EECS 530 - Electromagnetic Theory

Lecture 1 The Field Equations

Maxwell's Equations. - Point Form

ME1 → $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faraday's Law

ME2 → $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$ Modified Ampere's Law

ME3 → $\nabla \cdot \vec{D} = \rho_{v_{free}}$
ME4 → $\nabla \cdot \vec{B} = 0$ } Gauss' Laws.

$H \rightarrow A/m$
$E \rightarrow V/m$
$D \rightarrow C/m^2$
$B \rightarrow W/m^2$
$\rho_v = C/m^3$

Law of Conservation of Charges.

$\nabla \cdot J = -\frac{\partial \rho}{\partial t}$ } Equation of Continuity.

Proof: Suppose S is a closed surface & $\hat{n}$ is outward normal.

> Law of conservation of charge => $I = \oint\!\!\oint_S \vec{J} \cdot ds = -\frac{d}{dt} Q = -\frac{d}{dt} \int_V \rho dv$

> Assume stationary surface & apply divergence theorem.

Interdependance of MxEqs

$\nabla \cdot (ME2) \Rightarrow ME3.$
$\nabla \cdot (ME1) \Rightarrow ME4$ } using eqn. of continuity.

Maxwells Equations - Integral Form.

$ME1' \rightarrow \oint \vec{E} \cdot \vec{dl} = -\frac{d}{dt} \iint_S \vec{B} \cdot ds$

$ME2' \rightarrow \oint \vec{H} \cdot \vec{dl} = I + \frac{d}{dt} \iint_S \vec{D} \cdot \vec{ds}$

$ME3' \rightarrow \oiint \vec{D} \cdot \vec{ds} = q$

$ME4' \rightarrow \oiint \vec{B} \cdot \vec{ds} = 0$

These are more general and apply even when the surface is time varying!

Proofs.

$\iint_S (ME2) + \text{Stoke's} \Rightarrow ME2'$

$\iint_S (ME1) + \text{Stoke's} \Rightarrow ME1'$

$\iiint_V (ME3) + \text{Div. Thm} \Rightarrow ME3'$

$\iiint_V (ME4) + \text{Div. Thm} \Rightarrow ME4'$

Lecture 2 Time Varying Surfaces & Integral MEs.

$$\oint_C \vec{E} \cdot \vec{dl} = \iint_S -\frac{\partial \vec{B}}{\partial t} \cdot \vec{ds} + \oint_C (\vec{V} \times \vec{B}) \cdot \vec{dl}$$

Where $\vec{V}$ is the velocity vector of the contour. at each point.

$$\oint_C \vec{H} \cdot \vec{dl} = I + \iint_S \rho \vec{V} \cdot \vec{ds} + \iint_S \frac{\partial \vec{D}}{\partial t} \cdot \vec{ds} - \oint_C (\vec{V} \times \vec{D}) \cdot \vec{dl}$$

$\iint_S -\frac{\partial \vec{B}}{\partial t} \cdot \vec{ds} \rightarrow$ Transformer induction.

$\oint_C (\vec{V} \times \vec{B}) \cdot \vec{dl} \rightarrow$ Motional induction. $\rightarrow$ Is non zero if the contour is fixed even if the surface moves with time.

$$E_{emf} = \frac{F_{emf}}{q} = \vec{V} \times \vec{B}$$

$$\Rightarrow V_{emf} = \oint_C \vec{E}_{emf} \cdot \vec{dl} = \oint_C (\vec{V} \times \vec{B}) \cdot \vec{dl}$$

Voltage produced along the contour when it moves in a field $\vec{B}$.

Proofs

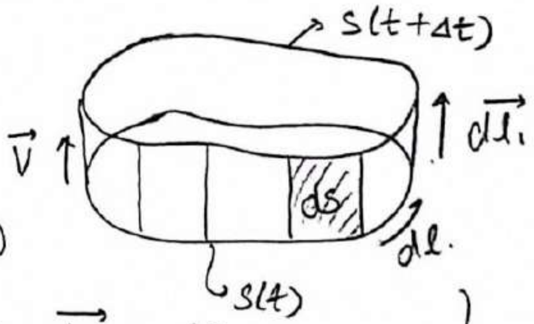
(I) $$\oint_C \vec{E} \cdot \vec{dl} = \frac{-d}{dt} \iint_{S(t)} \vec{B} \cdot \vec{ds} \qquad \tilde{V} = \frac{d}{dt} \iint_{S(t)} \vec{B} \cdot \vec{ds}$$

> Start with defn of derivative.

> Apply Taylor expansion on $\vec{B}(t + \Delta t) \rightarrow$ gives Transformer induction

> Second component is $\tilde{V}_2 = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \iint_{S(t+\Delta t)} \vec{B(t)} \cdot \vec{ds} - \iint_{S(t)} \vec{B(t)} \cdot \vec{ds} \right.$

> Consider a moving geometry.



Note: $\vec{dl_1} = \vec{V} \Delta t$

$$\vec{ds} = \vec{dl} \times \vec{dl_1} = \vec{dl} \times \underbrace{\vec{V} \Delta t}_{①}$$

> $\tilde{V}_2$ changes to $\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \underbrace{\oiint_{S_0} \vec{B}(t) \cdot \vec{ds}}_{\text{total surface}} - \underbrace{\iint_{\Delta S} \vec{B(t)} \cdot \vec{ds}}_{\text{outer ribbon}} \right\}$

** > Apply ①, $\lim_{\Delta t \to 0} \frac{1}{\Delta t} \iint_{\Delta S} \vec{B(t)} \cdot \vec{ds} = \oint_C \vec{B(t)} \cdot \vec{dl} \times V(t) = \oint_C (\vec{V} \times \vec{B}) \cdot \vec{dl}$

Since field passing through ΔS = field passing through C as $\Delta t \to 0$

> Apply Div Thm to $\oiint_{S_0}$ & $\nabla \cdot B = 0$

(II) Steps are same except : $\lim_{\Delta t \to 0} \frac{1}{\Delta t} \iiint_V \nabla \cdot D \, dV = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \iint_{S(t)} (\nabla \cdot D) \vec{V} \Delta t$

$= \iint_{S(t)} (\nabla \cdot D) \vec{V} \cdot ds = \iint_S \rho \vec{V} \cdot ds$ term remains.

Lecture 3 Constitutive Relationships.

Homogeneous & Isotropic : $\vec{D} = \epsilon \vec{E}$ $\quad \vec{B} = \mu \vec{H}$ $\quad \epsilon_0 = 8.85 \times 10^{-12}\ F/m$, $\mu_0 = 4\pi \times 10^{-7}\ H/m$

> Isotropic ⇒ $D \parallel E$ & $B \parallel H$.

> $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$; $\vec{P} = \epsilon_0 \chi_e \vec{E}$ $\quad \vec{P}$ = polarization vector, χ_e = electric susceptibility

$\Rightarrow \vec{D} = \epsilon_0 (1+\chi_e) \vec{E} = \epsilon_0 \epsilon_r \vec{E}$ $\quad \chi_e = \epsilon_r - 1$

> $\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M}$; $\vec{M} = \chi_m \vec{H}$

$\Rightarrow \vec{B} = \mu_0 (1+\chi_m) \vec{H} = \mu_0 \mu_r \vec{H}$

> $\mu_r > 1$ ⇒ paramagnetic.

* $\mu_r < 1$ ⇒ diamagnetic ⇒ induced magnetic moments are parallel to applied $\vec{H}$.

$\mu_r \gg 1$ ⇒ ferromagnetic ⇒ Nonlinear, Hysteresis

Anisotropic.

$\vec{D} = \bar{\bar{\epsilon}} \cdot \vec{E}$ $\quad \vec{B} = \bar{\bar{\mu}} \cdot \vec{H}$

$$\bar{\bar{\epsilon}} = \epsilon_0 \begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix}$$ Biaxial medium

$\epsilon_x = \epsilon_y \neq \epsilon_z$ ⇒ uniaxial medium.

Bianisotropic.

Magnetization under applied $\vec{E}$ & polarization under $\vec{H}$.

$$\vec{D} = \bar{\bar{\epsilon}} \cdot \vec{E} + \bar{\bar{\xi}} \cdot \vec{H}$$
$$\vec{B} = \bar{\bar{\zeta}} \cdot \vec{E} + \bar{\bar{\mu}} \cdot \vec{H}$$

Gram Schmidt

$\vec{B} = b - \frac{a^T b}{a^T a} a$. $\quad \vec{B}$ & a are orthogonal.

Dispersive Materials

Charges have finite mass $\Rightarrow$ time delay.

$\rightarrow$ Recall, $\vec{P} = \epsilon_0 . \chi_e \vec{E}$ $\quad \chi_e$ = constant $\Rightarrow$ ignores time delay.

In reality $\vec{P}(t) = \epsilon_0 \int_{-\infty}^{t} \chi_e(t-\tau) E(\tau) d\tau$

$$\Rightarrow \vec{P}(\omega) = \epsilon_0 \chi_e(\omega) \vec{E}(\omega).$$

$$\Rightarrow \vec{D}(\omega) = \epsilon_0 (1+\chi_e(\omega)) \vec{E}(\omega)$$

$\rightarrow$ $\epsilon(\omega) = \epsilon_0(1+\chi_e(\omega))$ is a complex number & a fn of frequency.

Similary, $\vec{M} = \chi_m(\omega) \vec{H}$

$\rightarrow$ Since $\chi_e(t)$ is causal, it is 0 below $t=0 \Rightarrow$ F.T is complex.

Causality $\Rightarrow$ Time delay.

Conducting Media

$\vec{J_c} = \sigma \vec{E}$ in the absence of $\vec{H}$ fields. If $\vec{H}$ is present $\vec{J} \neq \vec{E}$

$\boxed{\rho = \rho_0 e^{-(\sigma/\epsilon)t}}$ Charges vanish exponentially inside a medium.

Proof: Start from continuity & show $\sigma \nabla . E = \frac{\sigma}{\epsilon} \rho . = -\frac{\partial \rho}{\partial t}$

$\frac{\sigma}{\epsilon}$ is a measure of conductor quality. (Silver > Gold > Cu > Al)

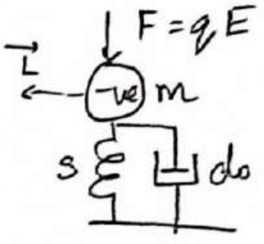
PEC $\Rightarrow \sigma = \infty \Rightarrow \vec{E}(r,t) = 0$ inside $\Rightarrow \vec{H}$ is purely static.

PMC $\Rightarrow \vec{H}(r,t) = 0$ inside $\Rightarrow \vec{E}$ is purely static. (Since $\vec{E} \neq 0$, there are no free charges inside a PMC

> In both cases there are no currents inside the material.

Lorentz Dielectric Model.

- Each molecule assumed independant of others.
- Electrostatic force is assumed to be linear. (Small signal)
- Equivalent mechanical model based.



$$m\frac{d^2\vec{L}}{dt^2} + d_o\frac{d\vec{L}}{dt} + S\vec{L} = q\vec{E}$$

$S \rightarrow$ represents linearized electrostatic force, $\vec{L}$ = displacement vector

$d_o \rightarrow$ damping.

Assume time harmonic excitation: $\vec{E}(t) = Re\left[\tilde{E}e^{-i\omega t}\right]$

$$\Rightarrow \tilde{L} = \frac{(q/m)\tilde{E}}{\omega_o^2 - \omega^2 - i\gamma\omega}$$

where $\omega_o^2 = \frac{S}{m}$ → natural resonance

$\gamma = \frac{d_o}{m}$ → damping factor.

→ N independant polarized molecules

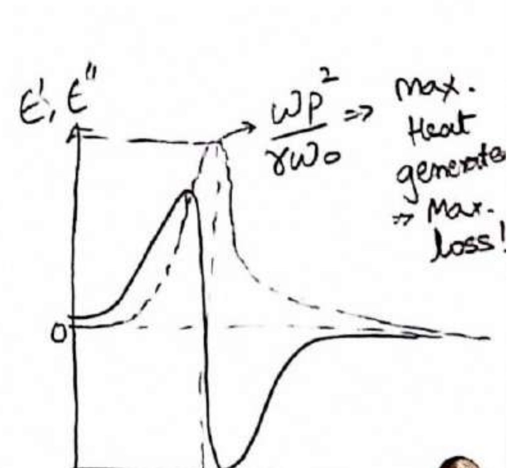
$$\Rightarrow \vec{P} = Nq\vec{L} = \frac{\omega_p^2}{\omega_o^2 - \omega^2 - i\gamma\omega}\epsilon_o\vec{E}$$

where $\omega_p^2 = \frac{Nq^2}{m\epsilon_o}$ = plasma frequency!

$$\Rightarrow \epsilon = \epsilon_o\left[1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2 - i\gamma\omega}\right]$$

$$\Rightarrow \epsilon' = \epsilon_o\left[1 + \frac{\omega_p^2(\omega_o^2 - \omega^2)}{(\omega_o^2 - \omega^2)^2 + \gamma^2\omega^2}\right]$$

$$\epsilon'' = \epsilon_o\left[\frac{\omega_p^2\gamma\omega}{(\omega_o^2 - \omega^2)^2 + \gamma^2\omega^2}\right]$$



Where $\epsilon = \epsilon' + i\epsilon''$

"polarizing"

- At the resonance frequency ω_0, the molecule is vibrating most vigorously due to applied $\vec{E}$. ϵ'' is at the peak & heat loss is very high.

- $\underline{\omega \ll \omega_0}$ $\Rightarrow$ $\epsilon' \cong \epsilon_0 \left(1 + \frac{\omega_p^2}{\omega_0^2}\right)$ → independant of ω.

  $\epsilon'' \cong \epsilon_0 \frac{\gamma \omega_p^2 \omega}{\omega_0^4}$ → loss $\left(\frac{\epsilon''}{\epsilon'}\right)$ is small & linear w.r.t ω

- $\underline{\omega \gg \omega_0}$ $\Rightarrow$ $\epsilon' \cong \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right)$ → As $\omega \to \infty$ $\epsilon' \simeq \epsilon_0$ $\epsilon'' \cong 0$. Since field do not penetrate at all.

  $\epsilon'' \simeq \epsilon_0 \frac{\gamma \omega_p^2}{\omega^3}$

Drude Dielectric Model for Metals.

$\epsilon''_{cond} = \frac{\sigma}{\omega}$ $\Rightarrow$ $\epsilon_{metal} = \epsilon_0 \left(1 + i\frac{\sigma}{\omega\epsilon_0}\right)$ when $f < 100\,GHz$.

- At higher frequency use Drude model:
- Good conductors $\Rightarrow$ electrons are not bound $\Rightarrow$ $S = 0 \Rightarrow \omega_0 = 0$!

$$\Rightarrow \epsilon = \epsilon_0 \left[1 - \frac{\omega_p^2}{\omega^2 + i\gamma\omega}\right] \Rightarrow \epsilon_r' = 1 - \frac{\omega_p^2}{\omega^2 + \gamma^2}$$

$$\epsilon_r'' = \frac{\omega_p^2 \gamma}{\omega(\omega^2 + \gamma^2)}.$$

- At low frequencies. $\omega \ll \gamma$ & $\omega_p \ll \gamma$

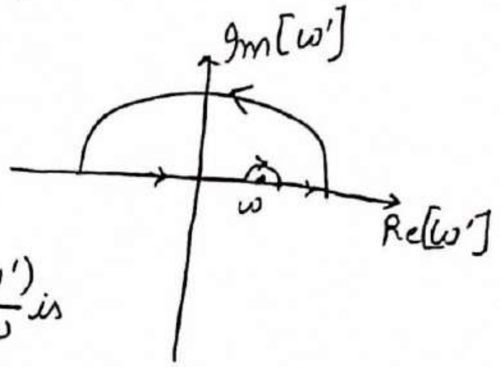
$$\Rightarrow \epsilon_r' \simeq 1, \quad \epsilon_r'' = \frac{\omega_p^2}{\omega\gamma} = \frac{1}{\epsilon_0}\frac{\sigma}{\omega} \text{ (from earlier)}$$

$$\Rightarrow \boxed{\sigma = \frac{\epsilon_0 \omega_p^2}{\gamma} = \frac{Nq^2}{m\gamma}.}$$

Kramer's Krönig's Relation.

> Unconditionally stable system of charges. $\Rightarrow \chi_e(t) \to 0$ as $t \to \infty$

$\Rightarrow$ F.T has no poles in UHP.
$\chi_e(\omega)$



> Take $I = \oint_C \frac{\chi_e(\omega')}{\omega' - \omega} d\omega'$

> Cauchy's theorem $\Rightarrow I = 0$ since $\frac{\chi_e(\omega')}{\omega' - \omega}$ is analytic over C.

> Jordan's lemma $\Rightarrow \frown = 0$

> $\oint_C f(z)\,dz = 2\pi i \sum_{n=1}^{N} \lim_{z \to z_n} (z - z_n) f(z_n)$ if $[z_n]_{n=1}^{N}$ are inside the contour.

$\oint_C f(z)\,dz = \pi i \sum_{n=1}^{N} \lim_{z \to z_n} (z - z_n) f(z_n)$ if $[z_n]_{n=1}^{N}$ are on the contour.

$$\Rightarrow \int_{\substack{\text{small}\\\text{circle}}} \frac{\chi_e(\omega')}{(\omega' - \omega)} d\omega' = -i\pi \chi_e(\omega)$$

$$\Rightarrow -i\pi \chi_e(\omega) + \int_{-\infty}^{\infty} \frac{\chi_e(\omega')}{\omega' - \omega} d\omega' = 0$$

$$\Rightarrow \chi_e'(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\chi_e''(\omega')}{\omega' - \omega} d\omega' \quad \& \quad \chi_e''(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\chi_e'(\omega')}{\omega' - \omega} d\omega'$$

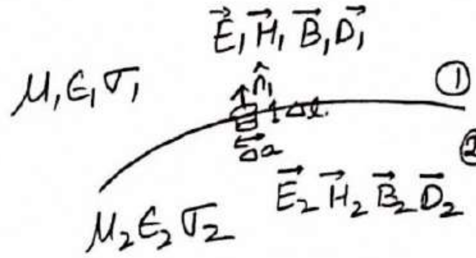
$$\Rightarrow \epsilon_r'(\omega) - \cancelto{1}{\epsilon_{r\infty}} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\epsilon_r''(\omega')}{\omega' - \omega} d\omega'$$

$$\epsilon_r''(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\epsilon_r'(\omega') - \epsilon_\infty}{\omega' - \omega} d\omega'$$

Lec 4 Boundary Conditions.

$\hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0$ | $\hat{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0$

$\hat{n} \times (\vec{H}_1 - \vec{H}_2) = J_s$ | $\hat{n} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s$



Proof :-

> $\iiint_{\Delta V}$ Faraday's law $\longrightarrow$ Stoke 2.0 $\rightarrow$ RHS $= 0$ as $\Delta V \to 0$ when $\lim \Delta l \to 0$

$$LHS = \lim_{\Delta l \to 0} \oiint_{\Delta S} \vec{E} \times ds = (E_1 \times \hat{n}_1 + E_2 \times \hat{n}_2)\Delta a = 0 \Rightarrow \hat{n}(\hat{E}_1 - \hat{E}_2) = 0$$

> For $\hat{n} \times (\vec{H}_1 - \vec{H}_2)$ as $\Delta l \to 0$ $\iiint_{\Delta V} \vec{J} dv$ reduces down to $J_s \Delta a$

> Use Gauss' laws with same procedure.

Lec 5 Generalized Coordinates.

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \qquad d\vec{r} = \vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

→ Let $U_1 = f_1(x, y, z)$ $U_2 = f_2(x, y, z)$ $U_3 = f_3(x, y, z)$.

Then $x = X(u_1, u_2, u_3)$ $y = Y(u_1, u_2, u_3)$ $z = Z(u_1, u_2, u_3)$.

→ $\hat{u}_1, \hat{u}_2, \hat{u}_3$ define unit vectors in a coordinate surface, tangent to coordinate surfaces.

→ Orthogonal ⇒ $\hat{u}_i \cdot \hat{u}_j = \begin{cases} 1 & i=j \\ 0 & i=j \end{cases}$ or $\hat{u}_1 \times \hat{u}_2 = \hat{u}_3$, $\hat{u}_2 \times \hat{u}_3 = \hat{u}_1$, $\hat{u}_3 \times \hat{u}_1 = \hat{u}_2$

→ Coordinate surfaces defined by $U_1 =$ constant, $U_2 =$ const. $U_3 =$ const

→ Coordinate surfaces meet in coordinate lines!

Differential length.

> First we define a metric or a scale factor for each u_i as h_i

$$h_i = \left[\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2\right]^{1/2}$$

> Use the metric to find $\hat{u}_i$

$$\hat{u}_i = \frac{1}{h_i}\frac{\partial \vec{r}}{\partial u_i}$$

$$\Rightarrow \vec{dl} = d\vec{r} = \sum_{i=1}^{3} h_i\, du_i\, \hat{u}_i = h_1 du_1 \hat{u}_1 + h_2 du_2 \hat{u}_2 + h_3 du_3 \hat{u}_3$$

Example : Spherical coordinates.

$$X = r\sin\theta\cos\phi \qquad Y = r\sin\theta\sin\phi \qquad Z = r\cos\theta.$$

$$h_r = 1\ ; \quad h_\theta = r\ ; \quad h_\phi = r\sin\theta.$$

$$\frac{\partial \vec{r}}{\partial r} = \frac{\partial}{\partial r}\left(X\hat{x} + Y\hat{y} + Z\hat{z}\right) = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\Rightarrow \hat{u}_1 = \hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\vec{dl} = h_r\, dr\, \hat{r} + h_\theta\, d\theta\, \hat{\theta} + h_\phi\, d\phi\, \hat{\phi} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\, d\phi\,\hat{\phi}$$

Differential area & volume

$$d\vec{S_1} = h_2 h_3\, du_2\, du_3\, \hat{u}_1$$
$$d\vec{S_2} = h_1 h_3\, du_1\, du_3\, \hat{u}_2$$
$$d\vec{S_3} = h_1 h_2\, du_1\, du_2\, \hat{u}_3$$

$$dV = h_1 h_2 h_3\, du_1\, du_2\, du_3$$

Gradient

$$\nabla\psi = \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial \psi}{\partial u_i}\hat{u}_i$$

Proof:- We know $d\psi = \nabla\psi \cdot dl$

$$d\psi = \frac{\partial\psi}{\partial u_1}du_1 + \frac{\partial\psi}{\partial u_2}du_2 + \frac{\partial\psi}{\partial u_3}du_3$$
$$= \left(\frac{1}{h_1}\frac{\partial\omega}{\partial u_1}\hat{u}_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}\hat{u}_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}\hat{u}_3\right)$$
$$\cdot \left(h_1 du_1 \hat{u}_1 + h_2 du_2 \hat{u}_2 + h_3 du_3 \hat{u}_3\right)$$

Divergence.

$$\nabla\cdot\vec{A} = \frac{1}{h_1h_2h_3}\sum_{i=1}^{3}\frac{\partial}{\partial u_i}\left(\frac{h_1h_2h_3}{h_i}\right)A_i$$

Proof

$$\nabla\cdot\vec{A} = \lim_{\Delta V\to 0}\frac{\oint\!\!\oint_S \vec{A}\cdot d\vec{s}}{\Delta V} = \lim_{\Delta V\to 0}\frac{\Sigma_i \vec{A}\cdot\Delta s_i}{\Delta V}$$

$$\sum_i \vec{A}\cdot\Delta s_i = \left(A_1 h_2 h_3\Big|_{u_1+\frac{du_1}{2}} - A_1 h_2 h_3\Big|_{u_1-\frac{du_1}{2}}\right) du_2 du_3$$

$$+ \ldots du_1 du_3 + \ldots du_1 du_2$$

x & ÷ by du_1 & se

$du_1 = \left(u_1 + \frac{du_1}{2}\right) - \left(u_1 - \frac{du_1}{2}\right)$

$$= \left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_3h_1A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right] du_1 du_2 du_3$$

$$\Rightarrow \nabla\cdot\vec{A} = \frac{\sum_i \vec{A}\cdot\Delta s_i}{dV} = \frac{1}{h_1h_2h_3}\sum_{i=1}^{3}\frac{\partial}{\partial u_i}\left(\frac{h_1h_2h_3 A_i}{h_i}\right)$$

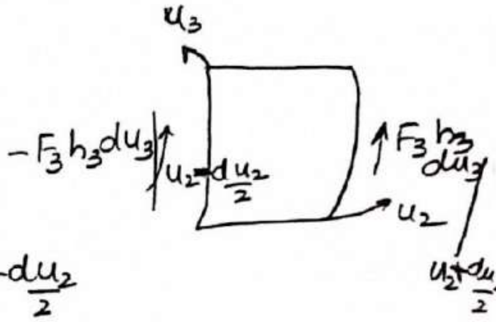
& expand.

Curl.

$$\nabla \times A = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{u}_1 & h_2 \hat{u}_2 & h_3 \hat{u}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

Proof

$$\hat{u}_i \cdot \nabla \times F = \lim_{\Delta S \to 0} \frac{\oint_{\Delta S} \vec{F} \cdot dl}{\Delta S} = \lim_{\Delta S \to 0} \frac{\sum \vec{F} \cdot \Delta l}{\Delta S}$$



In $u_2 u_3$ plane.

$$\sum \vec{F} \cdot \vec{\Delta l} = F_3 h_3 du_3 \Big|_{u_2 + \frac{du_2}{2}} - F_3 h_3 du_3 \Big|_{u_2 - \frac{du_2}{2}}$$

$$+ F_2 h_2 du_2 \Big|_{u_3 + \frac{du_3}{2}} - F_2 h_2 du_2 \Big| u_3 - \frac{du_3}{2}$$

$$= \left[\frac{\partial}{\partial u_2} (h_3 F_3) - \frac{\partial}{\partial u_3} (h_2 F_2) \right] du_2 \, du_3$$ (same trick as before)

Also, $\Delta S = h_2 h_3 \, du_2 \, du_3$.

$$\Rightarrow \hat{u}_1 \cdot \nabla \times \vec{F} = \frac{1}{h_2 h_3} \left[\frac{\partial}{\partial u_2} (h_3 F_3) - \frac{\partial}{\partial u_3} (h_2 F_2) \right]$$

Same for $u_1 u_3$ & $u_1 u_2$ planes gives the final form.

Laplacian $\nabla^2 \psi = \nabla \cdot \nabla (\psi)$

$$\nabla \cdot \nabla \psi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial u_3} \right) \right]$$

Note: $\nabla \cdot \nabla \vec{F} = \nabla (\nabla \cdot \vec{F}) - \nabla \times \nabla \times \vec{F}$

Lec 6 Magnetic Charge and Current.

→ Replace a medium with $\mu \xrightarrow{to} \mu_0$ with J_m. & use $\vec{M} = (\mu_r - 1)\vec{H}$ ↳ $\frac{\mu}{\mu_0}$

$$\Rightarrow \nabla \times \vec{E} = -\mu_0 \mu_r \frac{\partial \vec{H}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t} - \mu_0(\mu_r - 1)\frac{\partial \vec{H}}{\partial t}$$

$$= -\mu_0 \frac{\partial \vec{H}}{\partial t} - \vec{J_m}.$$

$$\Rightarrow J_m = \mu_0(\mu_r - 1)\frac{\partial \vec{H}}{\partial t} = \mu_0 \frac{\partial \vec{M}}{\partial t}$$ ⇒ J_m only exists for Time varying fields.

