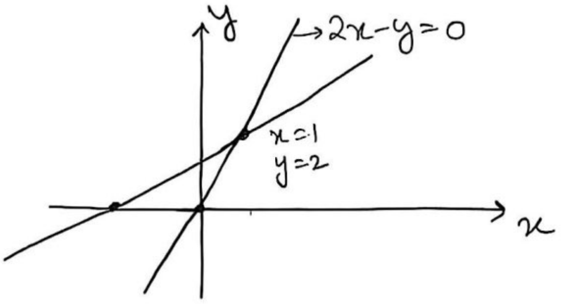
Linear Algebra - MIT 18.06

Lec 1

> $2x - y = 0$
> $-x + 2y = 3$ $\rightarrow$ $\underset{A}{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{b}{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}$

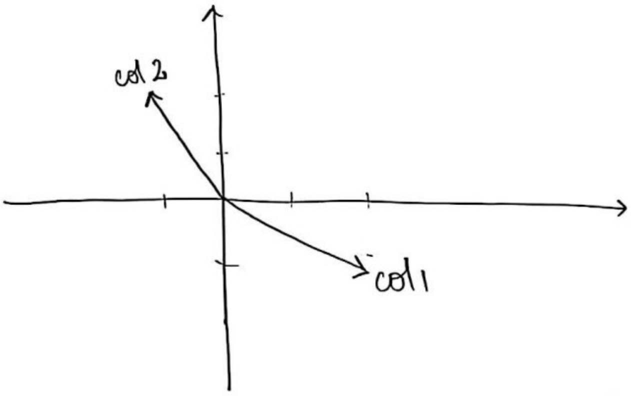
Row picture.



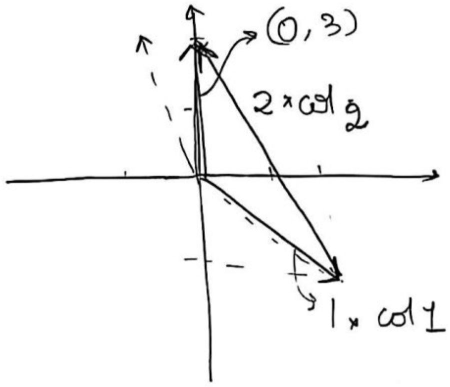
Column picture.

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

→ linear combination of 2 vectors should give resultant b vector.



$\xrightarrow[y=2]{x=1}$



> All combinations of x & y fill the plane.

Eg:

$$\begin{aligned} 2x \ -y \quad &= 0 \\ -x \ +2y - z &= -1 \\ -3y + 4z &= 4. \end{aligned}$$

$\Rightarrow A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2y & -1 \\ 0 & -3 & 4 \end{bmatrix}$

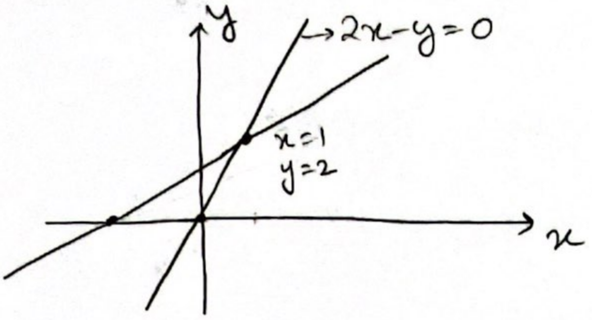
$b = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$

Linear Algebra - MIT 18.06

Lec 1

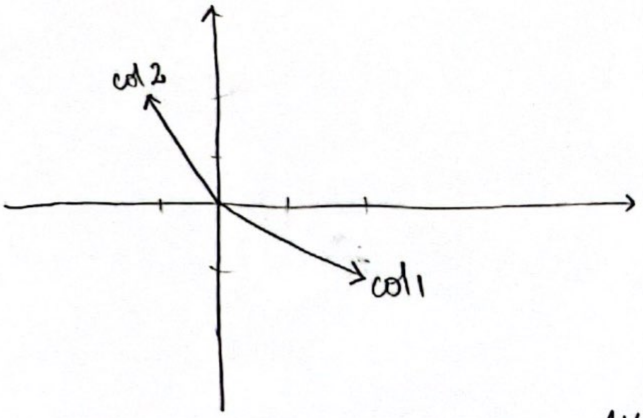
> $$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned} \rightarrow \underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}_{b}$$

Row picture:

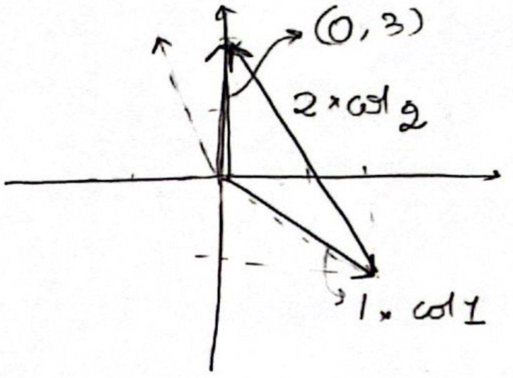


Column picture:

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$ → linear combination of 2 vectors should give resultan b vector.



$$\xrightarrow[y=2]{x=1}$$



> All combinations of x & y fill the plane.

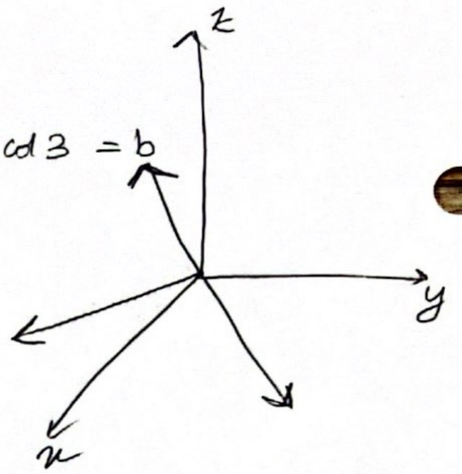
Eg:

$$\begin{aligned} 2x - y \quad &= 0 \\ -x + 2y - z &= -1 \\ -3y + 4z &= 4. \end{aligned} \Rightarrow A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2y & -1 \\ 0 & -3 & 4 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$x = 0,\ y = 0,\ z = 1.$



> Is $Ax = b$ always soluable?

Yes, if it is nonsingular or invertible!

=> If one column is a linear combination of the others.

Rec 1, MIT 18.085 - Lec 2

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = b$$

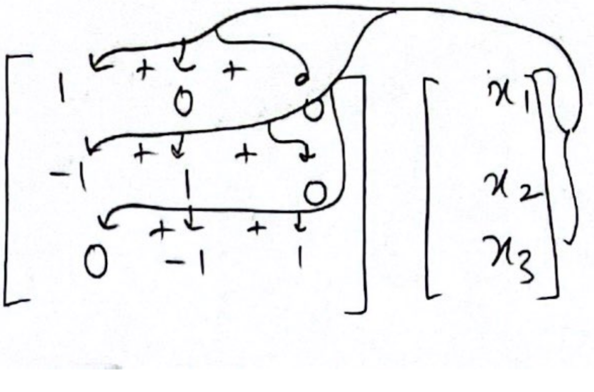
First order difference matrix.

$Ax = b$

=> $\boxed{x = A^{-1} b}$ given A is invertible.

$$x = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Sum matrix. $S = A^{-1}$ ↳ inverse of difference matrix.



> The **basis** for n dimensional space is a set of n independant vectors ⇒ the matrix must be invertible

> A vector space is a set of vectors whose linear combinatio could be taken.

> A sub space is a vector space inside a bigger vector space.

> For R^3 → vector space is 3D & subspaces are planes, lines and points (only the origin) & also the whole space too.

> A^TA is always symmetric & square.

Lec 2

Elimination.

$$x + 2y + z = 2$$
$$3x + 8y + z = 12$$
$$4y + z = 2 \rightarrow b$$

$Ax = b$

1st pivot.

$$\Rightarrow A = \begin{bmatrix} \boxed{1} & 2 & 1 \\ 3 & \boxed{8} & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

Eliminate x in equation ②.

2nd pivot

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$$

(2,1)

(3,1)

next make this zero.

$$\begin{bmatrix} 2 \\ 6 \\ -10 \end{bmatrix} \rightarrow c$$

$$\rightarrow \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{bmatrix}$$

U

3rd pivot

→ Upper triangular matrix.

→ Pivot's cannot be 0.

$A \rightarrow U$

$b \rightarrow c$

*> Determinant is product of pivots.

> If pivot position is 0, exchange rows. ⇒ temporary failure.

> Complete failure →

Back Substitution → Can be done in triangular matrices.

$x + 2y + z = 2$. $x = 2$

$2y - 2z = 6$ $\quad y = 1$

$5z = -10$ $\quad \Rightarrow z = -2$

Matrices - Elimination.

$$\underset{1\times 3}{\begin{bmatrix} 1 & 2 & 7 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}} = \underset{1\times 3}{\begin{bmatrix} a+2d+7g & b+2e+7h & c+2f+7i \end{bmatrix}}$$

✓ use the idea to do first step of elimination.

Subtract 3x row1 from row2.

$$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & +1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} -3\times(1 & 2 & 1) \\ +1(3 & 8 & 1) \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

sub. 2 row2 from 3.

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\left.\begin{aligned} &\Rightarrow E_{32}(E_{21} \cdot A) = U \\ &\Rightarrow (E_{32} \cdot E_{21})(A) = U \end{aligned}\right\} \text{Associative law.}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}$$

↳ Does the full

Permutation matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

P → exchanges rows.

P → exchanges columns.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Inverses.

$$\begin{bmatrix} 1 & 0 & 0 \\ +3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E^{-1} \quad E \quad = \quad I$$

Eg:

$$\begin{aligned} x - y - z + u &= 0 \\ 2x + 2z &= 8 \\ -y - 2z &= -8 \\ 3x - 3y - 2z + 4u &= 7 \end{aligned}$$

$$\left[\begin{array}{cccc|c} \boxed{1} & -1 & -1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 3 & -3 & -2 & 4 & 7 \end{array}\right]$$

Ans:

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & -1 & -4 \end{array}\right]$$

Lec 3 Basic way of Multiplication.

>
$$\underset{m\times n}{\underset{A}{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}} \underset{n\times p}{\underset{B}{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}} = \underset{C = AB}{\begin{bmatrix} & c_{ij} & \\ & & \end{bmatrix}_{m\times p}}$$

c_{ij} comes from row i & column j $=$ (row i of A). (col. j of B).

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots$$

$$= \sum_{k=1}^{n} a_{ik}b_{kj}$$

> Could do it column vector wise. from B → A ⇒ C

$$\overset{A}{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}} \overset{B}{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}} = \overset{C}{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}$$

columns of c are combinations of columns of A.

> could do it row wise from A → B ⇒ C

> another way is column x row $\underset{3\times1}{\begin{bmatrix} \\ \\ \end{bmatrix}} \underset{1\times2}{\begin{bmatrix} \quad & \quad \end{bmatrix}} \longrightarrow 3\times2$

> Block way.

$$\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right] = \left[\begin{array}{c|c} A_1B_1 + A_2B_3 & \\ \hline & \end{array}\right]$$

Inverses.

- $A^{-1}A = I = AA^{-1}$ (for square matrices)

$\Rightarrow$ A is invertible or nonsingular.

Singular case: $AX = 0 \Rightarrow A$ is singular. if X is non zero.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix}}_{A^{-1}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$a + 3b = 1$

$2a + 7b = 0$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$c + 3d = 0$

$2c + 7d = 1$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Gauss Jordan (solve 2 eqns. at once)

$$\Rightarrow \underbrace{\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]}_{\text{augmented}} \xrightarrow[\text{elimination.}]{\text{use.}} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

(under the first matrix: $\underbrace{\quad}_{A}$, $\underbrace{\quad}_{I}$)

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$\underbrace{I}_{\text{eliminate until this half is 'I'}}$, A^{-1}

$$\boxed{\begin{array}{l} E[A\ I] = [I\ X] \\ \text{since } EA = I. \\ \Rightarrow EI = A^{-1} \\ \Rightarrow X = A^{-1} \end{array}}$$

Inverse of A.B.

$> (AB)(B^{-1}A^{-1}) = AIA^{-1} = I.$

$> \boxed{(AB)^{-1} = B^{-1}A^{-1}}$

$> (AA^{-1})^T = (I)^T$

$\Rightarrow (A^{-1})^T A^T = I$

$\Rightarrow \boxed{(A^T)^{-1} = (A^{-1})^T}$

Most basic factorization of A

$$\boxed{A = LU}$$

$L \rightarrow$ lower triangular matrix.

$U \rightarrow$ triangular matrix after elimination!

$$\begin{array}{ccccc} E_{21} & & A & & U \\ \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} & & \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} & = & \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \end{array}$$

$$A = LU$$

$$\Rightarrow \boxed{L = E_{21}^{-1}} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$$

$$\Rightarrow A = LU = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}}_{\text{upper}}$$

> $E_{32}\, E_{31}\, E_{21}\, A = U$ assuming no row exchanges.

- $\Rightarrow$ $$A = \underbrace{E_{21}^{-1}\, E_{31}^{-1}\, E_{32}^{-1}}_{L}\, U$$

$$\overset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \overset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix}$$

$$\overset{E_{21}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{E_{32}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

because of this inverses are better!

- $EA = U$

  $A = LU$ → This is superior. If no row exchanges.

→ The multipliers go directly into L.

→ How many operations must be done in $n \times n$ for elimination.

> $n^2 + (n-1)^2 + \dots + 2^2 + 1^2 \simeq \frac{1}{3} \cdot n^3$ on A!

$\frac{1}{3}$ → like an integral.

> Cost on b or RHS is n^2.

> What if we have row exchanges?

Permutation matrices.

- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → doesn't do anything.

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

These are all row exchanges

> $\boxed{P^{-1} = P^T}$

> For 4X4 there are 24 ps. ⇒ $\boxed{n! \text{ of P matrices exist}}$

Lec 5

Need to use permutations to allow elimination.

⇒ $\boxed{PA = LU}$ → Accounts for row exchanges.

→ Symmetric matrix ⇒ $\boxed{A^T = A}$

→ $R^T R$ is always symmetric where R is any rectangular matrix.

since, $(R^T R)^T = R^T (R^T)^T = R^T R$.

Vector Spaces.

