with 0 angular momentum but leaves with some non zero angular momentum so it does contribute to the total torque. Luckily it is $\propto \omega$ so include it in ν

Torque due to gravity

In sector $d\theta$, mass $dM = m\, d\theta$

∴ Torque $d\tau = dM\, g r \sin\theta$. ↳ pendulum Torque.

↳ tilted g.

$$\tau = \int_0^{2\pi} g r\, m(\theta, t) \sin\theta \, d\theta$$

$g = g_0 \sin\alpha$ (g_0 = 9.8m/s²)

$$\boxed{\frac{d}{dt}(I\omega) = -\nu\omega + gr\int_0^{2\pi} m(\theta,t)\sin\theta\, d\theta} \quad \text{— (II)}$$

> We can show that $I(t)$ approaches a constant as $t \to \infty$. regardless of chaotic nature of wheel, since inflow balances outflow

$$\dot{M} = \int_0^{2\pi} \frac{\partial m}{\partial t} d\theta = \int_0^{2\pi} \left[Q - Km - \omega \frac{\partial m}{\partial \theta} \right] d\theta = Q_{total} - KM - \underset{m(2\pi)-m(0)}{0}$$

$$\dot{M} = Q_{total} - K\underset{\text{total mass.}}{M} \Rightarrow M \longrightarrow \frac{Q_{tot}}{K} \text{ as } t \longrightarrow \infty \left(\text{if } t \gg \frac{1}{K}\right)$$

$I = Mr^2 + I_{wheel}$. (parallel axis theorem)

$$(2) \Rightarrow \boxed{I\dot{\omega} = -\nu\omega + gr\int_0^{2\pi} m(\theta,t)\sin\theta\, d\theta} \quad \text{— (III)}$$

$$\frac{\partial m}{\partial t} = Q - Km - \overbrace{\omega \frac{\partial m}{\partial \theta}}^{\text{nonlinear quadratic term.}}$$

$$I\dot{\omega} = -\nu\omega + gr\int_0^{2\pi} m(\theta,t)\sin\theta\, d\theta \rightarrow \text{linear integrodifferential eqn.}$$

> Using Fourier Analysis we could reduce it to 3 nonlinear ODEs.

> Deriving amplitude equations!

Since $m(\theta,t)$ is 2π periodic in θ, write

$$m(\theta,t) = \sum_{n=0}^{\infty} a_n(t)\sin n\theta + b_n(t)\cos n\theta$$

Derive ODEs for $a_n, b_n, \ldots \infty$, $n = 0, 1, 2 \ldots$ → There are ∞ but we can seperate 3 of them

Let $Q(\theta) = \sum_{n=0}^{\infty} q_n \cos n\theta$ → no sine terms since Q is evenly distributed w.r.t θ. Water is symmetrically added at the top.

$$\frac{\partial}{\partial t}\left(\sum a_n \sin n\theta + b_n \cos n\theta\right) = -\omega \frac{d}{d\theta}\left(\sum a_n \sin n\theta + b_n \cos n\theta\right) +$$

$$+\sum q_n \cos n\theta - K \sum a_n \sin n\theta + b_n \cos n\theta$$

All the sines & cosines are orthogonal ⇒ this gives ∞ equations.
→ Coefficients are only time dependent!

Matching $\sin n\theta$.

$$\dot{a}_n = n\omega b_n - K a_n \quad \text{—(4)}$$

Matching $\cos n\theta$

$$\dot{b}_n = -n\omega a_n + q_n - K b_n \quad \text{—(5)}$$

Now substitute Fourier for $m(\theta, t)$ into (III):

$$\text{(III)} \rightarrow \quad I\dot{\omega} = -\nu\omega + gr \underbrace{\int_0^{2\pi} \left(\sum a_n \sin n\theta + b_n \cos n\theta\right) \sin\theta \, d\theta}.$$

Wow! All these fns are orthogonal to $\sin\theta$ except for $\sin\theta$. Therefore all but $\sin\theta$ integrate to 0.

$$= -\nu\omega + gr \int_0^{2\pi} a_1 \sin^2\theta \, d\theta \rightarrow \pi$$

$$I\dot{\omega} = -\nu\omega + \pi g r a_1 \quad \text{—(6)}$$

④ ⑤ & ⑥

$\therefore \quad \dot{a}_n = n\omega b_n - k a_n$

$$\dot{b}_n = -n\omega a_n + q_n - k b_n$$

$$I\dot{\omega} = -\nu\omega + \pi g r a_1$$

The next part is truly beautiful.

> $\dot{\omega}$ evolves wrt only a_1
> a_1 evolves wrt only b_1 & a_1
> b_1 evolves wrt only a_1 & b_1

Therefore these equations have decoupled from the rest !!!

$$\boxed{\begin{aligned} \dot{a}_1 &= \omega b_1 - K a_1 \\ \dot{b}_1 &= -\omega a_1 + q_1 - K b_1 \\ \dot{\omega} &= -\frac{\nu}{I}\omega + \frac{\pi g r}{I} a_1 \end{aligned}}$$

first harmonic of inflow.

These 3 coupled parameters are like an engine that determine ω. This ω then defines a_n & b_n for $n \geq 2$. They do not affect this engine on the "bottom

Lec 16 Waterwheel equations and Lorenz equations.

Fixed points: $\dot{a}_1 = 0 \Rightarrow a_1 = \frac{\omega b_1}{K}$ — ①

$\dot{b}_1 = 0 \Rightarrow \omega a_1 = q_1 - K b_1$ — ②

$\dot{\omega} = 0 \Rightarrow a_1 = \frac{\nu\omega}{\pi g r}$ — ③

Eliminate a_1 from ① and ② $\Rightarrow b_1 = \frac{K q_1}{\omega^2 + K^2}$ — ④

① & ③ $\Rightarrow \frac{\omega b_1}{K} = \frac{\nu\omega}{\pi g r}$, $\boxed{\omega = 0}$ or $\boxed{b_1 = \frac{K\nu}{\pi g r}}$ — ⑤

If $\omega = 0 \Rightarrow a_1 = 0$, $b_1 = q_1/K$. (no rotation & leakage balances inflow) → not saying stable or unstable

If $\omega \neq 0$

$$b_1 = \frac{K\nu}{\pi g r} = \frac{K q_1}{\omega^2 + K^2} \Rightarrow \boxed{\omega^2 = \frac{\pi g r q_1}{\nu} - K^2}$$

> Two solutions $\pm\omega$ (corresponding to steady rotation in either direction) exist iff RHS is positive. $\Rightarrow \frac{\pi g r q_1}{K^2 \nu} > 1$.

> This dimensionless group $\boxed{\frac{\pi g r q_1}{K^2 \nu}}$ is the Rayleigh number analog in convection.

> The rotating waterwheel is like a convection cell of air that goes up & down in cycles due to heating & cooling. The steady Solution is like heat exchange due to conduction & not convection.

> g, q_1 drive the wheel into rotation (gravity & inflow). K & ν are dampers & oppose the motion.

Edward Lorenz System / Lorenz Equations.

> Derived from simple model of convection.

> First example of a system with a Chaotic attractor. (self sustaining chaos)

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

$\sigma, r, b > 0$ parameters

$r \rightarrow$ Rayleigh number. $\frac{\text{conduction}}{\text{convection}}$.

$\sigma \rightarrow$ Prandtl number $\rightarrow$ Viscocity/thermal conduction

$b \rightarrow$ aspect ratio.

> Two quadratic nonlinearities. The waterwheel equations can be mapped to the Lorenz equations.

> Lorenz equations arise in laser dynamics, dynamos etc.

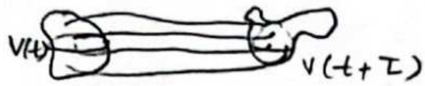
Simple properties of Lorenz equations.

1) Equations are symmetric under $(x, y) \longrightarrow (-x, -y)$

Means if $(x(t), y(t), +z(t))$ is a solution. Then, $(-x(t), -y(t), +z(t))$ is also a solution. Due to symmetry we might expect a pitchfork bifurcation.

2) System is dissipative, in sense that <u>volumes in phase space contract under the flow.</u>

> If we start with a volume of initial conditions where each point is a unique initial condition, this volume over time decreases.

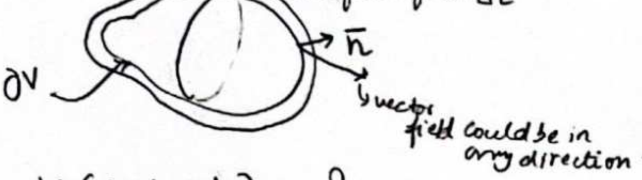


<u>To see this, ask</u>: How do volumes evolve?