Similarly Gauss' law ⇒ $\nabla \cdot (\mu_0 H + \mu_0(\mu_r - 1)H) = 0$

$$\Rightarrow \mu_0 \nabla \cdot \vec{H} = -\mu_0(\mu_r - 1)\nabla \cdot \vec{H}$$

Let $\rho_m = -\mu_0(\mu_r - 1)\nabla \cdot \vec{H} = -\mu_0 \nabla \cdot \vec{M}$

$$\Rightarrow \mu_0 \nabla \cdot \vec{H} = \rho_m$$

$$\Rightarrow \nabla \cdot \vec{H} = \frac{\rho_m}{\mu_0}$$

∴ Maxwell's Equations. (Homogeneous, Isotropic)

$$\nabla \times E = -\mu_0 \frac{\partial H}{\partial t} - J_m \qquad \nabla \cdot E = \frac{\rho}{\epsilon}$$

$$\nabla \times H = \epsilon \frac{\partial E}{\partial t} + J \qquad \nabla \cdot H = \frac{\rho_m}{\mu_0}.$$

In general we need not choose μ_0, any chosen μ can give a corresponding J_m & ρ_m.

$$\left.\begin{aligned} J_m &= \mu_0(\mu_r - \underbrace{\tilde{\mu}_r}_{\text{desired } \tilde{\mu}_r})\frac{\partial \vec{H}}{\partial t} \\ \rho_m &= -\mu_0(\mu_r - \tilde{\mu}_r)\nabla \cdot \vec{H} \end{aligned}\right\} \begin{aligned} &\Rightarrow \nabla \times E = -\mu \frac{\partial H}{\partial t} - J_m \\ &\Rightarrow \nabla \cdot H = \frac{\rho_m}{\mu} \\ &\text{where } \mu = \mu_0 \tilde{\mu}_r \end{aligned}$$

Duality Relations

$$E \longrightarrow H \qquad H \rightarrow -E$$
$$\mu \longrightarrow \epsilon \qquad J_m \rightarrow -J$$
$$\epsilon \longrightarrow \mu \qquad \rho_m \rightarrow -\rho$$
$$J \longrightarrow J_m$$
$$\rho \longrightarrow \rho_m$$

Duality that preserves μ & ϵ & also the units.

$$E \longrightarrow \sqrt{\frac{\mu}{\epsilon}}\, H$$

$$H \longrightarrow -\sqrt{\frac{\epsilon}{\mu}}\, E$$

$$J \longrightarrow \sqrt{\frac{\epsilon}{\mu}}\, J_m$$

$$J_m \rightarrow -\sqrt{\frac{\mu}{\epsilon}}\, J$$

$$\rho \rightarrow \sqrt{\frac{\epsilon}{\mu}}\, \rho_m$$

$$\rho_m \longrightarrow \sqrt{\frac{\mu}{\epsilon}}\, \rho$$

Boundary Conditions.

$$\hat{n} \times (\vec{H_1} - \vec{H_2}) = \vec{J_s} \qquad \hat{n} \times (\vec{E_1} - \vec{E_2}) = -\vec{J_{sm}} \qquad \vec{J_m} = \sigma_m \vec{H}.$$
$$\hat{n} \cdot (\vec{D_1} - \vec{D_2}) = \rho_s \qquad \hat{n} \cdot (\vec{B_1} - \vec{B_2}) = \rho_{sm}$$

PEC

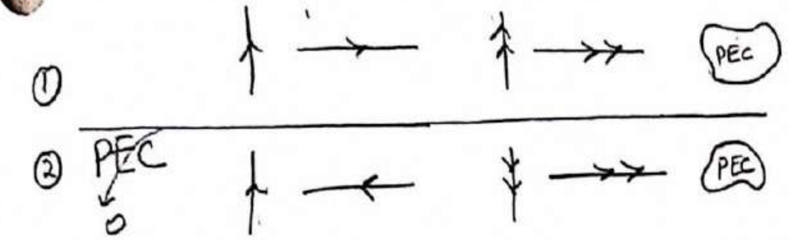
$$\hat{n} \times \vec{E_1} = 0$$
$$\hat{n} \times \vec{H_1} = \vec{J_s}$$
$$\hat{n} \cdot \vec{D_1} = \rho_s$$

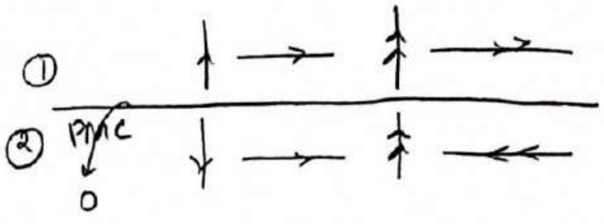
PMC

$$\hat{n} \times \vec{H_1} = 0$$
$$\hat{n} \times \vec{E_1} = -J_{sm}$$
$$\hat{n} \cdot \vec{B_1} = \rho_{sm}.$$

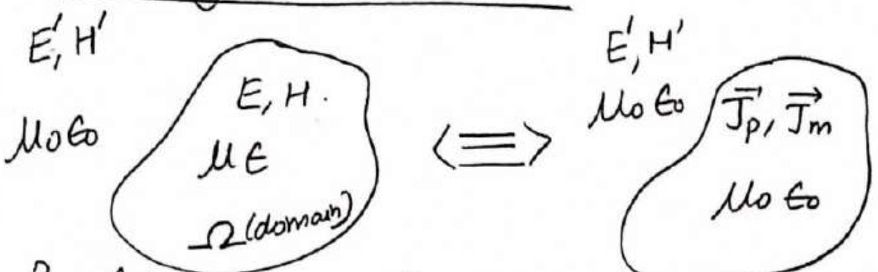
Image Theory

Boundary condition is satisfied $\Rightarrow$ solution in med① is unique!





Polarization Currents. (Same derivation as earlier for J_m)



$$\boxed{\begin{aligned}\vec{J}_p &= \epsilon_0(\epsilon_r - 1)\frac{\partial \vec{E}}{\partial t} = \frac{\partial \vec{P}}{\partial t} \\ \vec{J}_m &= \mu_0(\mu_r - 1)\frac{\partial \vec{H}}{\partial t} = \mu_0 \frac{\partial \vec{M}}{\partial t}\end{aligned}}$$

Proof of J_p

$$\nabla \cdot \vec{E} = \frac{\overset{\nabla \cdot \vec{D}}{\rho}}{\epsilon_0} - (\epsilon_r - 1)\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} - \frac{1}{\epsilon_0}\nabla \cdot \vec{P} = \frac{\rho + \rho_p}{\epsilon_0}$$

$$\text{Let } \rho_p = -\nabla \cdot P = -\epsilon_0(\epsilon_r - 1)\nabla \cdot \vec{E}$$

$$-\frac{\partial \rho_p}{\partial t} = \nabla \cdot J_p = \epsilon_0(\epsilon_r - 1)\frac{\partial}{\partial t}(\nabla \cdot \vec{E}) \Rightarrow J_p = \epsilon_0(\epsilon_r - 1)\frac{\partial \vec{E}}{\partial t}$$

$\rightarrow$ <u>J_p & J_m are not independant</u> (Since E & H are also coupled).

$$\frac{1}{\epsilon_0(\epsilon_r - 1)}\left[\nabla \times \vec{J}_p + \hat{n} \times \vec{J}_p\, \delta(|\vec{r} - \vec{r}'|)\right] = -\frac{\mu}{\mu - \mu_0}\frac{\partial}{\partial t}\vec{J}_m(\vec{r})$$

$\rightarrow$ Use duality to get the other one.

Proof $\psi(r) = \begin{cases} 1 & r \in \Omega \\ 0 & r \notin \Omega \end{cases} \Rightarrow$

$$J_p(r) = \epsilon_0(\epsilon_r - 1)\,\psi(r)\frac{\partial E}{\partial t} \quad — ①$$

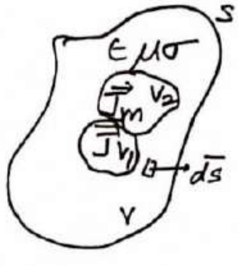
$$J_m(r) = \mu_0(\mu_r - 1)\,\psi(r)\frac{\partial H}{\partial t} \quad — ②$$

> $\nabla \times$ ① & use $\nabla \times (\psi \vec{A})$ & use Faraday + ② to get RHS $= \psi(r)\frac{\partial}{\partial t}(\nabla \times E)$

> Note, $\nabla \psi = -\hat{n}\,\delta(|\vec{r} - \vec{r}'|)$, $\vec{r}'$ is a position vector on the surface of Ω.

$$\Rightarrow \epsilon_0(\epsilon_r - 1)\left(\Delta\psi \times \frac{\partial E}{\partial t}\right) = -\hat{n} \times J_p \frac{\delta(|r - r'|)}{\cancel{\psi(r)} 1}$$

Lec 7 Flow of Energy.



$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} - \vec{J}_m \quad (1)$$

$$\nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \sigma\vec{E} + \vec{J} \quad (2)$$

J, J_m are impressed currents!

$-(1)\cdot\vec{H} + (2)\cdot\vec{E}$

$$\Rightarrow \vec{E}\cdot\nabla\times\vec{H} - \vec{H}\cdot\nabla\times\vec{E} = E\cdot\frac{\partial\vec{D}}{\partial t} + \sigma\vec{E}\cdot\vec{E} + J\cdot\vec{E} + H\cdot\frac{\partial\vec{B}}{\partial t} + \vec{J}_m\cdot\vec{H}$$

We know $\vec{H}\cdot\nabla\times\vec{E} - \vec{E}\cdot\nabla\times\vec{H} = \nabla\cdot(E\times H)$

& $\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D})$ & $\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{H}\cdot\vec{B})$

$$\Rightarrow \nabla\cdot(\vec{E}\times\vec{H}) + \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}) + \sigma|E|^2 = -\vec{J}\cdot\vec{E} - \vec{J}_m\cdot\vec{H}$$

$\iiint_V$ & div. thm

$$\Rightarrow \underbrace{\oiint_S (\vec{E}\times\vec{H})\cdot\vec{ds}}_{\text{Power leaving } S} + \underbrace{\frac{\partial}{\partial t}\iiint_V\left(\frac{\vec{E}\cdot\vec{D}}{2} + \frac{\vec{H}\cdot\vec{B}}{2}\right)dv}_{\text{power stored in } \epsilon \,\&\, \mu} + \underbrace{\iiint_V \sigma|E|^2 dV}_{\text{Ohmic loss.}} = \underbrace{-\iiint_{V_1}\vec{J}\cdot\vec{E}\,dV - \iiint_{V_2}\vec{J}_m\cdot\vec{H}\,dV}_{\text{power supplied.}}$$

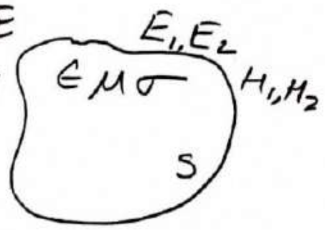
$$W_e = \frac{1}{2}\vec{E}\cdot\vec{D} = \frac{1}{2}\epsilon|E|^2 \;;\; W_m = \frac{1}{2}\bar{H}\cdot\bar{B} = \frac{1}{2}\mu|H|^2$$

$$\boxed{\vec{S}(t) = \vec{E}\times\vec{H}}$$ Poynting Vector!

$$\langle W_e\rangle = \frac{1}{4}\epsilon|E|^2 \;;\; \langle W_m\rangle = \frac{1}{4}\mu|H|^2$$

Uniqueness Theorem.

Assume E_1 & H_1, E_2 & H_2 are 2 solns that satisfy ME



$\Rightarrow \vec{E} = \vec{E_1} - \vec{E_2}$, $\vec{H} = \vec{H_1} - \vec{H_2}$ must satisfy ME

$\Rightarrow \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad \nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} + \sigma E$

Poynting Theorem

$$\frac{\partial}{\partial t} \iiint_V \left[\frac{1}{2} \epsilon |E|^2 + \frac{1}{2} \mu |H|^2 \right] dV = -\iiint_V \sigma |E|^2 dV - \oint\!\!\oint_S (\vec{E} \times \vec{H}) \cdot d\vec{S}$$

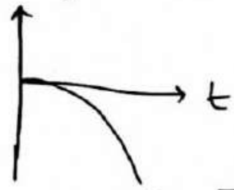
Assume $\hat{n} \times \vec{E} = 0$ or $\hat{n} \times \vec{H} = 0$ on S, then we know $d\vec{s} = \hat{n}\, ds$

$$\oint\!\!\oint_S (\vec{E} \times \vec{H}) \cdot ds = \oint\!\!\oint_S (\hat{n} \times \vec{E}) \cdot \vec{H}\, ds = -\oint\!\!\oint_S (\hat{n} \times H) \cdot \vec{E}\, ds = 0.$$

$\Rightarrow$ If $\hat{n} \times \vec{E_1} = \hat{n} \times \vec{E_2}$ or $\hat{n} \times \vec{H_1} = \hat{n} \times \vec{H_2}$ $\Rightarrow$ Surface $\int = 0$.
These 2 can be partially satisfied on a surface as well.

$\Rightarrow$ RHS is always non positive.

$\rightarrow$ If $\vec{E}(t=0) = \vec{H}(t=0) = 0$ $\Rightarrow$ If $\vec{E_1} = \vec{E_2}$, $\vec{H_1} = \vec{H_2}$ at $t = 0$, then $\frac{\partial}{\partial t}$ of something can only be non positive if that something is 0 or -ve. But LHS integrand is non negative $\Rightarrow$ It must be 0. So must RHS.



$\Rightarrow$ If $\vec{E}$, $\vec{H}$ start at 0 they must remain 0. Therefore $\vec{E_1} = \vec{E_2}$ and $\vec{H_1} = \vec{H_2}$ for all of time!

$\rightarrow$ The only condition to be satisfied is that $\hat{n} \times \vec{E_1} = \hat{n} \times \vec{E_2}$.
$\Rightarrow$ The tangential fields on the boundary must be unique!

$\rightarrow$ If we know the tangential $\vec{E}$ on S_1 & tan. $\vec{H}$ on S_2 where $S = S_1 + S_2$
we can uniquely define the $\vec{E}$ & $\vec{H}$ fields everywhere!

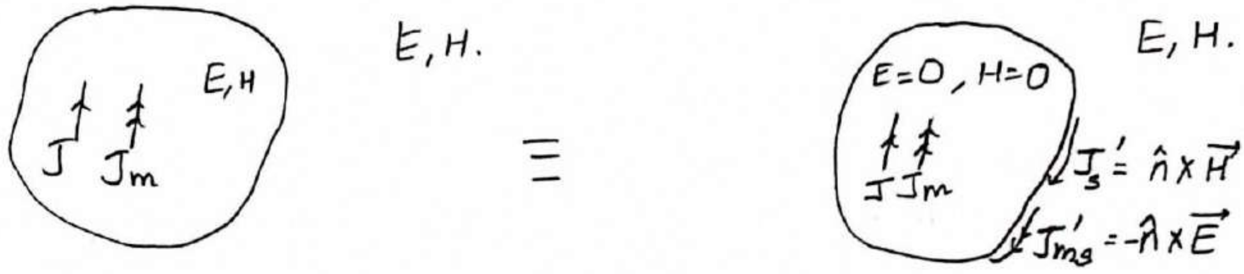
$\rightarrow$ For PEC & PMC $\hat{n} \times \vec{E_1} = n \times \vec{E_2} = 0$, $\hat{n} \times \vec{H_1} = \hat{n} \times \vec{H_2} = 0$ respectively.

> Therefore, by imposing the boundary conditions. we can uniquely find $\vec{E}_{tan}$ & $\vec{H}_{tan}$, which by uniqueness, completely describes $\vec{E}$ & $\vec{H}$ everywhere.

Equivalence Principle.

"Sources are not unique"

→



- E&H need not be 0, they can be arbitrary & this gives ∞ solutions for J_s' & J_{ms}'.

→ If we satisfy E_{tan}, H_{tan} on the boundary we know that E, H remain unchanged. In order to do that we set up J_s' & J_{ms}'. From uniqueness => E, H are same.

Lec 8 Electromagnetic Potentials.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ (1) \ ; \ \nabla \times \vec{H} = \frac{\partial D}{\partial t} + J \ (2) \ ; \ \nabla . B = 0 \ (3) \ ; \ \nabla . D = \rho \ (4)$$

$$\vec{B} = \nabla \times \vec{A} \ ; \ \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \Rightarrow \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \phi .$$

$A \rightarrow$ Magnetic Vector Potential; $\phi \rightarrow$ Scalar electric potential.

$$\left.\begin{aligned} \vec{B} &= \nabla \times \vec{A} \\ \vec{E} &= -\nabla \phi - \frac{\partial A}{\partial t} \end{aligned}\right\} \underset{\text{Homogeneous}}{\overset{\text{Isotropic}}{\Rightarrow}} \quad \begin{aligned} \vec{D} &= -\epsilon\left(\nabla \phi + \frac{\partial \vec{A}}{\partial t}\right) \ — (5) \\ \vec{H} &= \frac{1}{\mu} \nabla \times \vec{A} \ — (6) \end{aligned}$$

(5) & (6) in (2) & (5) in (4). Use $\nabla \times \nabla \times \vec{A} = \nabla \nabla . \vec{A} - \nabla^2 \vec{A}$.

Choose $\boxed{\nabla . \vec{A} = -\mu \epsilon \frac{\partial \phi}{\partial t}}$ to end up with. $\rightarrow$ gauge Condition.

$$\boxed{\begin{aligned} \nabla^2 \vec{A} - \mu \epsilon \frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu J \\ \nabla^2 \phi - \mu \epsilon \frac{\partial^2 \phi}{\partial t^2} &= -\frac{\rho}{\epsilon} \end{aligned}}$$

Replace $\rho \rightarrow \mu \epsilon J$;
$\phi \rightarrow A$;

Hertz Vector Potential.

$$A = \mu \epsilon \frac{d \overline{\pi}}{dt} \Rightarrow \phi = -\nabla . \pi$$

$$\overline{B} = \mu \epsilon \nabla \times \frac{\partial \overline{\pi}}{\partial t} \ (7) \ ; \ \overline{H} = \epsilon \nabla \times \frac{\partial \overline{\pi}}{\partial t} \ ; \ \overline{E} = \nabla \nabla . \overline{\pi} - \mu \epsilon \frac{\partial^2 \overline{\pi}}{\partial t^2} \ (8)$$

(7) & (8) in (1)

$$\Rightarrow \boxed{\nabla^2 \overline{\pi} - \mu \epsilon \frac{\partial^2 \overline{\pi}}{\partial t^2} = -\frac{1}{\epsilon} \int \vec{J} dt}$$

> In General we could have both Electric & Magnetic sources.

$$\nabla^2 \vec{A_m} - \mu\epsilon \frac{\partial^2 \vec{A_m}}{\partial t^2} = -\epsilon J_m$$

$$\nabla^2 \Phi_m - \mu\epsilon \frac{\partial^2 \Phi_m}{\partial t^2} = \frac{-\rho_m}{\mu} \longrightarrow \text{Scalar wave Eqn.}$$

$$\nabla^2 \overline{\Pi}_m - \mu\epsilon \frac{\partial^2 \overline{\Pi}_m}{\partial t^2} = -\frac{1}{\mu}\int J_m dt \longrightarrow \text{Vector Wave Eqn.}$$

$$\therefore \boxed{\begin{aligned} \vec{H} &= \frac{1}{\mu}\nabla\times\vec{A} - \frac{\partial A_m}{\partial t} - \nabla\Phi_m \\ \vec{E} &= -\frac{1}{\epsilon}\nabla\times\vec{A_m} - \frac{\partial A}{\partial t} - \nabla\Phi \end{aligned}}$$

$$\boxed{\begin{aligned} \vec{E} &= \nabla\nabla\cdot\overline{\Pi} - \mu\epsilon\frac{\partial^2\overline{\Pi}}{\partial t^2} - \mu\nabla\times\left(\frac{\partial \overline{\Pi}_m}{\partial t}\right) \\ \vec{H} &= \nabla\nabla\cdot\overline{\Pi}_m - \mu\epsilon\frac{\partial^2\overline{\Pi}_m}{\partial t^2} + \epsilon\nabla\times\left(\frac{\partial \overline{\Pi}}{\partial t}\right) \end{aligned}}$$

Solution of Wave Equation.

Point charge at $\vec{r} = 0 \Rightarrow \rho(\vec{r}) = Q\,\delta(r) \Rightarrow \frac{\partial^2\Phi}{\partial\theta} = \frac{\partial^2\Phi}{\partial\phi} = 0.$

$$\Rightarrow \nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right), \quad \text{Let } \psi(r) = r\,\phi(r)$$

$$\Rightarrow \nabla^2\psi = \frac{1}{r}\frac{\partial^2\psi}{\partial r^2} \Rightarrow \frac{1}{r}\left(\frac{\partial^2\psi}{\partial r} - \mu\epsilon\frac{\partial^2\psi}{\partial t^2}\right) = -\frac{Q}{\epsilon}\,\delta(\vec{r})$$

If $\vec{r} \neq 0 \Rightarrow \boxed{\frac{\partial^2\psi}{\partial r} - \mu\epsilon\frac{\partial^2\psi}{\partial t^2} = 0} \Rightarrow \psi(r+\Delta r, t+\Delta t)$

$$= f(t+\Delta t - (r+\Delta r)\sqrt{\mu\epsilon})$$

$f(t - r\sqrt{\mu\epsilon}) \rightarrow$ general soln

$$u_p = \left(\sqrt{\mu\epsilon}\right)^{-1}$$

$$\Rightarrow \Phi(r,t) = \frac{1}{r} f(t - r\sqrt{\mu\epsilon})$$

We know $\Phi(r,t) = \frac{Q(t)}{4\pi\epsilon r} \Rightarrow f(t) = \frac{Q(t)}{4\pi\epsilon}$

$$\Rightarrow \boxed{\Phi(r,t) = \frac{Q\left(t - \frac{r}{u_p}\right)}{4\pi\epsilon r}}$$

If located at $\vec{r}'$ $\Rightarrow \Phi(r,t) = \frac{Q\left(t - \frac{|\vec{r}-\vec{r}'|}{u_p}\right)}{4\pi\epsilon|\vec{r}-\vec{r}'|}$

For a charge distribution.

$$\Phi(r,t) = \frac{1}{4\pi\epsilon} \iiint_V \frac{\rho_v\left(r', t - \frac{|\vec{r}-\vec{r}'|}{u_p}\right)}{|\vec{r}-\vec{r}'|} d\vec{r}'$$

$$\rho \longrightarrow \mu\epsilon J \ \& \ \Phi \rightarrow A$$

$$\Rightarrow \vec{A}(r,t) = \frac{\mu}{4\pi} \iiint_V \frac{\vec{J}(\vec{r}', t - |\vec{r}-\vec{r}'|/u_p)}{|\vec{r}-\vec{r}'|} d\vec{r}'$$

Lec 9 Infinitesimal Current Element

$$J(\vec{r},t) = I_0 \, d\vec{l} \, \delta(\vec{r}) f(t)$$

$$\Rightarrow \vec{A}(\vec{r},t) = \frac{\mu I_0 \, d\vec{l}}{4\pi r} f\left(t - \frac{r}{u_p}\right)$$

$$\Rightarrow H = \frac{I_0}{4\pi} \nabla \times \left(\frac{d\vec{l}}{r} f\left(t - \frac{r}{u_p}\right)\right)$$

Using $\nabla \times (f A) = ((\nabla f) \times A) + f\cancel{(\nabla \times A)} \to 0$ since $\overline{dl}$ is const

Also note, $\nabla f(g(r)) = f' \nabla g$, $\nabla r = \hat{r}$ and $\nabla\left(\frac{1}{r}\right) = \frac{-\hat{r}}{r^2}$

$\left(\frac{1}{r}\right)' \cdot \nabla r = -\frac{1}{r^2}\hat{r}$

$$\Rightarrow \vec{H} = \frac{I_0}{4\pi} \nabla\left[\frac{1}{r} f\left(t - \frac{r}{u_p}\right)\right] \times \overline{dl}$$

$$= \frac{I_0}{4\pi}\left[\frac{1}{r} f'\left(t - \frac{r}{u_p}\right) \nabla\left(-\frac{r}{u_p}\right) + \nabla\left(\frac{1}{r}\right) \cdot f\left(t - \frac{r}{u_p}\right)\right] \times \overline{dl}$$

$$= \frac{I_0}{4\pi}\left[-f'\left(t - \frac{r}{u_p}\right) \cdot \frac{1}{u_p r} \cdot \hat{r} + -\frac{\hat{r}}{r^2} f\left(t - \frac{r}{u_p}\right)\right] \times \overline{dl}$$

$$\boxed{\vec{H} = \frac{I_0}{4\pi}\left[f'\left(t - \frac{r}{u_p}\right) \frac{1}{u_p r} + \underbrace{f\left(t - \frac{r}{u_p}\right) \cdot \frac{1}{r^2}}_{\text{Biot Savart term}}\right] \overline{dl} \times \hat{r}}$$

$$E = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$$

Gauge condition $\Rightarrow$ $\nabla \cdot \vec{A} = \frac{\mu I_0}{4\pi} \nabla \cdot \left(\frac{f(t - r/u_p)\overline{dl}}{r}\right)$

$$= \frac{-\mu I_0}{4\pi}\left[\left(\frac{f'(t - r/u_p)}{u_p r}\right) + \left(\frac{f(t - r/u_p)}{r^2}\right)\right] \hat{r} \cdot \overline{dl}$$

we know, $\frac{\partial \phi}{\partial t} = -\frac{\nabla \cdot \vec{A}}{\mu \epsilon}$.

$$\Rightarrow \phi(\bar{r}, t) = \frac{I_0}{4\pi\epsilon}\left[\frac{f(t - r/u_p)}{u_p r} + \left(\frac{\int f dt}{r^2}\right)\right] \hat{r} \cdot \overline{dl}$$

Find $\nabla \phi$

$$\nabla \phi = \frac{I_0}{4\pi\epsilon_0}\left\{-\hat{r}\left[\frac{f'}{u_p^2 r} + \frac{2f}{u_p r^2} + \frac{2\int f dt}{r^3}\right](\hat{r} \cdot \overline{dl}) + \left[\frac{f}{u_p r} + \frac{\int f dt}{r^2}\right] \nabla(\hat{r} \cdot \overline{dl})\right\}$$

$$\therefore \vec{E} = \frac{I_0}{4\pi\epsilon}\left\{\hat{r}\left[\frac{f'}{u_p^2 r} + \frac{2f}{u_p r^2} + \frac{2\int f dt}{r^3}\right](\hat{r}\cdot \bar{dl}) - \left[\frac{f}{u_p r} + \frac{\int f dt}{r^2}\right]\nabla(\hat{r}\cdot \bar{dl})\right\}$$

$$- \frac{\mu I_0}{4\pi}\frac{f'}{r}\bar{dl}$$

* We can show, $\nabla(\hat{r}\cdot\bar{l}) = \frac{1}{r}\hat{r}\times(\bar{dl}\times\hat{r})$

Use, $\bar{dl} = \hat{r}\times(\bar{dl}\times\hat{r}) + (\bar{dl}\cdot\hat{r})\hat{r}$ and recalling $\frac{1}{u_p^2} = \mu\epsilon$.

$$\vec{E} = \frac{I_0}{4\pi\epsilon}\left[\frac{3f}{u_p r^2} + \frac{3\int f dt}{r^3} + \frac{f'}{u_p^2 r}\right](\hat{r}\cdot\bar{dl})\hat{r} - \frac{I_0}{4\pi\epsilon}\left[\frac{f}{u_p r^2} + \frac{\int f dt}{\epsilon r^3} + \frac{f'}{u_p^2 r}\right]\bar{dl}$$

Far Field:

$$\vec{H}(r,t) = \frac{I_0\sqrt{\mu\epsilon}}{4\pi r} f'(t - r/u_p)\,\bar{dl}\times\hat{r}$$

$$\vec{E}(r,t) = \frac{\mu I_0}{4\pi r} f'(t - r/u_p)(\bar{dl}\times\hat{r})\times\hat{r}$$

$$\underbrace{\eta = \sqrt{\frac{\mu}{\epsilon}}}_{\text{Characteristic impedance of medium}} = \underbrace{\frac{|E|}{|H|}}_{\text{wave impedance.}}$$

General distribution of Current

Replace $\vec{H} \to d\vec{H}$; $r \to |r - r'|$; $\hat{r} \to \frac{(r-r')}{|r-r'|}$; $\vec{E} \to d\vec{E}$ & integrate.

Line current $\Rightarrow$ Replace $I_0 f'(t - |\bar{r} - \bar{r}'|/u_p) \to I'(\vec{r}', t - |r - r'|/u_p)$ & add $\int_c$

For other distributions of current we can use the following:

$$I(\vec{r}', t)\,\bar{dl}' \longrightarrow \vec{J}_s(r', t)\,ds' \quad \text{surface current}$$

$$I(\vec{r}', t)\,\bar{dl}' \longrightarrow \vec{J}(r', t)\,dv' \quad \text{volume current.}$$

The currents are now vectors & $\bar{dl}$ becomes a scalar $\Rightarrow \frac{1}{4\pi}\iiint_V J \times \frac{(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|} dv'$

Far Field $\vec{H}$ & $\vec{E}$ from J & Jm in V_1 & V_2

$$\vec{H}(\bar{r},t) = \frac{\sqrt{\mu\epsilon}}{4\pi} \iiint_{V_1} \frac{\bar{J}'(\bar{r}', t-|\bar{r}-\bar{r}'|/u_p)}{|\bar{r}-\bar{r}'|^2} \times (\bar{r}-\bar{r}')\,dv' + \frac{\epsilon}{4\pi} \iiint_{V_2} \left[\frac{\bar{J}_m'(\bar{r}', t-|\bar{r}-\bar{r}'|/u_p)}{|r-r'|^3} \times (\bar{r}-\bar{r}') \right] \times (\bar{r}-\bar{r}')\,d$$

$$\vec{E}(\bar{r},t) = \frac{\mu}{4\pi} \iiint_{V_1} \left[\frac{J'(r', t-|\bar{r}-\bar{r}'|/u_p)}{|\bar{r}-\bar{r}'|^3} \times (\bar{r}-\bar{r}') \right] \times (\bar{r}-\bar{r}')\,dv' - \frac{\sqrt{\mu\epsilon}}{4\pi} \iiint_{V_2} \frac{J_m'(r', t-|\bar{r}-\bar{r}'|/u_p)}{|r-r'|^2} \times (r-r')\,dv'$$

Lec 10 Time Harmonic EM Waves

→ $\bar{e}(\bar{r},t) = Re\left[\tilde{E}(\bar{r},\omega)e^{-i\omega t}\right]$ where $\tilde{E} = \tilde{E}_r + i\tilde{E}_i$.