$\mathbb{R}^2$ → all 2 dimensional real vectors $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} \pi \\ e \end{bmatrix}$...

$\mathbb{R}^n$ → all n dimensional real vectors → vector space.

> Subspace of $\mathbb{R}^2$ is any line in $\mathbb{R}^2$ that goes through the origin!
Subspaces of $\mathbb{R}^2$ also have the origin & $\mathbb{R}^2$ itself.
↳ zero vector.

Lec 6

Vector spaces requirements.

- All linear combinations of vectors in the vector space must remain in the vector space.
- All subspaces of R^3 must go through origin.
- Given 2 subspaces S & T, $\begin{cases} S \cup T \text{ is not a subspace (usually)} \\ S \cap T \text{ is a subspace (always)} \end{cases}$

> Column space of A

- $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$
  - Column space of A is a subspace of R^4 since columns are vectors of 4 dimension
  - C(A) = All linear combinations of the columns. Which is also a subspace.

- Does C(A) fill the full 4D space? No. We need 4 vectors that are independant.

- Does $Ax = b$ have a solution for every b? No. Because there are 4 eqs. with 3 unknowns. For some values of b, it can be solved.

- Which vectors b allow a solution? Vector [b] must lie on the subspace given by the linear combination of the columns $\Rightarrow \underline{\underline{C(A)}}$. Solution exists when b is in the column space of A.

- In A, note only 2 vectors are independant $\Rightarrow$ C(A) is a 2 dimensional subspace of R^4.

Null space of A

"All solutions $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $Ax = 0$."

> This is in $\mathbb{R}^3$.

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Contains, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} c \\ c \\ -c \end{bmatrix}$

$\Rightarrow$ nullspace is $c\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\Rightarrow$ Nullspace is a line in $\mathbb{R}^3$

> Nullspace is also a vector space.

> A solution to $Ax = b_1$ for some $b_1 \neq 0$ is not a vector space since $x = 0$ is not a solution.

Lec 7

$$A = \begin{bmatrix} \overset{\text{pivot}}{\boxed{1}} & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$\Rightarrow$ 2 pivots!

↳ Rank of the matrix is the no. of pivots.

> Elimination.

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & \overset{\text{pivot 2}}{\boxed{2}} & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

echelon form

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

pivot columns

free columns.
⇒ can select any values for these columns.

Let $x = \begin{bmatrix} x_1 \\ 1 \\ x_3 \\ 0 \end{bmatrix}$ & we know $x_1 + 2x_2 + 2x_3 + 2x_4 = 0$

$2x_3 + 4x_4 = 0.$

Substitute

⇒ $x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ & any multiple is part of the nullspace.

& $x = \begin{bmatrix} - \\ 0 \\ - \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ & any multiples...

⇒ All combinations of these special solutions. because null space is R^2 we only need these 2 independant sols. to define the whole N(A). $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ are part of nullspace

→ Rank = r = no. of pivot variables.

→ $n - r$ = free variables where n is no. of columns.

→ <u>Matrix R</u> → Reduced row echelon form has zeroes above & below the pivots & pivots are 1.

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In matlab: R = rref(A)

RREF : > Has I matrix in the pivot rows & columns.

> Solutions to $Ax=0$, $Ux=0$, $Rx=0$ are same. But back substitution from $Rx=0$ is much easier.

$$R = \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF is generally in the form $R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ after exchanging some rows.
& The nullspace matrix $N = \begin{bmatrix} -F \\ I \end{bmatrix}$ since RN
I is $r \times r$ identity matrix. The columns of N are the special solutions.

Lec 8

Eg

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \rightarrow \text{Solution?}$$

Augmented matrix = $[A \; b]$

$$= \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix}$$

elimination.

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 2 & 4 & b_3-3b_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3-b_2-b_1 \end{bmatrix}$$

$0 = b_3 - b_2 - b_1$ for a solution to exist.

Solvability.

Condition on b

otherwoise no combination of col. vectors can give b.

> $Ax=b$ solvable when b is in $C(A)$.

(OR)

> If a comb. of rows of A gives zero row, the same comb. of the entries of b must be zero.

To find complete soln to $Ax=b$.

one solution.

① $x_{particular}$: Set all free variables to zero.
Solve $Ax=b$ for the pivot variables.

$\Rightarrow x_2 = 0$ & $x_4 = 0$ since no pivots lie on col. 2 & 4.

$\Rightarrow x_1 + 2x_3 = 1$; $2x_3 = 3$ $\Rightarrow$ $x_3 = \frac{3}{2}$, $x_1 = -2$

$$x_{particular} = \begin{bmatrix}-2\\0\\3/2\\0\end{bmatrix}$$

② $x_{nullspace}$

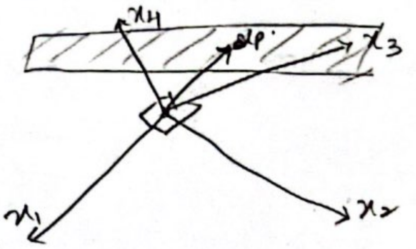
special solutions.

$$x_{complete} = \underbrace{\begin{bmatrix}-2\\0\\3/2\\0\end{bmatrix}}_{\text{particular.}} + \underbrace{c_1\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}}_{\text{nullspace}}$$

$x_{complete\ sol.} = x_{particular} + x_{nullspace}$.

$$\left.\begin{aligned} Ax_p &= b \\ Ax_n &= 0 \end{aligned}\right\} \Rightarrow A(x_p + x_n) = b.$$

Plot all Sols. x in $\mathbb{R}^4$



2 dimensional subspace.

but it must pass through. $x_p \Rightarrow$ it is not really a subspace!

> $m \times n$ matrix A of rank r. : (know $r \leq m$, $r \leq n$).
↳ aka no of pivots.

Full column rank ⇒ $r = n$ ⇒ no free variables ⇒ the null space of A is only the zero vector. The solution of $Ax = b$ ⇒ $x = x_{particular}$ if the solution exists. ⇒ 0 or 1 solution

Eg:- $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$ RREF = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Full row rank ⇒ $r = m$

> Can solve $Ax = b$ for every b. ⇒ sol. exists.

> Left with $\underset{\overset{\Updownarrow}{n-m}}{n-r}$ free variables.

Eg:

$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$ rank = 2
RREF = $\begin{bmatrix} 1 & 0 & \_\_ & \_\_ \\ 0 & 1 & \_\_ & \_\_ \end{bmatrix}$

F ↳ Free

Full rank : $r = m = n$

Eg:- $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ RREF = I ⇒ Wellbehaved!

* there are no constraints on b
⇒ solvable for any b!

> Null space = 0 vector.

> Only one solution exists.

> 2 vectors in $\mathbb{R}=2$ ⇒ covers the whole plane!

$r=m=n$.	$r=n<m$	$r=m<n$	$r<m<n$.
$R=I$	$R=\begin{bmatrix} I \\ 0 \end{bmatrix}$	$R=\begin{bmatrix} I & : & F \end{bmatrix}$	$R=\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
1 solution.	0 or 1 solution	∞ solutions	(0 or ∞ solutions)

Lec 9

Suppose A s.T $m<n$. Then there are non zero solutions to $Ax=0$. Since more unknowns than equations.

Reason: There will be free variables.

Independance.

> Vectors $x_1, x_2, \ldots, x_n$ are linearly independant if no linear combination gives zerovector, except the zerovector.

> If one of $x_1, x_2 \ldots$ are the zerovector $\Rightarrow$ dependant!

☆ In terms of vectors, they are independant if nullspace of A is the zero vector. Where A is the combination of thosevectors as columns. $\Rightarrow$ No free variables.

Span.

> Vectors $V_1\, V_2 \ldots V_l$ span a space $\Rightarrow$ the space consists of all combinations of these vectors.

Basis

> A basis for a space is a sequence of vectors $V_1, V_2 \ldots V_d$ with 2 properties. ① They are independant.
② They span the wholespace.

Eg: Space is $\mathbb{R}^3$

One basis is $\begin{bmatrix}1\\1\\2\end{bmatrix} \begin{bmatrix}2\\3\\4\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix}$

$\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix}$

> R^n, n vectors give basis if the $n \times n$ matrix of vectors as columns is invertible.

$\begin{bmatrix}1\\1\\2\end{bmatrix} \begin{bmatrix}2\\3\\5\end{bmatrix}$ are a basis for a plane in $\mathbb{R}^2$

> Given a space, every basis has the same no. of vectors. & this no. is called the "dimension".

Eg:

Space is C(A).

$$\begin{bmatrix}1 & 2 & 3 & 1\\ 1 & 1 & 2 & 1\\ 1 & 2 & 3 & 1\end{bmatrix}$$

→ not independant.

→ They do span the C(A).

→ Basis for the col. space = $\begin{bmatrix}1 & 2\\ 1 & 1\\ 1 & 2\end{bmatrix}$.

⇒ Rank = 2 = Dimension!

Rank (A) = # pivot columns = dimension of the $\overset{\text{column}}{\text{space}}$ C(A).

null spaces: $\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix} \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$ → The dimension of the null space = no. of free variables.

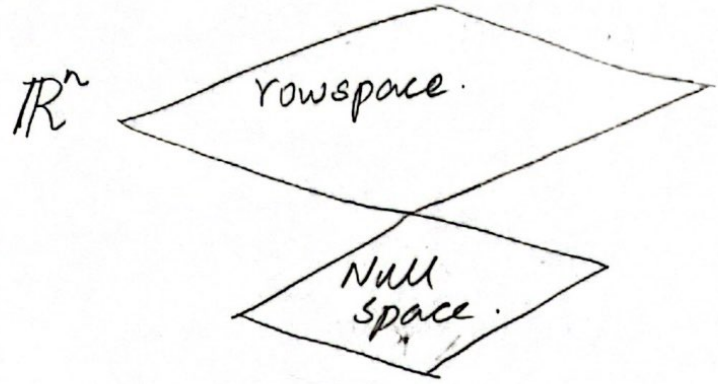
→ These form the basis!

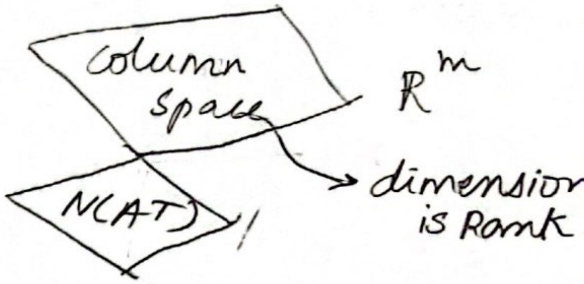
⇒ Dimension of null space = # free variables = $n - r$.

Lec 10 The 4 Subspaces.

1) Column space $C(A)$ in R^m → set of all vectors that span the space given by c
2) Null space $N(A)$ in R^n → set of all vectors that the matrix kills.
3) Row space = all combinations of rows = $C(A^T)$ → in R^n
4) Nullspace of A^T = $N(A^T)$ → aka left nullspace. → in R^m

4 spaces.





	C(A)	C(AT)	N(A)	N(AT)
Basis :	pivot cols.	pivot rows	special solutions.	
Dimension:	Rank R	R	N-R	M-R

Eg:-

$$\begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix} \rightarrow \begin{bmatrix}\overbrace{\begin{matrix}1&0\\0&1\end{matrix}}^{I}&\overbrace{\begin{matrix}1&1\\1&0\end{matrix}}^{F}\\0\;\;0&0\;\;0\end{bmatrix}$$

R= RREF

$C(R) \neq C(A)$.

> But row space is same since row operations were done to get to R!
=> row space is preserved

=> Basis for row space is first r rows of R.

th space: $N(A^T)$

$A^T y = 0 \Rightarrow y$ is in the null space of A^T.

$\Rightarrow y^T A = 0^T \Rightarrow \begin{bmatrix} \; y^T \; \end{bmatrix} \begin{bmatrix} \\ A \\ \\ \end{bmatrix} = \begin{bmatrix} \; 0 \; \end{bmatrix}$

refore
alled
eft
ullspace.

$$= \begin{bmatrix} \underset{m\times n}{A} & \vdots & \underset{m\times m}{I} \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} \underset{m\times n}{R} & \vdots & \underset{m\times m}{E} \end{bmatrix}$$ (Gauss Jordan)

$\Rightarrow EA = R.$

New vector space. → Matrix space

All 3x3 matrices!! since matrices follow all conditions of vectors we consider a vector space & subsequently subspaces of matrices.

Subspaces of M ⇒ upper triangular & symmetric matrices & diagonal matrices.

Lec 11

Let M = all 3x3 matrices.

⇒ symmetric. 3x3 } Both are
upper triangular 3x3 } examples of subspaces.

Basis for M = all 3x3's. → Need 9 matrices. $\begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix} - \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix} \cdots$

Dimension of M = 9

Symmetric 3x3 ⇒ Basis is less than 9.

Dimension of symmetric = 6.

Dimension of Upper triangular = 6

Basis of Upper triangular is 6 of the matrices from basis of M.

> $S \cap U \Rightarrow$ symmetric & uppertriangular
dimension of $(S \cap U) = 3$

$S \cup U \Rightarrow$ not a subspace.

$S+U \Rightarrow$ it is a subspace = any element of S + any element of U.

$S+U \rightarrow$ gives all 3×3 matrices $\Rightarrow M. \Rightarrow \dim(S+U) = 9.$

→ $\boxed{\dim(S) + \dim(U) = \dim(S \cap U) + \dim(S+U)}$

Eg: $\frac{d^2y}{dx^2} + y = 0$, $y = \cos x$, $\sin x$ & all linear combinations.
$\dim\left(\begin{smallmatrix}\text{soln.}\\ \text{space}\end{smallmatrix}\right) = 2$. since it is second order
= rank.

> $M =$ all 5×17 matrices.

> subset of rank 4 matrices can be either rank 4 or 5.

> rank of matrix$(A+B)$ is always $\leq$ rank A + rank B.

> subset of rank 1 or rank 2 . . . rank 4 are not subspaces

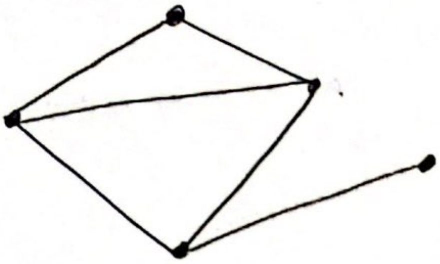
> In $\mathbb{R}^4$ let $V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$. S = all V in R^4 with $V_1 + V_2 + V_3 + V_4 = 0$ is a subspace. Dimension is 3. = null space of A is = $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$
rank of $N(A) = 1$, $\dim(N(A)) = n - r = 3$ dimensional.