> Points on the surface flow according to the vector field.
> What is the volume after Δt.

Let us think of $\vec{x} = (x, y, z)$, $\vec{u} = \dot{\vec{x}} \rightarrow$ velocity in phase space.

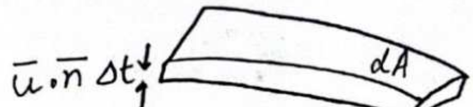
$\vec{u} \cdot \vec{n} = \dot{\vec{x}} \cdot \vec{n}$ = normal outward velocity on boundary $\underbrace{\partial V}_{\text{boundary of the vol.}} = S$



$V(t + \Delta t) = ?$

> The expansion (say) is only due to the normal component of the vector field at each point on the surface.

$V(t+\Delta t) = V(t)$ + volume swept out by tiny patches of the surface integrated over all patches.



$\Delta V = (\bar{u} \cdot \bar{n}\, \Delta t)\, dA$

$$V(t+\Delta t) \simeq V(t) + \Delta t \iint_{\partial V = S} \bar{u} \cdot \bar{n} \, dA$$

$$\Rightarrow \boxed{\dot{V}(t) = \iint_{\partial V} \bar{u} \cdot \bar{n} \, dA}$$

Divergence theorem!

$$\boxed{\dot{V}(t) = \iiint_{V} \nabla \cdot \bar{u} \, dV}$$

True for any vector field in 3D (maybe any dimension)

> In the Lorenz system we want to show RHS < 0.

$$\dot{V} = \iiint_{V} \nabla \cdot \bar{u} \, dV$$

note that $\nabla \cdot \bar{u} = \text{Trace}(\text{Jacobian}(\bar{u}))$

<u>Lorenz eqns</u>: $\dot{x} = \sigma(y-x)$; $\dot{y} = rx - y - xz$; $\dot{z} = xy - bz$

$$\bar{u} = (\dot{x}, \dot{y}, \dot{z}) \quad \therefore \quad \nabla \cdot \bar{u} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}$$

$$= -\sigma - 1 - b.$$

$\Rightarrow \nabla \cdot \bar{u} < 0$ since σ & b are +ve.

it is also a constant!

$$\Rightarrow \boxed{\dot{V} = -(\sigma + b + 1) V}$$

$\Rightarrow$ volumes shrink exponentially fast in the Lorenz system.

$$V(t) = V(0) \, e^{-(\sigma+b+1)t}$$

→ All trajectories end up on some limiting set of <u>zero volume</u>.

Could be a point, a cycle or a "strange attractor."

Lorenz called it an "infinite complex of surfaces." → FRACTAL.

Fixed points

→ Can show : $(x, y, z) = (0, 0, 0)$ for all values of the parameters.

→ Let us assume r is the only parameter. σ & b are constants.

→ $x = y = \pm\sqrt{b(r-1)}$, $z = r-1$ are fixed points if $r > 1$.

Let $c^+ = \sqrt{b(r-1)}$ & $c^- = -\sqrt{b(r-1)}$

→ As $r \to 1^+$, $(x, y, z) \to 0$ ⇒ these fixed points are born out of the origin.

→ Symmetry & the above point ⇒ pitchfork bifurcation (Supercritical ⇒ f.Ps born are stable

→ Local stability of origin:- Linearization.

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y.$$
$$\dot{z} = -bz$$ ⇒ $z(t)$ exponentially decays around the origin

→ In the linearization.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

> What does this imply for xy dynamics.

> $\tau = -\sigma - 1 < 0$

$\Delta = \sigma(1 - r)$ $r > 1$ ⇒ saddle point at the origin.

Saddle point with 2 directions coming in & 1 going out in 3D.

$\tau^2 - 4\Delta = (\sigma - 1)^2 + 4\sigma r > 0$ & $r < 1$ ⇒ stable node in all directions.

Lec 17 Chaos in Lorenz Equations

Global stability of origin for $r<1$

Every trajectory approaches the origin as $t \to \infty$. Origin is "globally stable" when $r<1$. $\Rightarrow$ There are no other attractors.

Proof: Define $V(x,y,z) = \frac{1}{\sigma}x^2 + y^2 + z^2 \rightsquigarrow$ Lyapunov fn

> A fn that monotonically decreases over time $\Rightarrow x, y, z$ must reach the origin over time.

> "Energy like" fn in a system with friction.

> $\underbrace{\text{Level sets}}_{\text{set where } V \text{ is constant}}$ of V are concentric ellipsoids centred at the origin.

Idea: Show $\frac{dV}{dt} < 0$ if $r<1$ & $r \neq 0$

Calculate: $\frac{1}{2}\frac{dV}{dt} = \frac{x\dot{x}}{\sigma} + y\dot{y} + z\dot{z}$

use Lorenz equations $\Rightarrow \frac{x}{\sigma}(\sigma(y-x)) + y(rx - y - xz) + z(xy - bz)$

$$= yx - x^2 + ryx - y^2 - \cancel{xyz} + \cancel{xyz} - bz^2$$

$$= xy(1+r) - x^2 - y^2 - bz^2 \longleftarrow \text{complete the square}$$

$$= -\left(x - \frac{r+1}{2}y\right)^2 - \underbrace{\left(1 - \left(\frac{r+1}{2}\right)^2\right)}_{\text{positive if } r<1} y^2 - bz^2$$

$\therefore$ RHS is negative definite! if $r<1$.

> When is $\frac{dV}{dt} = 0$?

Each square must be 0

$\Rightarrow z = 0$ and $y = 0 \Rightarrow x = 0$
$\Rightarrow$ only 0 at the origin.

$\Rightarrow V(x(t), y(t), z(t)) \longrightarrow 0 \Rightarrow (x, y, z)$ must all go to zero.

→ In fluid flow through a pipe if Reynolds' number is below 800 then the only solution is the laminar flow solution, but nobody has found the relevant Lyapunov function to prove it

Suppose $r > 1$ – Origin is a Saddle.

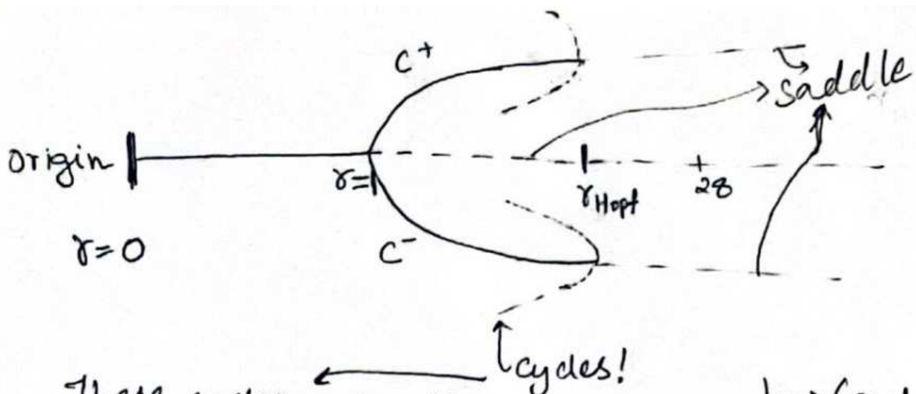
Stability of C^+, C^- is left as a homework exercise.

Can show they are linearly stable for $1 < r < \underbrace{\sigma\left(\frac{\sigma + b + 3}{\sigma - b - 1}\right)}_{r_{Hopf} = r_H}$

→ Also assume $\sigma > b + 1$. Lorenz used $\sigma = 10, b = 8/3$

$r_H = 24.74$.

→ When $r > r_H$, you might suppose there exists a small stable cycle around C^+, C^-. Actually no! The bifurcation turns out to be subcritical. Therefore trajectories must jump to a different attractor. But what can it be?



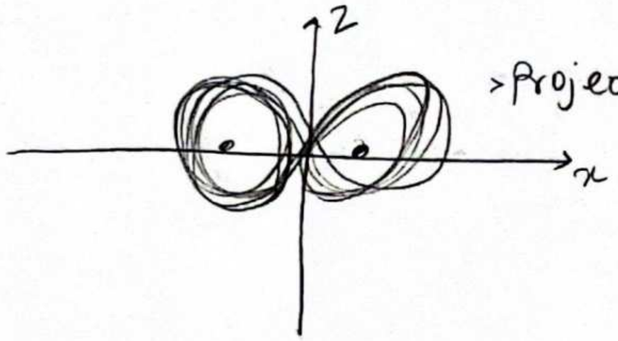
These cycles actually do hit the origin & the dynamics are too complex to analyze here.

→ Could the trajectories go to ∞ when $r > r_{Hopf}$. No. You can prove that there is a large sphere where all trajectories are moving in → trapping sphere

→ Can there be any stable limit cycles for $r > r_{Hopf}$. Lorenz showed there are none for $r = 28$.