→ Same for $\bar{h}, \bar{d}, \bar{b}, \bar{j}, \rho$

→ All operators are linear in Mx Eqs therefore they apply here.

$$\Rightarrow \nabla \times \tilde{E}(\bar{r},\omega) = i\omega\tilde{B}(\bar{r},\omega)$$

$$\nabla \times \tilde{H}(\bar{r},\omega) = -i\omega\tilde{D}(\bar{r},\omega) + \tilde{J}(\bar{r},\omega) + \sigma\tilde{E}(\bar{r},\omega)$$

$$\nabla\cdot\tilde{D}(\bar{r},\omega) = \tilde{\rho}(\bar{r},\omega)$$

$$\nabla\cdot\tilde{B}(\bar{r},\omega) = 0$$

Constitutive relations. $\tilde{D}(\bar{r},\omega) = \bar{\bar{\epsilon}}(\omega)\tilde{E}(r,\omega)$ → more general than in time domain since convolution is now multiplication.

$$\tilde{B}(\bar{r},\omega) = \underbrace{\bar{\bar{\mu}}(\omega)}_{\text{Complex Tensors in general.}}\tilde{H}(r,\omega)$$

Maxwell's Equations

$$\begin{rcases} \nabla\times\bar{E} = i\omega\mu\bar{H} - \bar{J}_m \\ \nabla\times\bar{H} = -i\omega\epsilon\bar{E} + \bar{J} \\ \nabla\cdot\bar{B} = \rho_m \\ \nabla\cdot\bar{D} = \rho \end{rcases}$$ Notation changed & ω dropped.

Continuity

$$\nabla\cdot\bar{J} - i\omega\rho = 0$$

$$\nabla\cdot\bar{J}_m - i\omega\rho_m = 0$$

$$\nabla \times \bar{H} = -i\omega\epsilon\bar{E} + \sigma\bar{E} + \bar{J}_{imp} \Rightarrow \epsilon_{eff} = \epsilon + i\frac{\sigma}{\omega}$$

$$= -i\omega\epsilon_{eff}\bar{E} + \bar{J}_{imp} \qquad\qquad = \epsilon' + i\left(\epsilon'' + \frac{\sigma}{\omega}\right).$$

Potentials.

$$\bar{B}(\bar{r},\omega) = \nabla \times \bar{A}(\bar{r},\omega) \quad ; \quad \bar{E}(\bar{r},\omega) = -\nabla\phi(\bar{r},\omega) + i\omega\bar{A}(\bar{r},\omega)$$

gauge condition: $\nabla \cdot A(\bar{r},\omega) = i\omega\mu\epsilon\phi(\bar{r},\omega)$

$\Rightarrow \phi(\bar{r},\omega) = -\frac{i\omega}{k^2}\nabla\cdot\bar{A}$. where $k = \omega\sqrt{\mu\epsilon}$ is the wave number.

Wave Equations.

$$\nabla^2\bar{A} + k^2\bar{A} = -\mu\bar{J} \qquad \nabla^2\bar{A}_m + k^2\bar{A}_m = -\epsilon\bar{J}_m$$

$$\nabla^2\phi + k^2\phi = -\frac{\rho}{\epsilon} \qquad \nabla^2\phi_m + k^2\phi_m = -\frac{\rho_m}{\mu}.$$

In general:
$$\bar{H} = \frac{1}{\mu}\nabla\times\bar{A} + \frac{i\omega}{k^2}\nabla(\nabla\cdot\bar{A}_m) + i\omega\bar{A}_m$$

$$\bar{E} = \frac{i\omega}{k^2}\nabla\nabla\cdot\bar{A} + i\omega\bar{A} - \frac{1}{\epsilon}\nabla\times\bar{A}_m$$

Hertz Potential.

$$\nabla^2\bar{\Pi} + k^2\bar{\Pi} = -\frac{i\bar{J}}{\omega\epsilon}$$

$$\nabla^2\bar{\Pi}_m + k^2\bar{\Pi}_m = -\frac{i\bar{J}_m}{\omega\mu}.$$

$$\Rightarrow \bar{H} = -i\omega\epsilon\nabla\times\bar{\Pi} + \nabla\nabla\cdot\bar{\Pi}_m + k^2\bar{\Pi}_m$$

$$\bar{E} = \nabla\nabla\cdot\bar{\Pi} + k^2\bar{\Pi} + i\omega\mu\nabla\times\bar{\Pi}_m$$

Poynting Theorem.

Time avg. Poynting Vector representing net power going through a point:

$$\bar{S}_{av}(\bar{r}) = \frac{1}{2} \mathrm{Re}\left[\bar{E}(\bar{r}) \times H^*(\bar{r})\right]$$

$$\Rightarrow \bar{S}_{av}(\bar{r}) = \mathrm{Re}\left[\bar{S}(r)\right]$$

$\mathrm{Im}\left[\bar{S}(r)\right]$ is related to the flow of reactive power density.

We can show,

$$P^c_{rad} = \frac{1}{2} \oint\!\!\oint_S (\bar{E} \times \bar{H}^*) \cdot ds = \underbrace{-\frac{1}{2} \iiint_V \left[\bar{E} \cdot \bar{J}^* + \bar{J}_m \cdot \bar{H}^*\right] dV}_{\text{total complex i/p power}}$$

$$+ i\omega \underbrace{\iiint_V \left[\frac{1}{2}\mu |H|^2 - \frac{1}{2}\epsilon^* |E|^2\right] dV}_{P_{loss} + 2\omega(W_e - W_m)}$$

$$P_{in}^c = -\frac{1}{2} \iiint_V \left[\bar{E} \cdot \bar{J} + \bar{J}_m \cdot \bar{H}^*\right] dV$$

$$W_e = \frac{1}{2} \iiint_V \frac{1}{2} \epsilon' |E|^2 dV \qquad W_m = \frac{1}{2} \iiint \frac{1}{2} \mu' |H|^2 dV$$

$$P_{loss} = \frac{1}{2} \iiint_V (\omega\mu'' |H|^2 + \omega\epsilon'' |E|^2)\, dV$$

$$\Rightarrow P_{in}^c = P^c_{rad} + P_{loss} + i2\omega(W_e - W_m)$$

Explicitly, $\mathrm{Re}[P_{in}^c] = \mathrm{Re}[P_{rad}^c] + P_{loss}$

shows We & Wm are 180° out of pha

$$\mathrm{Im}[P_{in}^c] = \mathrm{Im}[P_{rad}^c] + 2\omega(W_e - W_m)$$

Time Harmonic Polarization Currents.

→ Replace a medium with free space by placing:

$$\bar{J}_{ep}(\bar{r}) = -i\omega\epsilon_0(\epsilon_r - 1)\bar{E}(\bar{r})$$

$$\bar{J}_{mp}(\bar{r}) = -i\omega\mu_0(\mu_r - 1)\bar{H}(\bar{r})$$

→ ϵ_r is complex

→ If it is real $\bar{J}_{ep}$ & $\bar{E}(\bar{r})$ are in phase quadrature ⇒ Best case!

$$\bar{J}_{ep}(\bar{r}) = \underbrace{\omega\epsilon_0\epsilon_r''\bar{E}(r)}_{\substack{\text{In phase} \\ \text{"dissipative current"}}} - \underbrace{i\omega\epsilon_0(\epsilon_r' - 1)\bar{E}(r)}_{\substack{\text{Quadrature} \\ \text{"reactive current"}}}$$

Quality factor.

$$Q = \frac{\epsilon_r'}{\epsilon_r''}$$ → Quality of a dielectric

$$= \omega \cdot \frac{\text{max stored energy}}{\text{avg. power dissipated}} = \omega\,\frac{\frac{1}{2}\epsilon'|E|^2}{\frac{1}{2}\omega\epsilon''|E|^2} = \frac{\epsilon'}{\epsilon''}.$$

$$Q_m = \frac{\mu'}{\mu''}$$

Time Harmonic Retarded Potential.

$$\bar{j}(\bar{r},t) = \delta(\bar{r})\,\bar{dl}\,\mathrm{Re}\left[I_0 e^{-i\omega t}\right]$$ & use $F[f(t-t_0)] = F(\omega)e^{+i\omega t_0}$

$$\Rightarrow \bar{A}(r) = \frac{\mu I_0}{4\pi r}\,\bar{dl}\,e^{ikr} \qquad k = \frac{\omega}{u_p}$$

$$\bar{A}(\bar{r}) = \frac{\mu I_0}{4\pi}\,\bar{dl}\,\frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}$$

$$\Rightarrow \bar{H}(\bar{r}) = \frac{I_0 k^2}{4\pi}\left[-i + \frac{1}{k|\bar{r}-\bar{r}'|}\right]\frac{e^{ik|\bar{r}-\bar{r}'|}}{k|\bar{r}-\bar{r}'|}\,\bar{dl} \times \frac{(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|}$$

$$\& \ \bar{E}(\bar{r}) = \frac{I_0 k^2}{4\pi}\eta\,\frac{e^{ik|\bar{r}-\bar{r}'|}}{k|\bar{r}-\bar{r}'|}\left\{\left(\frac{3}{k|\bar{r}-\bar{r}'|} + \frac{i3}{k^2|\bar{r}-\bar{r}'|^2} + 1\right)\frac{(\bar{r}-\bar{r}')\cdot\bar{dl}}{|\bar{r}-\bar{r}'|^2}(\bar{r}-\bar{r}') - \left(\frac{1}{k|\bar{r}-\bar{r}'|} + \frac{i}{k^2|\bar{r}-\bar{r}'|} + 1\right.\right.$$

For volumetric distribution.

Replace $I_0 d\bar{l}$ with $\bar{J}(\bar{r}')$ & end with dv'

Far Field approximation.

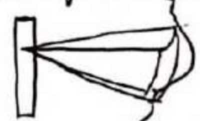
$|\bar{r}-\bar{r}'| \simeq r - \bar{r}'\cdot\hat{r}$ for phase terms

$|\bar{r}-\bar{r}'| \simeq r$ for other terms.

Eg:- If $\bar{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z}$ $\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

$\Rightarrow \bar{r}'\cdot\hat{r} = \sin\theta\cos\phi\, x' + \sin\theta\sin\phi\, y' + \cos\theta\, z'$.

$$\Rightarrow \bar{H}(\bar{r}) = -\frac{ike^{ikr}}{4\pi r}\iiint_V (\bar{J}(\bar{r}')\times\hat{r})\, e^{-ik\bar{r}'\cdot\hat{r}}\, dv'$$ → Looks like a Fourier Transform in space.



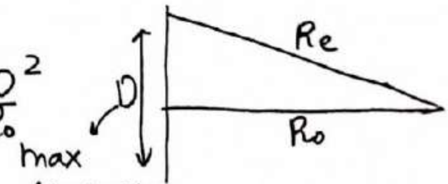
$$\bar{E}(\bar{r}) = \frac{-ik\eta}{4\pi}\frac{e^{ikr}}{r}\iiint_V (\bar{J}(\bar{r}')\times\hat{r})\times\hat{r}\, e^{-ik\bar{r}'\cdot\hat{r}}\, dv'$$

⇒ Thin & broad beam from such an antenna!

$$\Rightarrow \boxed{\bar{E}(\bar{r}) = \eta\, \bar{H}(\bar{r})\times\hat{r}}$$ TEM!

Farfield distance.

$$\Delta\phi = k(R_e - R_0) \simeq k\left[R_0\left(1+\frac{1}{8}\frac{D^2}{R_0^2}\right) - R_0\right] = \frac{1}{8}\frac{kD^2}{R_0}$$



$$\Rightarrow \Delta\phi < \frac{\pi}{8} \Rightarrow \boxed{R_0 \geq \frac{2D^2}{\lambda}}$$

Lec 11 Formal Solution of Helmholtz Equation.

→ Consider the scalar wave equation with an impulse excitation at the origin.

$$\nabla^2 g(\vec{r}) + k^2 g(\vec{r}) = -\delta(\vec{r}) \quad \text{—} ①$$

Impulse in space.

$d\vec{r}' = dx'dy'dz'$; $d\vec{r}' = \rho' d\rho' d\phi' dz'$; $d\vec{r}' = r^2 \sin\theta' d\theta' d\phi' dr$

$$\delta(\bar{r}) = \delta(x)\delta(y)\delta(z)$$

$$\delta(\vec{r}) = \frac{1}{\rho}\delta(\rho)\,\delta(\phi)\,\delta(z)$$

$$\delta(\vec{r}) = \frac{1}{r^2 \sin\theta}\delta(\phi)\,\delta(\theta)\,\delta(r)$$

In case of spherical symmetry

$$\boxed{\delta(\bar{r}) = \frac{\delta(r)}{4\pi r^2}} \quad \text{since} \iiint_V \delta(\bar{r})\,d\bar{r} = \int_0^R \int_0^{2\pi} \int_0^{\pi} \frac{\delta(r)}{4\pi r^2} r^2 \sin\theta\, d\theta\, d\phi\, dr = 1$$

∴ Since $g(r)$ is only a fn of r

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial g(r)}{\partial r}\right) + k^2 g(r) = -\frac{\delta(r)}{4\pi r^2}$$

Change of variables $\psi(r) = r g(r)$ & replacing $\psi(r) \to g(r)$

$$\frac{d^2}{dr^2}\left[r g(r)\right] + k^2\left[r g(r)\right] = -\frac{\delta(r)}{4\pi r}$$

General solution:

$$g(r) = A_1(k)\frac{e^{ikr}}{r} + \cancel{A_2(k)\frac{e^{-ikr}}{r}} \to 0$$ cannot have an incoming wave.

To find A_1, integrate ① over sphere centred at Origin.

$$\Rightarrow \iiint_V \nabla\cdot\nabla g(r)\,dv + k^2 \iiint_V g(r)\,dv = -1$$

Div Thm.:

$$\Rightarrow \oiint_S \nabla g(r)\cdot \bar{ds} + k^2 \iiint_V g(r)\,dv = -1$$

Apply $g(r) = A_1(k)\dfrac{e^{ikr}}{r}$

$$\Rightarrow A_1 \frac{\partial}{\partial R}\left(\frac{e^{ikR}}{R}\right) 4\pi R^2 + k^2 4\pi A_1 \int_0^R r e^{ikr}\,dr = -1$$

As $R \to 0$, $LHS \to -4\pi A_1$

$$\Rightarrow A_1 = \frac{1}{4\pi}$$

The green's function is a spatial impulse response to the wave eqn. Therefore any distribution of charge can be dealt with. It can be used to find potentials & vice versa for any distribution of charge or current.

$$\Rightarrow \boxed{g(r) = \frac{e^{ikr}}{4\pi r} = \frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|}} \rightarrow r \Leftrightarrow r' \Rightarrow \text{Reciprocity}$$

Green's Function

Green's fn for scalar wave equation with charge excitation.

Green's Theorem.

Apply divergence theorem to $\psi\nabla\phi$

$$\iiint_V \nabla\cdot(\psi\nabla\phi)\,dv = \oiint_S (\psi\nabla\phi)\cdot\vec{ds} \quad \vec{ds} \rightarrow \hat{n}\,ds$$

Recall, $\nabla\phi\cdot\hat{n} = \dfrac{\partial\phi}{\partial n}$ & $\nabla\cdot(\psi\nabla\phi) = \psi(\nabla\cdot\nabla\phi) + \nabla\phi\cdot\nabla\psi$

$$\Rightarrow \boxed{\iiint_V \nabla\psi\cdot\nabla\phi\,dv + \iiint_V \psi(\nabla\cdot\nabla\phi)\,dv = \oiint_S \psi\frac{\partial\phi}{\partial n}\,ds}$$

Green's First Identity.

If $\psi = \phi$ & $\nabla^2 \phi = 0$

$$\iiint_V |\nabla \phi|^2 dV = \oiint_S \frac{\partial \phi}{\partial \eta} \phi \, ds$$

→ Interchang ϕ & ψ in GFI & Subtract. ⇒

$$\boxed{\iiint_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi) \, dV = \oiint_S \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right) ds}$$

Green's Second Identity (or) Green's Theorem.

Solution to Scalar Helmholtz.

Recall, $\nabla^2 g(\bar{r}, \bar{r}') + k^2 g(\bar{r}, \bar{r}') = -\delta(\bar{r} - \bar{r}')$ — ②

& $\nabla^2 \phi(\bar{r}) + k^2 \phi(\bar{r}) = -\frac{\rho(\bar{r})}{\epsilon}$ — ③

Apply Green's Theorem:

$$\iiint_V \left(g(\bar{r}, \bar{r}') \nabla^2 \phi(\bar{r}) - \phi(\bar{r}) \nabla^2 g(\bar{r}, \bar{r}') \right) dV = \oiint_S \left[g(\bar{r}, \bar{r}') \frac{\partial \phi(\bar{r})}{\partial n} - \phi \frac{\partial g}{\partial n} \right] ds$$

Substitute ②&③ Laplacians. The LHS reduces to

$$\iiint_V \left\{ g(\bar{r}, \bar{r}') \left[-\frac{\rho(\bar{r})}{\epsilon} - \cancel{k^2 \phi(\bar{r})} \right] + \phi(\bar{r}) \left[+\delta(\bar{r} - \bar{r}') + \cancel{k^2 g(\bar{r}, \bar{r}')} \right] \right\} dV$$

$$= \iiint_V \left(-g(\bar{r}, \bar{r}') \frac{\rho(\bar{r})}{\epsilon} + \phi(\bar{r}) \delta(\bar{r} - \bar{r}') \right) dV = \phi(\bar{r}') - \frac{1}{\epsilon} \iiint_V \rho(\bar{r}) g(\bar{r}, \bar{r}') d$$

$$\therefore \phi(\bar{r}') = \frac{1}{4\pi\epsilon} \iiint \rho(r) \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} dV + \frac{1}{4\pi} \oiint \left[\frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} \frac{\partial \phi(r)}{\partial n} - \phi(r) \frac{\partial}{\partial n} \left(\frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} \right) \right] d$$

From reciprocity. $\bar{r}' \leftrightarrow \bar{r}$

$$\phi(\bar{r}) = \frac{1}{\epsilon}\iiint_V \rho(\bar{r}')\, g(\bar{r},\bar{r}')\,dv' + \underbrace{\oint\!\!\oint_S \left[g(\bar{r},\bar{r}')\frac{\partial \phi(\bar{r}')}{\partial n'} - \phi(\bar{r}')\frac{\partial g(\bar{r}-\bar{r}')}{\partial n'}\right] ds'}$$

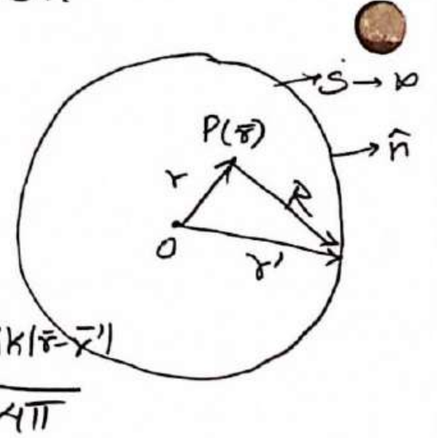
Can be interpreted as equivalent sources on the surface. ∴ even if $\rho(\bar{r}')$ is 0 $\phi(\bar{r})$ can be nonzero due to excitation from outside S.

Radiation Condition

As $r' \to \infty$, the surface contribution excited by $\rho(r')$ must go to 0 since $\rho(r')$ is too far away to affect the surface. This surface source contribution due to $\rho(\bar{r}')$ is the second term.

$$\text{Let } I = \oint\!\!\oint_S \left[g(\bar{r},\bar{r}')\frac{\partial \phi(\bar{r}')}{\partial n'} - \phi(\bar{r}')\frac{\partial g(\bar{r},\bar{r}')}{\partial n'}\right] ds'$$

$$g(\bar{r},\bar{r}') \simeq \frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi r'} \quad \text{since } r' \gg r.$$



Let S be spherical $\Rightarrow \hat{n} = \hat{r}'$

$$\Rightarrow \frac{\partial g(\bar{r},\bar{r}')}{\partial n'} \simeq \frac{\partial}{\partial r'}\left(\frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi r'}\right) = -\left(\frac{-ik}{r'} + \frac{1}{r'^2}\right)\frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi}$$

$$\Rightarrow I \simeq \oint\!\!\oint_S \left[\frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi r'}\frac{\partial \phi(\bar{r}')}{\partial r'} + \phi(\bar{r}')\left(\frac{-ik}{r'} + \underbrace{\cancel{\frac{1}{r'^2}}}_{0}\right)\frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi}\right] ds'$$

$$ds' = r'^2 \sin\theta'\, d\theta'\, d\phi' = r'^2\, d\Omega'$$

$$\Rightarrow I \simeq \oint\!\!\oint_S \left\{ r'\left[\frac{\partial \phi}{\partial r'} - ik\phi\right]\frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi}\right\} d\Omega' \longrightarrow \text{must go to } 0$$

$$\therefore \boxed{\lim_{r' \to \infty} r'\left[\frac{\partial \phi(\bar{r}')}{\partial r'} - ik\,\phi(\bar{r}')\right] = 0} \longrightarrow \text{Radiation boundary condition!}$$

Appendix

Curl: $(\nabla \times \vec{A}) \cdot \hat{n} \triangleq \lim_{\Delta S \to 0} \dfrac{\oint \vec{A} \cdot d\vec{l}}{\Delta S}$ $\hat{n}$ is direction of ΔS.

Div: $(\nabla \cdot \vec{A}) \triangleq \lim_{\Delta V \to 0} \dfrac{\oiint \vec{A} \cdot d\vec{s}}{\Delta V}$

Stokes Theorem: $\iint_S (\nabla \times \vec{A}) \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$

Divergence Theorem: $\iiint_V (\nabla \cdot \vec{A}) \, dV = \oiint_S \vec{A} \cdot d\vec{s}$

(Prove) **Volume Curl (Stokes 2.0):** $\iiint_V (\nabla \times \vec{A}) \, dv = -\oiint_S \vec{A} \times d\vec{s}$

Taylor Series: $f(x+h) = f(x) + f'(x)h + f''(x)\dfrac{h^2}{2!} + \dots$

Maclaurin: $f(x) = f(0) + f'(0)x + f''(0)\dfrac{x^2}{2!} + \dots$

Integral Relations

(Prove) $\iiint_V \nabla \psi \, dv = \oiint_S \psi \, d\vec{s}$

(Prove) $\int_S \hat{n} \times \nabla \psi \, ds = \oint_C \psi \, d\vec{l}$

Vector Identities

$\nabla(\phi\psi) = \phi \nabla \psi + \psi \nabla \phi$

$\nabla \cdot (\psi \vec{A}) = \psi (\nabla \cdot \vec{A}) + \nabla \psi \cdot \vec{A}$

$\nabla \times (\psi \vec{A}) = \psi (\nabla \times \vec{A}) + \nabla \psi \times \vec{A}$

$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A})$

$\vec{A} \times \vec{B} \times \vec{C} = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

$(\vec{A} \cdot \nabla)\vec{A} = \frac{1}{2} \nabla |A|^2 + (\nabla \times \vec{A}) \times \vec{A}$ → Only in Cartesian

$\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A}$

$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} (\nabla \times \vec{B})$

$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$

Proofs

① $$\iiint \nabla \times \bar{F}\, dv = -\oiint_S \bar{F} \times ds$$

Assume, G is a constant vector : Start with $\nabla \cdot (G \times F)$.

$$\nabla \cdot (G \times F) = [F \cdot \cancel{\nabla \times G}^{0} - G \cdot \nabla \times F]$$

$$\iiint_V \nabla \cdot (G \times F)\, dv = -G \cdot \iiint_V \nabla \times F\, dv = \oiint_S (\bar{G} \times \bar{F}) \cdot \bar{ds} = \oiint_S \bar{G} \cdot \bar{F} \times \bar{ds}$$

Since G is arbitrary $\iiint_V \nabla \times \bar{F}\, dv = -\oiint \bar{F} \times \bar{ds}$

② $$\iiint_V \nabla \psi\, dv = \oiint_S \psi\, ds$$

Assume, G is a constant vector ; Start with $\nabla \cdot (\psi G)$

$$\nabla \cdot (\psi G) = \psi (\cancel{\nabla \cdot G})^{0} + G \cdot \nabla \psi$$

$$\iiint_V \nabla \cdot (\psi G)\, dv = \iiint_V G \cdot \nabla \psi\, dv = \oiint_S \psi \bar{G} \cdot \bar{ds}$$

$$\Rightarrow \quad G \cdot \iiint_V \nabla \psi\, dV = G \cdot \oiint_S \psi\, \bar{ds}$$

$$\Rightarrow \quad \iiint_V \nabla \psi\, dv = \oiint_S \psi\, \bar{ds}$$ since G is arbitrary.

③ $$\iint_S \hat{n} \times \nabla \psi\, ds = \oint_C \psi\, \bar{dl}.$$

Start with Stokes Theorem for $\psi \vec{G}$ where G is constant.

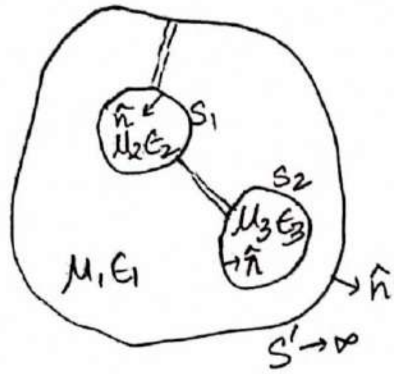
$$\iint_S \nabla \times (\psi \vec{G}) \cdot ds = \oint_C \psi \vec{G} \cdot \bar{dl} = G \cdot \oint_C \psi\, \bar{dl}.$$

$$\nabla \times \psi \vec{G} = \psi (\cancel{\nabla \times \vec{G}}) + \nabla \psi \times \vec{G} = \nabla \psi \times \vec{G} \Rightarrow \nabla \psi \times G \cdot ds = ds \times \nabla \psi \cdot G = \hat{n} \times \nabla \psi \cdot G$$

$$\Rightarrow \cancel{G} \cdot \iint \hat{n} \overset{0}{\times} \nabla \psi\, ds = \cancel{G} \cdot \oint_C \psi\, \bar{dl}$$ ⇒ Proved!

Lec 12

Solution of Helmholtz for a Complex Medium.



We know

$$\phi(r) = \frac{1}{\epsilon_1} \iiint_V \rho(\vec{r}')g(\vec{r},\vec{r}')\,dv'$$

$$+ \oint\!\!\oint_{S'+\Sigma S_i} \left[g(\vec{r},\vec{r}')\frac{\partial \phi}{\partial n'} - \phi \frac{\partial g(\bar{r},\bar{r}')}{\partial n'}\right] ds'$$

from rad con we know $\oint\!\!\oint_{S'} = 0$.

$$\Rightarrow \phi(r) = \underset{\text{incident potential}}{\phi_i(r)} + \underset{\text{scattered potential}}{\phi_s(r)}$$

$$\phi_i(\bar{r}) = \frac{1}{\epsilon_1} \iiint_V \rho(\vec{r}')\,g(\bar{r},\bar{r}')\,dv'$$

$$\phi_s(\bar{r}) = \oint\!\!\oint_{\Sigma S_i} \left[g(\bar{r},\bar{r}')\frac{\partial \phi(\bar{r})}{\partial n'} - \phi(\vec{r}')\frac{\partial g(\bar{r},\bar{r}')}{\partial n'}\right] ds'$$

Vector Helmholtz Equation soln. takes a similar form.

$$\vec{A}(\bar{r}) = \mu \iiint_V J(\bar{r})g(\bar{r},\bar{r}')\,dv' + \mu \oint\!\!\oint_{\Sigma S_i} \left(g(\bar{r},\bar{r}')\hat{n}'.\nabla \vec{A}(\bar{r}') - \bar{A}(\bar{r}')\hat{n}'.\nabla g(\bar{r},\bar{r}')\right) ds'$$

Also, $\hat{n}'.\nabla\bar{A} = \hat{n}'.\nabla A_x \hat{x} + \hat{n}'.\nabla A_y \hat{y} + \hat{n}'.\nabla A_z \hat{z}$.