> Row space is 1 dimensional. Basis for $S = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 8 \\ 1 \end{bmatrix}$

> Col. space of A is $\mathbb{R}^1$. $N(A^T) = \{0\}$

$3+1 = 4 = n.$

Graphs. → nodes + edges.



nodes = 5
edges = 6
⇒ 5×6 matrix to fully characterize it

↳ gives 6 degrees of seperation.

An insight from 3Blue1Brown.

$$\text{Let}\quad \underset{\text{matrix}}{\underset{\uparrow}{A}}\ \ \underset{\text{vector}}{\underset{\uparrow}{x}} = \underset{\text{vector}}{\underset{\uparrow}{b}}.$$

Think of A as a transformation or function that transforms vector x → vector b.

> $$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

coordinates where $\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ lands. after the transformations.

⇒ columns of A define where $\hat{i}$, $\hat{j}$, $\hat{3}$ land after the transformation & this fully characterizes this transformation.

> A.B is the same as 2 successive transformations B then

⇒ I matrix does not transform!

> Determinant ⇒ how much an area increases after the transformation is applied! Because a linear transformation ensures the coordinate lines are still straight, parallel & equally spaced. A negative determinant ⇒ the orientation of the space is inverted. Like flipping a paper. if $\hat{j}$ is to the right of $\hat{i}$ ⇒ orientation is inverted

> Det = 0 ⇒ a cube is squished into a plane since the volume is now "scaled" to 0. This means the final positions of the unit vectors $\hat{i}\,\hat{j}\,\hat{k}$ lie on a plane which comes from the fact that they are linearly dependant!

> orientation change in 3D ⇒ Right hand rule → Left hand rule.

> Rank: No. of dimensions in the o/p of a transformation
↳ No of dimensions in the column space.

> Column space :- Span of columns of matrix. If a transformation 3x3 lands on a plane ⇒ rank = 2 ⇒ span of the columns is that entire plane & since only 2 linearly independant columns exist.

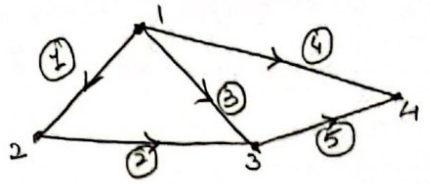
> Null space :- Space that lands on the origin after a transformation. If 2x2 has rank 1, then a line lands on the origin & this line is the null space. If 3x3 has rank 1 then a whose plane lands on the origin + all that info is lost. This plane is the null space.

$$\Rightarrow \quad A\vec{x} = 0$$

→ 3x2 matrix maps 2D vectors to 3D vectors.

2x3 matrix maps 3D vectors to 2D vectors

Lec 12 Graphs and Networks.



n = 4 nodes.
m = 5 edges.

Incidence matrix

$$A = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc|l} 1 & 2 & 3 & 4 & \\ -1 & 1 & 0 & 0 & \text{edge } ① \\ 0 & -1 & 1 & 0 & \text{edge } ② \\ -1 & 0 & 1 & 0 & \text{edge } ③ \\ -1 & 0 & 0 & 1 & \text{edge } ④ \\ 0 & 0 & -1 & 1 & \text{edge } ⑤ \end{array}$$

Loop ⇒ rows are linearly dependent.

$\boxed{2 \times m = \text{no. of zeroes.}}$

Null space ⇒ $Ax = 0$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{matrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{matrix}$$

Let x_1, x_2, x_3, x_4 be the potentials at the nodes.

⇒ A computes the potential differences.

⇒ Null space = $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $c\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ ⇒ basis for null space is the line $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

⇒ dim (null space) = **1**

> If we ground one of the nodes ⇒ $x_4 = 0$, then col 4 disappears & we get a nonsingular matrix that can be uniquely solved.

→ Rank = 3.

→ Null space of A^T ⇒ $A^T y = 0$. $\dim N(A^T) = m - r = 5 - 3 = \underline{\underline{2}}$

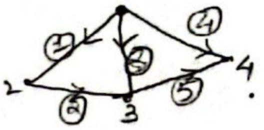
$\overset{n \times m \Rightarrow 4 \times 5}{}$

$$A^T = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$> \text{potential differences } x_2-x_1,\ x_3-x_2 \ldots \text{etc.} \xrightarrow[\text{Ohm's law.}]{\text{Some matrix 'C'}} \text{currents } y_1, y_2, y_3 \ldots$$

$$\downarrow$$

$$A^Ty = 0 \longrightarrow \text{Kirchoff's current law.}$$



$A^Ty = 0 \Rightarrow$ $-y_1 - y_3 - y_4 = 0$ ⇒ KCL at node 1

$y_1 - y_2 = 0$ → " node 2

$y_2 + y_3 - y_5 = 0$ → " node 3

$y_4 + y_5 = 0$ → " node 4.

Basis for nullspace:

> RREF gives 2x2 free vectors matrix ⇒ we need 2 vectors for the basis

$$\Rightarrow \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$ are enough for the basis.

Row space of A

⇒ col. space of A^T = 3 cols. → pivot columns (1, 2, 4) ⇒ They form a graph without a loop called a tree.

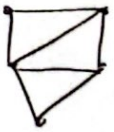
> Dim. $N(A^T) = m - r$.

⇒ # loops = # edges − (# nodes − 1)

$$\Rightarrow \boxed{\#\text{nodese} - \#\text{edges} + \#\text{loops} = 1}$$

Euler's formula.

Eg:



5−7+3 = 1

Summary.

$e = Ax$ for voltage differences.

$Ce = y$ → ohm's law

$A^T y = f$ → KCL with $f \neq 0$ for current sources.

$\Rightarrow$ $\boxed{A^T C A x = f}$ → steady state soln. of applied math in general.

> Trace of a matrix = sum of diagonal entries.

> Trace $(A^T A)$ = sum of degrees of all nodes.

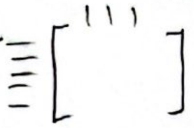
degree of a node ⇒ no. of edges connecting to the node.

Lec 13 Problems.

① U, V, W are nonzero vectors in $\mathbb{R}^7$. What are the possible dimension of the subspaces? Ans: 1, 2, 3 since they are nonzero.

② 5X3 U matrix has 3 pivots. ⓐ What is the null space?

Ans



. $N(U) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

ⓑ $10 \times 3 = \begin{bmatrix} U \\ 2U \end{bmatrix}$ What is the rank & RREF? Assume U is in RREF.

$\Rightarrow$ RREF $= \begin{bmatrix} U \\ 0 \end{bmatrix}$ → Rank = 3.

ⓒ $\begin{bmatrix} U & U \\ U & 0 \end{bmatrix}$? $\rightarrow \begin{bmatrix} U & U \\ 0 & -U \end{bmatrix} \rightarrow \begin{bmatrix} U & 0 \\ 0 & -U \end{bmatrix} \rightarrow \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}$

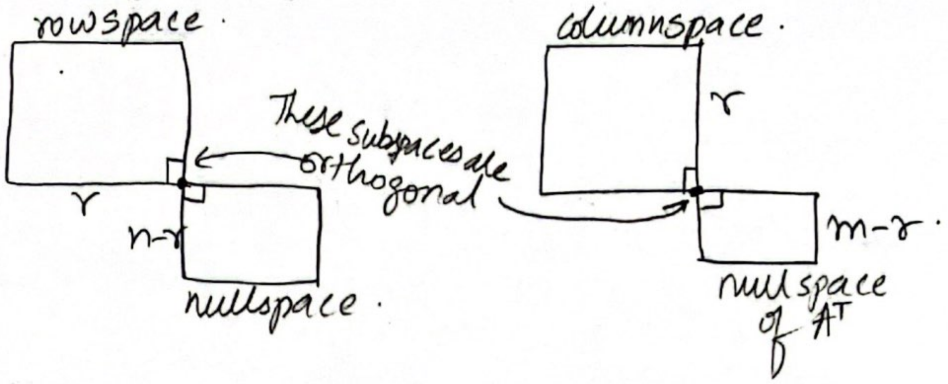
Rank = 6

ⓓ dim $N(C^T)$, $C \rightarrow 10 \times 6$ & $C^T \rightarrow 6 \times 10$ $\Rightarrow 10 - 6 = \boxed{4}$

Note: Watch "Exam #1 Problem Solving" video for a good summary.

Lec14 Orthogonality.

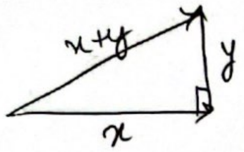
>



> Orthogonal vectors => 90°. b/w vectors. => Pythagoras.

> Dot product of vectors X & y = 0 => orthogonal.

$$\Rightarrow \boxed{x^T y = 0}$$



$$\Rightarrow \|x\|^2 + \|y\|^2 = \|x+y\|^2$$

$$x^T . x + y^T . y = (x+y)^T . (x+y)$$

$$= x^T y + x^T y + y^T x + y^T y.$$

$$\Rightarrow 2x^T y = 0 \Rightarrow \boxed{\langle x, y \rangle = 0}$$

> Inner product of $x, y = \langle x, y \rangle = x^T y$.

> The zero vector is orthogonal to all vector.

> Subspaces S & T are orthogonal if every vector in S is orthogonal to every vector in T.

> If two subspaces are orthogonal, they must only intersect at the origin.

> Row space is orthogonal to null space. → like a dot product b/w x & rows of A.

Why? $Ax = 0$ $\begin{bmatrix} \text{row 1 } A \\ \text{row 2 } A \\ \vdots \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

But all rows in the row space are linear

R^3 & $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix} \Rightarrow n = 3, r = 1$

$\dim(N(A)) = 2.$

⇒ null space is a plane ⊥^{lar} to [1,2,5] vector.

Null space & row space are orthogonal + their dimension add up to the whole space ⇒ they are orthogonal complimen

⇒ Null space contains all vectors ⊥ to row space.

$\underline{Ax = b}$? "Solve" when there is no solution. Because b might have noise.

> A^TA plays a key role. It is square symmetric.

$Ax = b.$

> $\boxed{A^TA\hat{x} = A^Tb.}$ + we hope $\hat{x}$ has a solution.

> Is ATA invertible? Sometimes.

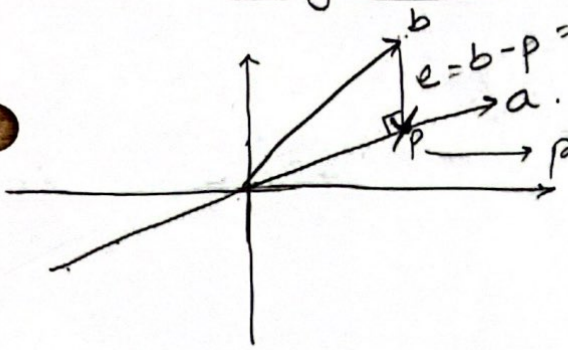
Eg: $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \Rightarrow A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$

* > $N(A^TA) = N(A)$.

* rank of A^TA = rank of A

> A^TA is invertible if A has columns that are independent.

Lec 15 Projections.



P = x.a.
↳ scalar.

$a^T(b - xa) = 0$ since a & e are ⊥.

⇒ $xa^Ta = a^Tb$.

✱⇒ $\boxed{x = \frac{a^Tb}{a^Ta}}$, $\boxed{p = ax.}$

✱⇒ $\boxed{p = a\frac{a^Tb}{a^Ta}}$

if b → 2b ⇒ p → 2p.
a → 2a ⇒ no change.

→ Some projection matrix.

⇒ $p = \mathbb{P} . b$

$$\mathbb{P} = \frac{aa^T}{a^Ta}$$

→ n×n
→ matrix
→ numbers.
→ 1×1.

> Column space of ℙ : C(ℙ) is the line through a because ℙ.b is always on a.

> Rank (ℙ) = 1

✱> $\mathbb{P}^T = \mathbb{P}$

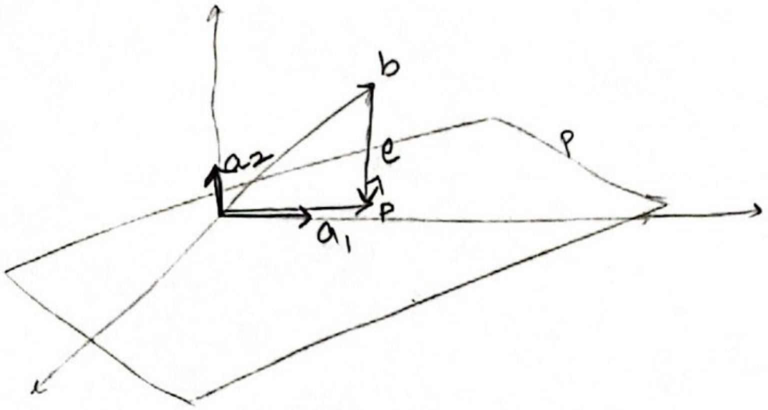
✱> $\mathbb{P}^2 = \mathbb{P}$ since projecting 2nd time doesn't change anything.

Why project?

Because $Ax=b$ may have no solution.

Solve $A\hat{x}=p$ ↳ projection of b onto column space so that it can have a solution.

→



> Project b onto plane p.
> plane p has basis a_1 & a_2
> P is col. space of $[a_1\ a_2]=A$.
> $e=b-p$
> e is $\perp^{lar}$ to the plane.

→ $p=\hat{x}_1a_1+\hat{x}_2a_2$

→ $p=A\hat{x}$. Find $\hat{x}$

→ $b-A\hat{x}$ is $\perp^{lar}$ to plane.