→ Can there be quasiperiodicity on an invariant Torus. No because an invariant Torus needs a fixed volume, but we know volume shrinks exponentially fast.

→ Therefore, we have eliminated all known possibilities. In fact what we end up getting is a chaotic strange attractor.



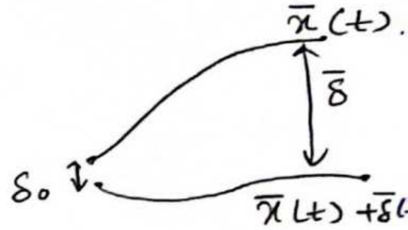
> Projection onto xz plane.

→ As $r \uparrow\uparrow$, we could leave chaos & get a limit cycle & as $r \uparrow\uparrow$ we go back & forth between chaos & limit cycles. This is just like a -ve resistance source on a transmission line.

→ Predictibility horizon for weather is $\simeq$ 10 days. For the solar system it is $\simeq$ 5 million years.

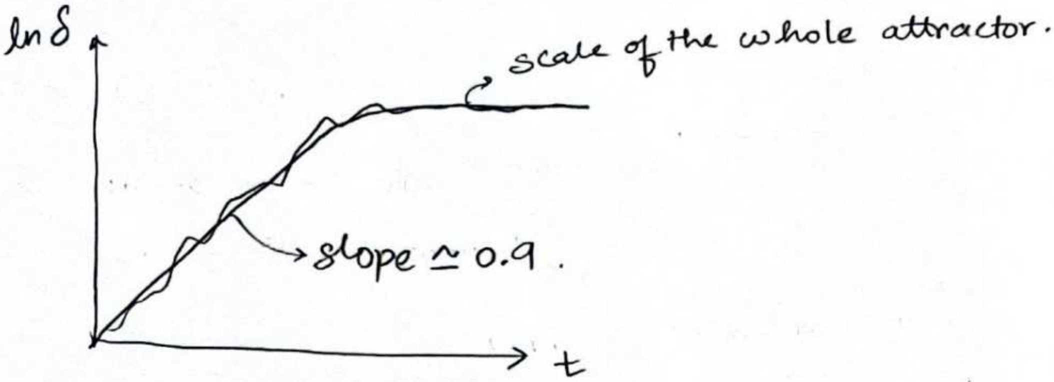
> **On the Lorenz attractor.**

$\delta(t) \to$ distance between trajectories.



$\delta(t) \cong \delta_0 e^{\lambda t}$ $\to$ exponential.

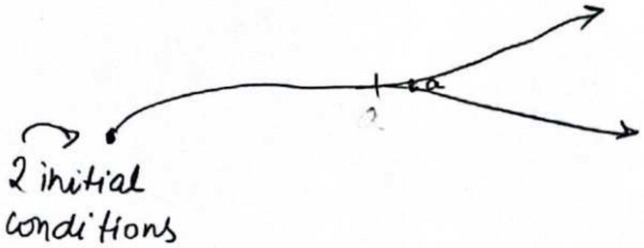
$\lambda \simeq 0.9$



$\to \lambda$ = Liapunov exponent.

$\to$ In reality we would have n Liapunov exponents for an n-dimensional system. In 3D like here every initial point $x_0(t)$ would have a sphere of radius δ_0 around it. This sphere grows into an ellipsoid because all the points in the sphere may diverge differently. The largest axis is usually used as the λ, but notice that it has 3 λs corresponding to the 3 axes of the ellipsoid.

$\to$ $\boxed{\text{positive Liapunov exponent} \Rightarrow \text{Chaos!}}$



a = tolerance
$\delta > a$ is when deviation becomes noticeable

$$\boxed{\delta_0 e^{\lambda t} \approx a}$$

$$t \approx \frac{1}{\lambda} \ln\left(\frac{a}{\delta_0}\right)$$

predictibility horizon or Liapunov time!
t is on the order of $\frac{1}{\lambda}$.

→ If we want to predict upto 10t we need to make the prediction 10^{10} times better. The logarithm kills the prediction.

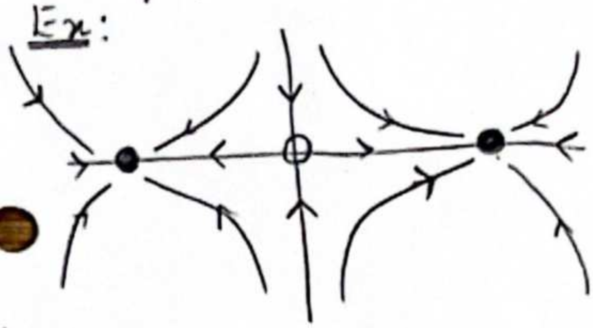
Lec 18 Strange Attractor for the Lorenz Equations

Definitions (conceptual)

Chaos: Aperiodic long term behaviour in a deterministic system (not probabilistic) that exhibits sensitive dependance on initial conditions. ↳ (positive Lyapunov exponent)

Attractor (A): 1) Invariant set (start in A ⇒ stay in A forever)
2) Attracts an open set of initial conditions. — Basin of attraction must be an open set
3) No proper subset of A satisfies 1) & 2) ⇒ "minimal".

Ex:



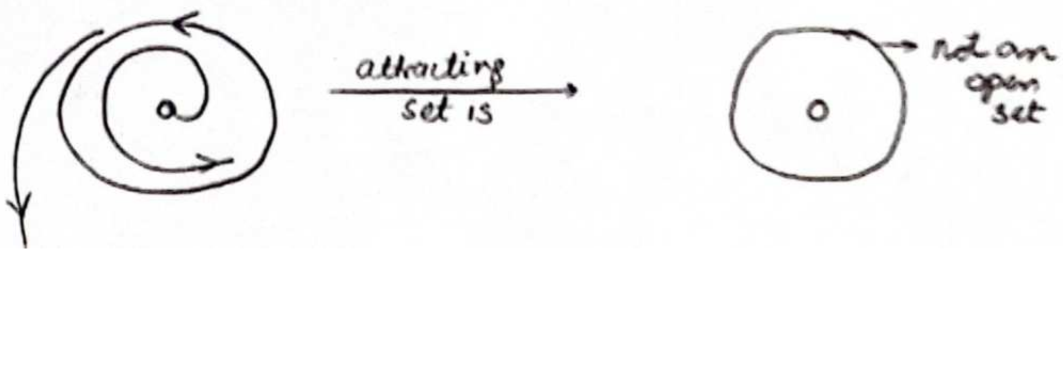
Open set → A disk drawn around a point in the set lies in the set. Just like open brackets ( )

The x axis is called an attracting set but it is not an attractor.

i > Saddle here is not an attractor (Fails 2))
⇒ Origin is not an attractor.
ii > x axis? Does not satisfy 3) since the 2 nodes satisfy 1) & 2). They are in fact attractors.

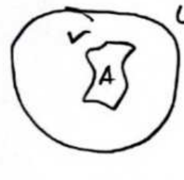
→ Is a stable limit cycle an attractor? Yes.

Is a half stable limit cycle an attractor? NO Because the set that it attracts is the area inside + the cycle itself − the unstable f.p at the center. Since it attracts itself it has a boundary & the set is closed!

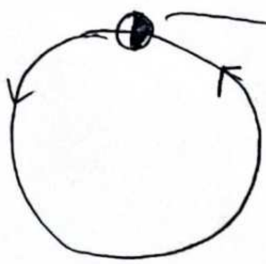


There is usually a 4^{th} property.

4) Trajectories that start near A, stay near A for all time $t>0$. Given any neighbourhood U of A, there exists another neighbourhood V of A, s.t if you start in V, then you stay in U.



Eg:



→ Is this an attractor?

> It satisfies 1), 2) & 3), but does not satisfy 4)

Strange attractor

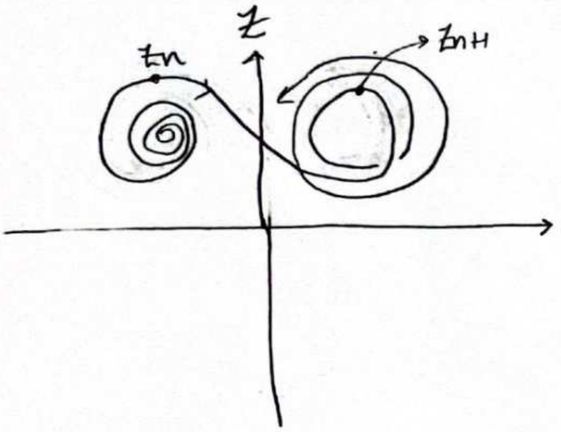
- Attractor that exhibits sensitive dependance on initial cond^{ns}

(or)

- An attractor whose local structure is a fractal (not smooth)

Dynamics of Lorenz attractor (§9.4)

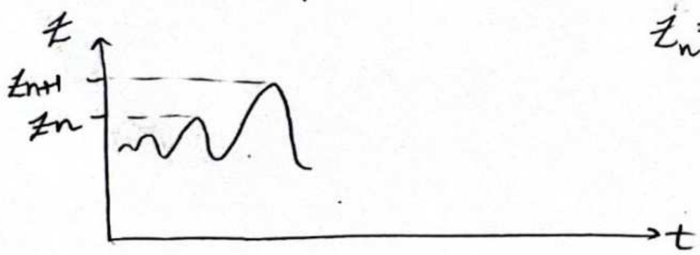
Reduction to a 1D map.