Hertz potential.

$$\nabla^2 \bar{\pi} + k^2 \bar{\pi} = \frac{-i J}{\omega \epsilon} \Rightarrow \bar{\pi}(\bar{r}) = \frac{i}{4\pi\omega\epsilon} \iiint_V \bar{J}(\bar{r}')\frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}\,dv'$$

$$B = -\frac{ik^2}{\omega}\nabla \times \bar{\pi} \;;\; E = \nabla\nabla.\bar{\pi} + k^2\bar{\pi}$$

- For a bounded medium, Greens functions can be obtained using other boundary conditions.

Reciprocity Theorem. (Lorentz reciprocity theorem)

Proof: (1) Start from time harmonic Maxwell's equations for 2 sets of sources J_a, J_{ma} and J_b, J_{mb}. Use identity $\nabla.(E\times H)$ to obtain: $\nabla.(\bar{E}_a\times\bar{H}_b - \bar{E}_b\times\bar{H}_a) = -(\bar{E}_a.\bar{J}_b - \bar{H}_a.\bar{J}_{mb}) + (\bar{E}_b.\bar{J}_a - \bar{H}_b.\bar{J}_{ma})$

(2) Volume integral & Div Thm.

Integral form of reciprocity

$$\oiint_S (\bar{E}_a\times\bar{H}_b - \bar{E}_b\times\bar{H}_a)\cdot d\bar{s} = -\iiint_V (\bar{E}_a.\bar{J}_b - \bar{H}_a.\bar{J}_{mb})dV + \iiint_V (\bar{E}_b.\bar{J}_a - \bar{H}_b.\bar{J}_{ma})dV$$

(3) As $s\to\infty$ show that LHS $\to 0$

$\Rightarrow \boxed{\langle a,b\rangle = \langle b,a\rangle} \rightarrow$ Unbounded region statement.

where $\langle a,b\rangle = \iiint_V (\bar{E}_a.\bar{J}_b - \bar{H}_a.\bar{J}_{mb})dV$

Given sources $\bar{J}_a = I_0\Delta\bar{l}_a\,\delta(\bar{r}-\bar{r}_1)$, $\bar{J}_b = I_0\Delta\bar{l}_b\,\delta(\bar{r}-\bar{r}_2)$ $\Rightarrow \bar{E}_a(\bar{r}_2).\Delta\bar{l}_b = \bar{E}_b(\bar{r}_1).\Delta\bar{l}_a$

$\Rightarrow$ If the positions of the source & observation are interchanged, the fields remain the same.

$\rightarrow$ In a source free bounded region RHS is 0 instead & we have

$$\oiint (\bar{E}_a\times\bar{H}_b - \bar{E}_b\times\bar{H}_a)\cdot d\bar{s} = 0$$ $\rightarrow$ Bounded source free region statement.

> The statement $\langle a,b\rangle = \langle b,a\rangle$ can be applied to more complex media that can be broken down into regions of homogeneous media.

> In general,

$$\oiint_{S_\infty+S} (\bar{E}_a \times \bar{H}_b - \bar{E}_b \times \bar{H}_a)\cdot\hat{n}_2\, ds = \iiint_V (\bar{H}_a\cdot\bar{J}_{mb} - \bar{E}_a\cdot\bar{J}_b)\,dV - \iiint_V (\bar{H}_b\cdot\bar{J}_{ma} - \bar{E}_b\cdot\bar{J}_a)\,dV$$

Plane Wave Propagation in Homogeneous Media

> In a homogeneous source free media we know:

$$\nabla^2\bar{E} + k^2\bar{E} = 0 \quad \text{and} \quad \bar{H} = \frac{-i}{k\eta}\nabla\times E \quad \text{where } k\eta = \omega\mu.$$

> Simplest possible solution is of the form $\vec{E} = \vec{E_0}\,\psi(\bar{r})$ where $\bar{E}_0$ is a constant vector "modulated" by $\psi(r)$ a scalar which is a fn of $\vec{r}$.

$$\Rightarrow \quad \bar{E}_0\nabla^2\psi(\bar{r}) + \bar{E}_0\, k^2\psi(\bar{r}) = 0$$

$$\Rightarrow \text{ Non vanishing solution} \Rightarrow \boxed{\nabla^2\psi(\bar{r}) + k^2\psi(\bar{r}) = 0}$$

> Using separation of variables $\Rightarrow \psi(\bar{r}) = X(x)Y(y)Z(z)$

$$\left(\frac{1}{X}\frac{\partial^2}{\partial x^2}X + \frac{1}{Y}\frac{\partial^2}{\partial y^2}Y + \frac{1}{Z}\frac{\partial^2}{\partial z^2}Z\right) + k^2 = 0$$

$$\Rightarrow \quad \frac{1}{X}\frac{\partial^2}{\partial x^2}X = -k_x^2, \quad \frac{1}{Y}\frac{\partial^2}{\partial y^2}Y = -k_y^2, \quad \frac{1}{Z}\frac{\partial^2}{\partial z^2}Z = -k_z^2$$

$\hookrightarrow X(x) = e^{ik_x x}$

$\quad Y(y) = e^{ik_y y}$

$\quad Z(z) = e^{ik_z z}$

$$\Rightarrow \quad k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2\mu\epsilon$$

$$K = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \ \ \& \ \ |K| = k$$

This $\bar{E}_0$ cannot be arbitrary. Must satisfy $\bar{E}_0 \cdot \hat{k} = 0$

$$\Rightarrow \boxed{\bar{E}(\bar{r}) = \bar{E}_0 \, e^{i\bar{k}\cdot\bar{r}}} \quad (18)$$

→ A constant vector "modulated" by a wave travelling along $\bar{k}$.

→ $\bar{k}$ is a spatial frequency

$$\bar{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z} \quad \& \quad |\bar{k}| = k = \omega\sqrt{\mu\epsilon}$$

$$\hat{k} = \frac{\bar{k}}{|\bar{k}|} = \frac{k_x}{k}\hat{x} + \frac{k_y}{k}\hat{y} + \frac{k_z}{k}\hat{z}$$

→ $\hat{k}\cdot\bar{r}$ = constant, are planes of constant phase and are defined by planes perpendicular to $\hat{k}$.

→ In this plane defined by $\hat{k}\cdot\bar{r}$ = constant, the magnitude and phase are constant.

→ Eq (18) may represent nonpropagating waves even for real ϵ & μ.

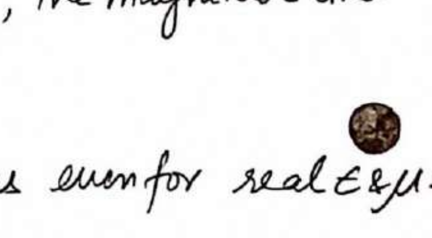
→ If $k_x^2 + k_y^2 > k^2 \Rightarrow k_z = i\sqrt{k_x^2 + k_y^2 - k^2} \Rightarrow \bar{E} = \bar{E}_0 e^{i(k_x x + k_y y)} e^{-z\sqrt{k_x^2 + k_y^2 - k^2}}$

⇒ exponential attenuation along $\hat{z}$ ⇒ evanescent along $+\hat{z}$.

→ $\nabla \cdot \bar{E} = \bar{E}_0 \cdot \nabla e^{i\bar{k}\bar{r}} = i\bar{E}_0 \cdot k\, e^{i\bar{k}\cdot\bar{r}} = 0 \Rightarrow \boxed{\bar{E}_0 \cdot \hat{k} = 0}$ must be satisfied. $\Rightarrow \bar{E}_0 \perp \hat{k}$

$$\boxed{\bar{H} = \frac{\hat{k} \times \bar{E}_0}{\eta} e^{i\bar{k}\cdot\bar{r}}} \Rightarrow \bar{E} \perp \bar{H} \perp \hat{k}$$

$$\frac{|E|}{|H|} = \eta = \sqrt{\frac{\mu}{\epsilon}}$$ ⇒ plane waves are TEM.



Wave solns. in this region are evanescent waves, they do not propagate in $\hat{z}$ since

Plane waves in Lossy Media.

> K is complex , $K = \omega\sqrt{\mu\epsilon} = k' + i\alpha$.
> since μ, ϵ maybe complex

μ, ϵ are complex $\Rightarrow$ lossy.

> If $\hat{k}$ is real $\Rightarrow$ uniform plane waves. Phase is $\xi k'$.

> $|E| = |\bar{E}_0| e^{-\alpha\xi}$ where $\xi = \hat{k}\cdot\bar{r}$ = constant

> Both magnitude & phase are constant in the direction of propagation if $\hat{k}$ is real.

Non uniform plane waves $\Rightarrow$ Mag & phase are constant in diff. directions.

- In general magnitude & phase are constant in different directions.

Ex: $k_z = \sqrt{k^2 - k_x^2 - k_y^2}$. Let $k_z = k_z' + i k_z''$

$\Rightarrow \bar{E} = \bar{E}_0 e^{i\bar{k}'\cdot\bar{r}} e^{-k_z'' z}$ where $\bar{k}' = k_x\hat{x} + k_y\hat{y} + k_z'\hat{z}$.

Define $\hat{k}' = \dfrac{k_x\hat{x} + k_y\hat{y} + k_z'\hat{z}}{\sqrt{k_x^2 + k_y^2 + k_z'^2}}$ $\Rightarrow$ phase is constant on $\hat{k}'\cdot\bar{r} = \xi$

but $|E| = |E_0| e^{+k_z' z}$ is constant along z direction.

→ Let $\phi = \bar{k}\cdot\bar{r}$ be complex phase. By defn. of $\nabla\phi$ we know it is normal to surfaces of constant ϕ. If $\bar{k}$ is real $\boxed{\nabla\phi = \bar{k}}$ which indicates equiphase planes are $\perp^{lar}$ to $\bar{k}$.

→ If $\bar{k} = \bar{k}' + i\bar{k}''$ $\Rightarrow$ $\nabla\phi = \bar{k}' + i\bar{k}''$ $\Rightarrow$ equiphase are $\perp^{lar}$ to k' and equiamplitude are $\perp^{lar}$ to k''.

- If the medium is lossless

$k^2 = \bar{k} \cdot \bar{k} = |\bar{k}'|^2 - |\bar{k}''|^2 + 2i\,\bar{k}' \cdot \bar{k}''$ must have a vanishing imaginary part $\Rightarrow \bar{k}'' = 0$ or $\bar{k}' \cdot \bar{k}'' = 0$. If $\bar{k}'' = 0$ we get a uniform plane wave. $\bar{k}' \cdot \bar{k}'' = 0$ gives a possible plane wave solution in a lossless medium that is nonuniform. This nonuniform wave in lossless medium has equiphase & equiamplitude $\perp^{lar}$ to each other and is called an <u>evanescent wave</u>. Lossy media also support evanescent waves but $\bar{k}'$ & $\bar{k}''$ need not be orthogonal.

Expansion of Fields as a Continuous Spectrum of plane waves.

- Any field can be written as a superposition of ∞ plane waves. In a source free region.

$$\bar{E}(\bar{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{E}_0(k_x, k_y)\, e^{i\bar{k}\cdot\bar{r}}\, dk_x\, dk_y$$, choice of x, y is arbitrary

$$\bar{H}(\bar{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\bar{k} \times \bar{E}_0(k_x, k_y)}{k\eta}\, e^{i\bar{k}\cdot\bar{r}}\, dk_x\, dk_y$$

$E_0, \bar{H}_0$ must be $\perp^{lar}$ but $\bar{E}, \bar{H}$ need not be. E_0 must satisfy $\bar{E}_0 \cdot \hat{k} = \xi$ and we can take an inverse to get $\bar{E}_0$ from $\bar{E}$ since the equation is a 2D Fourier Transform.

Power and Energy Densities.

$\rightarrow \quad \bar{E} = \bar{E}_0 e^{i\bar{k}\cdot\bar{r}} \qquad \bar{H} = \dfrac{\hat{k}\times\bar{E}_0}{\eta} e^{i\bar{k}\cdot\bar{r}}$

$$\langle W_e \rangle = \frac{1}{4}\epsilon' |\bar{E}|^2 = \frac{1}{4}\epsilon' |E_0|^2 e^{-2\bar{k}''\cdot\bar{r}}$$

$$\langle W_m \rangle = \frac{1}{4}\mu' |\bar{H}|^2 = \frac{1}{4}\mu' \frac{|\hat{k}|^2 |E_0|^2}{|\eta|^2} e^{-2\bar{k}''\cdot\bar{r}}$$

Lossless medium $\Rightarrow W_m = W_e = \frac{1}{4}\epsilon |E_0|^2$, since $\mu/\eta^2 = \epsilon$.

$\Rightarrow \langle W_e \rangle = \langle W_m \rangle$ for lossless medium & it is constant w.r.t position.

$\rightarrow$ Lossy $\Rightarrow$ fn of position & $\langle W_e \rangle \neq \langle W_m \rangle$

$$\vec{S}(\bar{r}) = \frac{1}{2} \bar{E}\times\bar{H}^* = \frac{1}{2} \frac{|E_0|^2 \hat{k}^* - (\bar{E}_0\cdot\hat{k}^*)E_0^*}{\eta^*} e^{-2\bar{k}''\cdot\bar{r}}$$

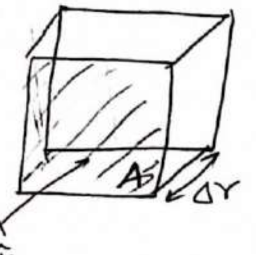
Lossless plane wave (uniform)

$$\Rightarrow \quad \vec{S}(\bar{r}) = \frac{|E_0|^2}{2\eta}\hat{k}$$

Energy delivered to a volume Δv along $\vec{S}$ over a time Δt is

$$\Delta W = (\vec{S}\cdot \hat{k} \overbrace{A}^{\text{area.}})\Delta t$$

$$\Delta r = \underbrace{v_e}_{\text{velocity of energy.}} \Delta t$$



Total Stored energy $\quad (W_e + W_m)\Delta v = \Delta W$

$$\Rightarrow \vec{S}\cdot\hat{k}\left(\cancel{A}\frac{\cancel{\Delta r}}{v_e}\right) = (W_e + W_m)\cancel{\Delta v}$$

$$\Rightarrow \boxed{v_e = \frac{S}{W_e + W_m}}$$

For a uniform plane wave. Substitute S, We,

$$\Rightarrow \quad v_e = \frac{1}{\eta\epsilon} = \frac{1}{\sqrt{\mu\epsilon}} \Rightarrow \text{Speed of light!}$$

Plane Waves in Anisotropic Media

$\bar{D} = \epsilon_0 \bar{\bar{\epsilon}}_r \bar{E}$ $\qquad$ $\bar{B} = \mu_0 \bar{\bar{\mu}}_r \bar{H}$.

We know, $\nabla \times \nabla \times \bar{E} - k^2 \bar{\bar{\epsilon}}_r \bar{E} = 0$.

$$\nabla \times \nabla \times = \begin{pmatrix} -\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}\right) & \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial x \partial z} \\ \frac{\partial^2}{\partial y \partial x} & -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) & \frac{\partial^2}{\partial y \partial z} \\ \frac{\partial^2}{\partial z \partial x} & \frac{\partial^2}{\partial z \partial y} & -\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right) \end{pmatrix}$$

We want solutions of the form $\bar{E} = \bar{E}_0 e^{i \bar{k} \cdot \bar{r}}$. Therefore, we must satisfy the following condition for $\bar{k}$.

$$\Delta = \begin{vmatrix} k_y^2 + k_z^2 - k^2 \epsilon_{xx} & -(k_y k_x + k^2 \epsilon_{xy}) & -(k_x k_z + k^2 \epsilon_{xz}) \\ -(k_y k_x + k^2 \epsilon_{yx}) & k_x^2 + k_z^2 - k^2 \epsilon_{yy} & -(k_y k_z + k^2 \epsilon_{yz}) \\ -(k_x k_z + k^2 \epsilon_{zx}) & -(k_y k_z + k^2 \epsilon_{zy}) & k_x^2 + k_y^2 - k^2 \epsilon_{zz} \end{vmatrix} = 0.$$

For a homogeneous isotropic medium this gives $k_x^2 + k_y^2 + k_z^2 = k^2$

Uniaxial Medium.

$$\bar{\bar{\epsilon}}_r = \begin{pmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{xx} & 0 \\ 0 & 0 & \epsilon_{zz} \end{pmatrix} \Rightarrow \Delta = -k^2(\epsilon_{xx} k^2 - k_x^2 - k_y^2 - k_z^2)\left[\epsilon_{xx}(\epsilon_{zz} k^2 - k_x^2 - k_y^2) - \epsilon_{zz} k_z^2\right] = 0$$

$\Rightarrow$ Sol 1:- $k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \epsilon_0 \epsilon_{xx} \rightarrow$ Similar to TEM since $D \parallel E$

This is a sphere in k_x, k_y, k_z space.

AKA "ordinary wave"

Sol2: $K_x^2 + K_y^2 + \frac{\epsilon_{zz}}{\epsilon_{xx}} k_z^2 = \omega^2 \mu \epsilon_0 \epsilon_{xx}$.

↳ spheroid in $k_x k_y k_z$

Called "Extraordinary wave".

→ Used to make wave plates & birefringent materials.

TE and TM field solutions of Helmholtz Equation.

> $\nabla^2 \bar{\pi} + k^2 \bar{\pi} = 0$ and $\nabla^2 \bar{\pi}_m + k^2 \bar{\pi}_m = 0$ → For source free region. $\bar{E}$ & H can be found from $\bar{\pi}$ & $\bar{\pi}_m$

> Assume xy plane is transverse and $\bar{\pi} = \pi_z \hat{z}$ generates TM mode

$\bar{\pi}_m$ generates TE mode

$$\bar{E} = \nabla\nabla\cdot\bar{\pi} + k^2\bar{\pi} + i\omega\mu\nabla\times\cancel{\bar{\pi}_m}^{\,0}$$

$$= \nabla\left(\frac{\partial \pi_z}{\partial z}\right) + k^2 \pi_z \hat{z}$$

$$\boxed{\bar{E} = \frac{\partial}{\partial z}\nabla_t \pi_z + \left(\frac{\partial^2 \pi_z}{\partial z^2} + k^2 \pi_z\right)\hat{z}}$$

$$\nabla_t = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}$$

$$\bar{H} = -i\omega\epsilon\,\nabla\times(\pi_z \hat{z})$$

$$\boxed{\bar{H} = -i\omega\epsilon\,\nabla_t \pi_z \times \hat{z}}$$

We are looking for solutions of the form $\pi_z(\bar{r}) = \Psi(x,y)e^{\pm i\beta z}$

↳ Waves propagating in z direction.

β is propagation constant along z.

$\Rightarrow$ We know $\nabla^2 \pi_z + k^2 \pi_z = 0$

$$\Rightarrow \nabla_t^2 \pi_z + \underbrace{\frac{\partial^2}{\partial z^2}}_{\mapsto -\beta^2} \pi_z + k^2 \pi_z = 0$$

$$\Rightarrow \boxed{\nabla_t^2 \Psi + k_c^2 \Psi = 0} \quad \text{where} \quad \boxed{k_c^2 = k^2 - \beta^2}$$

> This 2D wave equation can be expressed in a wide range of elementary functions and is the governing equation for TM waves. E & H fields are given by.

$$\boxed{\begin{aligned} \overline{E} &= \left[\pm i\beta \nabla_t \Psi + k_c^2 \Psi \hat{z}\right] e^{\pm i\beta z} \\ \overline{H} &= -i\omega\epsilon \left(\nabla_t \Psi \times \hat{z}\right) e^{\pm i\beta z} \end{aligned}}$$

$\longrightarrow$ notice TM!

> TM wave impedance = ratio of $E_{transverse}$ & H.

$$Z_{TM} = \frac{\hat{z} \times \overline{E}_t}{\overline{H}_t} = \frac{\beta}{\omega\epsilon}.$$

If wave is going towards $-\hat{z}$

$$Z_{TM} = \frac{-\hat{z} \times \overline{E}_t}{\overline{H}_t}$$

Similary for TE

$$\Pi_{mz}(r) = \Psi_m(x,y)\, e^{i\beta z}$$

$$\nabla_t^2 \Psi_m + k_c^2 \Psi_m = 0$$

$$\overline{H} = \left[i\beta \nabla_t \Psi_m + k_c^2 \Psi_m \hat{z}\right] e^{i\beta z}$$

$$\overline{E} = i\omega\mu \left(\nabla_t \Psi_m \times \hat{z}\right) e^{i\beta z}$$

$$Y_{TE} = -\frac{\hat{z} \times \overline{H}_t}{\overline{E}_t} = \frac{\beta}{\omega\mu}.$$

Hertz Potential

$\bar{\Pi}$ produces TM solution

$\bar{\Pi}_m$ produces TE solution

$$\nabla^2\bar{\Pi} - \mu\epsilon \frac{\partial^2 \bar{\Pi}}{\partial t^2} = \frac{-1}{\epsilon} \cancel{\int \bar{J}\, dt} \xrightarrow{} 0 \text{ in source free}$$

$$\bar{A} = \mu\epsilon \frac{\partial \bar{\Pi}}{\partial t}$$

$$\phi = -\nabla\cdot\bar{\Pi}$$

In general

$$\bar{E} = \nabla\nabla\cdot\bar{\Pi} + k^2\bar{\Pi} + i\omega\mu\nabla\times\bar{\Pi}_m$$

$$\bar{H} = -i\omega\epsilon\nabla\times\bar{\Pi} + \nabla\nabla\cdot\bar{\Pi}_m + k^2\bar{\Pi}_m$$

TM

$$\bar{\Pi} = \Pi_z\hat{z}$$

$$\Rightarrow \bar{E}_{TM} = \frac{\partial}{\partial z}\nabla_t\Pi_z + \left(\frac{\partial^2\Pi_z}{\partial z^2} + k^2\Pi_z\right)\hat{z}$$

$$\bar{H}_{TM} = -i\omega\epsilon\nabla_t\Pi_z\times\hat{z}$$

TE

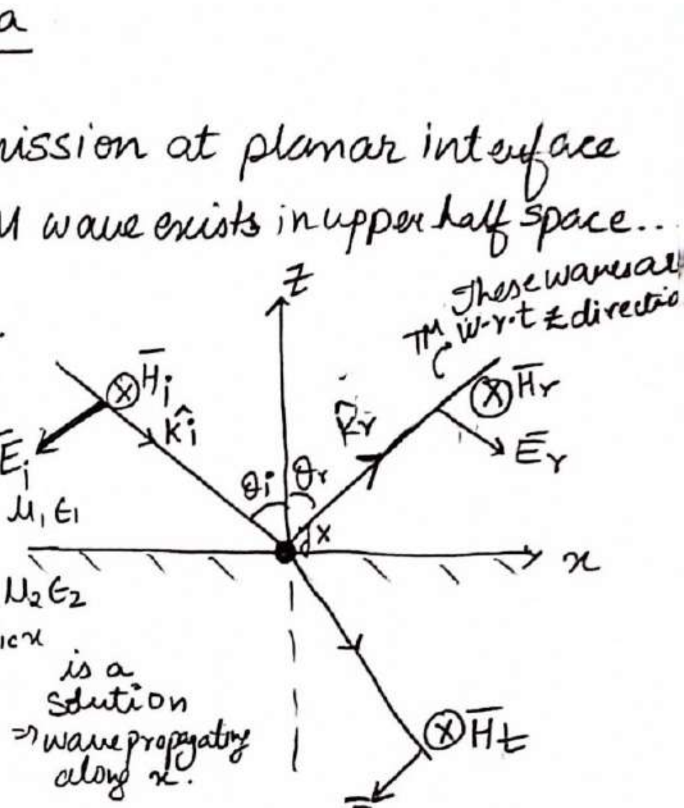
$$E_{TE} = i\omega\mu\nabla_t\Pi_{mz}\times\hat{z}$$

$$\bar{H}_{TE} = \frac{\partial}{\partial z}\nabla_t\Pi_{mz} + \left(\frac{\partial^2\Pi_{mz}}{\partial z^2} + k^2\Pi_{mz}\right)\hat{z}$$

In general $\Pi_z = \psi(x,y)\, e^{\pm i\beta z}$.

TE and TM Waves in 1-D Media

- Plane wave reflection and transmission at planar interface between 2 dielectrics. Assume TM wave exists in upper half space..



TM wave assuming no variation along y. ($\Rightarrow \frac{\partial}{\partial y} = 0$) must satisfy the following wave equations.

$\therefore$ in ①: $\left(\frac{\partial^2}{\partial x^2} + k_{1c}^2\right)\Psi_1 = 0$

$\hookrightarrow \psi = Ae^{ik_{1c}x}$ is a solution $\Rightarrow$ wave propagating along x.

in ②: $\left(\frac{\partial^2}{\partial x^2} + k_{2c}^2\right)\Psi_2 = 0$

where $k_{1c}^2 = k_1^2 - \beta_1^2$ $\quad k_{2c}^2 = k_2^2 - \beta_2^2$. The solns. are of the form

- $e^{\pm i k_c x}$. Assuming propagation in $+x$, $+z$ & $-z$ we get,

$$\Pi_1(z) = \left[A_1 e^{-i\beta_1 z} + B_1 e^{+i\beta_1 z}\right] e^{ik_{1c}x} \qquad \text{since, } \Pi = \psi e^{\pm i\beta z}$$

$$\Pi_2(z) = \left[A_2 e^{-i\beta_2 z} + B_2 e^{+i\beta_2 z}\right] e^{ik_{2c}x}$$

Fields in medium ① & ② can be obtained from Π

$$\left.\begin{aligned}
E_{1x} &= A_1 \beta_1 k_{1c}\left[e^{-i\beta_1 z} - \frac{B_1}{A_1} e^{i\beta_1 z}\right] e^{ik_{1c}x}. \\
E_{1z} &= k_{1c}^2 A_1 \left[e^{-i\beta_1 z} + \frac{B_1}{A_1} e^{i\beta_1 z}\right] e^{ik_{1c}x} \\
H_{1y} &= -\omega\epsilon_1 k_{1c} A_1 \left[e^{-i\beta_1 z} + \frac{B_1}{A_1} e^{i\beta_1 z}\right] e^{ik_{1c}x}
\end{aligned}\right\} \begin{matrix}(21)\\ \& \\ (22)\end{matrix}$$

- $1 \to 2 \Rightarrow$ fields in medium 2.