→ $a_1^T(b-A\hat{x})=0$ & $a_2^T(b-A\hat{x})=0$

$$\Rightarrow \underset{2\times 1}{\begin{pmatrix}a_1^T\\a_2^T\end{pmatrix}} \underset{1\times 1}{(b-A\hat{x})} = \underset{2\times 1}{\begin{bmatrix}0\\0\end{bmatrix}}$$

$$\Rightarrow A^T\underbrace{(b-A\hat{x})}=0$$

e is in the $N(A^T)$. since $A^Te=0$

→ ⇒ e is $\perp^{lar}$ to column space of A. ⇒ YES! as we have seen.

$$\Rightarrow \underbrace{A^TA}_{n\times n}\hat{x}=A^Tb$$

$$\Rightarrow \boxed{\hat{x}=(A^TA)^{-1}.A^Tb}$$

$$\Rightarrow P=A\hat{x}=A.(A^TA)^{-1}.A^Tb.$$

$$\boxed{p = A(A^TA)^{-1}A^T b}$$

$$\Rightarrow \boxed{\mathbb{P} = A(A^TA)^{-1}A^T}$$

> If A is square & invertible, $\mathbb{P} = I$ since $A(A^TA)^{-1}A^T = AA^{-1}.A^{T^{-1}}A$
which makes sense since column space is all of $\mathbb{R}^n$ & we are projecting b in all of $\mathbb{R}^n$.

<u>Again.</u>

> $\mathbb{P}^T = \mathbb{P}$

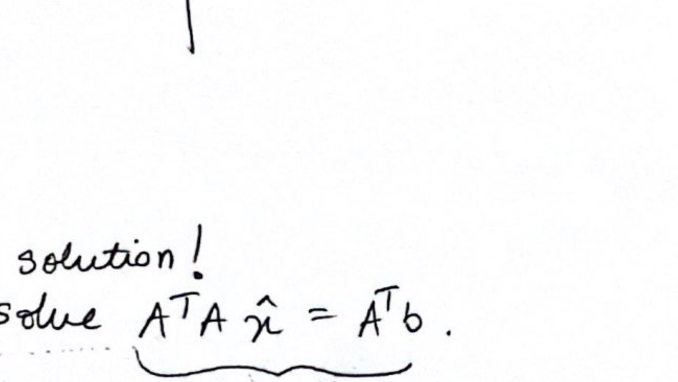
> $\mathbb{P}^2 = \mathbb{P} \Rightarrow A(A^TA)^{-1}A^T . A(A^TA)^{-1}A^T$

$$= A(A^TA)^{-1}\underbrace{(A^TA)(A^TA)^{-1}}_{I}A^T$$

$$= A(A^TA)^{-1}A^T.$$

<u>Least squares fitting by a line.</u>

<u>Eg</u>:-



Fit a line through (1,1) (2,2) (3,2)

$$\Rightarrow \begin{aligned} C + D &= 1 \\ C + 2D &= 2 \\ C + 3D &= 2 \end{aligned}$$

$$\Rightarrow \underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} C \\ D \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

$\Rightarrow$ No solution!

$\longrightarrow$ solve $\underbrace{A^TA\hat{x} = A^Tb}_{\text{one of the most imp. equations.}}$.

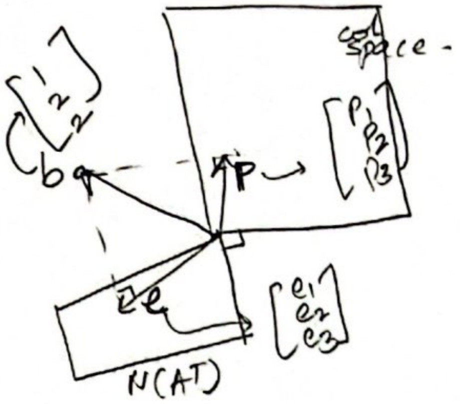
Lec 16

> If b is in column space $Pb = b. \Rightarrow Ax = b$.(since. $b = Ax$.)

> If b is ⊥ to col. space $Pb = 0$

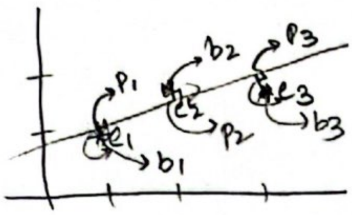
> ⊥ to col. space ⇒ in null space of $A^T \Rightarrow A^Tb = 0$
$\Rightarrow Pb = 0$.

>



$$p + e = b.$$
$$\mathbb{P}b + \mathbb{P}_N b$$
$$\boxed{\mathbb{P}_N = I - \mathbb{P}}$$

> Going back to line fitting.



$$\Rightarrow \begin{bmatrix}1&1\\1&2\\1&3\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix}=\begin{bmatrix}1\\2\\2\end{bmatrix}$$
$$A \quad x \quad = \quad b$$

col space of A doesnt include b.

Minimize, length of $Ax - b = \underset{\text{error vector}}{e}$. ⇒ minimize $\|Ax-b\|^2 = \|e\|^2$.
⇒ minimize $e_1^2 + e_2^2 + e_3^2$.
Called linear regression!

> In reality we need to ignore outliers which cannot be done using least squares!

> P_1, P_2, P_3 lie on the line. ⇒ $Ax = p$ has a solution. It lies in the col. space.

> Notice this result is also in the earlier col space & N(AT) figure.

> Find $\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$, P.

> $A^T A \hat{x} = A^T b$. → estimate.

> $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$\Rightarrow \left.\begin{matrix} 3C + 6D = 5 \\ 6C + 14D = 11 \end{matrix}\right\} \text{ AKA normal equations.}$$

> Going back: $e_1^2 + e_2^2 + e_3^2 = (C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2$

need to minimize $\Rightarrow \frac{\partial}{\partial C} = 0$ & $\frac{\partial}{\partial D} = 0$ gives the normal equations.

> Solving we get $\boxed{D = \frac{1}{2}. \ \& \ C = \frac{2}{3}.}$

$\Rightarrow$ Best line is $\boxed{\frac{2}{3} + \frac{1}{2} t = y}$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -1/6 \\ +2/6 \\ -1/6 \end{bmatrix}$$

→ Note $b = p + e$.

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 7/6 \\ 10/6 \\ 13/6 \end{bmatrix}}_{p} + \underbrace{\begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}}_{e}$$

→ p & e are perpendicular.

→ e is $\perp$ to anything in the col. space $\Rightarrow$ col. space of $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

$\Rightarrow$ e is $\perp$ to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as well.

If A has independant columns then $A^T A$ is invertible!..

roof

- Suppose $A^T A x = 0$. then x must be 0 since null space must be only the zero vector. => $A^T A$ is invertible.

DEA $x^T A^T A x = 0 \Rightarrow (Ax)^T (Ax) = 0 \Rightarrow Ax = 0.$

=> If A has ind. cols. & $Ax = 0$ then $x = 0$!

- columns are definitely independant if they are perpendicular unit vectors => $\begin{matrix}1\\0\\0\end{matrix}, \begin{matrix}0\\1\\0\end{matrix}, \begin{matrix}0\\0\\1\end{matrix}$. => They are called $\underbrace{\text{ortho}}_{\perp}\underbrace{\text{normal}}_{\text{unit}}$ vectors

Eg:- $\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}, \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$

Q Find the quad. eqn. through origin that is the best fit for $\underset{t\ \ y}{(1,1)}\ \underset{t\ \ y}{(2,5)}\ \underset{t\ \ y}{(-1,-2)}$?

Sol: $Ct + dt^2 = y.$

$$A = \begin{pmatrix}1 & 1\\2 & 4\\-1 & 1\end{pmatrix} \text{squares of first col.}$$

$$\underset{A}{\begin{pmatrix}1 & 1\\2 & 4\\-1 & 1\end{pmatrix}}\underset{\hat{x}}{\begin{pmatrix}c\\d\end{pmatrix}} = \underset{b}{\begin{pmatrix}1\\5\\-2\end{pmatrix}} \longrightarrow \text{same as plugging in the 3 pts into the equation.}$$

- can't solve $A\hat{x} = b$.
- Solve $A^T A \hat{x} = A^T b$

$$A^T A = \begin{pmatrix}1 & 2 & -1\\1 & 4 & +1\end{pmatrix}\begin{pmatrix}1 & 1\\2 & 4\\-1 & 1\end{pmatrix} = \begin{pmatrix}6 & 8\\8 & 18\end{pmatrix}$$

$$A^T b = \begin{pmatrix}1 & 2 & -1\\1 & 4 & 1\end{pmatrix}\begin{pmatrix}1\\5\\-2\end{pmatrix} = \begin{pmatrix}13\\19\end{pmatrix}$$

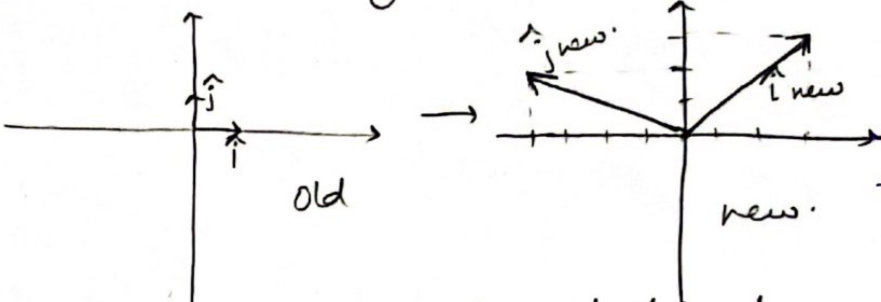
$$\begin{pmatrix}6 & 8\\8 & 10\end{pmatrix}(\hat{x}) = \begin{pmatrix}13\\19\end{pmatrix} \Rightarrow d = 5/22,\quad C = 41/22$$

Eigenvalues & Eigen vectors. - 3B1B.

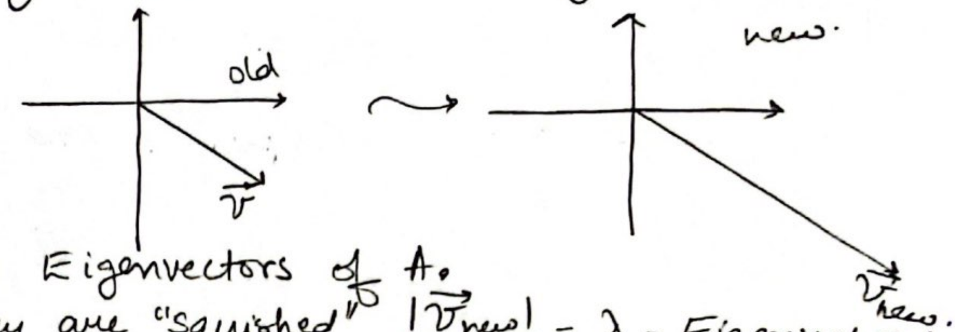
> A is some transformation => a matrix.

> A warps $\hat{i}$ & $\hat{j}$ to some new values say 3,3 & -4,2

$$\Rightarrow \quad A = \begin{bmatrix} 3 & -4 \\ 3 & 2 \end{bmatrix}$$



> There are a few sneaky vectors that are only stretched or squished by A.



> These vectors are called Eigenvectors of A.

> The value by which they are "squished" $\frac{|\vec{v}_{new}|}{|\vec{v}|} = \lambda =$ Eigenvalue!

> Eigenvectors are vectors that dont change their span after a given transformation.

Computing.

> This means A transforms $\vec{v}$ the same as λ stretches it.

$$\Rightarrow \quad A\vec{v} = \lambda \vec{v}$$

$$\rightarrow \quad A\vec{v} = (\lambda I)\vec{v}$$

$$\Rightarrow \quad (A - \lambda I)\vec{v} = 0$$

└ ① ⟶ nontrivial solutions of $A'\vec{v}$ only exist if $\det(A') = 0$

=> $\det(A - \lambda I) = 0$ give us the eigenvalues!

sub. λ in ① give eigenvectors!

→ A could have $\underbrace{\text{1 eigenvalue \& } \infty \text{ eigenvectors}}_{\text{everything is magnified.}}$ or $2\ \lambda$ & $\infty\ \vec{v}$ and so on.

↳ $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

→ Transformations such as rotations have no eigen vectors.

→ A transformation given by a diagonal matrix. $\begin{vmatrix} -5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{vmatrix}$
Scales only $\hat{i}$, $\hat{j}$, $\hat{k}$ ⇒ eigenvectors are all the basis vectors. Eigenvalues are the diagonal elements.

⇒ This is why we love diagonal matrices.

Also, if $A = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow A^n = \begin{bmatrix} 5^n & 0 \\ 0 & 3^n \end{bmatrix}$.

> We try to change the basis vectors to eigen vectors by applying a change of basis transformation B. Then the original transformation becomes a simple scaling operation.

Ex: Let $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ be a transformation. If we change the basis to $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ Then the new transformation is given by

$$A_{new} = B^{-1} A B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

↳ much easier to work with!

→ This new basis where basis vectors are eigen vectors is called the Eigenbasis!

→ Not all A have eigenvectors that can span the full space ⇒ Cannot

Lec 17

> Orthonormal vectors.

$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

orthonormal matrix

> $$Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$$

> $$Q^T Q = \begin{bmatrix} - & q_1^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix} \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & \cdots & \\ & \cdots & & \end{bmatrix}$$

$$\Rightarrow \boxed{Q^T Q = I}$$

> Orthogonal is used when Q is a square matrix. It's inverse exists.

$\Rightarrow \boxed{Q^T = Q^{-1}} \longrightarrow$ square orthonormal ⇒ orthogonal.

> $$Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Q^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Eg:-

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$Q = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Why Q?

Project onto Its column space

$$\Rightarrow P = Q\underbrace{(Q^TQ)^{-1}}_{I}Q^T$$

$$P = QQ^T$$

→ If Q is square, col. space is the whole space & P should be I.
But we already know $QQ^T = I$.

→ P should by symmetric & QQ^T is symmetric.

→ $P = P^2 \Rightarrow (Q\underbrace{Q^T)(Q}_{I}Q^T) = QQ^T = P.$

→ $A^TA\hat{x} = A^Tb$

$\Rightarrow \boxed{\hat{x} = Q^Tb} \Rightarrow \boxed{\hat{x}_i = q_i^T b.}$ → dot product of q_i & b.

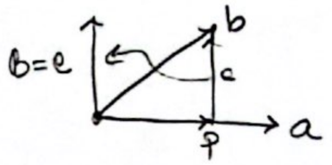
Gram - Schmidt

> Make a matrix orthonormal.

> independant vector a & b $\xrightarrow{\text{to}}$ orthogonal vectors A, B → orthonorma
vectors.