$r=28$
$\sigma=10$
$b=8/3$

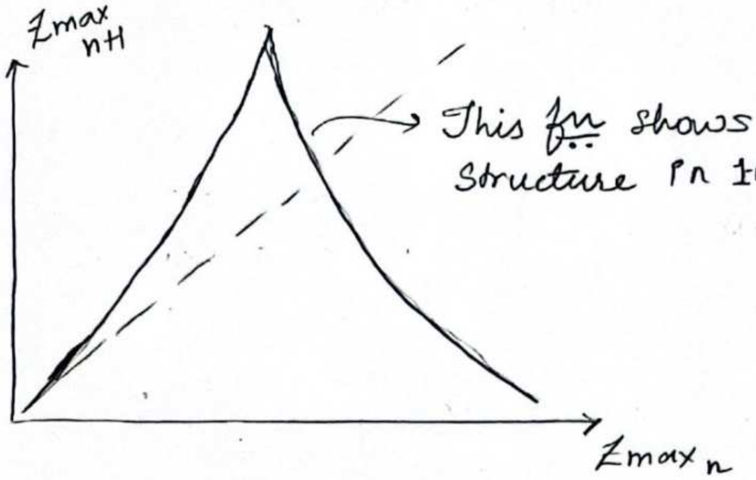
> Once the trajectory reaches a critical value in z, it leaves one spiral & goes to the next. This continues forever.

> Depending on value of z when it leaves the value on the 2^{nd} spiral where it lands is determined.



z_n = n^{th} relative local maximum of the fn $z(t)$.

> How does z_n determine z_{n+1}



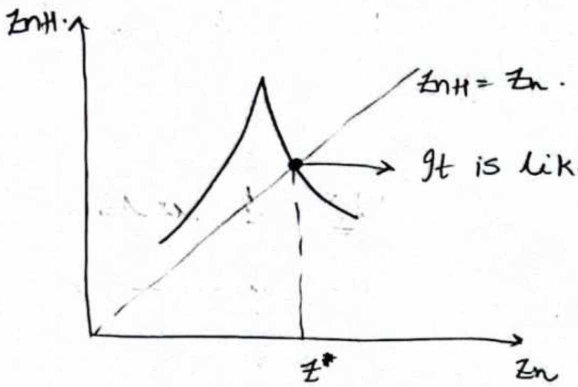
This fn shows some underlying structure in 1D.

→ Looks like $Z_{n+1} = f(Z_n)$ since essentially the curve has no thickness. In reality it is not a fn since for Z_n we could have more than one Z_{n+1}. Remember we are in 3D.

Key: $|slope| > 1$

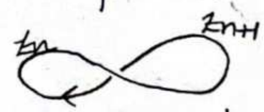
$|f'(z)| > 1 \quad \forall z.$

→ Iterate $f(z)$ to study the dynamics.



It is like a fixed point since $f(z^*) = z^*$

$\Rightarrow$ the trajectory is only in the xy plane.



Is z^* stable? No!

Let $Z_n = z^* + \eta_n$ → tiny perturbation.

Linearizing around z^*

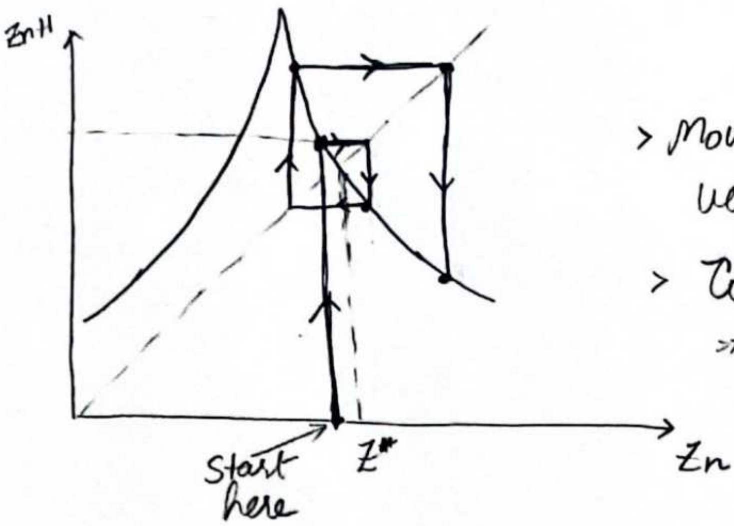
$$Z_{n+1} = \cancel{z^*} + \eta_{n+1} = f(Z_n) = f(z^* + \eta_n) = \cancel{f(z^*)} + \eta_n f'(z^*) + \cdots$$

equal!

$\therefore \eta_{n+1} = \eta_n f'(z^*)$

$\hookrightarrow |f'(z^*)| > 1 \Rightarrow |\eta_{n+1}| > |\eta_n|$

$\Rightarrow z^*$ is unstable.



Cobweb diagram.

> Move Horizontally to the diagonal & vertically to the curve.

> Could the cobweb close?

$\Rightarrow$ Can a periodic orbit be stable.

→ Periodic orbit gives a sequence $z_1, z_2, z_3 \ldots$ with $z_{n+p} = z_n \ \forall n$, where p = period. This is all to explore if the strange attractor is not a long period limit cycle. Need to show that the cobweb is unstable.

$z_2 = f(z_1)$

$z_3 = f(z_2) = f(f(z_1)) = f^2(z_1)$

$\Rightarrow z_4 = f^3(z_1)$

$\Rightarrow \boxed{z_{n+p} = f^p(z_n)}$

Defn: z is a point of period p if $f^p(z) = z$. p is smallest positive integer with this property.

→ We want to show that any period p would be unstable

→ When $p=2$, suppose $f(f(z)) = z \Rightarrow f^2(z) = z$. Therefore, a point of period 2 is a fixed point for $f^2(z)$.

✱ → In general, a point of period p is a fixed point for f^p.

→ Let's look at stability of f^2 at z.

$\Rightarrow$ we look at $(f^2)' \Rightarrow \frac{d}{dz}(f(f(z))) = f'(f(z))f'(z)$.

We see that $|(f^2)'| = \underbrace{|f'(f(z))|}_{>1} \cdot \underbrace{|f'(z)|}_{>1}$

$$\Rightarrow |(f^2)'| > 1$$

& $(f^2)'$ determines the stability

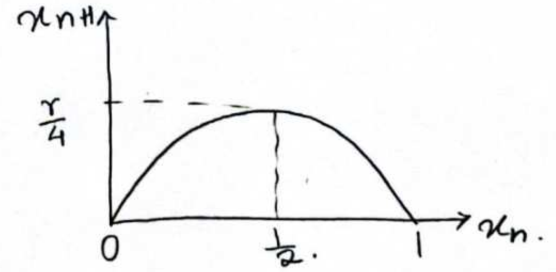
→ Therefore there are no stable trajectories & therefore no stable limit cycles. This is not a real proof since f is not a function.

Lec 19 - One dimensional maps

$x_{n+1} = f(x_n)$ we leave differential equations for a while and focus on 1D maps as a simpler model of chaos.

Logistic map: $x_{n+1} = r\, x_n (1 - x_n)$.