Apply continuity of E_{tan} & H_{tan} at $z = 0$.

$$\Rightarrow A_1\beta_1 k_{1c}\left(1-\frac{B_1}{A_1}\right)e^{ik_{1c}x} = A_2\beta_2 k_{2c}\left(1-\frac{B_2}{A_2}\right)e^{ik_{2c}x}$$

$$\epsilon_1 k_{1c}A_1\left(1+\frac{B_1}{A_1}\right)e^{ik_{1c}x} = \epsilon_2 k_{2c}A_2\left(1+\frac{B_2}{A_2}\right)e^{ik_{2c}x}.$$

Since the equality must be valid independant of x we must have no dependance on x. Therefore,

$$\boxed{k_{1c} = k_{2c}}$$

"Phase matching condition" The variation of phase must be continuous along the boundary.

$$\Rightarrow A_1\beta_1\left(1-\frac{B_1}{A_1}\right) = A_2\beta_2\left(1-\frac{B_2}{A_2}\right) \;\&\; \epsilon_1 A_1\left(1+\frac{B_1}{A_1}\right) = \epsilon_2 A_2\left(1+\frac{B_2}{A_2}\right). \quad (26)$$

Assume incident waves:

$$H_i(x,z) = \hat{y}H_0 e^{ik_1\hat{k}\cdot\bar{r}} = \hat{y}H_0 e^{ik_1(-\cos\theta_i z + \sin\theta_i x)}$$

where $H_0 = -\omega\epsilon_1 k_{1c}A_1$

$$E_i(x,z) = -\eta_1 H_0(\sin\theta_i\hat{z} + \cos\theta_i\hat{x})e^{ik_1(-\cos\theta_i z + \sin\theta_i x)}$$

This excites a mode given by $\beta_1 = k_1\cos\theta_i$, $k_{1c} = k_1\sin\theta_i$ by comparing with Eqs (21) & (22).

$$\Rightarrow \beta_2 = \sqrt{k_2^2 - k_{1c}^2} \quad \text{since } k_{1c} = k_{2c}$$

$$= \sqrt{k_2^2 - k_1^2\sin^2\theta_i}$$

Remember that β comes from $\Pi_z(\bar{r}) = \Psi(x,y)e^{\pm i\beta z}$ and $k = \frac{\omega}{c} = \omega\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda}$. They are equal only for TEM solutions for $\bar{\Pi}$.

medium (2) is semiinfinite no upward going wave is excited $\Rightarrow B_2 = 0$

From (26) $\Rightarrow$
$$\frac{\beta_1}{\epsilon_1}\,\frac{1-B_1/A_1}{1+B_1/A_1} = \frac{\beta_2}{\epsilon_2}$$

$$\boxed{\Gamma_{TM} = \frac{B_1}{A_1} = \frac{\epsilon_2/\beta_2 - \epsilon_1/\beta_1}{\epsilon_2/\beta_2 + \epsilon_1/\beta_1}}$$

Ratio of reflected wave to incident wave aka the magnetic reflection coefficient for TM waves only.

$$H_{1y} = H_0\left(e^{-i\beta_1 z} + \Gamma_{TM}\, e^{i\beta_1 z}\right) e^{ik_1 \sin\theta_i x}.$$

$$E_{1x} = -\frac{\beta_1}{\omega\epsilon_1} H_0\left[e^{-i\beta_1 z} - \Gamma_{TM}\, e^{i\beta_1 z}\right] e^{ik_1 \sin\theta_i x}$$

$$E_{1z} = -\frac{k_1 \sin\theta_i}{\omega\epsilon_1} H_0\left[e^{-i\beta_1 z} + \Gamma_{TM}\, e^{i\beta_1 z}\right] e^{ik_1 \sin\theta_i x}.$$

$$\frac{\beta_1}{\omega\epsilon_1} = \eta_1 \cos\theta_i \Rightarrow \Gamma_{TM} = -\frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

where $\cos\theta_t = \frac{\beta_2}{k_2} = \sqrt{1 - \frac{k_1^2}{k_2^2}\sin^2\theta_i}$

In medium②, $B_2 = 0$ since semiinfinite.

$$\Rightarrow A_2 = \frac{\epsilon_1 A_1}{\epsilon_2}(1 + \Gamma_{TM})$$

$$\Rightarrow H_{2y} = H_0 (1 + \Gamma_{TM}) e^{-i\beta_2 z} e^{ik_1 \sin\theta_i x}$$

{ E_{2x} & E_{2z} can be found similarly

$$\frac{H_{2y}(z=0)}{H_{1y}(z=0)} = \tau_{TM} = 1 + \Gamma_{TM} = \text{magnetic transmission coefficient.}$$

If we had a TE wave in medium ① we can use duality

$$E_{1y} = -E_0\left[e^{-i\beta_1 z} + \Gamma_{TE}\, e^{i\beta_1 z}\right] e^{ik_1 \sin\theta_i x}$$

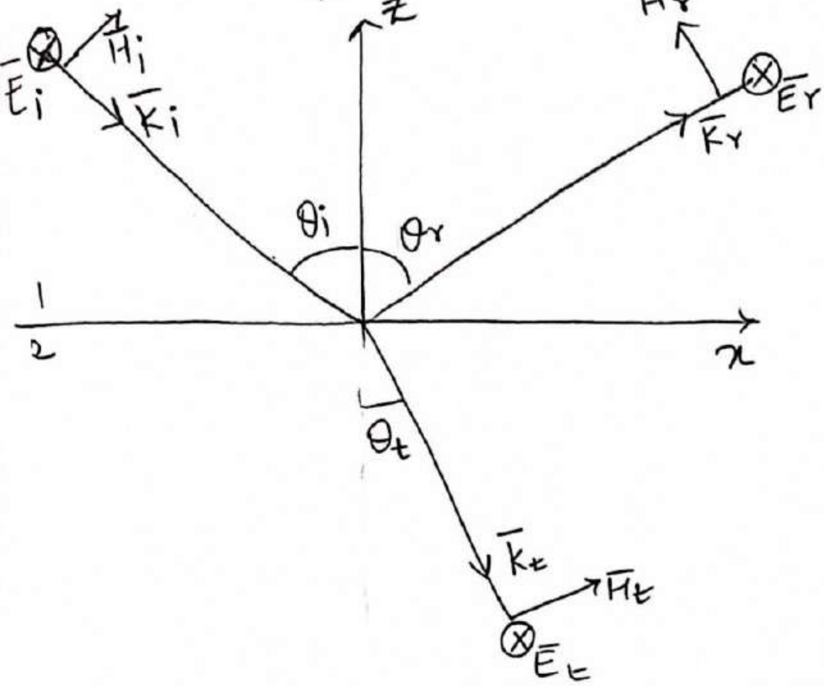
$$H_{1x} = -\frac{\cos\theta_i}{\eta_1} E_0\left[e^{-i\beta_1 z} - \Gamma_{TE}\, e^{i\beta_1 z}\right] e^{ik_1 \sin\theta_i x}$$

$$H_{1z} = -\frac{\sin\theta_i}{\eta_1} E_0\left[e^{-i\beta_1 z} + \Gamma_{TE}\, e^{i\beta_1 z}\right] e^{ik_1 \sin\theta_i x}$$

$\boxed{H_{2x} + H_{2z} \text{ can be found similarly}}$

$$E_{2y} = -E_0 (1 + \Gamma_{TE}) e^{-i\beta_2 z} \quad \& \quad \boxed{\Gamma_{TE} = \frac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_i}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i}}$$

TE perpendicular



$$\bar{E}_i = E_0 e^{ik_1(x\sin\theta_i - z\cos\theta_i)}\hat{y}$$

$$\bar{H}_i = \frac{E_0}{\eta_1} e^{ik_1(x\sin\theta_i - z\cos\theta_i)}(\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)$$

$$\bar{E}_r = \Gamma_{TE} E_0 e^{ik_1(\sin\theta_i x + \cos\theta_i z)}\hat{y}$$

$$\bar{H}_r = \Gamma_{TE}\frac{E_0}{\eta_1} e^{ik_1(\sin\theta_i x + \cos\theta_i z)}(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)$$

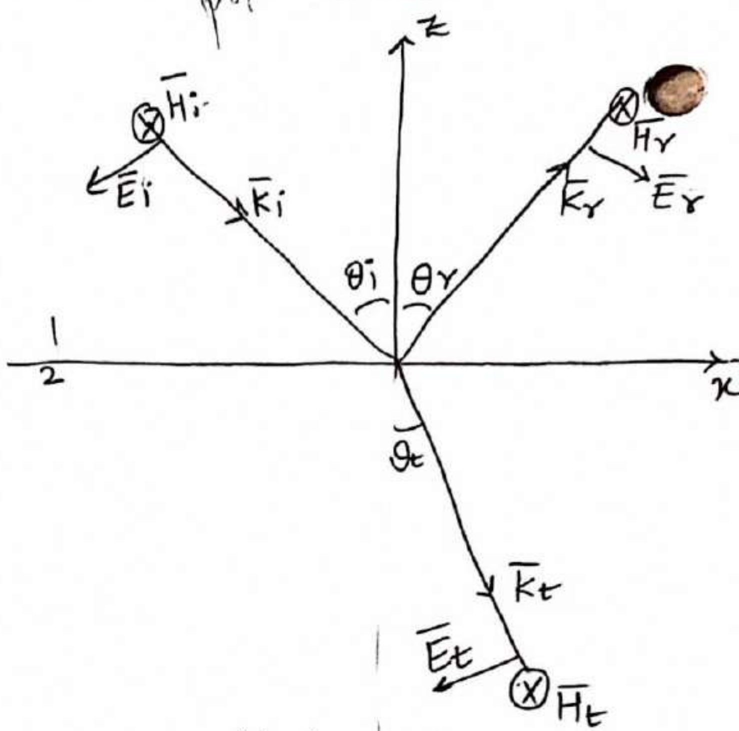
$$\bar{E}_t = \tau_{TE} E_0 e^{ik_2(x\sin\theta_t - z\cos\theta_t)}\hat{y}$$

$$\bar{H}_t = \frac{1}{\eta_2}\tau_{TE} E_0 e^{ik_2(x\sin\theta_t - z\cos\theta_t)}(\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)$$

$$\Gamma_{TE} = \frac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_i}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i}\ ;\ \tau_{TE} = 1 + \Gamma_{TE}$$

> Notice $\bar{E}, \bar{H}$ are TEM waves of the form $\bar{E}_0 e^{i\bar{k}\cdot\bar{r}}$

TM parallel



$$\bar{E}_i = E_0 e^{ik_1(x\sin\theta_i - z\cos\theta_i)}(-\hat{x}\cos\theta_i - \hat{z}\sin\theta_i)$$

$$\bar{H}_i = \frac{E_0}{\eta_1} e^{ik_1(x\sin\theta_i - z\cos\theta_i)}\hat{y}$$

$$\bar{E}_r = \Gamma_{TM} E_0 e^{ik_1(x\sin\theta_i + z\cos\theta_i)}(\hat{x}\cos\theta_i - \hat{z}\sin\theta_i)$$

$$\bar{H}_r = \frac{E_0}{\eta_1}\Gamma_{TM} e^{ik_1(x\sin\theta_i + z\cos\theta_i)}\hat{y}$$

$$\bar{E}_t = \tau_{TM} E_0 e^{ik_2(x\sin\theta_t - z\cos\theta_t)}(-\hat{x}\cos\theta_t - \hat{z}\sin\theta_t)$$

$$\bar{H}_t = \tau_{TM}\frac{E_0}{\eta_2} e^{ik_2(x\sin\theta_t - z\cos\theta_t)}\hat{y}$$

$$\Gamma_{TM} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{\beta_1\epsilon_2 - \beta_2\epsilon_1}{\beta_1\epsilon_2 + \beta_2\epsilon_1}$$

→ TE & TM is w.r.t $\hat{z}$ direction.

→ $\frac{\beta_1}{\omega\epsilon_1} = \eta_1\cos\theta_i$ & $\frac{\beta_2}{\omega\epsilon_2} = \eta_2\cos\theta_t$

Note: $Z_{TM} = \frac{\hat{z} \times \bar{E}_t}{\bar{H}_t} = \frac{\beta}{\omega \epsilon}$ → True for any mode.

current reflection has -ve sign too!

$Z_{TE} = \frac{-\bar{E}_t}{\hat{z} \times \bar{H}_t} = \frac{\omega \mu}{\beta}$ → True for any mode.

$\Gamma_{TM} = -\frac{Z_{TM2} - Z_{TM1}}{Z_{TM2} + Z_{TM1}}$ $\qquad \Gamma_{TE} = \frac{Z_{TE2} - Z_{TE1}}{Z_{TE2} + Z_{TE1}}$ → True for any mode

These equations give the same results as earlier.

In wave guides

$$Z_{TE} = \frac{\omega \mu_0}{\beta_{mn}} = \frac{\omega \mu_0}{\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \geq \eta_0$$

$$Z_{TM} = \frac{\beta_{mn}}{\omega \epsilon_0} = \frac{\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}{\omega \epsilon_0} \leq \eta_0$$

In wave guides If $\omega \leq \omega_c$, $\beta = i\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2} = i\alpha$

$\therefore Z_{TE} = \frac{\omega \mu_0}{i\alpha} = -i\frac{\omega \mu_0}{\alpha} = -i\omega L$ → inductive since $\propto \omega$

$Z_{TM} = \frac{i\alpha}{\omega \mu_0} = \frac{-1}{i\omega C}$ → capacitive since $\propto \frac{1}{\omega}$

For TM solution use $\bar{\Pi}$ & for TE solution use $\bar{\Pi}_m$

→ Here, $\cos\theta_t = \dfrac{\beta_2}{k_2} = \sqrt{1 - \dfrac{k_1^2}{k_2^2}\sin^2\theta_i} = \sqrt{1 - \dfrac{n_1^2}{n_2^2}\sin^2\theta_i}$ —(1)

since $\beta_2 = k_2 \cos\theta_t$;

where $n_1 = \sqrt{\mu_1 \epsilon_1}$ and $n_2 = \sqrt{\mu_2 \epsilon_2}$.

$\therefore \dfrac{n_1^2}{n_2^2}\sin^2\theta_i = \sin^2\theta_t \Rightarrow \boxed{\sin\theta_t = \dfrac{n_1}{n_2}\sin\theta_i}$ Snell's law of refraction.

Derived from phase matching condition.

→ From phase matching, $k_{1c} = k_{2c}$

$\Rightarrow k_1 \sin\theta_i = k_2 \sin\theta_t = k_1 \sin\theta_r$.

$\Rightarrow \sin\theta_i = \sin\theta_r$ & $\theta_i, \theta_r \in [0, 90] \Rightarrow$ sin is single valued.

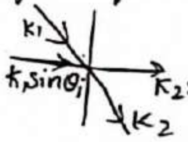
$\Rightarrow \boxed{\theta_i = \theta_r}$ Snell's law of reflection.

Note: $\boxed{\begin{array}{l} \beta_1 = k_1\cos\theta_i,\ k_{1c} = k_1 \sin\theta_i \\ \beta_2 = \sqrt{k_2^2 - k_{1c}^2} = \sqrt{k_2^2 - k_1^2\sin^2\theta_i} \end{array}}$

β_1 is k_1 along z because we have defined TM & TE along z.

since $k_{2c} = k_{1c}$

$k_1 \sin\theta_i = k_2 \sin\theta_t$ from phase matching.



⇒ phase component along boundary must be equal.

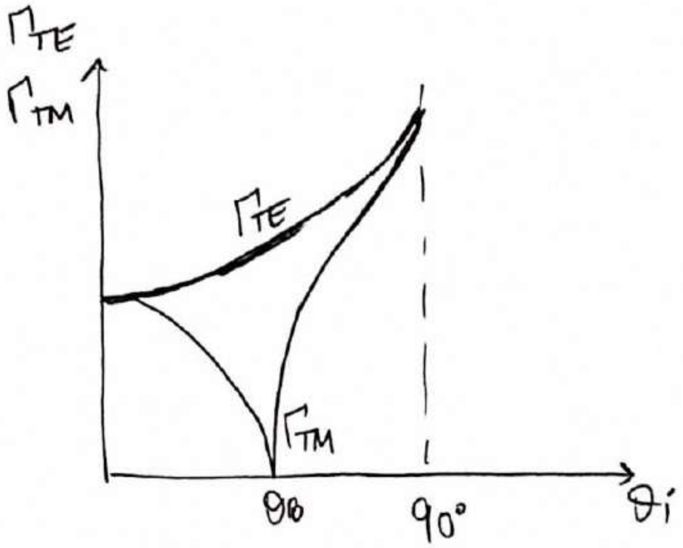
From (1) if $n_1 > n_2 \Rightarrow$ for some θ_{ic}, $\cos\theta_t$ is imaginary $\Rightarrow \beta_2$ is imaginary and all the power is reflected back. The wave in medium 2 is evanescent. This is total internal reflection.

$\boxed{\theta_{ic} = \sin^{-1}\dfrac{n_2}{n_1}}$ → Again derived from phase matching.

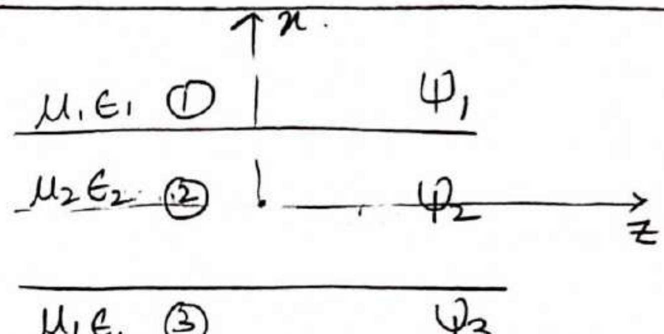
→ $\Gamma_{TM} = 0 \Rightarrow$ At some angle of incidence all the power goes into medium 2. This angle is called Brewster angle θ_B and it only exists if H field is in the xy plane. ⇒ TM mode.

$\boxed{\theta_B = \sin^{-1}\sqrt{\dfrac{\epsilon_2}{\epsilon_1 + \epsilon_2}}}$

The reflected wave for an arbitrarily polarized incident wave is purely TE ⇒ E field is in xy plane. Therefore this can be used to polarize a wave or even detect land mines since PEC would not



Dielectric Plate Waveguide



We are looking for wave solutions that propagate along z direction.

$\Pi_{lz}(x,z) = \psi_l(x)\, e^{i\beta z}$ where $l = 1,2,3$.

ψ must satisfy $\frac{d^2\psi_l}{dx^2} + K_{lc}^2\psi_l = 0$ ② since $\frac{d^2}{dy^2} = 0$, we assume no variation along y direction.

$K_{lc}^2 = K_l^2 - \beta^2$. Phase match here implies $\beta_1 = \beta_2 = \beta_3$

> since β is the propagation constant along z & must be same for a wave in all 3 media. This is different from $k_{1c} = k_{2c} = k_{3c}$ because the wave is not at an oblique incidence.

> We dont want to allow wave propagation along $\pm x$ in mediums ① & ③ so we look for solutions with imaginary K_{1c} & K_{3c}.

$$k_{1c} = k_{3c} = i\nu.$$

> Solutions to ② can be obtained as even mode or odd mode.

→ Considering the odd mode solutions.

$$\Pi^{o}_{2z}(x,z) = A\sin(k_{2c}x)\,e^{i\beta z} \qquad |x| < \frac{t}{2}$$

$$x > t/2 \Rightarrow \Pi^{o}_{1z}(x,z) = B e^{-\nu x} e^{i\beta z}$$

$$x < t/2 \Rightarrow \Pi^{o}_{3z}(x,z) = -B e^{+\nu x} e^{i\beta z}$$

(x is -ve)

These decay exponentially along x in mediums ① & ③.

Recall, $k_{2c}^2 = k_2^2 - \beta^2 = \omega^2\mu_2\epsilon_2 - \beta^2$ and $\nu^2 = \beta^2 - k_1^2 = \beta^2 - \omega^2\mu_1\epsilon_1$.

Field components can be derived:

$E_{2z} = Ak_{2c}^2\sin(k_{2c}x)e^{i\beta z}$	$E_{1z} = -B\nu^2 e^{-\nu x}e^{i\beta z}$	$E_{3z} = B\nu^2 e^{\nu x}e^{i\beta z}$
$H_{2y} = iA\omega k_{2c}\epsilon_2\cos(k_{2c}x)e^{i\beta z}$	$H_{1y} = iB\omega\epsilon_1(-\nu)e^{-\nu x}e^{i\beta z}$	$H_{3y} = iB\omega\epsilon_1(\nu)e^{\nu x}e^{i\beta z}$

Continuity of E_{tan} & H_{tan} assuming no currents are supported at the interface gives 2 equations. Divide them to get

$$\frac{k_{2c}}{\epsilon_2}\tan\left(k_{2c}\frac{t}{2}\right) = \frac{\nu}{\epsilon_1} \quad — ③$$

We also have $k_{2c}^2 + \nu^2 = \omega^2(\mu_2\epsilon_2 - \mu_1\epsilon_1)$. — ④

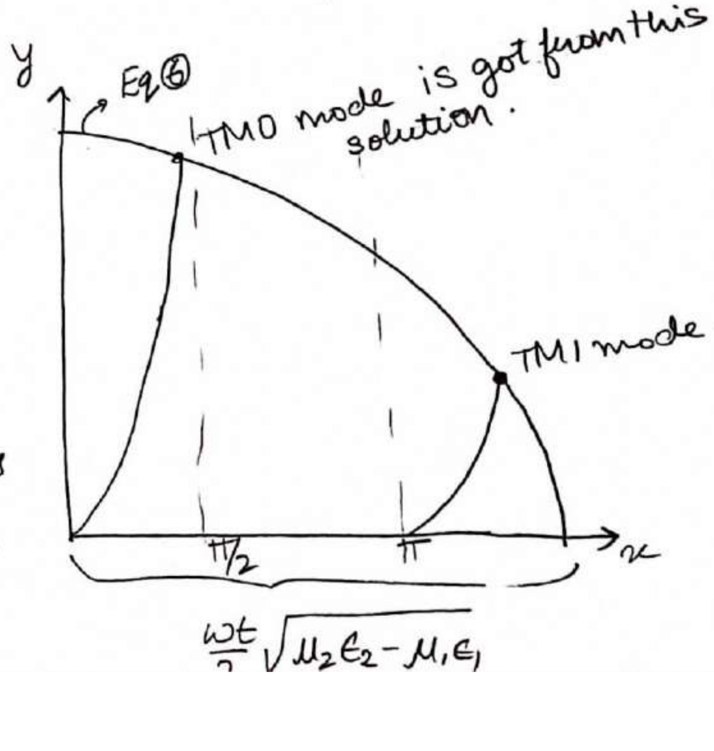
Solving for k_{2c} & ν can give β.

Let $X = k_{2c}t/2$ and $Y = \nu t/2$

$$\Rightarrow X\frac{\epsilon_1}{\epsilon_2}\tan X = Y \quad — ⑤$$

$$X^2 + Y^2 = \left(\frac{\omega t}{2}\sqrt{\mu_2\epsilon_2 - \mu_1\epsilon_1}\right)^2 \quad — ⑥$$

→ Even for $\omega \to 0$ or $t \to 0$, as circle shrinks there is a solution ⇒ TM0 mode always exists!



→ There is no cut off for TM0 mode propagation.

→ In order to define some kind of cutoff we say it is when $\nu = 0 \Rightarrow$ no decay along $\pm x$. This is for modes above TM0.

$$\Rightarrow \nu = 0 \Rightarrow \beta = k_1 = \omega\sqrt{\mu_1\epsilon_1}.$$

$$\Rightarrow \tan k_{2c}\frac{t}{2} = 0 \quad \Rightarrow k_{2c}\frac{t}{2} = n\pi, \quad n = 0, 1, 2 \ldots$$

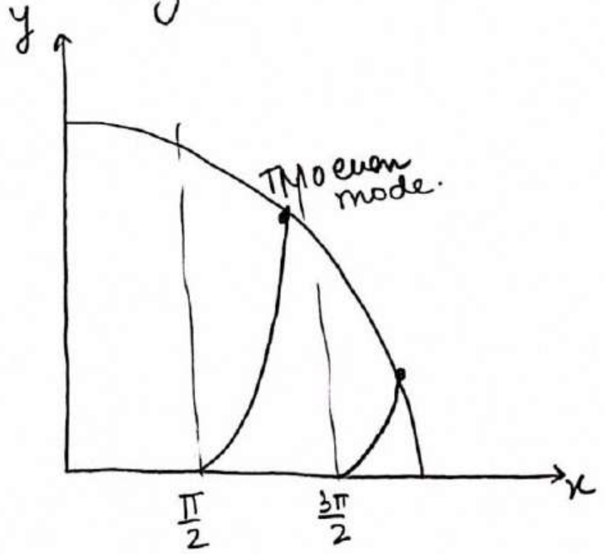
$$\Rightarrow \frac{\omega t}{2}\sqrt{\mu_2\epsilon_2 - \mu_1\epsilon_1} = n\pi \quad \Rightarrow f_c^{TM_n^o} = \frac{n}{t\sqrt{\mu_2\epsilon_2 - \mu_1\epsilon_1}} \quad n = 1, 2 \ldots$$

note $n=0$ is excluded since it does not have a cut off.

Same procedure for Even modes.

$$\left.\begin{aligned} \Pi_{2z}^e(x,z) &= A\cos(k_{2c}x)e^{i\beta z} \\ \Pi_{1z}^e(x,z) &= Be^{-\nu x}e^{i\beta z} \\ \Pi_{3z}^e(x,z) &= Be^{+\nu x}e^{i\beta z} \end{aligned}\right\} \ldots \text{ find } E \,\&\, H \text{ fields.}$$

Boundary conditions $\Rightarrow \; -\frac{k_{2c}t}{2}\cot\left(\frac{k_{2c}t}{2}\right) = \frac{\epsilon_2}{\epsilon_1}\frac{\nu t}{2}$



Cut off exists, $\nu = 0 \Rightarrow \cot\left(\frac{k_{2c}t}{2}\right) = 0$

$$f_c^{TM_n^e} = \frac{(2n+1)}{2t\sqrt{\mu_2\epsilon_2 - \mu_1\epsilon_1}} \quad n = 0, 1, 2 \ldots$$

> Combining even and odd modes, combined cut off is

$$\boxed{f_c^{TM_n} = \frac{n}{2t\sqrt{\mu_2\epsilon_2 - \mu_1\epsilon_1}}} \quad n = 0, 1, 2, \ldots$$

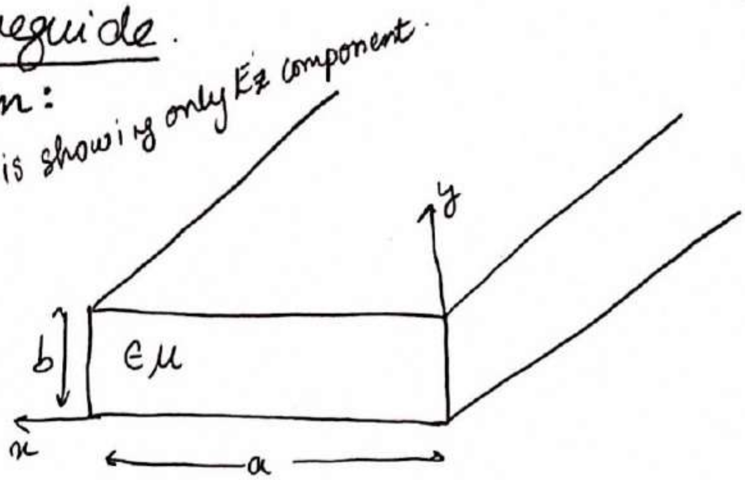
→ TE modes are duals of TM. f_c is exactly the same for TE.

→ Same analysis is done for a dielectric coated PEC in notes & HW#8

The Rectangular Waveguide.

① > TM wave soln for E are of the form: This is showing only E_z component.

$$E_z(x,y,z) = k_c^2 \psi(x,y)\, e^{i\beta z}$$



Boundary condition mandates $E_z = 0$ at the surface & therefore

$\boxed{\psi(x,y) = 0.}$ when $x, y \in S$. → Dirichlet's boundary condition.

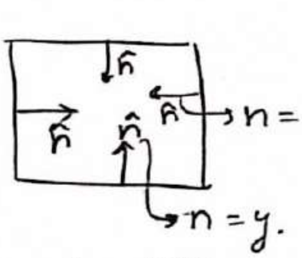
For PMC $\frac{\partial \psi}{\partial n} = 0$

② > TE wave solns. for E are of the form.

$$\hat{n} \times E = i\omega\mu\, \hat{n} \times (\nabla_t \psi_m \times \hat{z})\, e^{i\beta z} = -i\omega\mu \frac{\partial \psi_m}{\partial n} e^{i\beta z} \hat{z}$$

∴ For TE $\boxed{\frac{\partial \psi_m}{\partial n} = 0}$ when $x, y \in S$. → Neumann's boundary condition.

$n = x$ or y depending on the wall.



For PMC

$\psi_m = 0$

Consider TM solutions.

We are seeking solutions for $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right)\psi(x,y) = 0.$

$\psi(0,y) = \psi(a,y) = \psi(x,0) = \psi(x,b) = 0$ from DBC

Using seperation of variables $\psi(x,y) = X(x)Y(y)$ we get $k_c^2 = k_x^2 + k_y^2$

and $X(x) = A\sin(k_x x) + B\cos(k_x x)$; $Y(y) = C\sin(k_y y) + D\cos(k_y y)$

Boundary conditions translate to $X(0) = X(a) = Y(0) = Y(b) = 0$.

$\Rightarrow B = D = 0$ and $\sin(k_x a) = 0$, $\sin(k_y b) = 0$ which give discrete solutions (eigenvalues) for k_x and k_y.

$$\left.\begin{aligned} k_x &= \frac{m\pi}{a} \quad m = 1,2,3\ldots \\ k_y &= \frac{n\pi}{b} \quad n = 1,2,3\ldots \end{aligned}\right\} \therefore \boxed{\psi(x,y) = AC\, \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)}$$

m or n cannot be 0 for TM

$$\beta_{mn}^2 = k^2 - k_c^2 = k^2 - k_x^2 - k_y^2 \Rightarrow \boxed{\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$$

<u>Guide wavelength</u> $\lambda_g = \frac{2\pi}{\beta}$.

$$\lambda_{g_{mn}} = \left[\sqrt{\frac{1}{\lambda^2} - \left(\frac{m}{2a}\right)^2 - \left(\frac{n}{2b}\right)^2}\right]^{-1}$$

→ β is propagation constant inside the structure & is different for different modes and frequencies (& material of course)

→ K is propagation constant inside the medium without a structure and is only a function of frequency & material. $K = \omega\sqrt{\mu\epsilon}$.

→ K_c is defined only for the structure and is independent of frequency. It is different for different modes

→ When the boundary conditions of the structure are enforced, the waves can only propagate in certain modes (shapes or spatial distributions). The K now changes to β. For TEM these 2 are the same because the wave does not need to break down into modes.