$q_1 = \frac{A}{\|A\|}$

$q_2 = \frac{B}{\|B\|}.$

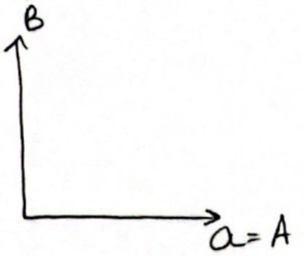


> Let A = a ; Use e since it is ⊥ to p.

$$\Rightarrow \boxed{B = b - \frac{A^Tb}{A^TA} \cdot A.}$$

> Check: $A^TB = A^T\left(b - \frac{A^Tb}{A^TA}A\right) = A^Tb - \frac{A^Tb \cdot A^TA}{A^TA.} = 0$

> What about getting a 3rd ortho~~normal~~gonal vector C.



$$\boxed{C = c - \frac{A^Tc}{A^TA}A - \frac{B^Tc}{B^TB}\cdot B}$$

> Eg: $a = \begin{bmatrix}1\\1\\1\end{bmatrix}$ $b = \begin{bmatrix}1\\0\\2\end{bmatrix}$

Let $A = a$, $B = \underset{b}{\begin{bmatrix}1\\0\\2\end{bmatrix}} - \frac{3}{3}\underset{a=A}{\begin{bmatrix}1\\1\\1\end{bmatrix}} = \underset{B}{\begin{bmatrix}0\\-1\\1\end{bmatrix}}$

$q_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$ $q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}$

$$\Rightarrow Q = \begin{bmatrix}1/\sqrt{3} & 0\\1/\sqrt{3} & -1/\sqrt{2}\\1/\sqrt{3} & 1/\sqrt{2}\end{bmatrix}$$

> Col. space of $[a\ b]$ & Q are the same.

→ If $A = [a\ b]$. Then $\boxed{A = QR}$

Let $A = \begin{bmatrix}a_1 & a_2\end{bmatrix} = \begin{bmatrix}q_1 & q_2\end{bmatrix}\begin{bmatrix}a_1^Tq_1 & a_2^Tq_1\\ \underset{0}{\cancel{a_1^Tq_2}} & a_2^Tq_2\end{bmatrix}$

$\Rightarrow$ $\boxed{\text{R is uppertriangular}}$.

Lec 18 Determinants.

Properties.

(I) $\det(I) = 1$

(II) Exchange rows ⇒ reverse the sign of the determinant.

$$\Rightarrow \det(\underset{\text{permutation matrix}}{P}) = \begin{cases} 1 & \text{even no. of row exchanges} \\ -1 & \text{odd " " " "} \end{cases}$$

· $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

(IIIa) multiply a row by 't' ⇒ $t \times \det(A)$.

(IIIb) $$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

$\Rightarrow \det(A+B) \neq \det A + \det B$.

> Linear for each row while keeping all other rows same.

(IV) 2 equal rows = det is 0.

derived

Since from prop (II) exchange the equal rows ⇒ $\det(A) = -\det(A')$

But $A = A'$ ⇒ $\det(A) = 0$

(V) Subtract some $l \times$ row i from row k, the determinant doesn't change. ⇒ $\text{Det}(A) = \text{Det}(U)$. Elimination doesn't affect det.

derived.

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} \overset{\text{by IIIb}}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \underbrace{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}_{0} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

VI
derived

Row of zeroes ⇒ det is 0.

Rule 3b with t = 0.

VII
derived

$$\det(U) = \det\begin{bmatrix} d_1 & \cdots & \cdots \\ 0 & d_2 & \cdots \\ 0 & 0 & d_3 & \cdots \\ & \cdots & \end{bmatrix} = \boxed{d_1 \times d_2 \times \cdots d_n.}$$

⇒ Elimination & product of pivots. (if no row exchanges).

Proof:- Could use elimination to go from U to a purely diagonal matrix. Then use rule IIIa to pull out $d_1, d_2 .. d_n |I|$ & use rule Ⓘ.

VIII
derived.

det (A) is 0 when A is singular! → we get a row of zeroes.

det (A) ≠ 0 when A is invertible.

IX

$$\boxed{\det(AB) = [\det(A)] \cdot [\det(B)]}$$

$$\Rightarrow \det(A^{-1}) = ?$$

$$A^{-1}A = I$$

$$\Rightarrow \det(A^{-1}) \det(A) = 1$$

$$\Rightarrow \boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

Eg:-

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$> \det(A^2) = [\det(A)]^2$$

⇒ $\det(2A) = 2^n \det(A)$ if A is $n \times n$.

↳ Think of it as a volume. If side is doubled volume is 2^3.

Ⓧ $\det(A^T) = \det(A)$.

⇒ all properties mentioned in rows also applies to columns.

Proof

$$|A^T| = |A|$$

$$\downarrow \qquad \downarrow$$

$$|U^T L^T| = |LU|$$

$$|U^T||L^T| = |L||U|$$

equal since product of diag elements are equal

Lec 19

$$\rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \underbrace{\begin{vmatrix} a & \cancel{0} \\ c & 0 \end{vmatrix}}_{0} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \underbrace{\begin{vmatrix} 0 & \cancel{b} \\ \cancel{0} & d \end{vmatrix}}_{0}$$

$$\boxed{\text{I} \rightarrow \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \text{II} \Rightarrow \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1}$$

$$= ad\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - cb\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \underline{ad - bc}.$$

$$\rightarrow \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \cdots$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31}$$

$$+ a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

→ Big formula.

$$\det A = \sum_{n!\text{ terms.}} \pm\, a_{1\alpha}a_{2\beta}a_{3\gamma}\ldots a_{n\omega}$$

$(\alpha, \beta, \gamma \ldots \omega)$ = permutation of $(1, 2 \ldots, n)$.

Cofactors.

$$\begin{aligned}\det = {} & a_{11}\left[\det\begin{pmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{pmatrix}\right]\\ & - a_{12}\left[\det\begin{pmatrix} a_{21} & a_{23}\\ a_{31} & a_{33}\end{pmatrix}\right]\\ & + a_{13}\left[\det\begin{pmatrix} a_{21} & a_{22}\\ a_{31} & a_{32}\end{pmatrix}\right]\end{aligned}$$

> cofactor of a_{ij} is $= \pm \det\begin{pmatrix} n\text{-1 matrix with} \\ \text{row } i \,\&\, \text{col } j \text{ erased}\end{pmatrix} = C_{ij}$
>
> $+$ if $i+j$ is even.
>
> $-$ if $i+j$ is odd.

> Cofactor without sign is called minor. $\longrightarrow$ $\begin{vmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{vmatrix}$

Eg:- Tridiagonal determinant.

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} \longrightarrow A_4$$

$|A_n| = |A_{n-1}| - |A_{n-2}|.$

$|A_1| = 1$, $|A_2| = 0 \Rightarrow |A_3| = -1$

$\longrightarrow |A_4| = -1$, $|A_5| = 0$, $|A_6| = 1$, $|A_7| = 1$ ...

Lec 20

A^{-1} formula.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{\det(A)} \cdot C^{T}}$$

C ↳ Cofactor matrix.

C → products of (n-1) entries.

$\det(A)$ → products of n entries.

> Check $AA^{-1} = I \Rightarrow AC^T = (\det A)I$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ C_{12} & & \vdots \\ C_{13} & & \vdots \\ \vdots & & \\ C_{1n} & & C_{nn} \end{bmatrix} = \begin{bmatrix} \overset{\text{det}}{a_{11}C_{11}+a_{11}C_{12}+\dots} & \overset{\nearrow \text{ they come from terms cancelling.}}{0} & 0 & 0 \\ 0 & \text{det} & \cdot & - \\ 0 & 0 & \text{det} & \\ \cdots & & & \ddots \end{bmatrix}$$

$$= (\det A)I$$

> $Ax = b$

$$\Rightarrow \boxed{x = \frac{1}{\det A} C^T b}$$

Cramer's rule.

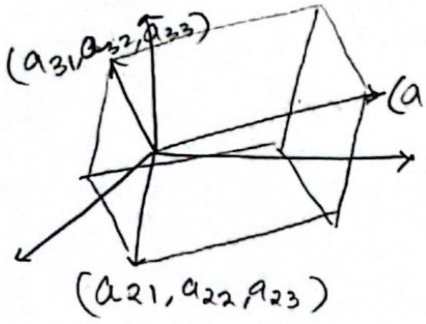
$$x_1 = \frac{\det B_1}{\det A} \quad \text{where} \quad B_1 = \begin{bmatrix} b & \begin{matrix} n-1 \text{ remaining} \\ \text{columns} \\ \text{of } A \end{matrix} \end{bmatrix}$$

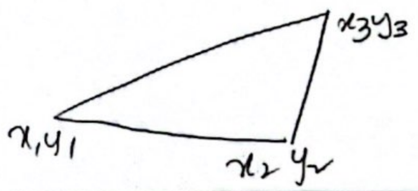
$$x_2 = \frac{\det B_2}{\det A}$$

det of B_1 goes back to $C^T b$.

B_j = A with column j replaced by b.

> Determinant A = volume of a box.





$$\Rightarrow \boxed{\text{area} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}$$

Lec 21 Eigenvalues + Eigenvectors.

$$\boxed{Ax = \lambda x.}$$

> Eigenvectors with $\lambda = 0$ are the vectors in the null space.

> What about projection matrix?

Any x in plane of projection is an Eigenvector with $\lambda = 1$
Any $x \perp$ to plane of projection is " with $\lambda = 0$.

Eg:- $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\Rightarrow \lambda = 1$

$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \lambda = -1$

> Fact: Sum of λ's $= a_{11} + a_{22} + \ldots + a_{nn} \rightarrow$ called Trace
> determinant of matrix is product of eigenvalues. of the matri
> nxn matrix has n Eigenvalues.

Solving for $Ax = \lambda x$.

$$\underbrace{(A - \lambda I)}_{\text{singular.}} x = 0 \quad \text{Characteristic eqn. or Eigenvalue equation.}$$

$$\Rightarrow \boxed{\det |A - \lambda I| = 0} \text{ gives } \lambda.$$

> If we add 3I to a matrix, 3 gets added to the eigen values & eigenvectors remain unchanged.

> Rotation matrices eg: $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has no real eigen values & eigenvectors.

$\hookrightarrow \lambda_1 + \lambda_2 = 0$ & $\lambda_1 \lambda_2 = 1$

$\Rightarrow \boxed{\begin{matrix} \lambda_1 = i \\ \lambda_2 = -i \end{matrix}} \rightarrow$ complex conjugates!

→ Eg : $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ $\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2$

⇒ Triangular matrices → Eigenvalues are on the diagonal

⇒ $\lambda_1 = \lambda_2 = 3$

$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_2 = ?$
↳ cannot get 2 eigenvectors

⇒ No second independant eigenvector!

→ $AV = \lambda V$

⇒ $A^2 V = A(\lambda V) = \lambda(AV) = \lambda^2 V$

⇒ A^2 has same eigenvectors.
& eigenvalues are λ^2.

→ A^{-1} ⇒ $A^{-1}v = A^{-1}\frac{Av}{\lambda} = A^{-1}A\frac{v}{\lambda} = \frac{1}{\lambda}v$.

⇒ v is eigenvector of A^{-1} & eigenvalue is $\frac{1}{\lambda}$.

Lec 22. Diagonalization.

› Suppose we have n ind. eigenvectors. Put them in a matrix

$S = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$

› $AS = A\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix}$

$$= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & & & 0 \\ & & \lambda_3 & & & 0 \\ \vdots & & & 0 & 0 & \lambda_n \end{bmatrix} = S\Lambda$$

↳ diagonal eigenvalue matrix.

⇒ $\boxed{AS = S\Lambda}$

> $\Rightarrow \boxed{S^{-1}AS = \Lambda}$ if A has n ind. eigen vectors. 4

$\Rightarrow \boxed{A = S\Lambda S^{-1}}$

> $A^2 = (S\Lambda S^{-1})(S\Lambda S^{-1})$

$\boxed{A^2 = S\Lambda^2 S^{-1}}$ → eigenvectors are same
→ eigen values are squared.

> $A^k = S\Lambda^k S^{-1}$ → Eigenthings are good for dealing with powers.

<u>Theorem</u>

$A^K \longrightarrow 0$ as $K \longrightarrow \infty$ if $|\lambda_i| < 1$

> A is sure to have n ind. eigenvectors if all the λ's are differen

> <u>Repeated λs</u> ... ⇒ may or maynot have n independant vectors.

<u>Eg:</u> Eq. $u_{k+1} = A u_k$.

Solve?

Sol: $u_1 = Au_0$, $u_2 = Au_1 = A^2 u_0$

$\Rightarrow \boxed{u_k = A^k u_0}$

To really solve.

Decompose u_0 as a combination of eigenvectors.

$$u_0 = c_1 x_1 + c_2 x_2 + \dots + c_n x_n = Sc$$

$$A u_0 = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 \dots + c_n \lambda_n x_n$$

$$\Rightarrow A^{100}u_0 = c_1 \lambda_1^{100} x_1 + c_2 \lambda_2^{100} x_2 + \dots + c_n \lambda_n^{100} x_n .$$

$$= S\Lambda^{100} c$$

Eg: 0, 1, 1, 2, 3, 5, 8 . . . . . $F_{100} = ?$

$F_{k+2} = F_{k+1} + F_k$ take $F_{k+1} = F_{k+1}$. to make it a matrix.

Let $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} \Rightarrow \lambda^2 - \lambda - 1 = 0$$

$\Rightarrow \lambda = \dfrac{1 \pm \sqrt{5}}{2} \Rightarrow \lambda_1 = \frac{1}{2}(1+\sqrt{5}) = 1.618 \rightarrow$ gives us how much the Fib no. are growing by. (Goldenratio)

$\lambda_2 = \frac{1}{2}(1-\sqrt{5}) = -0.618$

$$F_{100} \approx c_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{100}$$

$$> u_{100} = A^{100} u_0 = \underbrace{c_1 \lambda_1 x_1}_{F_{100}} + \underbrace{c_2 \lambda_2 x_2}_{\approx 0}$$

Eigen vectors = $\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}, \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$, $u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$c_1 x_1 + c_2 x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find $c_1, c_2 \dots$

Lec 23 Diff. eqns.

Eg:
$$\frac{du_1}{dt} = -u_1 + 2u_2 \qquad u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\frac{du_2}{dt} = u_1 - 2u_2$$

$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \rightarrow$ singular $\Rightarrow \lambda = 0 \Rightarrow \lambda = -3$

$\lambda = 0 \rightarrow$ steady state
$\lambda = -3 \rightarrow$ decaying exponential } Sols. are of the form $e^{\lambda t}$.