$x \in [0,1]$
$0 \le r \le 4$

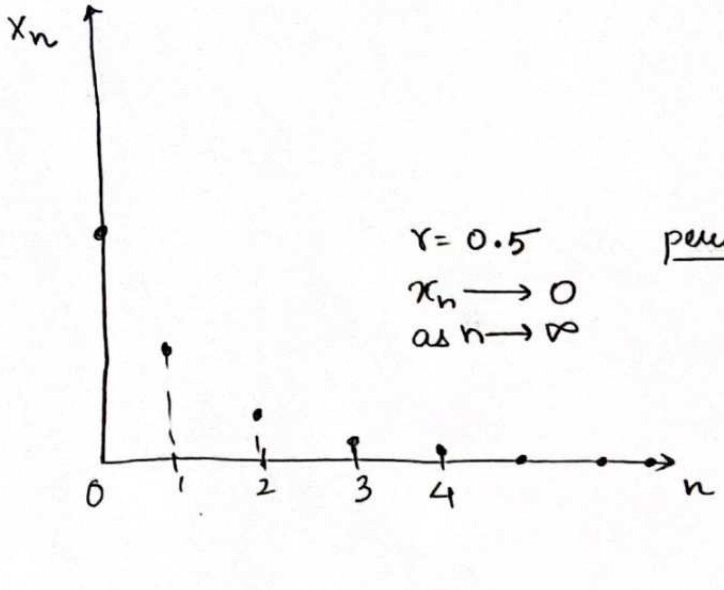


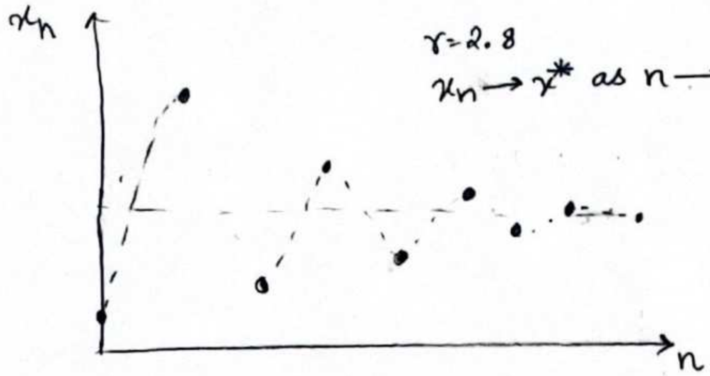
> Robert May (1976) Nature Magazine 261, 459
> Could say it is the simplest possible Nonlinearity.
→ If we have a nonlinear system of differential equations we cannot find basis functions since their linear combination will not necessarily be a solution. No basis function or eigenfn
⇒ no Fourier Transform, Laplace transform etc. Complex stuff can happen!

→ $x_1 = r\, x_0 (1 - x_0)$

$x_2 = r\, x_1 (1 - x_1)$

⋮

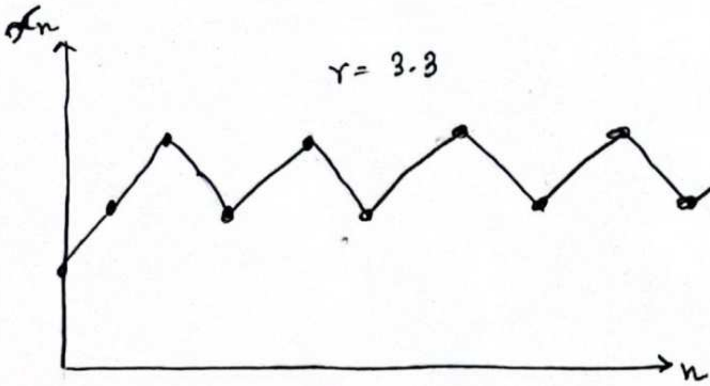




$r = 2.8$

$x_n \to x^*$ as $n \to \infty$

(some x^* for all $x_0 > 0$ & $x_i \neq 1$)



Stable period 2.

$\Rightarrow x_{n+2} = x_n$

Stable period 4

$\Rightarrow x_{n+4} = x_n$

→ Since period doubled as $r\uparrow$, & we can find any power of 2 as a periodic orbit as $r\uparrow$.

→ Where do the bifurcations from period 2^n to 2^{n+1} occur.

→ Let r_n to be the value of r where a stable 2^n cycle first occurs.

<u>Find:</u> Period 2 is born when $r_1 = 3$.

Period 4 " " " $r_2 = 3.449$

Period 6 $r_3 = 3.54409$

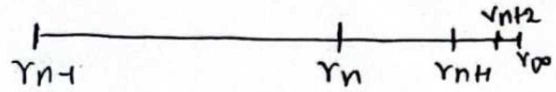
$r_4 = 3.5644$

$r_5 = 3.568759$

→ $r_\infty = 3.569946\ldots \Rightarrow$ period is 2^∞.

→ r_n converges (essentially) geometrically to r_∞.

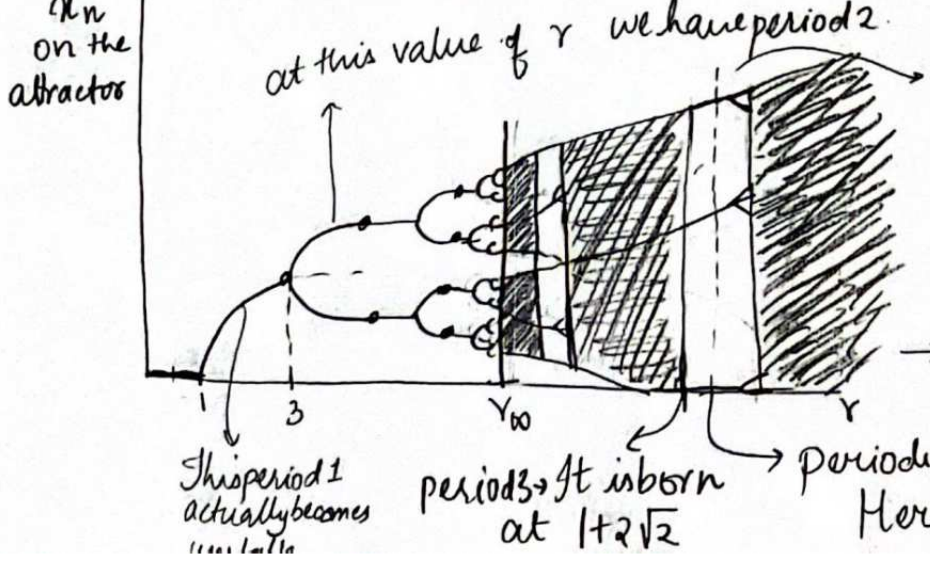
approaches a geometric series as $r_n \uparrow$



$$\frac{r_n - r_{n-1}}{r_{n+1} - r_n} \longrightarrow \underline{\underline{4.6692\ldots}} \quad \text{as } n \to \infty \quad \to \text{geometric as } n \to \infty$$

→ $\boxed{\delta}$ → Poster Boy/Girl for Chaos!!

<u>What happens after r_∞?</u>



→ <u>Orbit diagram</u>

→ Inside this period 3 window there is actually a full copy of the orbit diagram. This is true for every window & not just for this period 3 window.

→ <u>Every integer has a window.</u>

↳ Theorem!

→ Try to derive the first few r_ns

$x_{n+1} = r x_n (1-x_n)$ $\qquad x^* = 0$ is always a f.p $\forall r$.

Near $x^* = 0$, map behaves like $x_{n+1} \cong r x_n$

$\Rightarrow x_1 = r x_0$

$x_2 = r x_1 = r^2 x_0$

$\vdots$

$x_n = r^n x_0 \Rightarrow x^* = 0$ is linearily stable if $r < 1$.

→ Stability of x^* depends on looking at $|f'(x^*)| \Rightarrow |f'(x^*)| < 1$ $\Rightarrow$ stable. Same as Lorenz map.

$f(x) = r x (1-x)$

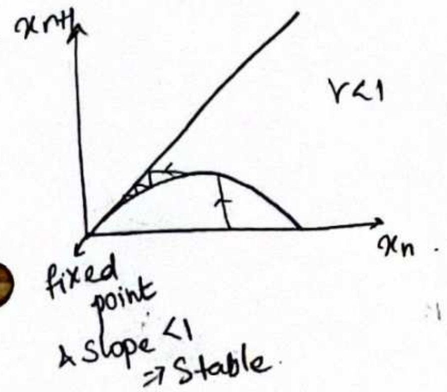
$= r x - r x^2$

$f'(x) = r - 2 r x$

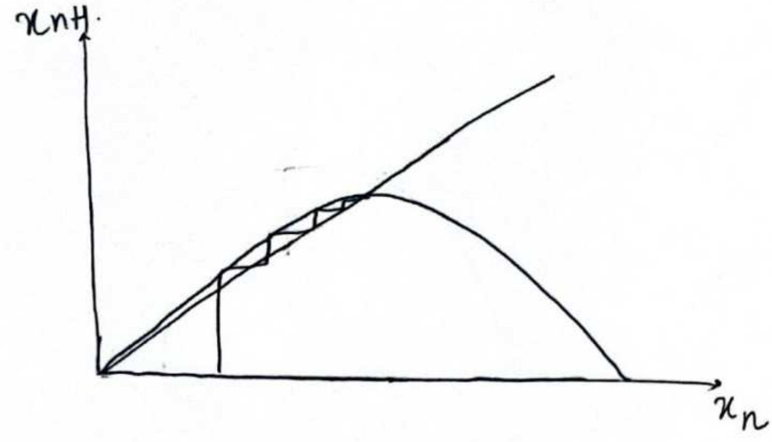
$f'(0) = r \Rightarrow r < 1 \Rightarrow$ linearly stable $\qquad$ since r is always positive.