<u>Field quantities for TM</u>

$$E_x = i\beta_{mn}\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{i\beta_{mn}z}$$

$$E_y = i\beta_{mn}\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{i\beta_{mn}z}$$

$$E_z = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi}{b}y\right)e^{i\beta_{mn}z}$$

$$H_x = -i\omega\epsilon\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{i\beta_{mn}z}$$

$$H_y = i\omega\epsilon\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{i\beta_{mn}z}$$

wave impedance

$$Z_{TM} = \frac{\hat{z}\times\bar{E}_t}{\bar{H}_t} = \frac{\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}{\omega\epsilon}$$

$\beta = 0$

$$\Rightarrow f_{c_{TM}}^{(mn)} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$a > b \Rightarrow$ <u>TM_{11}</u> is the lowest propagating mode. m or n can't be 0 otherwise all fields

→ TE soln. for PMC ⟺ TM soln. for PEC. TE soln. for PEC ⟺ dual of TM soln. of PEC.

→ TE solutions are duals but use sin → cos & cos → −sin since ψ is different.

$$\psi_{mn}(x,y) = \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \Rightarrow m \text{ or } n \text{ can be zero for TE}$$

∴ $a > b \Rightarrow TE_{10}$ is the lowest propagating mode & is overall the dominant mode.

$$f_{c_{TE_{10}}} = \frac{1}{2\pi a\sqrt{\mu\epsilon}}$$

→ λ_g is longer than λ_0 (free space wavelength) since $\beta_{mn} < k$.

→ The cosines in TE & sines in TM for ψ fn are all orthogonal & complete. Therefore any arbitrary field distribution in the cross section can be written in terms of TE or TM modes. (These cosines & sines are also orthogonal to each other). You can also show that they individually form a complete set. By properly exciting the different modes we can get any arbitrary field distribution in the cross section. Each mode can be modulated independantly for multiplexing at one freq, but exciting them in practice is not so easy.

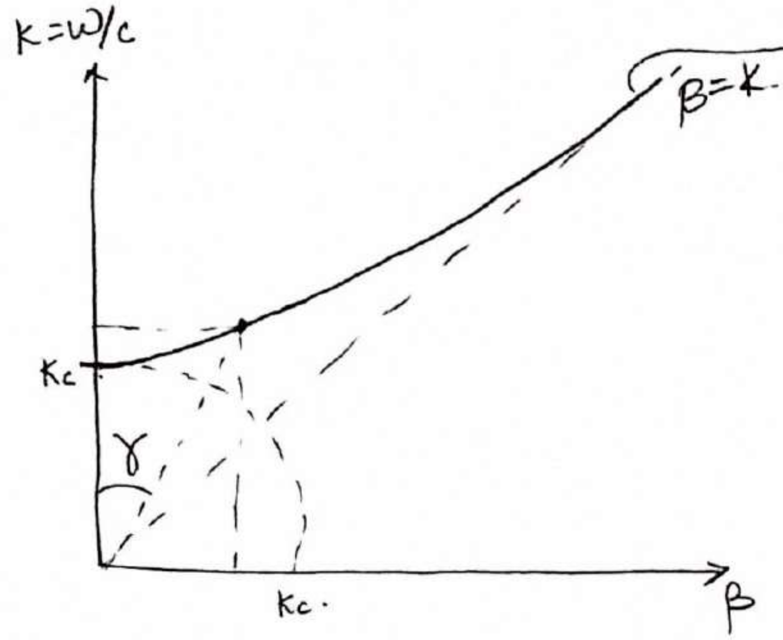
K-β diagram (Dispersion diagram)

We know $k^2 - \beta^2 = k_c^2$ ⇝ hyperbola in k-β plane.

If $k_c > k$ ⇒ β is imaginary

Let $\beta = i\alpha$ ⇒ $k^2 + \alpha^2 = k_c^2$ ⇝ circles on $k\beta$ plane.

$$v_p = \frac{dz}{dt} = \frac{\omega}{\beta} \Rightarrow v_p = \frac{\omega}{\sqrt{k^2 - k_c^2}} \Rightarrow v_p > c \text{ for propagating modes.}$$



At high frequency $K \simeq \beta$.

$$\tan\gamma = \frac{\beta}{K} = \frac{c}{v_p}$$

$$\Rightarrow \boxed{v_p = \frac{c}{\tan\gamma}}$$

$$\boxed{v_p = \frac{c}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}}$$

$$\Rightarrow v_g = c\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

Group velocity.

- 2 tones proof is used in the lecture notes.

$$v_g = \frac{dz}{dt} = \frac{d\omega}{d\beta}$$

$$v_g = v_p + \beta\frac{dv_p}{d\beta} = v_p - \lambda_g \frac{d v_p}{\partial \lambda_g} \quad \text{since } d\beta = \frac{v_p}{c^2}\, d\omega$$

$$\therefore \boxed{v_p v_g = c^2}$$

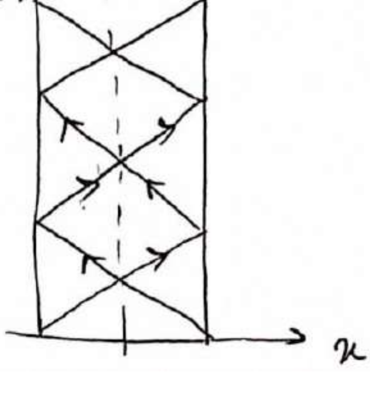
→ True for any waveguide. Therefore v_p is always greater than c since v_g must be < c. Also true for all modes.

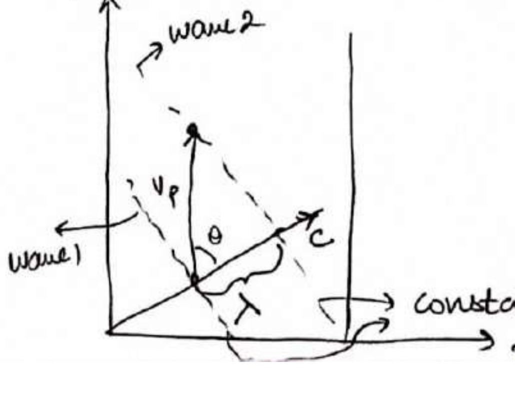
TE_{10} mode.

$$E_y(x) = -i\omega\mu\frac{\pi}{a}\sin\left(\frac{\pi}{a}x\right)e^{i\beta z} \qquad \frac{e^{i(\frac{\pi}{a}x+\beta z)} - e^{-i(\frac{\pi}{a}x-\beta z)}}{2i}$$

∴ TE mode can be interpreted as 2 planewaves $K_1 = \frac{\pi}{a}\hat{x} + \beta\hat{z}$

$$K_2 = \frac{\pi}{a}-\hat{x} + \beta\hat{z}$$





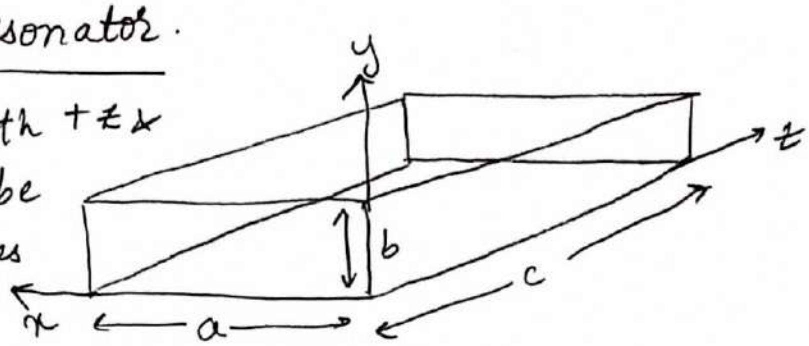
At cutoff the 2 rays are just bouncing back & forth ⇆ Therefore v_p is ∞ & v_g is 0.

→ constant phase plane.

> As wave 1 reaches wave 2 along $\hat{c}$ direction the phase point along z direction must have travelled $\frac{c}{\sin\theta}$ as fast. Therefore the speed can be faster than light for phase point.

Rectangular Cavity Resonator.

This structure can support both $+z$ & $-z$ waves. The fields can be decomposed into TM & TE modes w.r.t z axis.



For TM modes.

$$E_x(x,y,z) = \left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\frac{\partial}{\partial z}\left[Ae^{i\beta_{mn}z}+Be^{-i\beta_{mn}z}\right]$$

The boundary condition at $z=0$ and $z=c$ enforces

$$\Rightarrow E_x(x,y,0) = E_x(x,y,c) = 0. \quad \Rightarrow A = B \text{ \& } \sin(\beta_{mn}c) = 0$$

$$\Rightarrow \beta_{mn} = \frac{l\pi}{c}.$$

$$\psi = \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cos\left(\frac{l\pi}{c}z\right)$$

$$\therefore \quad k^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{c}\right)^2 \rightarrow$$

Frequencies for which we can satisfy this relationship excite resonant modes.

Fields.

$$E_x(x,y,z) = -\left(\frac{l\pi}{c}\right)\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{l\pi z}{c}\right)$$

$$E_y(x,y,z) = -\left(\frac{l\pi}{c}\right)\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{l\pi z}{c}\right)$$

$$E_z(x,y,z) = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{l\pi z}{c}\right)$$

$$H_x(x,y,z) = -i\omega\epsilon\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{l\pi z}{c}\right)$$

$$H_y(x,y,z) = -i\omega\epsilon\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{l\pi z}{c}\right)$$

TE fields are duals, but use $\sin \to \cos$ & $\cos \to -\sin$ since ψ_m is different

$$\psi_m = \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{l\pi}{c}z\right)$$

> The complex power in a region is given by:

$$P_{in}^{c} = P_{rad}^{c} + P_{loss} + i2\omega(W_e - W_m)$$

> Assuming $P_{rad} = P_{loss} = 0$ & no sources $\Rightarrow P_{in} = 0$. At resonance we can see $W_m = W_e = \frac{\epsilon}{4}\iiint_V |E|^2 dV = \frac{\mu}{4}\iiint_V |H|^2 dV$

> Total energy stored is constant $\Rightarrow \frac{\partial}{\partial t}(W_e + W_m) = 0$.

$$W = 2W_e = \frac{\epsilon}{2}\iiint |E|^2 dV.$$

> $k_c = 2\pi f_c \sqrt{\mu\epsilon}$

Flow Chart to solve Rectangular Waveguide Problems.

TM shown here.

Step ① Define the Hertz vector potential direction based on the solution you are looking for.

Eg: $\overline{\Pi}^z = \Pi_z \hat{z}$

where Π_z must satisfy the scalar wave eqn.

$$\nabla^2 \Pi_z + k^2 \Pi_z = 0.$$

$$\Rightarrow \nabla_t^2 \Pi_z + \frac{\partial^2 \Pi_z}{\partial z^2} + k^2 \Pi_z = 0 \Rightarrow \Pi(z) = \psi(x,y)\, e^{i\beta z}.$$

We could also have $\overline{\Pi}^x = \Pi_x \hat{x}$

Since propagation is still along z we have $\Pi_x = \psi_{mn}\, e^{i\beta z}$.

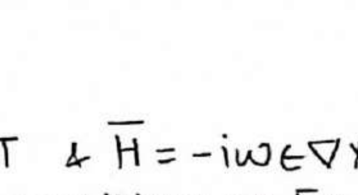
Step②

In the general case use $\overline{E} = \nabla\nabla\cdot\Pi + k^2\Pi$ & $\overline{H} = -i\omega\epsilon\nabla\times\Pi$ to find $\overline{E}$ & $\overline{H}$. See what the boundary conditions on E & H are and how they translate in terms of ψ & $\frac{d\psi}{dn}$.

Step③

Use seperation of variables & define $\psi(x,y) = X(x)\,Y(y)$

General forms of $X(x)$ & $Y(y)$ must be chosen to satisfy the boundaries.



Eg: $X(x) = A\sin\left(k_x\left(x+\frac{a}{2}\right)\right) + B\cos\left(k_x\left(x+\frac{a}{2}\right)\right)$

$Y(y) = C\sin(k_y y) + D\cos(k_y y)$.

Step④ Apply the B.C to find $\psi(x,y)$ and use it to find $\overline{E}$ & $\overline{H}$.

Lec 16 Modal Expansion of Field Quantities

→ The solution to $\nabla_t^2 \psi + K_c^2 \psi = 0$ gives the eigen function ψ of the operator ∇_t^2 whose eigenvalue is K_c^2.

→ Since the wave equation is linear, any linear combination of eigenfunctions is a valid solution.

→ In fact the eigenfunction ψ is actually a set of eigenfunctions ψ_{mn} corresponding to different modes, each having an eigenvalue $K_{c_{mn}}$.

→ We are going to show that for any arbitrary cross section (that is constant in z), the eigenfunctions derived form an <u>orthogonal</u> and <u>complete</u> set.

→ In case of TM modes of Rectangular waveguide $\psi_{mn} = \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$. This means that each ψ_{mn} is orthogonal to the rest of the ψ_{mn}s & any arbitrary distribution of potential across the cross section can be decomposed into a linear combination of these eigenfunctions ψ_{mn}. Just like data compression using "wavelets".

→ What do we mean by orthogonal?

$\langle \psi_{mn}, \psi_{m'n'} \rangle = 0$ if $m' \neq m$ <u>or</u> $n' \neq n$.

$= \iint_S \psi_{mn} \psi_{m'n'} \, ds = 0$

3 powerful statements. - True for any waveguide

① $\iint_{\text{cross section}} \psi_{mn}(x,y)\, \psi_{m'n'}(x,y)\, ds = 0$

② ψ_{mn} form a complete set. Also eigenvalues k_{mn}^2 or k_c^2 are real and positive.

③ $\iint_{\text{cross section}} \nabla_t \psi_{mn}(x,y) \cdot \nabla_t \psi_{m'n'}(x,y)\, ds = 0 \Rightarrow$ Gradients are also orthogonal!

Proofs → ①

Choose any 2 modes ψ_{mn} & $\psi_{m'n'}$. They must satisfy Helmholtz

$$\because \left[\left(\nabla_t^2 + k_{mn}^2 \right) \psi_{mn} = 0 \right] \times \psi_{m'n'}$$
$$\left[\left(\nabla_t^2 + k_{m'n'}^2 \right) \psi_{m'n'} = 0 \right] \times \psi_{mn}$$

& subtract.

$$\Rightarrow \psi_{m'n'} \nabla_t^2 \psi_{mn} - \psi_{mn} \nabla_t^2 \psi_{m'n'} = \left(k_{m'n'}^2 - k_{mn}^2 \right) \psi_{mn} \psi_{m'n'}$$

According to Green's second identity $\iiint_V [\psi \nabla^2 \phi - \phi \nabla^2 \psi]\, dv = \oiint_S \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right) ds$

$$\Rightarrow \iint_S \psi_{m'n'} \nabla_t^2 \psi_{mn} - \psi_{mn} \nabla_t^2 \psi_{m'n'}\, ds = \oint_C \psi_{m'n'} \frac{\partial \psi_{mn}}{\partial n} - \psi_{mn} \frac{\partial \psi_{m'n'}}{\partial n}$$

RHS is identically 0 since Dirichlet or Neumann holds for TE & TM case.

$$\Rightarrow \left(k_{m'n'}^2 - k_{mn}^2 \right) \iint_S \psi_{mn} \psi_{m'n'}\, ds = 0$$

. If $m \neq m'$ or $n \neq n'$, $k_{m'n'}$ & k_{mn} are not equal, which mandates $\langle \psi_{mn} \psi_{m'n'} \rangle = 0$.

Proof ~ 2

- Consider an arbitrary continuous function $f(x,y)$ on the cross section, that satisfies the boundary conditions
- For the set of eigenfunctions to be complete, there must exist coefficients a_{mn} such that

$$f(x,y) = \sum_m \sum_n a_{mn} \psi_{mn}(x,y) \longrightarrow \text{linear combination.}$$

multiply both sides by $\psi_{mn}(x,y)$ & integrate.

$$\iint_S f(x,y)\, \psi_{mn}(x,y) = \iint_S \sum_m \sum_n a_{mn}\, \psi_{mn}^2(x,y)$$

Since the area S is constant throughout the waveguide we can take the Integral inside & use the orthogonality property that when $m \neq n$, the value is 0. The summation must dissappear after taking the integral inside. For example if $\psi_{10}(x,y)$ is multiplied on RHS & integral is moved in only the term with a_{10} as the coefficient survives!

$\therefore$ a_{mn} can be pulled out using the orthogonality property of ψ_{mn}.

$$\boxed{a_{mn} = \frac{\iint_S f(x,y)\, \psi_{mn}(x,y)\, dS}{\iint_S \psi_{mn}^2(x,y)\, dS}}$$

To show that K_{mn}^2 are real and positive:

Consider a mode & its complex conjugate. They must both be valid solutions since HelmHoltz equation is invariant to taking a complex conjugate.

$$\left.\begin{array}{l} \therefore \left[\left(\nabla_t^2 + K_{mn}^2\right)\Psi_{mn} = 0\right] \times \Psi_{mn}^* \\ \left[\left(\nabla_t^2 + K_{mn}^{2*}\right)\Psi_{mn}^* = 0\right] \times \Psi_{mn} \end{array}\right\} \text{subtract \& apply Green's Theorem.}$$

We can show $\left(K_{mn}^{2*} - K_{mn}^2\right) \underbrace{\iint_S |\Psi_{mn}|^2 ds}_{\text{Never zero.}} = 0.$

Therefore $K_{mn}^2 = K_{mn}^{2*} \Rightarrow K_{mn}^2$ is purely real!

To show that it is positive, we use Green's First identity.

$$\iiint_V (\nabla\psi \cdot \nabla\phi + \psi\nabla^2\phi)\, dV = \oiint \psi \frac{\partial\phi}{\partial n} ds.$$

$$\therefore \iint_S \left(\nabla_t \Psi_{mn} \cdot \nabla_t \Psi_{mn} + \Psi_{mn}\nabla_t^2\Psi_{mn}\right) ds = \underbrace{\oint_C \Psi_{mn} \frac{\partial \Psi_{mn}}{\partial n} dl}_{0 \text{ due to D \& N B.C.}}.$$

Replacing $\nabla_t^2\Psi_{mn}$ with $-K_{mn}^2\Psi_{mn}$

$$\Rightarrow \iint_S \nabla_t\Psi_{mn} \cdot \nabla_t\Psi_{mn} - K_{mn}^2 \iint_S \Psi_{mn}^2 ds = 0$$

$$\Rightarrow K_{mn}^2 = \frac{\iint (\nabla_t\Psi_{mn} \cdot \nabla_t(\Psi_{mn}))ds}{\iint \Psi_{mn}^2 ds.}$$

If Ψ_{mn} is real $\Rightarrow K_{mn}^2 = \dfrac{\iint |\nabla_t\Psi_{mn}|^2 ds \rightarrow \text{positive.}}{\iint \Psi_{mn}^2 ds \rightarrow \text{positive}} \Bigg\}$ $\therefore K_{mn}^2$ is positive.

> ψ_{mn} need not be real. Assuming $\psi_{mn} = \psi_{mn}' + i\,\psi_{mn}''$.

> It can be shown that similar to above for ψ_{mn}'' also K_{mn}^2 is positive.

Proof ~ 3

Green's $\Rightarrow$ $$\iint_S \left(\nabla_t \psi_{mn} \cdot \nabla_t \psi_{m'n'} + \psi_{mn} \nabla_t^2 \psi_{m'n'}\right) ds$$

$$= \underbrace{\oint \psi_{mn} \frac{\partial \psi_{m'n'}}{\partial n}\, dl}_{0}.$$

Replace $\nabla_t^2 \psi_{m'n'}$ with $-K_{m'n'}^2 \psi_{m'n'}$

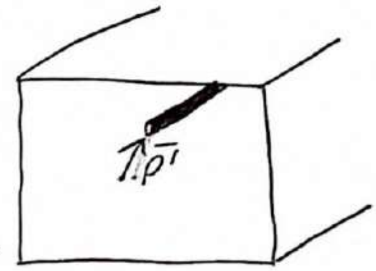
$$\Rightarrow \iint_S \nabla_t \psi_{mn} \cdot \nabla_t \psi_{m'n'}\, ds = K_{m'n'}^2 \underbrace{\iint_S \psi_{mn} \psi_{m'n'}\, ds}_{0 \text{ from orthogonality}}$$

$$\Rightarrow \boxed{\iint_S \nabla_t \psi_{mn} \cdot \nabla_t \psi_{m'n'}\, ds = 0}$$

Expansion of 2D Scalar Green's Function in terms of Eigenfunctions.

> We want to find the scalar potential inside an arbitrary waveguide with an arbitrary current distribution (flowing in z). First we find the green's function for an impulse distribution of current in the cross section.

$$(\nabla_t^2 + k_c^2)\, g(\bar{\rho},\bar{\rho}') = -\delta(\bar{\rho}-\bar{\rho}') \quad —\ (1)$$



> $g(\bar{\rho},\bar{\rho}')$ must be a valid soln. to B.C & hence can be expressed in terms of eigenfunctions.

> Similarly $\delta(\bar{\rho},\bar{\rho}')$ can also be represented.

$\bar{\rho} = x\hat{x} + y\hat{y}$

$\bar{\rho}' = x'\hat{x} + y'\hat{y}$

$$\therefore\quad g(\bar{\rho},\bar{\rho}') = \sum_m \sum_n A_{mn}\,\psi_{mn}(\bar{\rho})$$

$$\delta(\bar{\rho}-\bar{\rho}') = \sum_m \sum_n B_{mn}\,\psi_{mn}(\bar{\rho}).$$

$$B_{mn} = \frac{\iint_S \delta(\bar{\rho}-\bar{\rho}')\,\psi_{mn}(\bar{\rho})\,ds}{\iint_S \psi_{mn}^2(\bar{\rho})\,ds} = \frac{\psi_{mn}(\bar{\rho}')}{\iint_S \psi_{mn}^2(\bar{\rho})\,ds}$$

Substituting in (1)

$$\sum_m \sum_n A_{mn}(\nabla_t^2 + k_c^2)\,\psi_{mn}(\bar{\rho}) = -\sum_m \sum_n \frac{\psi_{mn}(\bar{\rho}')\,\psi_{mn}(\bar{\rho})}{\iint_S \psi_{mn}^2(\bar{\rho})\,ds}$$

but $\nabla_t^2 \psi_{mn} = -k_{mn}^2 \psi_{mn}$

$$\Rightarrow \sum_m \sum_n A_{mn}(k_c^2 - k_{mn}^2)\,\psi_{mn}(\bar{\rho}) = -\sum_m \sum_n \frac{\psi_{mn}(\bar{\rho}')\,\psi_{mn}(\bar{\rho})}{\iint_S \psi_{mn}^2(\bar{\rho})\,ds}.$$

→ Since this equality must be true for all $\bar{\rho}$ & ψ, the summands must be equal.

$$\Rightarrow \quad A_{mn} = \frac{1}{K_{mn}^2 - K_c^2} \cdot \frac{\psi_{mn}(\bar{\rho}')}{\iint_S \psi_{mn}^2(\bar{\rho}')\, d\rho}$$

$$\therefore \quad \boxed{g(\bar{\rho},\bar{\rho}') = \sum_m \sum_n \frac{\psi_{mn}(\bar{\rho}')\, \psi_{mn}(\bar{\rho})}{\left(K_{mn}^2 - K_c^2\right) \iint_S \psi_{mn}^2(\rho)\, d\rho}}$$

→ K_{mn} is the wave number of the mn^{th} mode of fields that can exist inside the waveguide.

→ Now however the propagation constant β is specified by $K_c^2 = K^2 - \beta^2$ where K_c^2 is specified by the source. It is the wave number of the current wave propagating inside.

→ $K_c^2 \neq K_{mn}^2$ in this case. K_{mn}^2 are the eigenvalues of the waveguide (without excitation). Eigenvalues don't make sense in the presence of excitation since eqn is no longer in the form $A\vec{x} = \nu\vec{x}$.

→ If $K_{mn}^2 = K_c^2$, $g(\bar{\rho},\bar{\rho}')$ blows up since the waveguide is excited at its resonance.

→ For rectangular waveguide → TM modes

$$g(\bar{\rho},\bar{\rho}') = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x'\right)\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{n\pi}{b}y'\right)}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - K_c^2\right]\left(\frac{ab}{4}\right)}$$

To find the potential function for an arbitrary current distribution $J(\bar{\rho}')$, we can find the "sum" of potential generated by impulses spread over the cross section.

$$\psi_{mn}(\bar{\rho}) = \iint_{S'} g(\bar{\rho},\bar{\rho}')\, J(\bar{\rho}')\, ds'$$

→ We found green's function due to impulse current excitation in terms of eigenfunctions. (eigenfunctions are potentials in the absence of currents or excitations). Then we found how to write the potential for an arbitrary distribution of current from the green's function..

Note:

$$\delta(\bar{\rho}-\bar{\rho}') = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4}{ab} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{m\pi}{a}x'\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{n\pi}{b}y'\right)$$

Try this in MATLAB.

When $\bar{\rho}=\bar{\rho}'$ $\delta(\bar{\rho}-\bar{\rho}')=1 \Rightarrow \displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4}{ab} \sin^2\left(\frac{m\pi}{a}x\right) \sin^2\left(\frac{n\pi}{b}y\right)$

We know $\displaystyle\sum_{m}\sum_{n} \sin^2\left(\frac{m\pi}{a}x\right) \sin^2\left(\frac{n\pi}{b}y\right) = \frac{ab}{4}$

Therefore it works at $\bar{\rho}=\bar{\rho}'$. If $\bar{\rho}\neq\bar{\rho}'$ then the $\sum\sum$ of the sinusoids must give 0.

Calculus of Variations for Estimation of Eigenvalues.

→ Method of seperation of variables only works when the geometry of the structure falls on the coordinate surfaces. For other structures, a numerical method is used.

Recall that the eigenvalues of the scalar wave equation are:

$$K_{mn}^2 = K_c^2 = \frac{\iint_R \nabla_t \psi \cdot \nabla_t \psi \, ds}{\iint_R \psi^2 ds}$$

R is the waveguide cross section.

Take the following functional:

$$\frac{N(f)}{D(f)} = \frac{\iint_R \nabla_t f \cdot \nabla_t f \, ds}{\iint_R f^2 ds}$$

** → Among all functions f that are continuous and have a continuous first derivative in R, the function that minimizes the above functional is an eigen function of the ∇_t^2 operator. The minimum value of the functional thus obtained is the eigenvalue K_{c1}^2.

→ Therefore to find the eigenfunction, we must find a function that minimizes $\frac{N(f)}{D(f)}$.

Proof

> Suppose ψ_1 is the minimizing function.
> Let g be another admissible function & ϵ be an arbitrarily small constant

$$\therefore \quad N(\psi_1 + \epsilon g) > k_{c1}^2 D(\psi_1 + \epsilon g)$$

since $\frac{N}{D}$ here is not the minimum k_{c1}^2.

$$\& \quad N(\psi_1) = k_{c1}^2 D(\psi_1).$$

$$N(\psi_1 + \epsilon g) = \iint_R \nabla_t(\psi_1 + \epsilon g) \cdot \nabla_t(\psi_1 + \epsilon g)\, ds$$

$$= \iint_R \nabla_t(\psi_1) \cdot \nabla_t(\psi_1)\, ds. + 2\epsilon \iint \nabla_t \psi \cdot \nabla_t g\, ds + \epsilon^2 \iint \nabla_t g \cdot \nabla_t g\, ds$$

$$D(\psi_1 + \epsilon g) = \iint_R (\psi_1 + \epsilon g)^2\, ds = \iint_R \psi^2\, ds + 2\epsilon \iint \psi_1 g\, ds + \epsilon^2 \iint g^2\, ds.$$

$$\therefore N(\psi_1 + \epsilon g) = N(\psi_1) + \epsilon^2 N(g) + 2\epsilon N(\psi_1, g)$$

$$D(\psi_1 + \epsilon g) = D(\psi_1) + \epsilon^2 D(g) + 2\epsilon D(\psi_1, g)$$

Where $$N(\psi_1, g) = \iint_R \nabla_t \psi_1 \cdot \nabla_t g\, ds$$

$$D(\psi_1, g) = \iint_R \psi_1 g\, ds.$$

$\therefore \quad 2\epsilon\left[N(\psi_1, g) - K_{c_1}^2 D(\psi, g)\right] + \epsilon^2\left[N(g) - K_{c_1}^2 (D(g))\right] \geq 0$ (27)

since $N(g) - K_{c_1}^2 D(g) > 0$ for all g,

and Eq (27) must be valid for all ϵ and any function g,

$$\boxed{N(\psi_1, g) - K_{c_1}^2 D(\psi_1, g) = 0}$$

Otherwise ϵ can be chosen appropriately to make the whole term -ve for a given g.