$\underline{\lambda_1 = 0} \searrow x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow Ax_1 = 0x_1$

$\underline{\lambda_2 = -3} \searrow x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow Ax_2 = -3x_2$

Solution: $u(t) = C_1 e^{\lambda_1 t} x_1 + C_2 e^{\lambda_2 t} x_2$ ↗ general solution of $\frac{du}{dt} = Au$

Check: $\boxed{\frac{du}{dt} = Au}$, plug in $e^{\lambda_1 t} x_1 \Rightarrow LHS = \lambda_1 e^{\lambda_1 t} x_1$, $RHS = A e^{\lambda_1 t} x_1$ } $\Rightarrow LHS = RHS$.

$$u(t) = C_1 . 1 . \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \underset{1}{\cancel{e^{-3t}}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$C_1 = \frac{1}{3}, \quad C_2 = \frac{1}{3}$

steady state $= u(\infty) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

① Stability ⇒ $u(t) \rightarrow 0$ ⇒ need all $e^{\lambda t} \rightarrow 0$

⇒ λs must all be < 0.

More generally speaking $Re\{\lambda_i\} < 0$ for all i

② Steady state ⇒ $\lambda_1 = 0$ and other λs have $Re\{\lambda\} < 0$.

③ Unstable if any $Re\{\lambda\} > 0$.

→ Eigenvalues change sign for $-A$ ⇒ it becomes unstable.

Note :- 2x2 stability ⇒ $Re\,\lambda_1 < 0$ & $Re\,\lambda_2 < 0$.

⇒ if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ⇒ $\boxed{a+d = \lambda_1 + \lambda_2 < 0}$ trace must be negative

> But trace could be negative & still be unstable.

> Also $\det(A)$ should be positive.

⇒ $\boxed{\text{Stability in case of 2x2 is ensured when trace} < 0 \text{ + det} > 0}$

> Given $\frac{du}{dt} = Au$. A is coupled & eigenvalues uncouple it by diagonalizing.

Set $u = SV$

$$\Rightarrow \frac{dv}{dt} = S^{-1}ASV = \underline{\underline{\Lambda V}}$$

$$\left.\begin{array}{l} \frac{dv_1}{dt} = \lambda_1 v_1 \\ \vdots \end{array}\right\} \Rightarrow \begin{array}{l} v(t) = e^{\Lambda t} v(0) \\ \boxed{u(t) = Se^{\Lambda t}S^{-1}u(0)} \end{array}$$

$$e^{At} = Se^{\Lambda t}S^{-1}$$

But what does e^{At} mean?

Matrix exponentials.

$$e^{At} = I + At + \frac{(At)^2}{2} + \dots + \frac{(At)^n}{n!}$$

$$(I - At)^{-1} = I + At + (At)^2 + \dots + (At)^n$$

$$e^x = \sum \frac{x^n}{n!}$$

$$\frac{1}{1-x} = \sum x^n$$

↳ only when At has $\lambda < 1$. for all λ.

Try to prove $e^{At} = Se^{\Lambda t}S^{-1}$

$$e^{At} = I + At + \frac{(At)^2}{2} + \dots + \frac{(At)^n}{n!}$$

$$= \underbrace{I}_{SS^{-1}} + S\Lambda S^{-1}t + \frac{S\Lambda^2S^{-1}t^2}{2} + \dots + \frac{S\Lambda^nS^{-1}t^n}{n!}$$

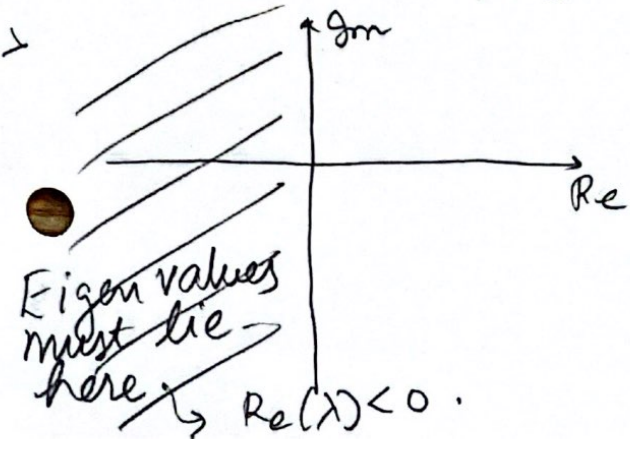
$$= S\left[I + \Lambda t + \frac{\Lambda^2 t^2}{2} + \dots + \frac{\Lambda^n t^n}{n!}\right]S^{-1}$$

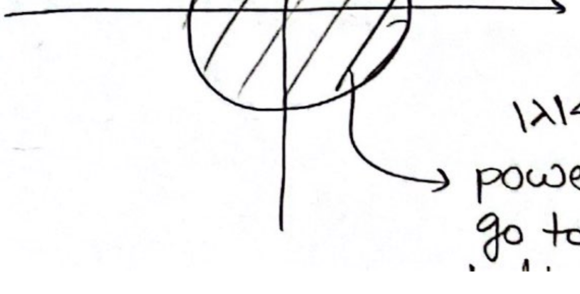
$$e^{At} = Se^{\Lambda t}S^{-1}$$

> It only works when A can be diagonalized ⇒ all eigenvectors are independant

What is $e^{\Lambda t}$? ↷ diagonal matrix

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \\ & & & e^{\lambda_n t} \end{bmatrix}$$





Eg:- given $y'' + by' + ky = 0$ solve it.

Sol:- Create a dummy equation $y' = y'$

$$\Rightarrow \quad u = \begin{bmatrix} y' \\ y \end{bmatrix} \qquad u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$$

In general going from 5th order to $5 \times 5 \Rightarrow \begin{bmatrix} \text{coefficients} & & & & \\ 1 & 0 & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{bmatrix}$

Lec 24. Markov matrices

$$A = \begin{bmatrix} 0.1 & 0.01 & 0.3 \\ 0.2 & 0.99 & 0.3 \\ 0.7 & 0 & 0.4 \end{bmatrix}$$

Properties

① All entries ≥ 0

② Sum of column entries $= 1 \Rightarrow$ one of the eigenvalues is 1.

③ Powers of a Markov matrices are also Markov.

Keypoints

① $\lambda = 1$ is an eigenvalue.

② All other magnitudes of $\lambda : |\lambda_i| < 1$

Recall: $u_k = A^k u_0 = c_1 \lambda_1^k \vec{x}_1 + c_2 \lambda_2^k \vec{x}_2 + \ldots$

$\Rightarrow$ If $\lambda_1 = 1$ & the rest are < 1

As $t \to \infty$, $k \to \infty \Rightarrow$ steady state is $\boxed{c_1 x_1}$

③ Eigenvalue $x_1 \geq 0 \Rightarrow$ steady state is positive.

$\lambda = 1 \Rightarrow A - 1.I = \begin{bmatrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & -0.6 \end{bmatrix}$

- This matrix must be singular for $\lambda=1$ to be an eigenvector.

All columns add to zero ⇒ matrix is singular.

Since combination of 2 rows gives -ve to the remaining row.

⇒ $(1, 1, 1)$ vector is in the $N(A^T)$.

and x_1 vector is in the $N(A)$.

→ Eigenvalues of A = Eigenvalues of A^T $\left\{ \text{since } \det(A - \lambda I) = \det(A^T - \lambda I^T) = \det(A^T - \lambda I). \right\}$

→ Here, $x_1 = \begin{bmatrix} 0.6 \\ 33 \\ 0.7 \end{bmatrix}$

→ <u>Applications of Markov matrices.</u>

- > Markov chains!

Eg: $\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=k+1} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=k}$

- The steady state is the eigenvector corresponding to $\lambda = 1$

Say $\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$

$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_1 = \begin{bmatrix} 200 \\ 800 \end{bmatrix}$

- → $\lambda_1 = 1$, $\lambda_2 = \text{Trace} - \lambda_1 = 0.7$

$$\lambda=1 \Rightarrow \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

After ∞ steps. pop is $2x + 1x = 100$
$\Rightarrow u_{cal} = 666.6..,\ u_{mass} = 333.3..$

$=0.7$
$\Rightarrow$

$$\begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

After 100 steps.?

$$u_K = C_1 1^K \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 (0.7)^K \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

↳ could solve C_1 & C_2 $\Rightarrow C_1 = \frac{1000}{3},\ C_2 = \frac{2000}{3}$

$\Rightarrow$

$$u_K = \frac{1000}{3} 1^K \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{2000}{3} (0.7)^K \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

---

* Projections with orthonormal basis : $q_1, \dots, q_n$.

Any

$$v = x_1 q_1 + x_2 q_2 + \dots + x_n q_n.$$

What is x_1?

Take inner product of LHS & RHS with q_1.

$\Rightarrow q_1^T v = x_1.$

$$\begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = v \Rightarrow Qx = v \Rightarrow x = Q^{-1} v$$

$\Rightarrow \boxed{x = Q^T v}$ since Q is orthonormal.

$\Rightarrow x_1 = q_1^T v.$

Fourier Series.

Let $f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots$

- vectors are functions.
- basis vectors are basis functions > 1, cos x, sin x, cos 2x....
- This works because the basis functions are "orthogonal".

What are orthogonal functions?

$$f^T(x)\, g(x) = ?$$

recall : $V^T . W = V_1 W_1 + \dots + V_n W_n$ for vectors.

$$\Rightarrow f^T(x)\, g(x) = \int_0^{2\pi} f(x)\, g(x)\, dx$$

or the period!

$$\text{Eg} \int_0^{2\pi} \sin x \cos x \, dx = \frac{1}{2} (\sin x)^2 \Big|_0^{2\pi} = 0.$$

→ $a_1 = ?$ Take < > of LHS & RHS with cos(x).

$$\Rightarrow a_1 \underbrace{\int_0^{2\pi} (\cos x)^2 dx}_{\pi} = \int_0^{2\pi} f(x) \cos x \, dx.$$

$$\boxed{\Rightarrow a_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos x \, dx.}$$

- wow!

Lec 25

Symmetric matrices

Antisymmetric or skew symmetric matrices ($A^T = -A$) have only imaginary Eigenvalues.

> Symmetric matrices that are real have real Eigen values.

> Eigen vectors are perpendicular. (can be chosen)

⇒ eigenvector matrix S can be converted to an orthonormal matrix Q.
↳ scale each ⊥lar vector to 1.

> Usual case : $A = S \Lambda S^{-1}$

Symmetric : $A = Q \Lambda Q^{-1}$

$$\boxed{A = Q \Lambda Q^T}$$

→ spectral theorem.
or principle axis theorem.

Why real eigenvalues?

complex conjugate.

$$Ax = \lambda x \Rightarrow \bar{A}\bar{x} = \bar{\lambda}\bar{x}$$

$$\Rightarrow A\bar{x} = \bar{\lambda}\bar{x} \text{ since A is real.}$$

$$\Rightarrow \bar{x}^T A^T = \bar{x}^T \bar{\lambda}$$

$$\Rightarrow \bar{x}^T A = \bar{x}^T \bar{\lambda} \text{ since A is symmetric}$$

$$\left.\begin{aligned} \bar{x}^T A x &= \lambda \bar{x}^T x \\ \bar{x}^T A x &= \bar{x}^T \bar{\lambda} x \end{aligned}\right\} \text{These two are identical.}$$

$\Rightarrow \lambda = \bar{\lambda}$ (if $\bar{x}^T x \neq 0$)

$\Rightarrow \lambda$ is real!

Is $\bar{x}^T x \neq 0$?

x could be complex!

$$\begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_3 \end{bmatrix} = \bar{x}_1 x_1 + \bar{x}_2 x_2 + \dots$$

$$= |x_1|^2 + |x_2|^2 + \dots$$

$\Rightarrow \bar{x}^T x$ is +ve $\Rightarrow$ if x is nonzero, $\bar{x}^T x$ is $\neq 0$.

> What if A was complex?

→ it works if $A = \bar{A}^T$ ⇒ conjugate transpose symmetry.

⇒ if $A = \bar{A}^T$

→ xs are ⊥lar & λs are real!

→ $A = A^T$

$$\Rightarrow A = Q\Lambda Q^T = \begin{bmatrix} q_1 & q_2 & \cdots \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \end{bmatrix}$$

$$= \underset{\text{real } \lambda}{\lambda_1} \underbrace{q_1 q_1^T}_{\text{projection matrix.}} + \lambda_2 q_2 q_2^T + \cdots$$

⇒ Every symmetric matrix is a combination of perp. projection matrices.

→ If λ is real, are they +ve or −ve? For stability reasons.

→ For symmetric matrices, the signs of the pivots are the signs of the eigenvalues. (Product of pivots = product of λs.)

Positive definite matrices

→ ① Symmetric matrices with all positive eigenvalues.

→ ② all pivots are also positive.

Eg: $\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$ pivots : $5, \frac{11}{5}$ since det = 11

$\lambda^2 - 8\lambda + 11 = 0 \Rightarrow \lambda = 4 \pm \sqrt{5}$.

All sub determinants are positive!

Properties

-> Every posdef matrix is invertible (since det > 0).

> The only posdef projection matrix is $P = I$.

-> If D is diagonal with positive entries, it is also posdef.

-> If S is sym. with det S > 0, it might not be posdef.

Lec 26 Complex matrices. & FFT

> $z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ in $\mathbb{C}^n$. $z^T z$ does not give the length!

$\boxed{\bar{z}^T z}$ does give the length!

$\Rightarrow \boxed{z^H z} \longrightarrow$ Hermitian!

> Inner product : $\boxed{y^H x}$ & not $y^T x$.

> symmetric => $A^T = A$.

> Hermitian => $\boxed{A^H = A}$ => diagonal entries are real.

↳ Real eigenvalues & $\perp^{lar}$ eigenvectors.

$$\begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$$

Perpendicular vectors => $\bar{q}_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

=> $Q^H Q = I$ where $Q = [q_1\ q_2 \cdots q_n]$.

> Q is now called unitary & not orthogonal since its columns satisfy $q_i^H q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$.