$f(x^*) = x^* \Rightarrow r \cancel{x^*} (1-x^*) = \cancel{x^*} \Rightarrow x^* = 1 - \frac{1}{r}$

$f'(x^*) = r - 2r\left(1 - \frac{1}{r}\right) < 1 \Rightarrow 2 - r < 1 \Rightarrow \boxed{r > 1}$ → notice this on orbit diagram

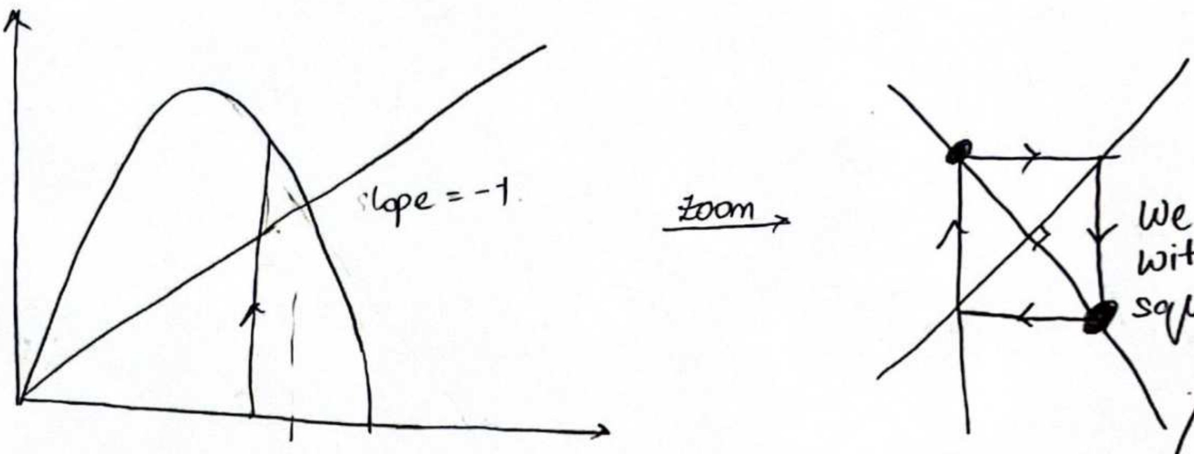
→ 

Also note that from here we see global asymptotic stability not just linear stability.



<u>What happens at $r=3$.</u>

$$f'(x^*) = 2 - r = -1$$



We end up with a square!

period 2 !

→ When $r=3$ we have $\underbrace{f'(x^*) = -1}$ (⇒ period doubling) a condition that leads to period doubling.

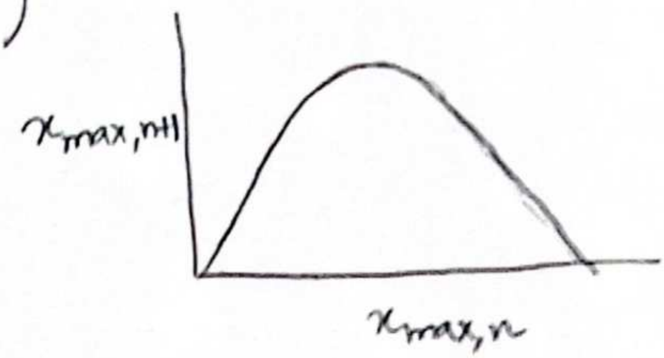
$f'(x^*) = -1$ called flip bifurcation & -1 here is the eigenvalue because it is the multiplier to the deviation for each iteration

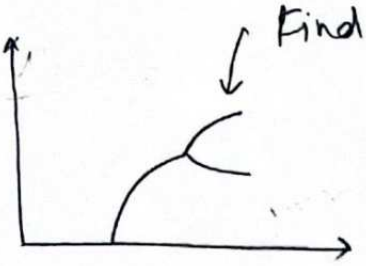
$$\eta_{n+1} = \underbrace{f'(x^*)}_{\text{eigenvalue}} \eta_n .$$

Rossler System

$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + z(x - c)$$

1D map of this is unimodal.



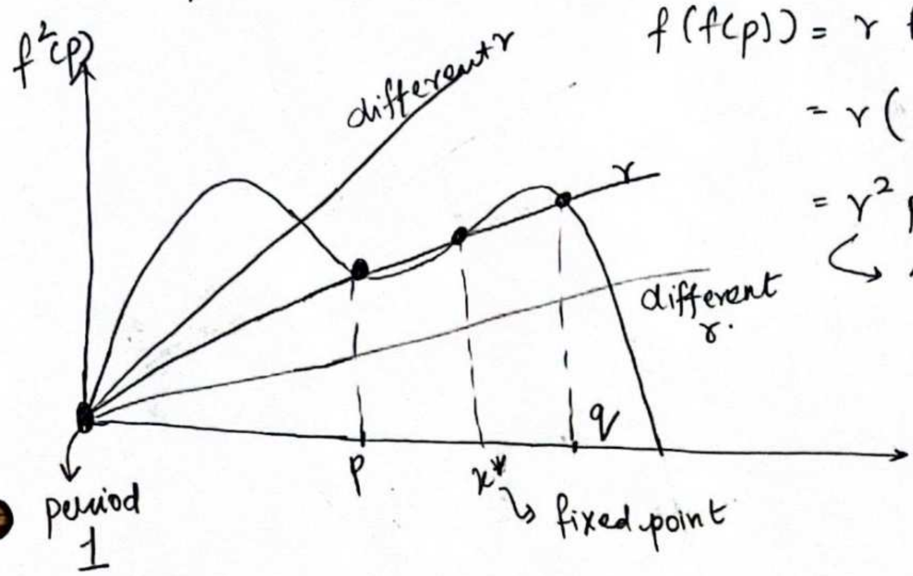


$$\left.\begin{array}{l} f(p) = q \\ f(q) = p \end{array}\right\} \Rightarrow f(f(p)) = p$$

→ Period 2 point is a fixed point of the map $f^2(p)$

$$f(f(p)) = r\, f(p)\,(1 - f(p))$$
$$= r(r\,p(1-p))(1 - r\,p(1-p))$$
$$= r^2 p(1-p)(1 - r\,p(1-p))$$

↳ 4th degree polynomial.



$p = f^2(p)$ gives p & q → factoring out the 2 trivial roots
⇒ use quadratic formula.

→ p & q could be explicitly found.

→ When do they go unstable to give period 4 $r_2 = 1 + \sqrt{6} = 3.449\ldots$

→ Cannot really go further than r_3 analytically.

→ p, q are fixed points of f^2. If x_n is at p x_{n+2} is at p, which means at x_{n+1} it must be at q since at x_{n+3} it must be back at q since q is also a f.p of f^2.

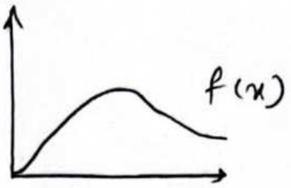
Lec 20 - Universal aspects of Period doubling. → Hardest part of the subject

Hard paper. M. J. Feigenbaum (1978) J. Stat. Physics. vol. 19, Pg 25.

Easier paper. (1983) Physica D vol. 7, Pg 16

1) Studied period doubling in various 1 humped (unimodal) maps

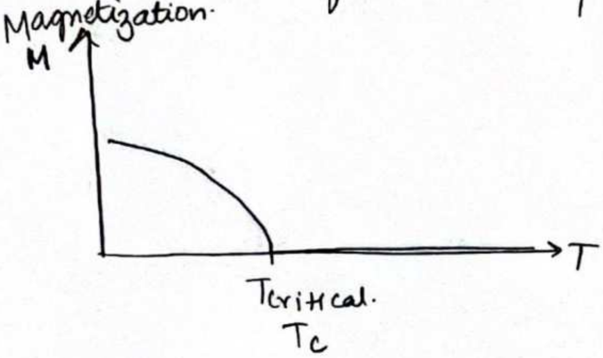
$x_{n+1} = r f(x_n)$



→ He found amazing quantitative laws independant of precise form of $f(x)$!

2) Connections to statistical physics -

a) Analogy to "universal exponents" observed in 2nd order phase transitions in magnets, fluids, superfluids ...



$m \propto |T - T_c|^{\beta}$ → exponent rate at which phase transition occurs.