Notice that ϵ can be chosen to be very small & negative to make sure ϵ^2 is $\simeq 0$ and this would make the whole term negative. (This is why proofs can be simple & elegant)

$$\Rightarrow \iint_R \nabla_t \psi_1 \cdot \nabla_t g \, ds - K_{c_1}^2 \iint_R \psi_1 g \, ds = 0 \quad (31)$$

Green's First identity

$$\Rightarrow \iint_R \left(\nabla_t \psi_1 \cdot \nabla_t g + g \nabla_t^2 \psi_1\right) ds = \oint_C \underbrace{g \frac{\partial \psi_1}{\partial n}}_{0 \text{ since } g \,\&\, \psi_1 \text{ both satisfy D\&N BC}} dl.$$

$$\Rightarrow \iint_R g\left(\nabla_t^2 \psi_1 + K_{c_1}^2 \psi_1\right) ds = 0$$

→ For all arbitrary g functions, the integral can only be identically 0 if $\boxed{\nabla_t^2 \psi_1 + k_{c1}^2 \psi_1 = 0}$ — (33)

Therefore the function that minimizes the functional is an eigen function of ∇_t^2 with k_{c1}^2 as the eigenvalue.

→ If g is non zero, from (31) we know $\frac{\partial \psi_1}{\partial n} = 0$ since from (33) L.H.S is zero.

Therefore $\boxed{\frac{\partial \psi_1}{\partial n} = 0}$ is the natural boundary condition.

→ TM ⇒ ψ must be 0 on the boundary, therefore the class of admissible functions f must also be 0 on the boundary. Because $\frac{\partial \psi}{\partial n} = 0$ is not guaranteed by $\psi = 0$ on the boundary. Therefore admissible functions f must be 0 to satify LHS = 0 in (31)

→ TE ⇒ Since $\frac{\partial \psi}{\partial n} = 0$ is guaranteed by Neumann, the condition $\frac{\partial f}{\partial n} = 0$ need not be satisfied for f to be an admissible function.

→ The next eigenfunction can be found by minimizing the functional subject to the condition that it is orthogonal to ψ_1.

$$\therefore \iint_R f\psi_1 \, ds = 0$$

$\Rightarrow$ $K_{c_2}^2$ thus obtained is larger than $K_{c_1}^2$.

Let us assume this solution is ψ_2.

$$\Rightarrow N(\psi_2) = K_{c_2}^2 D(\psi_2).$$

$$\& \; D(\psi_2, \psi_1) = 0 \Rightarrow \text{orthogonal.}$$

→ Similar to before we have,

$$N(\psi_2 + \epsilon g) \geq K_{c_2}^2 D(\psi_2 + \epsilon g)$$

→ Now the set of admissible functions are constrained by

$$D(\psi, g) = 0$$

→ Take a general function p within the class of admissible functions not subject to the orthogonality constraint.

We may write $g = p - c\psi_1$ where $c = \dfrac{D(\psi, p)}{D(\psi_1)}$.

The p is chosen s.t $\langle \psi_1, g \rangle = 0$.

$$\therefore N(\psi_2 + \epsilon p - \epsilon c\psi_1) \geq K_{c_2}^2 D(\psi_2 + \epsilon p - \epsilon c\psi_1)$$

$$\Rightarrow N(\psi_2, p) - K_{c_2}^2 D(\psi_2, p) - c\left[N(\psi_2, \psi_1) - K_{c_2}^2 D(\psi_2, \psi_1)\right] = 0$$

We know $D(\psi_2, \psi_1) = 0$ since they are orthogonal.

From earlier we know, $N(\psi_1, P) = K_{c1}^2 D(\psi_1, P)$.

If $P = \psi_2$, we have $N(\psi_1, \psi_2) = 0$.

Therefore, last term in (39) $= 0$

$\Rightarrow N(\psi_2, P) - K_{c2}^2 D(\psi_2, P) = 0$ $\longrightarrow$ Same as earlier.

Using Green's First Identity we can show that,

$\nabla_t^2 \psi_2 + K_{c2}^2 \psi_2 = 0$ $\Rightarrow$ K_{c2}^2 is an eigenvalue & ψ_2 is an eigenfunction.

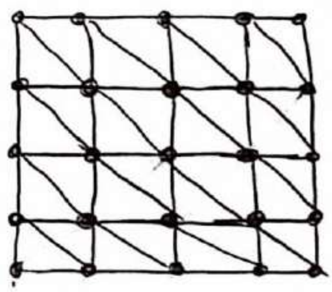
Also $\dfrac{d\psi_2}{dn} = 0$.

This process can be continued...

Numerical Solution

$\longrightarrow$ Finite element method.

The eigenfunction on the cross section is approximated by a piecewise linear function.



$$\psi(x,y) = \underbrace{\sum_{j=1}^{3} N_j^e(x,y)}_{\text{linear basis function.}} \cdot \overbrace{\psi_j^e}^{\text{node values.}}$$

$$N_j^e(x,y) = \frac{1}{2\Delta^e}\left(a_j^e + b_j^e x + c_j^e y\right), \quad j = 1,2,3$$

$$a_1^e = x_2^e y_3^e - y_2^e x_3^e, \quad b_1^e = y_2^e - y_3^e, \quad c_1^e = x_3^e - x_2^e$$

Where Δ^e = area of triangle $= \frac{1}{2}\left(b_1^e c_2^e - b_2^e c_1^e\right)$

→ The expressions for $a_2^e \ldots c_3^e$ are obtained from the ones above (for a_1^e, b_1^e, c_1^e) by cyclic interchange $(1\to2, 2\to3, 3\to1)$

→ Substituting back into the functional & finding the minima by setting $\delta/\delta \psi_j^e = 0$, the following matrix is obtained.

$$\boxed{\tilde{A}\,\tilde{\Psi} = k_c^2\,\tilde{B}\,\tilde{\Psi}}$$

Where $\tilde{A}_{ij}^e = \frac{1}{4\Delta^e}\left(b_i^e b_j^e + c_i^e c_j^e\right)$

$\tilde{B}_{ij}^e = \frac{\Delta^e}{12}\left(1+\delta_{ij}\right)$ → Kronecker delta function.

Since some of the nodes (many in fact) are shared b/w adjacent elements, the vector $\tilde{\psi}$ for the unconnected elements must be related to the vector ψ for the connected elements using a connection matrix C.

$$\tilde{\psi} = C\,\psi$$

$$\Rightarrow \boxed{A\,\psi = k_c^2\,B\,\psi}$$

where $A = C^t \tilde{A} C$,
$B = C^t \tilde{B} C$

Generalized eigenvalue problem which can be solved for k_c numerically.

Wave Functions in Cylindrical Coordinate Systems.

$$\nabla^2\psi + K^2\psi = 0$$

In cylindrical coordinates,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = 0.$$

M.O.S.O.V $\Rightarrow$ $\psi = R(\rho)\,\Phi(\phi)\,Z(z).$

Substituting & dividing by $R\Phi Z$ we get

$$\frac{1}{\rho R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{1}{\rho^2\Phi}\frac{\partial^2\Phi}{\partial\phi^2} + \underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{\text{only a fn of } z} + K^2 = 0.$$

$$\Rightarrow \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -k_z^2 \quad \text{①}$$

$$\therefore \quad \frac{\rho}{R}\frac{d}{d\rho}\rho\frac{dR}{d\rho} + \underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}}_{\text{only a fn of } \phi} + k_\rho^2\rho^2 = 0 \quad \text{where } K_\rho^2 = K^2 - K_z^2$$

$$\Rightarrow \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -\nu^2 \quad \text{②}$$

$$\Rightarrow \boxed{\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \left[(k_\rho\rho)^2 - \nu^2\right]R = 0} \quad \text{③}$$

Solutions to Eq ① & ② are

$$Z(z) = A e^{ik_z z} + B e^{-ik_z z}$$
$$\Phi(\phi) = C e^{i\nu\phi} + D e^{-i\nu\phi}$$

They can also be in terms of a linear combination of sines & cosines.

$A \sin\nu\phi + B\cos\nu\phi$

Eq ③ can be rewritten as

$$(K_\rho \rho) \frac{d}{d(K_\rho \rho)} \left((K_\rho \rho) \frac{dR}{dK_\rho \rho} \right) + \left[(K_\rho \rho)^2 - \nu^2 \right] R = 0$$

→ Bessel Equation in $K_\rho \rho$ with order ν.

→ Solution is a linear combination of Bessel and Neumann functions of νth order.

Bessel fn is a.k.a Bessel fn of first kind

Neumann fn is a.k.a Bessel fn of second kind.

$$\therefore \quad R_\nu(K_\rho \rho) = A J_\nu(K_\rho \rho) + B N_\nu(K_\rho \rho).$$

→ Solution can also be expressed as a linear combination of Hankel functions which have a neat physical interpretation.

$$H_\nu^{(1)}(K_\rho \rho) = J_\nu(K_\rho \rho) + i N_\nu(K_\rho \rho).$$

$$H_\nu^{(2)}(K_\rho \rho) = J_\nu(K_\rho \rho) - i N_\nu(K_\rho \rho)$$

→ In the domain if ϕ spans the entire domain we have.

$\psi(\rho, \phi, z) = \psi(\rho, \phi + 2\pi, z)$. In this case the order ν becomes an integer n and the possible solutions for Φ are of the form.

$$\Phi(\phi) = A e^{in\phi} + B e^{-in\phi}$$ → periodic in ϕ with period 2π

Properties of Bessel Functions.

→ If $\rho = 0$ is within the domain of interest, only $J_n(K_\rho \rho)$ is a valid solution since $N_n(K_\rho \rho)$ & H_n blow up (singular)

→ Asymptotic behaviour.

$$\lim_{K_\rho \rho \to \infty} J_n(K_\rho \rho) \simeq \sqrt{\frac{2}{\pi K_\rho \rho}} \cos\left(K_\rho \rho - \frac{2n+1}{4}\pi\right)$$

↳ asymptotes to a cosine function. that is weakly damped & has a phase shift

$$\lim_{K_\rho \rho \to \infty} N_n(K_\rho \rho) \simeq \sqrt{\frac{2}{\pi K_\rho \rho}} \sin\left(K_\rho \rho - \frac{2n+1}{4}\pi\right)$$ → sin fn, weakly damped & has a phase shift.

$$\lim_{K_\rho \rho \to \infty} H_n^{(1)}(K_\rho \rho) \simeq \sqrt{\frac{2}{\pi K_\rho \rho}} e^{i\left[K_\rho \rho - \frac{(2n+1)}{4}\pi\right]}$$ → outgoing cylindrical wave.

$$\lim_{K_\rho \rho \to \infty} H_n^{(2)}(K_\rho \rho) \simeq \sqrt{\frac{2}{\pi K_\rho \rho}} e^{-i\left[K_\rho \rho - \frac{(2n+1)}{4}\pi\right]}$$ → incoming cylindrical wave.

→ For small values of $(K_\rho \rho)$, $J_n(K_\rho \rho) \propto (K_\rho \rho)^n$ ⇒ n is large ⇒ the plot is flat & if n is small the plot asymptotes towards the origin at an angle.

→ $N_n(K_\rho \rho)$ all go to $-\infty$ as $K_\rho \rho$ approaches 0.

Any solution to Helm Holtz can be expanded in terms of a linear combination of elementary cylindrical functions

$R(\rho)$, $Z(z)$ & $\Phi(\phi)$.

$$\therefore \Psi(\rho, \phi, z) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \left[A_n(k_z) J_n(k_\rho \rho) + B_n(k_z) N_n(k_\rho \rho) \right] e^{in\phi} e^{ik_z z} dz$$

↳ would become an integral if ν is not integers.

Remember, $k_\rho = \sqrt{k^2 - k_z^2}$

→ If $|k_z| < k$ => Bessel functions have real arguments & the elementary wave functions can be decomposed into incoming and outgoing cylindrical waves.

→ If $|k_z| > k$, the argument of Bessel functions becomes purely imaginary (lossless case) and the radial wave function becomes exponentially decaying or evanescent. In this case we use the modified Bessel functions, I_n and K_n given by

$$J_n(i\alpha\rho) = (i)^n I_n(\alpha\rho)$$

$$H_n^{(1)}(i\alpha\rho) = \frac{2}{\pi}(i)^{n+1} K_n(\alpha\rho)$$

where $\alpha = \sqrt{k_z^2 - k^2}$

More useful properties of Bessel Functions.

- Here I have represented $k_\rho \rho$ as just ρ.

$$R_\nu(\rho) = J_\nu(\rho) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\underbrace{\Gamma(\nu+m+1)}_{\text{Gamma function.}}} \left(\frac{\rho}{2}\right)^{\nu+2m}$$

is the solution to $\frac{d^2R_\nu}{d\rho^2} + \frac{1}{\rho}\frac{dR_\nu}{d\rho} + \left(1 - \frac{\nu^2}{\rho^2}\right)R_\nu = 0$.

→ A second solution is obtained by replacing $\nu \to -\nu$. This cannot be done if ν is an integer. In that case, the solution is the Bessel function of first kind as given by

$$J_n(\rho) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(n+m)!} \left(\frac{\rho}{2}\right)^{n+2m}.$$

Notice here replacing $n \to -n$ does not give an independant solution. Infact, $J_{-n}(\rho) = (-1)^n J_n(\rho)$

→ The Gamma function is an analytic continuation of the factorial fn.

→ Bessel fn of 2nd kind: $N_\nu(\rho) = \frac{1}{\sin\nu\pi}\left[J_\nu(\rho)\cos\nu\pi - J_{-\nu}(\rho)\right]$ → Independent Solution of the D-E.

Recurrence relations: $R_{\nu-1} + R_{\nu+1} = \frac{2\nu}{\rho} R_\nu$ → In MATLAB, $\nu \to \infty$ is used to find large values which are then brought down. Since asymptotic form is simple.

$$\frac{dR_\nu}{d\rho} = \frac{1}{2}\left[R_{\nu-1} - R_{\nu+1}\right]$$ → Used to compute derivative.

$$\frac{d}{d\rho}\left[\rho^\nu R_\nu(\rho)\right] = \rho^\nu R_{\nu-1}$$

$$\frac{d}{d\rho}\left[\rho^{-\nu} R_\nu(\rho)\right] = -\rho^{-\nu} R_{\nu+1}$$

Wronskian relationships.

$$W = J_\nu(\rho) N_\nu'(\rho) - J_\nu'(\rho) N_\nu(\rho) = \frac{2}{\pi \rho}.$$

$$W_2 = J_\nu(\rho) H_\nu^{(1)'}(\rho) - J_\nu'(\rho) H_\nu^{(1)}(\rho) = \frac{i2}{\pi \rho}$$

Orthogonality Relation

Wierdly in case of Bessel functions, the orthogonality is between a Bessel function of one order and the same Bessel function of the same order but with a scaled argument.

Let $X_{\nu p}$ and $X_{\nu q}$ represent two distinct zeroes of $J_\nu(\rho)$.

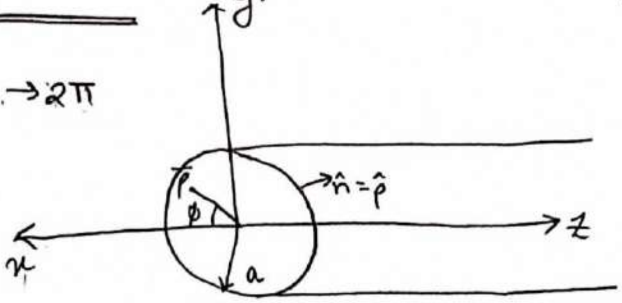
Then $\int_0^1 J_\nu(X_{\nu p}\rho) J_\nu(X_{\nu q}\rho)\, \rho \, d\rho = 0$. ie. if the arguments are scaled by zeroes (that are distinct) and a weighting function ρ is used, we get orthogonality in $[0,1]$ domain.

Therefore any function in the domain $[0,1]$ can be expressed as,

$$R(\rho) = \sum_{p=1}^{\infty} a_p J_\nu(X_{\nu p}\rho)$$

& $$a_p = \frac{\int_0^1 R(\rho) J_\nu(X_{\nu p}\rho) \rho \, d\rho}{\frac{1}{2} J_{\nu+1}^2(X_{\nu p})}.$$

The Cylindrical Waveguide.



→ Domain covers entire ϕ from $0 \to 2\pi$
$\Rightarrow \nu \to n$
→ Domain includes $\rho = 0$.
$\Rightarrow$ only $J_n(k_\rho \rho)$ can be used.

$$\therefore \quad \psi(\rho, \phi) = A_n J_n(k_\rho \rho) \begin{cases} \sin n\phi \\ \cos n\phi \end{cases}$$

→ TM case.

→ This form of ψ satisfies $\nabla_t^2 \psi + k_\rho^2 \psi = 0$. The sin & cos are both valid solutions & therefore for every mode there is a sin & cos solution that propagates. Since they only differ by a $90°$ rotation & the waveguide is symmetric in ϕ we can consider one of them. These 2 modes of sin & cosine are therefore degenerate, but orthogonal.

→ Using cosine & applying B.C $\Rightarrow J_n(k_\rho a) = 0$ (Dirichlet)

Let x_{np} be the zeroes of J_n where p is the ascending order of zeroes. → computed on MATLAB (known result).

$$\Rightarrow k_{np} = \frac{x_{np}}{a} = k_\rho \quad ; \quad \beta_{np} = k_z$$

$$\therefore \boxed{\beta_{np}^{TM} = \sqrt{k^2 - \left(\frac{x_{np}}{a}\right)^2}} \quad \& \quad \boxed{\psi_{np}(\rho, \phi) = J_n\left(\frac{x_{np}}{a}\rho\right) \cos n\phi}$$

$$\boxed{f_c^{(np)} = \frac{x_{np}}{2\pi a \sqrt{\mu\epsilon}}}$$

TM_{01} is the lowest mode.

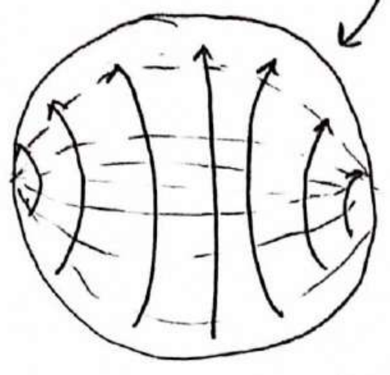
TE: Neumann B-C $\Rightarrow J_n'(k_\rho a) = 0$

Denoting the zeroes of $J_n'(k_\rho a)$ by x_{np}' we have

$$k_\rho^{TE} = k_{np}' = \frac{x_{np}'}{a} \Rightarrow \beta_{np}^{TE} = \sqrt{k^2 - \left(\frac{x_{np}'}{a}\right)^2}$$

$$\Rightarrow \boxed{f_c^{(np)} = \frac{x_{np}'}{2\pi a\sqrt{\mu\epsilon}}}$$ Lowest mode is TE11.

→ The dominant mode overall is TE11.



→ TM_{11} & TE_{01} have the same β and are therefore degenerate modes since they propagate together & a small disturbance can couple power from one to the other. In an ideal situation, the modes are orthogonal.

Next modes are TM_{01}, TE_{21}, (TM_{11} & TE_{01}), TE_{31}, etc.

→ Field expressions → TM

$$E_\rho(\rho,\phi,z) = A_n i\beta k_\rho J_n'(k_\rho\rho)\cos(n\phi)e^{i\beta z}$$

$$E_\phi(\rho,\phi,z) = -\frac{A_n i\beta n}{\rho} J_n(k_\rho\rho)\sin(n\phi)e^{i\beta z}$$

$$E_z(\rho,\phi,z) = A_n k_\rho^2 J_n(k_\rho\rho)\cos(n\phi)e^{i\beta z}$$

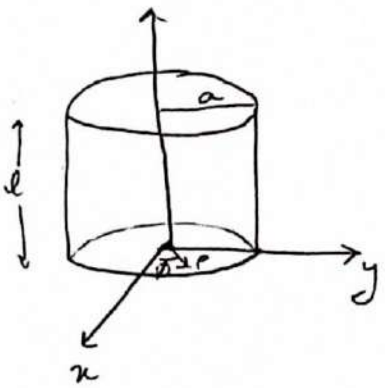
$$H_\rho(\rho,\phi,z) = \frac{A_n i\omega\epsilon n}{\rho} J_n(k_\rho\rho)\sin(n\phi)e^{i\beta z}$$

$$H_\phi(\rho,\phi,z) = A_n i\omega\epsilon k_\rho \rho J_n'(k_\rho\rho)\cos(n\phi)e^{i\beta z}$$

$$H_z(\rho,\phi,z) = 0$$

Cylindrical Cavity

- The ρ component of E field is



$$E_\rho(\rho,\phi,z) = A_n B K_\rho J_n'(K_\rho \rho)\cos(n\phi)\sin(\beta z)$$

The choice of the sin function ensures $E_\rho = 0$ when $z=0$. But we also need $E_\rho = 0$ when $z = l$.

$$\Rightarrow \quad \beta = \frac{q\pi}{l}, \quad q = 1,2,3.$$

$$\therefore \quad \left(\frac{q\pi}{l}\right)^2 + \left(\frac{x_{np}}{a}\right)^2 = \omega^2\mu\epsilon$$

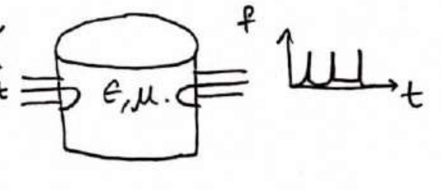
$$\Rightarrow \boxed{f_{npq}^{TM} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{x_{np}}{a}\right)^2 + \left(\frac{q\pi}{l}\right)^2}}$$

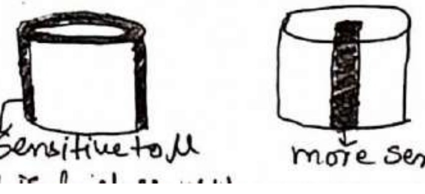
$n = 0,1,2\ldots$
$p = 1,2,3\ldots$
$q = 1,2,3\ldots$

Similarly,

$$\boxed{f_{npq}^{TE} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{x'_{np}}{a}\right)^2 + \left(\frac{q\pi}{l}\right)^2}}$$

→ By filling the structure with a material & sweeping the frequency we can find ϵ & μ. Different structures are more sensitive to ϵ & μ.

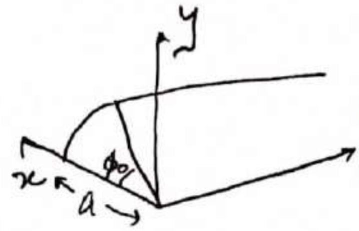




Angular Sector Waveguide

→ ν is not an integer but J_ν is still the only admissible soln.

$\rho \le a$
$0 \le \phi \le \phi_0$.
$-\infty < z < \infty$



$\Rightarrow \psi(\rho,\phi) = \left[A_m \sin\nu\phi + B_m \cos\nu\phi\right] J_\nu(k_\rho \rho)$.

TM $\quad \psi(\rho,0) = \psi(\rho,\phi_0) = 0 \quad \Rightarrow \quad B_m = 0$ & $\nu = \dfrac{m\pi}{\phi_0}$, $m = 1,2,3\ldots$

Also $\psi(a,\phi) = 0 \quad \Rightarrow \quad K_{mn} = k_\rho = \dfrac{x_{\nu n}}{a}$, $\quad n = 1,2,3\ldots$

$$= \frac{x_{\frac{m\pi}{\phi_0} n}}{a}.$$

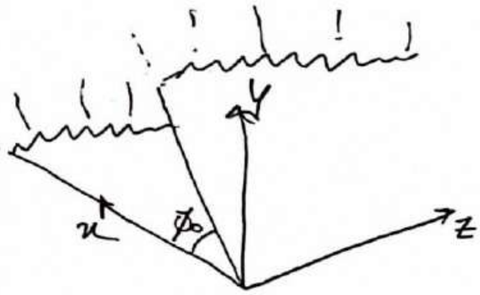
Where $x_{\nu n}$ is the n^{th} zero of $J_\nu(x)$. $\quad \nu \to \dfrac{m\pi}{\phi_0}$.

$$\therefore \quad \psi_{mn}(\rho,\phi) = A_{mn} \sin\left(\frac{m\pi}{\phi_0}\phi\right) J_{\frac{m\pi}{\phi_0}}\left(\frac{x_{\nu n}}{a}\rho\right)$$

Large Angular Sector Waveguide.

Consider it as a limiting case for $a \to \infty$ of earlier case. The domain then supports infinitely many modes with a continuous eigenvalue.

$0 < \rho < \infty$
$0 \le \phi \le \phi_0$



When $k_\rho a \to \infty \quad \lim\limits_{k_\rho a \to \infty} J_\nu(k_\rho a) = \sqrt{\dfrac{2}{\pi k_\rho a}} \cos\left(k_\rho a - \dfrac{(2\nu+1)\pi}{4}\right)$

Zeroes gives eigenvalues. Therefore $x_{\nu n} = k_{\rho_n} a = \left(n + \dfrac{\nu}{2} - \dfrac{1}{4}\right)\pi$.

The seperation between 2 modes $\Delta k_{\rho_n} = k_{\rho n+1} - k_{\rho n} = \pi/a. \to 0$ as $a \to \infty$

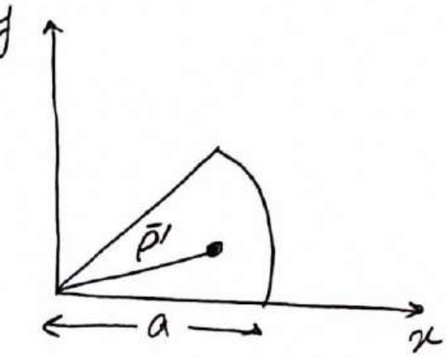
Therefore there is a continuum of modes as $a \to \infty$.

$$\psi(\rho,\phi) = \lim_{a\to\infty} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi}{\phi_0}\phi\right) J_\nu\left(\frac{x_{\nu n}}{a}\rho\right)$$

$$\Rightarrow \boxed{\psi(\rho,\phi) = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{\phi_0}\phi\right) \int_0^\infty A_m(k_\rho) J_\nu(k_\rho \rho)\, dk_\rho}$$

Green's Function for Angular Sector Waveguide.

$$g(\bar\rho,\bar\rho') = \sum_m \sum_n \frac{\psi_{mn}(\rho')\,\psi_{mn}(\rho)}{(k_{mn}^2 - k_\rho^2)\iint \psi_{mn}^2\, \rho\, d\rho\, d\phi}$$



This green's function is constructed from the eigenvalues and eigenfunctions of the waveguide.

Recall this is Eq(23) in Lec 16.

$$\rightarrow \int_0^{\phi_0} \sin^2\left(\frac{m\pi}{\phi_0}\phi\right) d\phi = \frac{\phi_0}{2} \quad \& \quad \int_0^a \rho J_\nu^2\left(\frac{x_{\nu n}}{a}\rho\right) d\rho = \frac{a^2}{2} J_{\nu+1}^2(x_{\nu n})$$

$$\Rightarrow g(\bar\rho,\bar\rho') = \frac{4}{a^2\phi_0}\sum_m\sum_n \frac{\sin\left(\frac{m\pi}{\phi_0}\phi\right)\sin\left(\frac{n\pi}{\phi_0}\phi'\right) J_{\frac{m\pi}{\phi_0}}\left(\frac{x_{\nu n}}{a}\rho\right) J_{\frac{m\pi}{\phi_0}}\left(\frac{x_{\nu n}}{a}\rho'\right)}{(k_{mn}^2 - k_\rho^2)\, J_{\nu+1}^2(x_{\nu n})}$$

where $k_{mn} = \dfrac{x_{\nu n}}{a}$ & $\nu = \dfrac{m\pi}{\phi_0}$

Also note that $\displaystyle\lim_{x_{\nu n}\to\infty} J_{\nu+1}(x_{\nu n}) = (-1)^n \sqrt{\frac{2}{\pi x_{\nu n}}}$

$$\therefore g(\bar\rho,\bar\rho') = \frac{2}{\phi_0}\sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{\phi_0}\phi\right)\sin\left(\frac{m\pi}{\phi_0}\phi'\right)\int_0^\infty \frac{\alpha}{\alpha^2 - k_\rho^2} J_{\frac{m\pi}{\phi_0}}(\alpha\rho) J_{\frac{m\pi}{\phi_0}}(\alpha\rho')\, d\alpha$$

By setting $\phi_0 = 2\pi$ we get the solution for a circular waveguide with a metallic vane (fin)

$$g(\bar{\rho},\bar{\rho}') = \frac{2}{a^2\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{m\phi}{2}\right)\sin\left(\frac{m\phi'}{2}\right)J_{m/2}\left(\frac{x_n}{2}\rho\right)J_{m/2}\left(\frac{x_n}{2}\rho'\right)}{\left(K_{mn}^2 - K_\rho^2\right)J^2_{(m+3)/2}(x_n)}$$

where x_n is the n^{th} zero of $J_{m/2}(x)$.

→ The $K_{mn}^2 - K_\rho^2$ makes the solution blowup & also computing the double summation can be tedious. To overcome these we look at a different form for the Green's Function.

Formal Solution for Green's Function of Angular Sector.

→ The Green's function is obtained from delta functions.