Fourier Matrix

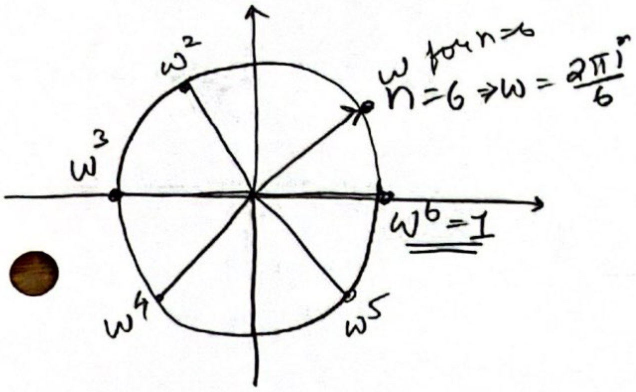
$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2} \end{bmatrix}$$

$(F_n)_{ij} = w^{ij}$ where $w^n = 1$

where $i, j = 0, \dots, n-1$

$\Rightarrow \boxed{w = e^{i(2\pi/n)}}$

$= \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.



$n = 4 \Rightarrow w^4 = 1 \Rightarrow w = e^{2\pi i/4} = i \Rightarrow i, i^2, i^3, i^4$

$$F_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

→ Very useful when trying to find the 4 point fourier transform.

> Columns of F_4 are orthonormal. since < >s are all zeroes. Remember to take conjugates.

→ $F_4^H F_4 = I$ since cols. are orthonormal.

Inverse of $F_4 = F_4^{-1} = F_4^H$.

→ Note that $(W_{64})^2 = W_{32}$

⇒ $$[F_{64}] \xrightarrow{?} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix}$$

⇒ $$[F_{64}] = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & & & \\ & & 1 & \\ 1 & & & 1 \\ & 1 & & \ddots 1 \end{bmatrix}}_{P \longrightarrow \text{permutation!}}$$

$(64)^2$ computations ══ → $2(32)^2 + \underset{\hookrightarrow \text{From } D}{32}$

where $D = \begin{bmatrix} 1 & & & \\ & w & & \\ & & w^2 & \\ & & & \ddots \; w^{31} \end{bmatrix}$

→ Now use recursions to further reduce no. of computations.

⇒ no. of computations $= 2[2(16)^2 + 16] + 32$

$= 6 \times 32$

⇒ Final count is $\frac{1}{2} n \log_2 n$.

$n = 1024 \Rightarrow n^2 > 10^6$

$\frac{1}{2} n \log n = 5 \cdot 1024$

⇒ 200 times faster!

Lec 27 Positive definite Matrices.

Tests for positive definiteness. $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

→ Any of these tests are valid.

① All eigen values are positive.

$\Rightarrow \lambda_1 > 0 \quad \lambda_2 > 0$

② All determinants are >0 (just consider the leading submatrices)

$\Rightarrow a > 0$ & $ac - b^2 > 0$

③ Pivots are positive.

$a > 0, \quad \frac{ac-b^2}{a} > 0$

**④ $\boxed{x^T A x > 0}$ → More like the definition itself.

Eg: $\begin{bmatrix} 2 & 6 \\ 6 & > \underline{18} \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 6 \\ 6 & 19 \end{bmatrix}$. If $\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \Rightarrow$ positive semidefinite.

$\Rightarrow \lambda_1 = 0$.

& $\lambda_2 = 20$.

pivots = 2 since rank = 1 & matrix is singular.

let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\Rightarrow X^T A X = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2x_1 + 6x_2 \\ 6x_1 + 18x_2 \end{bmatrix}$$

$$= 2x_1^2 + 12x_1x_2 + 18x_2^2$$

$$= ax^2 + 2bxy + cy^2 \longrightarrow \text{quadratic form!}$$

- Is $ax^2 + 2bxy + cy^2 > 0$ for all x, y (or x_1, x_2) $\Rightarrow$ it is tve definite.

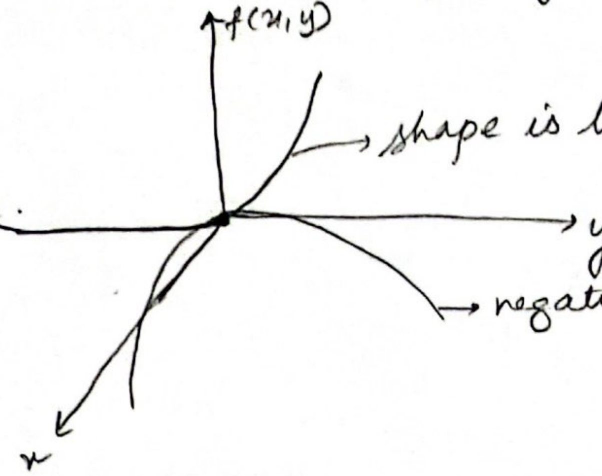
> Let us graph!

* graphs of $f(x,y) = x' A x$

$= ax^2 + 2bxy + cy^2$

Let $A = \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$

not positive definite.

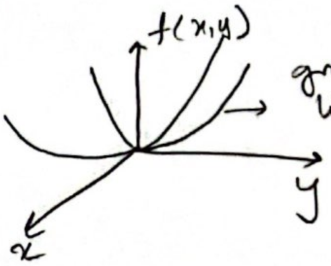


→ shape is like a saddle!

→ negative in some directions!

Let $A = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ det = 4 & trace = 22 ⇒ positive λs.
$(\Pi\lambda)$ $(\Sigma\lambda)$

→ $X^TAX > 0$ except at $x=0$.



→ graph is like a bowl.

⇒ $f(x,y) = 2x^2 + 12xy + 20y^2$

⇒ first derivatives are zero at origin.

⇒ second derivates are +ve at origin.

In linear algebra terms → The matrix of second derivates is positive definite.

Minimum: $f(x_1, x_2 \dots x_n)$

Minimum ⇒ Matrix of 2nd derivatives is positive definite.

→ Going back to $f(x,y) = 2x^2 + 12xy + 20y^2$ → write as a square.

$= 2(x + 3y)^2 + 2y^2$ which is always positive!

↳ This would be negative if $c < 18$.

If $f(x,y) = 1$ it would be an ellipse equation ⇒ cross section of the bowl.

> $\overset{A}{\begin{bmatrix}\boxed{2} & 6\\ 6 & 20\end{bmatrix}} \xrightarrow{\text{elimination}} \overset{U}{\begin{bmatrix}\boxed{2} & 6\\ 0 & \boxed{2}\end{bmatrix}}$

$L = \begin{bmatrix}1 & 0\\ ③ & 1\end{bmatrix}$

Notice that elimination is the same as completing the square.

$\underline{2}(x+3y)^2 + \underline{2}y^2 = 1$

pivots!

> Pivots go outside the squares & therefore positive pivots ensure positive squares!

> What is the matrix of 2^{nd} derivatives?

> $\begin{bmatrix} f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{bmatrix}$ where $f_{xy} = \frac{d^2f}{dxdy} = \frac{d^2f}{dydx}$.

> Condition for a fn in 2 variables to have a minimum is first derivates $=0$ & $f_{xx}f_{yy} - f_{xy}^2 > 0$. We can now do this for n variables!

Eg:

$A = \begin{bmatrix} 2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}$

$\det[2] = 2$

$\det\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix} = 3$

$\det\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix} = 4$

Pivots.

$2, \frac{3}{2}, \frac{4}{3}$

Eigenvalues.

They are all positive.

$2-\sqrt{2}, 2, 2+\sqrt{2}$

$f = x^TAx = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$.

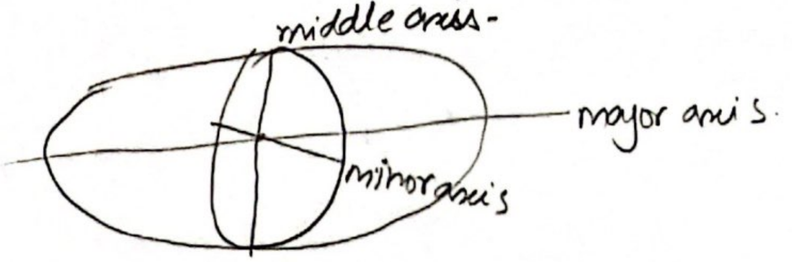
> 3D "Bowl".

> Cutting it at height 1 ⇒ $f=1$ gives an ellipsoid.

> Sphere is an ellipsoid with all equal eigenvalues.

> Rugby ball has 2 equal eigenvalues.

> 3 principle directions ellipsoid is like a squished rugby ball.



→ Length of these axes is given by eigenvalues.

→ Direction of the axes is given by eigenvectors.

→ $\boxed{A = Q \Lambda Q^T}$ gives the principle axis theorem!

Lec 28

> A^TA is a positive (semi) definite matrix.

> If A is positive definite ⇒ A^{-1} is also posdef since $\lambda \rightarrow \frac{1}{\lambda}$.

> What about A+B? ⇒ If A & B are posdef. So is A+B.
since $x^T(A+B)x > 0$.

> Suppose A is m×n → rectangular. A^TA is square symmetric
Is it positive definite?

> $x^TA^T Ax \longrightarrow (Ax)^T Ax$ is always ≥ 0. since it is length^2.
$|Ax|^2 \geq 0$ & it is $=0$ if $x=0$.

> $A^TA > 0$ if null space of A contains only zero vectors.
⇒ rank of A is = smallest dimension!

Similar matrices.

› A and B are $n \times n$ and they are 'similar' means that for some matrix M. $\boxed{B = M^{-1} A M}$.

Ex: $S^{-1}AS = \Lambda \Rightarrow A$ & Λ are similar.

Ex:- $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$, A & Λ are similar.

$$B = \underset{M^{-1}}{\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}} \underset{M}{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 9 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}$$

→ $\boxed{\text{A \& B have the same eigenvalues!}}$

→ Every matrix with λs same as A will have a corresponding

→ But why? $Ax = \lambda x$.

› $B = M^{-1}AM$.

› $A\, MM^{-1} x = \lambda x$.

$= MAMM^{-1}x = \lambda M^{-1}x$

$\Rightarrow BM^{-1}x = \lambda M^{-1}x$.

⇒ λ is an eigenvalue of B!

→ Eigenvector of B is M^{-1}(eigenvector of A).

→ No. of eigenvectors are same for A & B.

hat about if $\lambda_1 = \lambda_2$? Matrix may not be diagonalizable.

Eg: $\lambda_1 = \lambda_2 = 4$.

- one family has $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ } Both these families have $\lambda s = 4$

another family is $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$

- For the $4I$ case, any M will give $4I$ again since

$$M^{-1} 4I M = 4(M^{-1} I M) = 4M^{-1}M = 4I.$$

- So the family of NIs cannot be translated to the "bigger" family despite being similar.

> $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ is called Jordan Form. It is not diagonalizable. It is closest to a diagonal matrix.

<u>More members of the family.</u>

Best one

$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$, $\begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix}$ . . . $\begin{bmatrix} a & \sim \\ \sim & 8-a \end{bmatrix}$ s.t. trace = 8, det = 16.

> Members of the big family are not diagonalizable since the diagonal form is $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ & we cant get to it using $M^{-1}AM$.

Eg :-

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

$\lambda = 0, 0, 0, 0$

rank = 2

$\Rightarrow$ no. of e-vectors = 2

dim $N(A) = 2$

Jordan block

$$\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\lambda = 0, 0, 0, 0$.

rank = 2

$\Rightarrow$ no. of e-vectors = 2

dim $N(A) = 2$

} But these two are not similar!

Jordan Block.

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

> Has only one eigenvector.

> Every square A is similar to a Jordan Matrix J.

$$J = \begin{bmatrix} \boxed{J_1} & & \\ & \boxed{J_2} & \\ & & \boxed{J_d} \end{bmatrix}$$

#blocks = #eigenvectors since each block has only 1 eigenvector.

> <u>goodcase</u>: J is Λ $\rightarrow$ n λs, n blocks & so on.

<u>Note</u>: <u>Geometric interpretation of quadratic form.</u>

> Let $\vec{x}$ is a vector $\Rightarrow$ $A\vec{x}$ is a new vector.

> $x^T A x$ is the inner product of the old & new vectors.

> If A is positive definite $\Rightarrow$ $\langle\ \rangle$ is positive

$\Rightarrow$ dot product is positive $\Rightarrow$ $\cos\theta$ is +ve $\Rightarrow$ $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

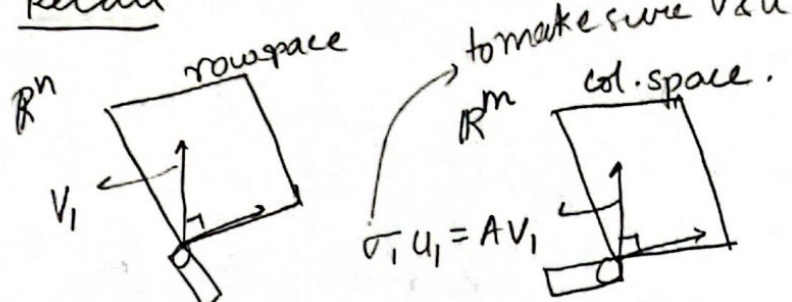
$\Rightarrow$ The transformation A applied on x cannot rotate it more than 90°.

Lec 29 Singular Value Decomposition (SVD).

$\underset{\text{Any matrix!}}{A} = U\Sigma V^T$ where Σ is a diagonal matrix & U, V are orthogonal.

If A is symmetric pos def $\Rightarrow$ $A = Q\Lambda Q^T$.

> Recall



> An orthogonal basis in the row space becomes an orthogonal basis in the column space.

> $$A\left[V_1\ V_2 \dots V_r\right] = \left[u_1\ u_2 \dots u_r\right]\begin{bmatrix}\sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_r\end{bmatrix}$$

$$\Rightarrow \underset{\text{orthonormal basis in rowspace}}{AV} = \underset{\text{orthonomal in col. space.}}{U}\Sigma$$

Eg:- $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

$V_1 V_2$ in rowspace R^2

u_1, u_2 in col. space R^2

$\sigma_1 > 0$ $\sigma_2 > 0$.

$AV_1 = \sigma_1 u_1$

$AV_2 = \sigma_2 u_2$

$A = U \Sigma V^{-1} = U \Sigma V^T$ since V is square orthogonal.

> $A^TA = V\Sigma^T(U^TU)\Sigma V^T$

$\boxed{A^TA = V\Sigma^T\Sigma V^T} = V \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \end{bmatrix} V^T$

- Since A^TA is symmetric posdef, V will be $Q \rightarrow \Rightarrow$ Eigenvectors & $\sigma_1^2 \sigma_2^2 \ldots$ will be eigenvalues of A^TA.

- |||ly Us are eigenvectors of AA^T.