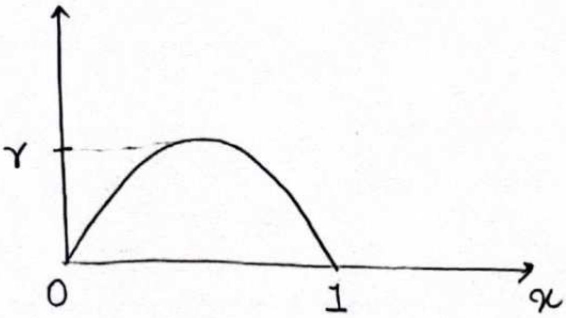
→ Completely different physical systems have the same phase transition exponents. This was explained using "renormalization"

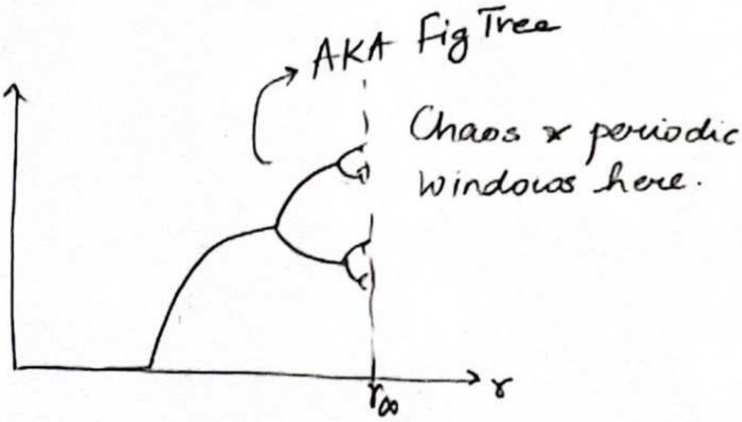
b) → Feigenbaum used renormalization to explain this universality in dynamical chaotic systems.

3) Predictions about route to chaos :- Confirmed in very different experiments in fluids, chem oscillators, semiconductors etc.

<u>Computer experiment</u>

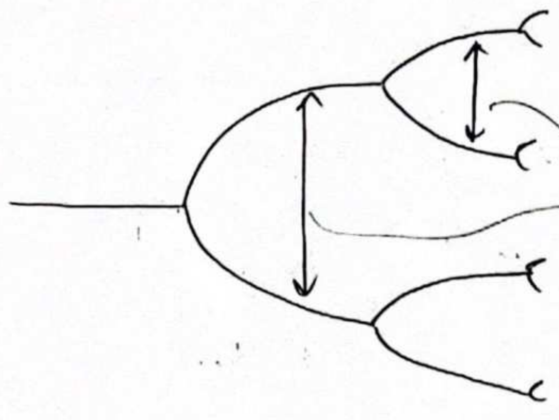
Consider sine map $f(x) = r \sin \pi x$. $x \in [0,1]$.



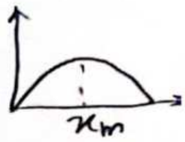


→ Period doubling still occurs. The specific r_ns depend on $f(x)$, but $\dfrac{r_n - r_{n-1}}{r_{n+1} - r_n} \longrightarrow 4.6692\ldots = \delta$ again!

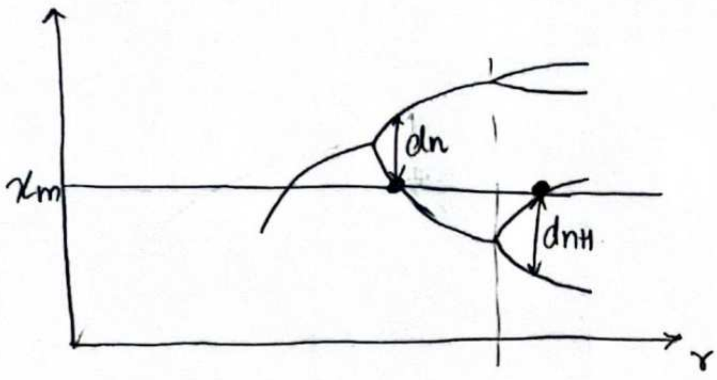
→ There's also universal scaling in the x direction.



He wanted to find what the relationship b/w these 2 heights is.

→ Say x_m is where the maximum is on the map. 

→ We want to look at the fig tree near x_m.



$d \to$ distance from x_m to the branch. We will later show that all periods cut the x_m line.

Ratio $\frac{d_n}{d_{n+1}}$ always converges to $-2.5029\ldots = \alpha$.

Lec 21 Feigenbaum's renormalization analysis of period doubling

Define superstable fixed points and cycles.

$x_{n+1} = f(x_n)$. Fixed pt. $f(x^*) = x^*$

$x_n = x^* + \eta_n \qquad \eta_{n+1} = f'(x^*)\eta_n + \ldots$ → linear growth

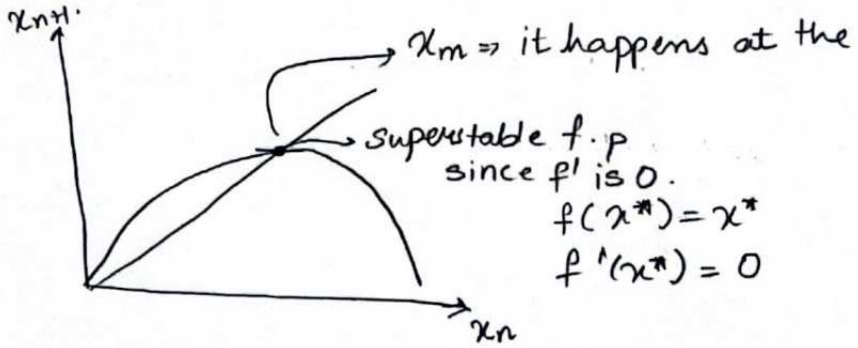
x^* is linearly stable $\Longleftrightarrow |f'(x^*)| < 1$

If $f'(x^*) = 0 \Rightarrow x^*$ is super stable

$\eta_{n+1} = \frac{1}{2} f''(x^*)\eta_n^2 + \ldots \longrightarrow$ quadratic growth.

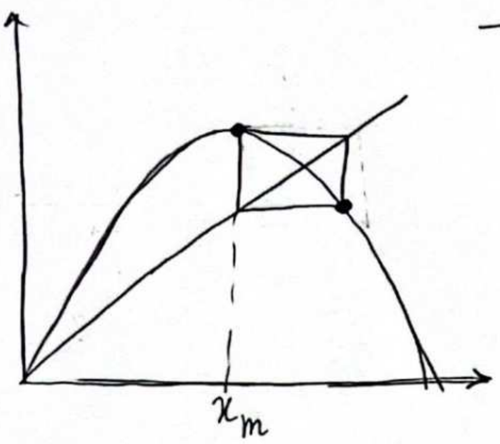
Say $\eta_{n+1} = \eta_n^2$, if $\eta_1 = 10^{-1} \Rightarrow \eta_2 = 10^{-2} \Rightarrow \eta_3 = 10^{-4}, 10^{-8}, 10^{-16}\ldots$

This is much faster than exponential.



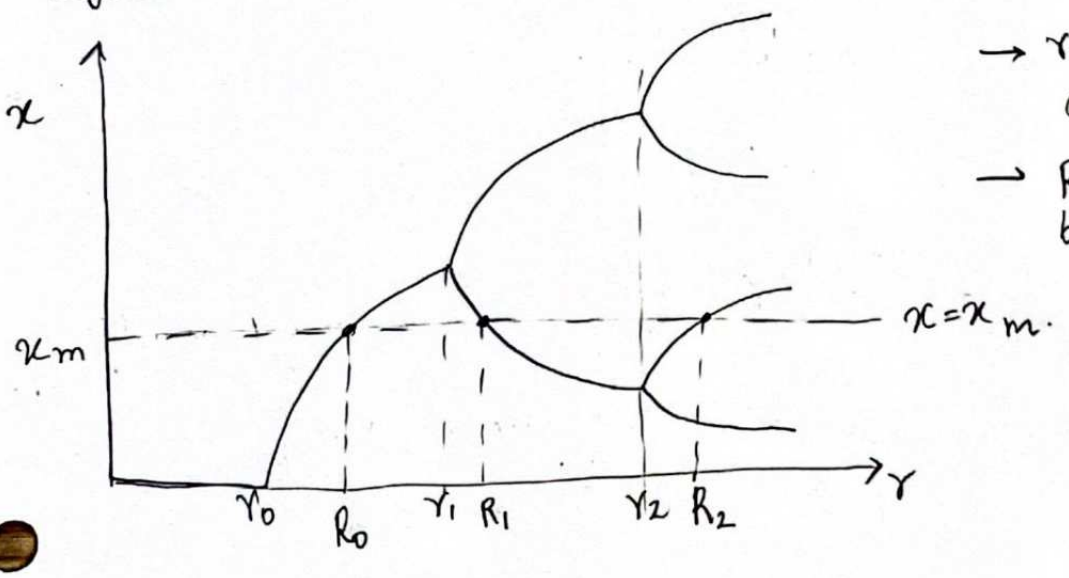
Superstable P-cycle: x_m is a fixed pt. of f^P. Equivalently x_m is one of the points in the P cycle.