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+k_\rho^2\right)g(\bar{\rho},\bar{\rho}') = -\frac{\delta(\rho-\rho')\,\delta(\phi-\phi')}{\rho}$$

For TM case $g(\bar{\rho},\bar{\rho}')$ must vanish on the waveguide surface.

$$g(\bar{\rho},\bar{\rho}') = \sum_m g(\rho,\rho')\, F_m(\phi,\phi') \longrightarrow \text{M.O.S.O.V.}$$

Also $\delta(\phi-\phi') = \frac{2}{\phi_0}\sum_{m=1}^{\infty}\sin\left(\frac{m\pi}{\phi_0}\phi'\right)\sin\left(\frac{m\pi}{\phi_0}\phi\right)$

Substitute above.

$$\Rightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}-\left(\frac{m\pi}{\phi_0}\right)^2\frac{1}{\rho^2}+k_\rho^2\right)g(\rho,\rho') = \frac{-\delta(\rho-\rho')}{\rho}$$

→ $g(\bar{\rho},\bar{\rho}')$ must be continuous for the field to be continuous. However the derivative of g must be discontinuous since its second derivative is of a delta form. The derivative has a discontinuity of $\frac{-1}{\rho'}$. Let $g(\bar{\rho},\bar{\rho}')$ for $\rho<\rho'$ be $g_<(\rho,\rho')$ & $\rho>\rho'$ be $g_>(\rho,\rho')$.

$\Rightarrow g_< = A J_\nu(k_\rho\rho)$ since the domain includes $\rho=0$.

$g_> = B J_\nu(k_\rho\rho) + C H_\nu^{(1)}(k_\rho\rho)$ where $\nu = \frac{m\pi}{\phi_0}$

At $\rho=\rho'$ $\quad g_< = g_>$

A continuous fn can have a discontinuous derivative

Also $\frac{\partial g_>}{\partial \rho} - \frac{\partial g_<}{\partial \rho} = -\frac{1}{\rho'}$

At $\rho = a \Rightarrow g_>(a) = B J_\nu(k_\rho a) + C H_\nu^{(1)}(k_\rho a) = 0.$

B.C at $\rho = \rho' \Rightarrow$

$$\left.\begin{aligned} (A-B) J_\nu(k_\rho \rho') - C H_\nu^{(1)}(k_\rho \rho') &= 0 \\ (A-B) J_\nu'(k_\rho \rho') - C H_\nu^{(1)'}(k_\rho \rho') &= \frac{1}{k_\rho \rho'} \end{aligned}\right\} \quad (48)$$

Solving (48) we have,

$$C = \frac{-J_\nu(k_\rho \rho')}{k_\rho \rho' \left[J_\nu(k_\rho \rho') H_\nu^{(1)'}(k_\rho \rho') - J_\nu'(k_\rho \rho') H_\nu^{(1)}(k_\rho \rho') \right]}$$

$$A - B = \frac{-H_\nu^{(1)}(k_\rho \rho')}{k_\rho \rho' \left[J_\nu(k_\rho \rho') H_\nu^{(1)'}(k_\rho \rho') - J_\nu'(k_\rho \rho') H_\nu^{(1)}(k_\rho \rho') \right]}$$

Wronskian $\Rightarrow$ $W(J_\nu(x), H_\nu^{(1)}(x)) = J_\nu(x) H_\nu^{(1)'}(x) - J_\nu'(x) H_\nu^{(1)}(x)$

$$= \frac{2i}{\pi x}$$

$$\therefore \quad C = \frac{-\pi}{2i} J_\nu(k_\rho \rho')$$

$$B = -\frac{i\pi}{2} \frac{J_\nu(k_\rho \rho') H_\nu^{(1)}(k_\rho a)}{J_\nu(k_\rho a)}$$

$$A = \frac{\pi i}{2} \left[H_\nu^{(1)}(k_\rho \rho') - \frac{J_\nu(k_\rho \rho') H_\nu^{(1)}(k_\rho a)}{J_\nu(k_\rho a)} \right]$$

$$\therefore\ g_<(\bar{\rho},\bar{\rho}') = \frac{\pi i}{\phi_0}\sum_{m=1}^{\infty} J_\nu(k_\rho \rho)\left[H_\nu^{(1)}(k_\rho\rho') - \frac{J_\nu(k_\rho\rho')\,H_\nu^{(1)}(k_\rho a)}{J_\nu(k_\rho a)}\right]$$

$$\times \sin\nu\phi\,\sin\nu\phi'$$

$$g_>(\bar{\rho},\bar{\rho}') = \frac{\pi i}{\phi_0}\sum_{m=1}^{\infty} J_\nu(k_\rho \rho')\left[H_\nu^{(1)}(k_\rho\rho) - \frac{J_\nu(k_\rho\rho)\,H_\nu^{(1)}(k_\rho a)}{J_\nu(k_\rho a)}\right]$$

$$\times \sin\nu\phi\,\sin\nu\phi'.$$

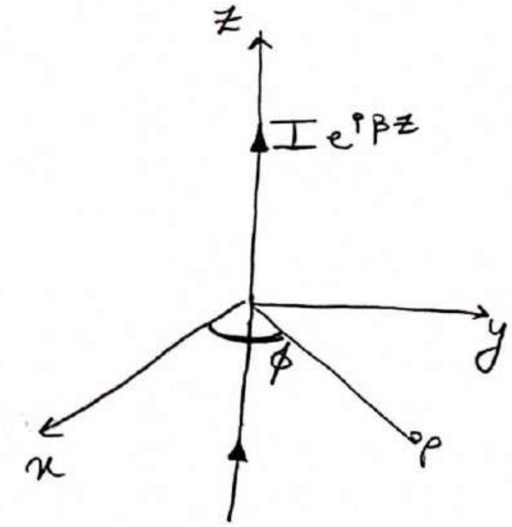
→ Note that reciprocity works here.
If $\rho \leftrightarrow \rho'$, then $g_< \leftrightarrow g_>$.

where $\nu = \frac{m\pi}{\phi_0}$

2D Green's function for Unbounded Homogeneous Medium

Consider an infinitely long filament along z axis

$\bar{J}(\bar{r}) = I e^{i\beta z}\delta(\bar{\rho})\,\hat{z}$



→ Only Π_z is produced ⇒ only TM.

$$\nabla^2\Pi_z + k^2\Pi_z = \frac{-i}{\omega\epsilon} I e^{i\beta z}\delta(\bar{\rho}).$$

$\Pi_z(\rho,\phi,z) = \Psi(\rho,\phi)e^{i\beta z}$ since Π_z is along z.

$$\nabla_t^2\Psi(\rho,\phi) + k_\rho^2\Psi(\rho,\phi) = -\frac{i}{\omega\epsilon} I\,\delta(\bar{\rho}) \quad \text{where } k_\rho^2 = k^2 - \beta^2$$

→ Hankel functions are an admissible solution everywhere except $\bar{\rho}=0$ since RHS is zero and as $\bar{\rho}=0$ $\bar{H}$ goes to ∞ since a current impulse is used. Also $H^{(1)}$ is used since wave is outgoing.

$$\Rightarrow \boxed{\Psi(\rho,\phi) = A\,H_0^{(1)}(k_\rho\rho)}$$

$$\bar{H}(\rho,\phi) = -i\omega\epsilon K_\rho A H_0^{(1)\prime}(K_\rho\rho)\,\hat{\rho}\times\hat{z}$$

$$= i\omega\epsilon k_\rho A H_0^{(1)\prime}(k_\rho\rho)e^{i\beta z}\hat{\phi}$$

According to Ampere's Law $\lim_{\rho\to 0}\oint_c H_\phi(\rho,\phi)\rho\, d\phi = I.$

↳ no displacement current.

$$\Rightarrow \lim_{\rho\to 0}\int_0^{2\pi} A i\omega\epsilon K_\rho H_0^{(1)\prime}(K_\rho\rho)e^{i\beta z}\rho\, d\phi = I$$

Small argument expansion of Hankel function.

$$\Rightarrow \lim_{X\to 0} H_0^{(1)}(X) \simeq -\frac{i.2}{\pi}\ln\left(\frac{2}{\gamma X}\right)$$

where $\gamma=1.78$

$$\Rightarrow \quad H_0^{(1)\prime}(K_\rho\rho) \simeq \frac{i2}{\pi}\cdot\frac{1}{K_\rho\rho}$$

$$\therefore \quad i\omega\epsilon K_\rho A\int_0^{2\pi}\frac{i2}{\pi}\frac{1}{K_\rho\rho}\rho\, d\phi = -4\omega\epsilon A = I$$

$$\Rightarrow \boxed{A = \frac{-I}{4\omega\epsilon}}$$

$$\therefore \bar{H}(\rho,\phi) = \frac{-iK_\rho I}{4}H_0^{(1)\prime}(K_\rho\rho)e^{i\beta z}\hat{\phi}$$

$$\bar{E}(\rho,\phi) = \frac{-K_\rho^2 I}{4\omega\epsilon}H_0^{(1)}(K_\rho\rho)e^{i\beta z}\hat{z} - \frac{iI\beta K_\rho}{4\omega\epsilon}H_0^{(1)\prime}(K_\rho\rho)\hat{\rho}e^{i\beta}$$

If the current filament is moved to $\bar{\rho}'$ instead.

$$\bar{H}(\rho,\phi) = -\frac{iK_\rho}{4}\iint_S J(\bar{\rho}')\, H_0^{(1)'}(K_\rho|\bar{\rho}-\bar{\rho}'|)\,\frac{\hat{z}\times(\bar{\rho}-\bar{\rho}')}{|\bar{\rho}-\bar{\rho}'|}\,\rho' d\rho' d\phi'$$

Basically azimuthal symmetry is lost, so $\hat{\phi} = \hat{z}\times\hat{\rho} = \hat{z}\times\frac{\bar{\rho}}{|\rho|}$

is used to replace $\hat{\phi}$ with $\hat{z}\times\frac{(\bar{\rho}-\bar{\rho}')}{|\bar{\rho}-\bar{\rho}'|}$

Similarly

$$\bar{E}(\rho,\phi) = \frac{-K_\rho}{4\omega\epsilon}e^{i\beta z}\left\{\iint_S\left[K_\rho H_0^{(1)}(K_\rho|\bar{\rho}-\bar{\rho}'|)\hat{z} + i\beta H_0^{(1)'}(K_\rho|\bar{\rho}-\bar{\rho}'|)\hat{\rho}\right]J(\bar{\rho}')\rho' d\rho' d\phi'\right\}$$

Where $|\bar{\rho}-\bar{\rho}'| = \sqrt{(x-x')^2-(y-y')^2}$

Far field expressions.

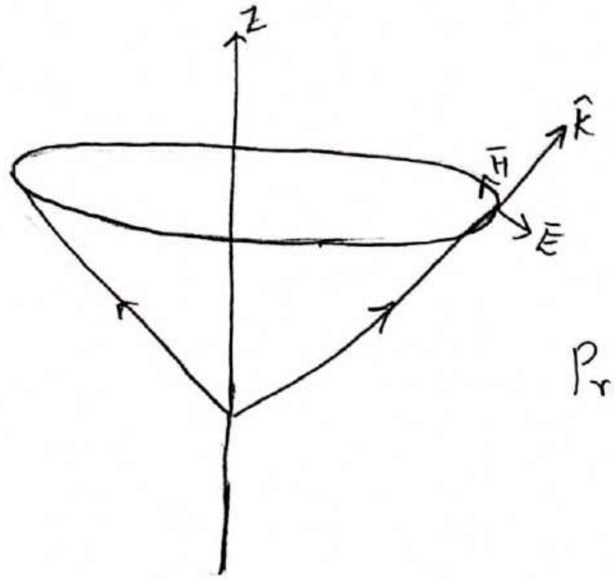
$$H_\phi(\rho,\phi) \simeq K_\rho I\sqrt{\frac{1}{8\pi K_\rho\rho}}\,e^{i(K_\rho\rho-\pi/4)}e^{i\beta z}$$

$$E_z(\rho,\phi) \simeq -\eta\frac{K_\rho^2}{K} I\sqrt{\frac{1}{8\pi K_\rho\rho}}\,e^{i(K_\rho\rho-\pi/4)}e^{i\beta z}$$

$$E_\rho(\rho,\phi) \simeq \eta\frac{K_\rho\beta}{K} I\sqrt{\frac{1}{8\pi K_\rho\rho}}\,e^{i(K_\rho\rho-\pi/4)}e^{i\beta z}$$

$\Rightarrow \frac{|E|}{|H_\phi|} = \eta = \sqrt{\frac{\mu}{\epsilon}}$ in the far field.

Conical shaped waves are produced that are TEM & TM w.r.t $\hat{z}$



The total power per unit length radiated from the filament by integrating the Poynting vector over the surface of the cylinder of height 1m.

$$P_{rad} = \frac{1}{2}\int_0^{2\pi}\int_0^1 Re\,(E \times H^*)\cdot\hat{\rho}\,\rho\, d\phi\, dz$$

$$= \frac{1}{2} Re\left\{ \frac{i\pi k_\rho^3 |I|^2}{8\omega\epsilon}\rho\, H_0^{(1)}(k_\rho\rho)\left[H_0^{(1)'}(k_\rho\rho)\right]^*\right\}$$

Using Wronskian

$J_\nu N_\nu'(x) - J_\nu'(x) N_\nu(x) = \frac{2}{\pi x}$ $\Rightarrow$ $\boxed{P_{rad} = \frac{k_\rho^2}{8\omega\epsilon}|I|^2}\ \frac{W}{m}$

Independant of ρ & $\propto I^2$.

Fourier Representation of Two Dimensional Green's Function

Let $\tilde{\psi}(k_x, y) = \int_{-\infty}^{\infty} \psi(x,y)\, e^{-ik_x x}\, dx$ be the F.T of ψ.

$$\therefore \quad \psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\psi}(k_x,y)\, e^{ik_x x}\, dk_x.$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1\cdot e^{ik_x x}\, dx.$$

$$\therefore \nabla^2\psi(\rho,\phi) + k_\rho^2\psi(\rho,\phi) = \frac{-i}{\omega\epsilon} I\,\delta(\rho) \quad \text{can be written}$$

as : $$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\left(-k_x^2 + \frac{d^2}{dy^2}\right)\tilde{\psi}(k_x,y) + k_\rho^2\tilde{\psi}(k_x,y)\right]e^{ik_x x}\,dk_x$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-iI}{\omega\epsilon}\,\delta(y)\,e^{ik_x x}\,dk_x.$$

The equality implies integrands are equal.

$$\left[\frac{d^2}{dy^2} + (k_\rho^2 - k_x^2)\right]\tilde{\psi}(k_x,y) = -\frac{iI}{\omega\epsilon}\,\delta(y)$$

Which is an ODE of 2nd order.

$$\therefore \tilde{\psi}(k_x,y) = \begin{cases} A(k_x)\,e^{ik_y y} & y>0 \\ B(k_x)\,e^{-ik_y y} & y<0 \end{cases}$$

Where $k_y = \sqrt{k_\rho^2 - k_x^2}$. Continuity of $\tilde{\psi}$ & jump discontinuity of $\tilde{\psi}'$ at $y=0$ are used to find $A(k_x)$ & $B(k_x)$.

$$\psi(k_x,0^+) = \psi(k_x,0^-)$$

$$\frac{d}{dy}\psi(k_x,0^+) - \frac{d}{dy}\psi(k_x,0^-) = \frac{-iI}{\omega\epsilon}$$

$$\Rightarrow A = B = \frac{-I}{2k_y\omega\epsilon}$$

$$\therefore \psi(x,y) = \frac{-I}{4\pi\omega\epsilon}\int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{K_\rho^2 - K_x^2}}\, e^{i(K_x x + K_y |y|)}}_{\text{plane wave propagating in } xy \text{ plane along } \frac{K_x}{K_\rho}\hat{x} + \frac{K_y}{K_\rho}\hat{y}.}\, dK_x.$$

Also $\psi(x,y) = \frac{-I}{2\omega\epsilon} H_0^{(1)}\left(K_\rho\sqrt{x^2+y^2}\right)$

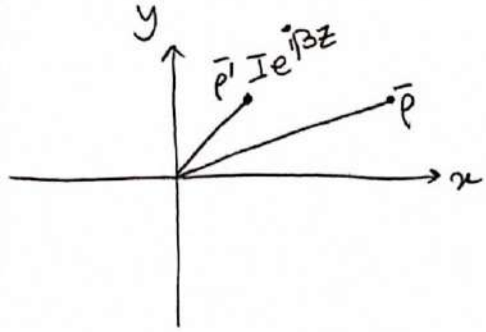
$$\Rightarrow \boxed{H_0^{(1)}\left(K_\rho\sqrt{x^2+y^2}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{\sqrt{K_\rho^2-K_x^2}}\, e^{i\left(K_x x + \sqrt{K_\rho^2-K_x^2}\,|y|\right)}\, dK_x}$$

This integral form is used to derive the various asymptotic forms of $H_0^{(1)}$. Branch cuts & saddle point methods are used to do cool stuff.

Series Solution for 2D Green's Function of Homogeneous Media

- From 2D Helmholtz and M.O.S.O.V,

$$g(\bar{\rho},\bar{\rho}') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} g(\rho,\rho')\, e^{im(\phi-\phi')}$$



The entire ϕ domain is included in the region of interest. But there is no azimuthal symmetry

Let $g_<$ for $\rho \le \rho'$ $\quad\quad \nu \longrightarrow m.$

$g_>$ for $\rho \ge \rho'$

$$\Rightarrow \quad g_< = A J_m(k_\rho \rho)$$

$$g_> = C H_m^{(1)}(k_\rho \rho)$$

Continuity & jump discontinuity are used to show that,

$$A = \frac{\pi i}{2} H_m^{(1)}(k_\rho \rho')$$

$$C = \frac{\pi i}{2} J_m(k_\rho \rho')$$

$$\Rightarrow \quad g_<(\rho,\rho') = \frac{i}{4}\sum_{m=-\infty}^{\infty} H_m^{(1)}(k_\rho \rho') J_m(k_\rho \rho)\, e^{im(\phi-\phi')}$$

$$g_>(\rho,\rho') = \frac{i}{4}\sum_{m=-\infty}^{\infty} J_m(k_\rho \rho') H_m^{(1)}(k_\rho \rho)\, e^{im(\phi-\phi')}$$

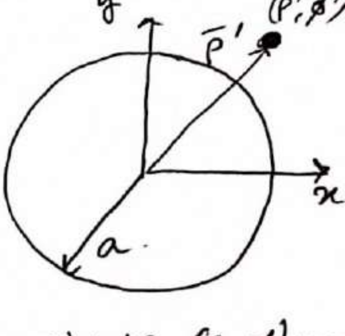
$$\therefore\ H_0^{(1)}(k_\rho|\rho-\rho'|) = \begin{cases} \sum_{m=-\infty}^{\infty} H_m^{(1)}(k_\rho\rho') J_m(k_\rho\rho)\, e^{im(\phi-\phi')} & \rho<\rho' \\ \sum_{m=-\infty}^{\infty} J_m(k_\rho\rho') H_m^{(1)}(k_\rho\rho)\, e^{im(\phi-\phi')} & \rho\geq\rho'. \end{cases}$$

This is a very useful expansion of the Hankel function for scattering problems.

Can be considered as Fourier coefficients of $H_0^{(1)}$.

Eq (45)*

2D Green's Function for Homogeneous Medium in the presence of a Metallic Cylinder.



→ In the absence of cylinder, the potential produced by the filament is,

$$\psi_i(\rho,\phi) = -\frac{I}{4\epsilon\omega} H_0^{(1)}(k_\rho|\bar{\rho}-\bar{\rho}'|).$$

The scattered wave must satisfy the B.C: $\psi_i(a,\phi)+\psi_s(a,\phi)=0$

→ Expressing ψ_s in terms of elementary cylindrical wave functions

$$\psi_s(\rho,\phi) = \sum_{m=-\infty}^{\infty} A_m H_m^{(1)}(k_\rho\rho)\, e^{im(\phi-\phi')}$$

→ Using (45) to expand ψ_i for $\rho<\rho'$ & $\rho=a$;

$$\psi_i(a,\phi) = \frac{-I}{4\epsilon\omega}\sum_{m=-\infty}^{\infty} H_m^{(1)}(k_\rho\rho') J_m(k_\rho a)\, e^{im(\phi-\phi')}.$$

$$\to \text{BC}:\Rightarrow \sum_{m=-\infty}^{\infty}\underbrace{\left[A_m H_m^{(1)}(k_\rho a) - \frac{I}{4\epsilon\omega} H_m^{(1)}(k_\rho\rho') J_m(k_\rho a)\right]}\, e^{im(\phi-\phi')} = 0.$$

This term must be 0 since $e^{im(\phi-\phi')}$ are all orthogonal & their linear combinations will not cancel. Proof: × both sides by $e^{im'(\phi-\phi')}$ & $\int_0^{2\pi}$.

$$\Rightarrow A_m = \frac{I}{4\epsilon\omega}\frac{J_m(k_\rho a) H_m^{(1)}(k_\rho\rho')}{H_m^{(1)}(k_\rho a)}$$

. Therefore, the total potential is

$$\psi_t(\rho,\phi) = \begin{cases} \frac{I e^{i\beta z}}{4\epsilon\omega}\sum_{m=-\infty}^{\infty}\left[\frac{J_m(k_\rho a)}{H_m^{(1)}(k_\rho a)} H_m^{(1)}(k_\rho\rho) - J_m(k_\rho\rho)\right] H_m^{(1)}(k_\rho\rho')\, e^{im(\phi-\phi')} & \rho\leq\rho' \\ \frac{I e^{i\beta z}}{4\epsilon\omega}\sum\left[\frac{J_m(k_\rho a)}{H_m^{(1)}(k_\rho a)} H_m^{(1)}(k_\rho\rho') - J_m(k_\rho\rho')\right] H_m^{(1)}(k_\rho\rho)\, e^{im(\phi-\phi')} & \rho\geq\rho' \end{cases}$$

Vector Calculus properties.

$A \rightarrow$ vector field.
$\psi \rightarrow$ scalar field.

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Gradient

$$\nabla\psi = \frac{\partial\psi}{\partial x}\hat{i} + \frac{\partial\psi}{\partial y}\hat{j} + \frac{\partial\psi}{\partial z}\hat{k}$$

Properties

$$\nabla(\psi\phi) = \psi\nabla\phi + \phi\nabla\psi$$

$$\nabla(\vec{u}\cdot\vec{v}) = (\vec{u}\cdot\nabla)\vec{v} + (\vec{v}\cdot\nabla)\vec{u} + \vec{u}\times(\nabla\times\vec{v}) + \vec{v}\times(\nabla\times\vec{u})$$

Divergence

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Properties

* $\nabla\cdot(\psi\vec{A}) = \psi(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla\psi)$

$$\nabla\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(\nabla\times\vec{u}) - \vec{u}\cdot(\nabla\times\vec{v})$$

$$\nabla\cdot(\psi\nabla\phi) = \psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi$$

Curl.

$$\nabla\times\vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Properties.

$$\nabla\times(\psi\vec{A}) = (\nabla\psi)\times\vec{A} + \psi(\nabla\times\vec{A})$$

$$\nabla\times(\vec{u}\times\vec{v}) = \vec{u}(\nabla\cdot\vec{v}) - \vec{v}(\nabla\cdot\vec{u}) + (\vec{v}\cdot\nabla)\vec{u} - (\vec{u}\cdot\nabla)\vec{v}$$

$$\nabla\times(\psi\nabla\phi) = \nabla\psi\times\nabla\phi$$

$$(\vec{A}\cdot\nabla)\vec{A} = \frac{1}{2}\nabla|A|^2 + (\nabla\times A)\times A$$

Laplacian.

$$\nabla^2\psi = \nabla\cdot\nabla\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

Distributive properties.

$$\nabla(\psi+\phi) = \nabla\psi + \nabla\phi$$

$$\nabla(\vec{A}+\vec{B}) = \nabla\vec{A} + \nabla\vec{B}$$

$$\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot B$$

$$\nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A} + \nabla\times\vec{B}$$

Product rule.

$$\nabla(\psi\phi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla(\psi\vec{A}) = \nabla\psi \overset{\text{tensor product}}{\otimes} \vec{A} + \psi\nabla\vec{A}$$

$$\nabla\cdot(\psi\vec{A}) = \psi(\nabla\cdot\vec{A}) + (\nabla\psi)\cdot\vec{A}$$

$$\nabla\times(\psi\vec{A}) = \psi\nabla\times\vec{A} + (\nabla\psi)\times\vec{A}$$

Quotient rule

$$\nabla\left(\frac{\psi}{\phi}\right) = \frac{\phi\nabla\psi - (\nabla\phi)\psi}{\phi^2}$$

$$\nabla\cdot\left(\frac{\vec{A}}{\phi}\right) = \frac{\phi\nabla\cdot\vec{A} - (\nabla\phi)\cdot\vec{A}}{\phi^2}$$

$$\nabla\times(\vec{A}/\phi) = \frac{\phi\nabla\times\vec{A} - (\nabla\phi)\times\vec{A}}{\phi^2}$$

Scalar Triple product, cyclic.

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

$$\vec{A}\cdot\vec{B}\times\vec{C} = (\vec{A}\times\vec{B})\cdot\vec{C} \rightarrow \text{swapping operations.}$$

$$\left.\begin{aligned}\vec{A}\cdot\vec{B}\times\vec{C} &= -\vec{A}\cdot(\vec{C}\times\vec{B})\\ &= -\vec{B}(\vec{A}\times\vec{C})\\ &= -\vec{C}(\vec{B}\times\vec{A})\end{aligned}\right\}\text{ swapping 2 vectors}$$

Vector Triple product

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}$$

Vector Triple product contd.

$$(A\times B)\times C = -C\times(A\times B) = -(C\cdot B)A + (C\cdot A)B$$

$$A\times(B\times C) + B\times(C\times A) + C\times(A\times B) = 0.$$

$$\star\ \nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - (\nabla\cdot\nabla)A$$

Identities.

$$\nabla\times(\nabla\psi) = 0$$

$$\nabla\cdot(\nabla\times A) = 0$$

$$\nabla^2\psi = \nabla\cdot(\nabla\psi)$$

Second derivatives.

$$\psi\nabla^2\phi - \phi\nabla^2\psi = \nabla\cdot(\psi\nabla\phi - \phi\nabla\psi)$$

$$\nabla^2(\psi\phi) = \phi\nabla^2\psi + 2(\nabla\phi)\cdot(\nabla\psi) + (\nabla^2\phi)\psi$$

$$\nabla^2(\psi A) = A\nabla^2\psi + 2(\nabla\psi\cdot\nabla)A + \psi\nabla^2 A$$

$$\nabla^2(A\cdot B) = A\cdot\nabla^2 B - B\cdot\nabla^2 A + 2\nabla\cdot((B\cdot\nabla)A + B\times(\nabla\times A)) \rightarrow \text{Green's vector identity.}$$

Third derivatives

$$\nabla^2(\nabla\psi) = \nabla(\nabla\cdot(\nabla\psi)) = \nabla(\nabla^2\psi)$$

$$\nabla^2(\nabla\cdot A) = \nabla\cdot(\nabla(\nabla\cdot A)) = \nabla\cdot(\nabla^2 A)$$

$$\nabla^2(\nabla\times A) = -\nabla\times(\nabla\times(\nabla\times A)) = \nabla\times(\nabla^2 A)$$

Integration properties.

$$\oiint_S A\cdot ds = \iiint_V (\nabla\cdot A)\, dV$$ Divergence Theorem.

$$\oiint_S \psi\, dS = \iiint_V \nabla\psi\, dV$$

$$\oiint_S A\times ds = -\iiint_V \nabla\times A\, dV$$

$$\oiint_S \psi\nabla\phi\cdot ds = \iiint_V (\psi\nabla^2\phi + \nabla\phi\cdot\nabla\psi)\, dV$$ Green's first identity

$$\oiint_S (\psi\nabla\phi - \phi\nabla\psi)\cdot ds = \oiint_S \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right) dS = \iiint_V (\psi\nabla^2\phi - \phi\nabla^2\psi)\, dV$$

$$\iiint_V A\cdot\nabla\psi\, dV = \oiint_S \psi A\cdot ds - \iiint_V \psi\nabla\cdot A\, dV$$

$$\iiint_V \psi\nabla\cdot A\, dV = \oiint_S \psi A\cdot ds - \iiint_V \nabla\psi\cdot A\, dV$$

$$\iint_S (\nabla\times A)\cdot dS = \oint_C A\cdot dl.$$

$$\iint_S \nabla\psi\times ds = -\oint_C \psi\, dl.$$

$$\iiint_V \nabla\psi\, dV = \oiint_S \psi\, dS$$

$$\oint_C A\cdot dl = -\oint_C A\cdot dl.$$

counterclockwise clockwise.