> σs are the tre square roots of A^TA.

Eg: $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$ $A^T = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}$.

$A^TA = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} \longrightarrow$ eigenvectors $\rightarrow \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \longrightarrow 32 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

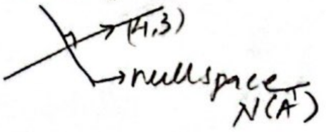
$\rightarrow \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \longrightarrow 18 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$AA^T = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix} \Rightarrow \lambda = 32, 18$; eigenvectors $\rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

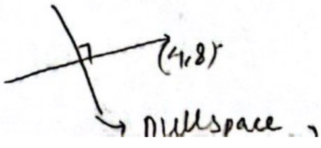
↳ There is an error here!

Eg:- > $A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \rightarrow$ singular! row space is a line! → multiples of $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$



> Col. space is multiples of $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$.

$\Rightarrow V_1 = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$ $U_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.



$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}}_{V^T}$$

→ from nullspace

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}$$

→ null space ←

* *

$v_1 \ldots v_r$ is an orthonormal basis for rowspace.

$u_1 \quad u_r$ " " " " col. space.

$v_{r+1}, \ldots v_n$ " " " " $n(A)$.

$u_{r+1}, \ldots u_n$ " " " " $n(A^T)$. *

*

Eg: SVD of $C = \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix}$.

Sol: $C = U \Sigma V^T$

- $C^TC = V \Sigma^T \Sigma V^T$ & $CV = U\Sigma$

$C^TC = \begin{bmatrix} 26 & 18 \\ 18 & 74 \end{bmatrix} \longrightarrow \det(C^TC - \lambda I) = \det\begin{pmatrix} 26-\lambda & 18 \\ 18 & 74-\lambda \end{pmatrix} \Rightarrow \lambda = 20, 80$

$C^TC - 20I = \begin{bmatrix} 6 & 18 \\ 18 & 54 \end{bmatrix} \Rightarrow v_1 = \begin{pmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}$

$C^TC - 80I = \begin{bmatrix} -54 & 18 \\ 18 & -6 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$

$V = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$

$\Sigma = \begin{bmatrix} 2\sqrt{5} & 0 \\ 0 & 4\sqrt{5} \end{bmatrix}$

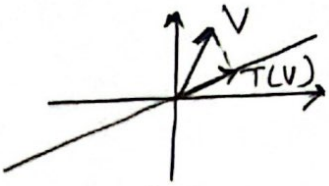
- $CV = \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} = \begin{bmatrix} -\sqrt{10} & 2\sqrt{10} \\ \sqrt{10} & 2\sqrt{10} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2\sqrt{5} & 0 \\ 0 & 4\sqrt{5} \end{bmatrix}$

$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Lec 30: Linear Transformations.

Example 1 Projection

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

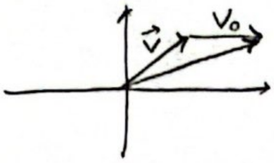


→ Projection is a linear transform.

Linear:
$$\left.\begin{aligned} T(v+w) &= T(v) + T(w) \\ T(cv) &= cT(v) \end{aligned}\right\} T(cv+dw) = cT(v) + dT(w)$$

Example: 2

Add vector $\vec{v}_0$ to a vector $\vec{v}$



→ Not linear!

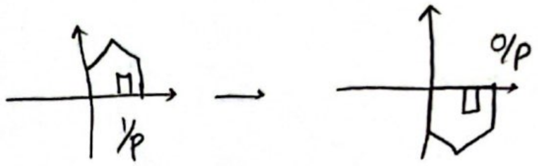
Example: 3

Rotation! → Linear.

Example: 3

$T(v) = Av$. where A is a matrix, is linear.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow$



Start : T is a linear transformation.

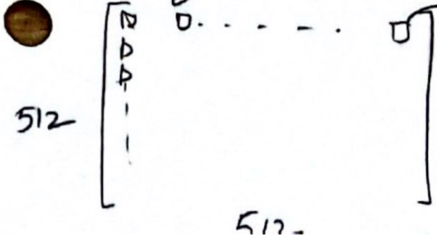
$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ → any matrix A of 2x3 size is good.

> Coordinates come from a basis. $v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$ basis!

> Eigenbasis leads to a diagonal matrix Λ of eigenvalues.

Lec 31: Change of basis

Image compression.



→ say each pixel is 8 bits.
⇒ 0 → 255

} ⇒ $x \longrightarrow \mathbb{R}^n$
where $n = (512)^2$

$x = \begin{bmatrix} \vdots \\ 200 \\ 121 \end{bmatrix}$

Standard basis

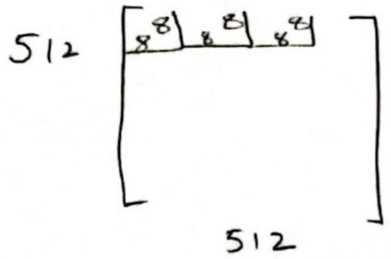
→ $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

→ Say we have vector $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ as the basis! ⇒ All white.

Say we have vector $\begin{bmatrix} +1 \\ -1 \\ +1 \\ -1 \\ \vdots \end{bmatrix}$ → checkerboard!

→ JPEG basis is the Fourier Basis! (8 × 8)

⇒



→ Breaks down 512 × 512 in 64 × 64 of 8×8 blocks. Computation is done on these 64 × 64 blocks.

Fourier basis → $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ w \\ w^2 \\ \vdots \\ w^{n-1} \end{bmatrix}$ → in general!

Wavelet basis → $\underset{w_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \underset{w_2}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}} \underset{w_3}{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \underset{w_4}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}} \underset{w_5}{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \underset{w_6}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \underset{w_7}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}} \underset{w_8}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}}$

→ Given a pixel value $P = C_1 W_1 + C_2 W_2 + C_3 W_3 + \cdots$ which is the same as changing from the standard basis in which P is written to the new wavelet basis in W.

$$P = \begin{bmatrix} W_1 & W_2 & W_3 & W_4 & \cdots \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_6 \end{bmatrix}$$

→ Note that wavelet transform is orthogonal!

→ We can make it orthonormal.

⇒ $W_{norm}^{-1} = W_{norm}^{T}$

→ Solve $P = Wc$ ⇒ $\boxed{C = W^{-1}p}$

⇒ Goodbasis

① Fast computation for Wc & $W^{-1}p$.

② A few basis vectors should be enough to reconstruct it.

Change of basis

Columns of W → new basis vectors.

⇒ Vector X in old basis → C in new basis

⇒ $\boxed{X = Wc.}$ ⤳ Every vector changes based on this

> Let a transformation T. wrt basis $V_1, \dots V_8$ it has a matrix A. wrt basis $W_1 \dots W_8$ it has matrix B.

> The two matrices A & B are similar

⇒ $\boxed{B = M^{-1}AM}$ Where M is the change of basis matrix

↳ Every matrix changes based on this.

$[M = W]$

> What is A? Using basis $V_1 \dots V_8$ we know T completely by knowing $T(V_1), T(V_2) \dots T(V_8)$ because every $X = c_1V_1 + \dots c_8V_8$. Then $T(x) = c_1 T(V_1) + c_2 T(V_2) + \dots + c_8 T(V_8)$.

> In the new basis: $T(V_1) = a_{11}V_1 + a_{21}V_2 + \dots + a_{81}V_8$

$T(V_2) = a_{12}V_1 + a_{22}V_2 + \dots + a_{82}V_8$

$$\Rightarrow [A] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{81} \\ a_{12} & \cdots & & \\ & & & \\ a_{18} & & \cdots & a_{88} \end{bmatrix}$$

> For Eigenbasis $T(v_i) = \lambda_i v_i$. What is A?

A is a diagonal matrix of λ_i $\Rightarrow A = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 \ddots & \\ 0 & & & \lambda_8 \end{bmatrix}$

Lec 32 is exam review

Note: Look at the example video for change of basis.

Eg: From 3Blue 1Brown.

> Let basis $A_{old} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ → standard basis. Another basis is given by $A = \begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}$. These are the 2 basis vectors in the new basis.

$$\boxed{\vec{v}_{new} = A^{-1}\vec{v}_{old}}$$

> $A\begin{bmatrix}x_j\\ y_j\end{bmatrix} = \begin{bmatrix}x_o\\ y_o\end{bmatrix}$ ⇒ $\begin{bmatrix}x_j\\ y_j\end{bmatrix} = A^{-1}\begin{bmatrix}x_o\\ y_o\end{bmatrix}$

vector in new basis A

Vector in old basis A_{old}

$$\boxed{\vec{v}_{old} = A\vec{v}_{new}}$$

↳ vector with cols as basis vecto

> Say we have some transformation T in our old basis. Let the corresponding matrix be $B_{old} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ ↳ 90° rotation!

> To see the same 90° in the new basis we compute B as

$$\boxed{B = \left[A^{-1}\right]\left[B_{old}\right]\left[A\right]}$$

Converts new vector from old basis to new basis

computes transformation in old basis to get the new vector in old bais

converts vector in new basis to old basis

Lec 33: Pseudo inverses

2 sided inverse: $AA^{-1} = I = A^{-1}A$.

$r = m = n \Rightarrow$ full rank square matrix.

left inverse:

- full column rank $\Rightarrow r = n < m. \Rightarrow$ null space $= \{0\}$.
- independant columns $\Rightarrow$ 0 or 1 sol. to $Ax = b$.
- A^TA is an invertible symmetric matric.

$$\Rightarrow \underbrace{(A^TA)^{-1}A^T}_{\substack{A^{-1}_{left} \\ n\times m}} \underset{m\times n}{A} = \underset{n\times n}{I} \Rightarrow \text{left inverse!}$$

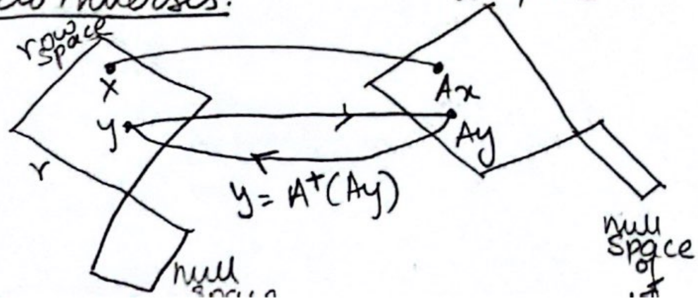
right inverse:

- full row rank : $r = m < n$
- $n(A^T) = \{0\} \Rightarrow$ independant rows. $\Rightarrow \infty$ solutions to $Ax = b$
- $$A\underbrace{A^T(AA^T)^{-1}}_{} = I$$

$$A \; A^{-1}_{right} = I$$

- $A(A^{-1}_{left})$ is $A(A^TA)^{-1}A^T$ which is the projection onto col.space
- $\underbrace{A^T(AA^T)^{-1}}_{A^{-1}_{right}}A$ is projection onto row space.

Pseudo inverses.



- Mapping from rowspace to col.space Ax is one to one.

> If x, y are in row space $\Rightarrow Ax \neq Ay$.

$\Rightarrow$ If A is limited to row & column space it can be inverted.

$$y = \underbrace{A^{+}(Ay)}_{\hookrightarrow \text{ notation of pseudoinverses.}}$$

<u>Proof</u>: If $x \neq y$ & both in row space the $Ax \neq Ay$ in col. space.

Suppose $Ax = Ay \Rightarrow A(x-y) = 0$

$\Rightarrow$ $x-y$ is in the null space but $x-y$ must be in the row space! $\Rightarrow$ $x-y$ must be the 0 vector.

> <u>Finding the pseudoinverse A^{+}.</u>

method ① Start from SVD: $A = U\Sigma V^T$

$$\Sigma \longrightarrow \underset{n \text{ cols.}}{\begin{bmatrix} \sigma_1 & \cdots & & 0 & & \\ & 0 & \sigma_r & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}} \; m \text{ rows}$$

Σ has a rank of r. Therefore Σ^{-1} does not exist.

$$\text{But } \Sigma^{+} = \underset{n \times m}{\begin{bmatrix} 1/\sigma_1 & & & & & \\ & 1/\sigma_2 & & & & \\ & & \ddots & & & \\ & & & 1/\sigma_r & & \\ & & & & 0 \; 0 & \\ & & & & & 0 \end{bmatrix}}$$

$$\Sigma\Sigma^{+} = \underset{\substack{m \times m \\ \hookrightarrow \text{ projection into col. space.}}}{\begin{bmatrix} 1 & & & & \\ & 1 & & 0 & \\ 0 & & \ddots & & \\ & & & 1 & \\ & & & & 0 \; 0 \; 0 \end{bmatrix}} \qquad \underset{n \times m}{\Sigma^{+}} \underset{m \times n}{\Sigma} = \underset{n \times n}{\begin{bmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ 0 & & & 0 & \\ & & & & 0 \cdots 0 \end{bmatrix}} \; \text{projection onto row space}$$

$$\boxed{A^{+} = V\Sigma^{+}U^{T}}$$

→ If $A^TA = AA^T$, the matrix has orthogonal eigen vectors.
› This test is passed for symmetric, skewsymmetric & orthogonal matrices.

→ **SVD:** $A = U\Sigma V^T$ .. U & V^T are orthogonal matrices ⇒ they correspond to rotation. Σ is diagonal ⇒ corresponds to scaling.

⇒ SVD breaks up transformation $A \longrightarrow \underset{V^T}{\text{rotation}} + \underset{\Sigma}{\text{scaling}} + \underset{U.}{\text{rotation}}$

→ Orthogonal matrices are pure rotations ⇒ $|\lambda| = 1$.

→ A real symmetric matrix is always orthogonally diagonalizable.
⇒ there's a basis $\mathbb{R}^n$ consisting of mutually perp. eigenvectors of the matrix. ⇒ it can be decomposed into $Q\underset{\hookrightarrow \text{diag.}}{\Lambda}Q^T$

→ Eigen values of projection matrix P are either 0 or 1.
Say you want to project onto a vector $a = \begin{bmatrix}3\\2\end{bmatrix}$. ⇒ $P = \dfrac{aa^T}{a^Ta}$ is the projection matrix. The eigenvector corresponding to $\lambda = 1$ is a because a projected onto itself doesn't move. All vectors $\perp^{\text{lar}}$ to a correspond to $\lambda = 0$, because these are all killed by P.

→ A^TA is invertible if $r = n$: independant cols of A.