Eg: superstable 2 cycle.



→ Finding these points is very fast in convergence.

Figtree.



→ r_n is where a stable 2^n cycle is born.

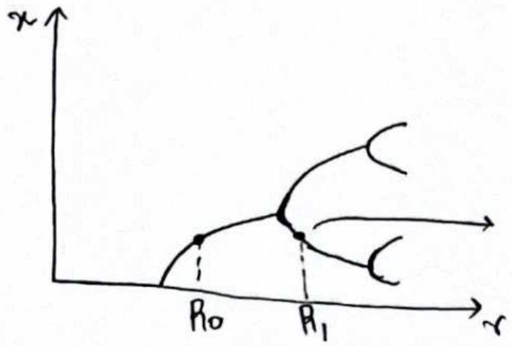
→ R_n is where the 2^n cycle becomes superstable.

→ R_n occurs between r_n and r_{n+1}.

→ Spacing between the successive R_ns also shrinks geometrically at a rate $\delta = 4.669$ as $n \to \infty$. Since R_n is easier to compute we can study that instead.

Renormalization

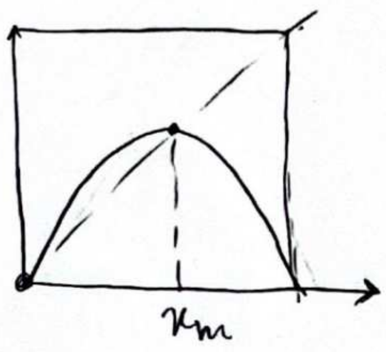
→ Picture looks self similar.



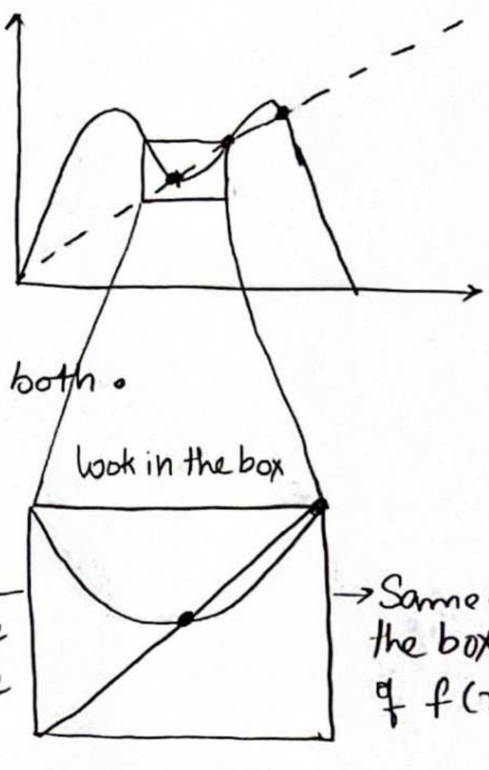
› Compare the situations at R_0 and R_1 and "renormalize" one into the other.

in f this is a point on the superstable cycle. However in f^2 it is a superstable fixed point.

→ Graph of $f(x, R_0)$



Graph of $f^2(x, R_1)$



→ x_m is a superstable fixed point for both.

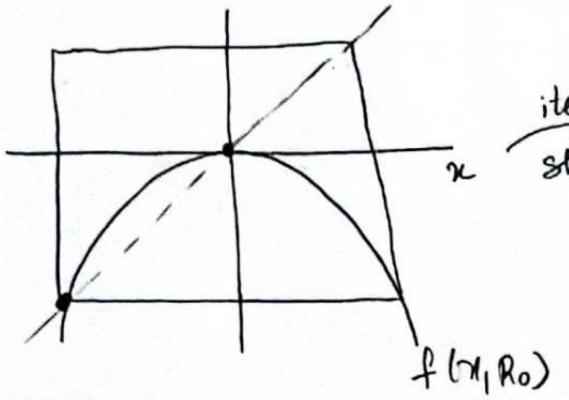
Need to flip ↕ & scale to make the pictures look the same.

→ Same as the box of $f(x, R_0)$.

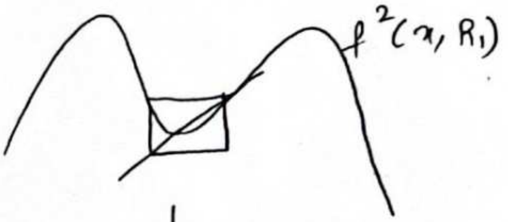
→ $f^2(x, R_1)$ has the same local dynamics as $f(x, R_0)$ except shrunken and flipped.

→ Helpful to translate origin to x_m on diagonal.

⇒ subtract x_m from both x & f ↳ since f is x in the future.



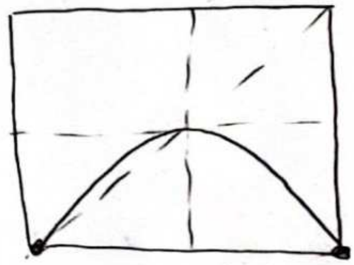
iterate & shift



↓

Rescale by $\alpha \simeq -2.5$ ↳ Notice!

↓



This shifting & scaling is the renormalization procedure.

In equations.

→ Pictures suggest $f(x, R_0) \simeq \alpha f^2\left(\frac{x}{\alpha}, R_1\right)$

× scales up ÷ scales up

→ Doing it again,

$$f(x, R_0) \simeq \alpha^2 f^4\left(\frac{x}{\alpha^2}, R_2\right)$$

$$\simeq \alpha^n f^{2^n}\left(\frac{x}{\alpha^n}, R_n\right)$$

→ Renormalization n times!

→ Feigenbaum found numerically that the limit as $n \to \infty$ of

$$\lim_{n\to\infty} \alpha^n f^{2^n}\left(\frac{x}{\alpha^n}, R_n\right) = g_0(x)$$

↳ converged to this $g_0(x)$ which looks

→ <u>What is $g_0(x)$?</u>

→ A universal function with a superstable fixed point

Only works if α is chosen to be $-2.5029...$

→ Universal in the sense that the limit function is 'almost' independant of f. (as long as it is 1 humped)

→ <u>Can now see where universality comes from:</u>

Limiting function $g_0(x)$ is universal because it only depends on the behaviour of the original function f near the maximum x_m.

> The boxes are 'zooming' in around the x_m point with each iteration. Therefore, it doesn't really depend on the fn but only on the nature of the maximum. Global aspects of f are lost & only the order of the maximum survives.

> Different $g_0(x)$s are found for fs with a 4^{th} degree max or 6^{th} degree max etc...

→ To get other universal functions $g_i(x)$ start with $f(x, R_i)$

↳ has a superstable 2^i cycle.

Look for a function $g_i(x) = \lim_{n\to\infty} \alpha^n f^{2^n}\left(\frac{x}{\alpha^n}, R_{n+i}\right)$

→ We want $i = \infty$ ⇒ where we have the onset of chaos.

Notice here R_∞ iterates to $R_{\infty+1}$ which is just R_∞.

→ $R = R_\infty$ is most interesting, since then $f(x, R_\infty) \approx \alpha f^2(\frac{x}{\alpha}, R_\infty)$ ↑ no shift

⇒ R does not change!

→ Limiting function $g_\infty(x)$ is just called $g(x)$.

$g(x)$ — it satisfies: $\boxed{g(x) = \alpha g^2\left(\frac{x}{\alpha}\right)}$

These are like boundary conditions.

↳ The functional equation for g and α.

→ But we also need $g'(0) = 0$ since max is now at origin.

→ Also need: g has a quadratic max.

→ Can choose $g(0) = 1$ → normalizing x since scaling x still satisfies the fnal. eqn.

⇒ If $g(x)$ solves the functional eqn. So does $\mu(g(\frac{x}{\mu}))$ for any μ

→ Plug in $g(x) = 1 + c_2 x^2 + c_4 x^4 \ldots$ to solve the functional equation (since $g(x)$ is even → can be shown).

→ <u>Finding α</u>:

→ $g(0) = \alpha g^2(0) \Rightarrow \alpha g(1) = 1 \Rightarrow \boxed{\alpha = \frac{1}{g(1)}}$

Substituting in the functional equation we get,

$c_2 = -1.527, \ldots \quad c_4 = 0.1048 \ldots$

We have found α, now we need to find δ!

Lec 22 Renormalization: Function Space and a hands on Calculation