①

Differentiable Manifolds

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EH 5850
Fri: 11-2 | Tue: 5-6
Mon: 12-1 |
HW due on Wed.
Midterm (Oct 19?)
Final (Dec 15? 4-6pm)

- ✱ John Lee
- * Loring Tu
- * (Munkres)
- ✱ Milnor (Intro to Diff Mani
- * Guillemin ...

Motivation: Linearization.

> Constructions of new manifolds

> Inability to embed manifolds in spaces?

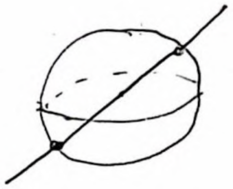
Eg: Klein bottle $\not\hookrightarrow \mathbb{R}^N$

Eg: $S^n \subset \mathbb{R}^{n+1}$

$\parallel$

$\{x \in \mathbb{R}^{n+1} \mid \|x\|^2 = 1\}$

$\rightsquigarrow$



$\mathbb{RP}^n \to$ real projective space.
(space of lines through the origin)

Here, $S^n \rightsquigarrow S^n/\sim$ $x \sim y$ if $y = -x$

> To differentiate, just need a good "local" structure.

> $\mathbb{CP}^n = \{1\text{-dim } \mathbb{C}\text{-lines in } \mathbb{C}^{n+1}\}$

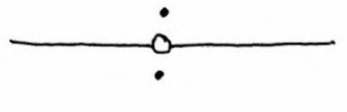
A sphere in $\mathbb{C}^{n+1}$ is S^{2n+1} & intersects complex plane in a circle S^{2n-1}.

$\Rightarrow$ points in $\mathbb{CP}^n$ are covered by circles.

Def^n: A top space M (Hausdorff, 2nd countable (∃ countable basis of the top)) is called an n-dim topological manifold if every pt. $p \in M$ has a nbd U which is homeomorphic to $\mathbb{R}^n$.

$$\begin{tikzcd} S^n \arrow[d] \\ \mathbb{RP}^n \end{tikzcd}$$

Non ex (double origin real line).



Not Hausdorff.

Lec 2 (08/31/2022)

Def Collection $\mathcal{X}$ of sets $\subset M$; $\mathcal{X}$ is called locally finite if each $p \in M$ has a nbd U s.t U only intersects finitely many $C \in \mathcal{X}$.

Def: M top. space is called paracompact if every open cover $\mathcal{X}$ of M admits a locally finite subcover

Def: A cover of M is a collection $\mathcal{X}$ s.t $\bigcup_{C \in \mathcal{X}} C = M$

Def: Subcover of a cover $\mathcal{X}$ is a cover $\mathcal{X}^*$ s.t every $U \in \mathcal{X}^*$ is contained in some $U \in \mathcal{X}$. $\mathcal{X}^*$ is also called a refinement.

Def: A cover $\mathcal{X}$ is open if $\forall C \in \mathcal{X}$, C is open.

(Thm) Topological manifolds are paracompact

$$\swarrow \quad \downarrow \quad \searrow$$

Hausdorff Second Countable Locally Euclidean.

Prop: A 2nd countable loc. compact Hf space M admits an exhaustion by compact sets.

Def: M is called locally compact if for every pt. $p \in M$ and nbds U of p $\exists$ nbd $V \subset U$ s.t $\overline{V} \subset U$ is compact

Lemma :- Top mf are locally compact.

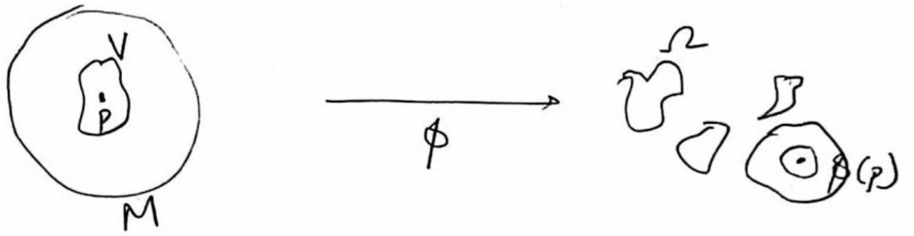
Pf : Locally Euclidean ⇒ $\mathbb{R}^n$ is locally compact.

Top Mf M

- Loc Eucl.

$\forall p \in M \ \exists\ U$ nbd of p & a homeo $U \to \mathbb{R}^n$ exists.

Good enough that $\forall p \in M \ \exists\ V$ nbd & a homeo $\phi : V \to$ open set in $\mathbb{R}^n$.



Def : An exhaustion is a sequence of sets $K_1, \dots K_n, \dots$ S.T $\bigcup K_n = M$

& $K_1 \subset K_2 \cdots \subset K_n \subset K_{n+1} \subset \dots$

Proof of prop.

1 $\exists$ a basis of $\overset{\nearrow \text{ closure is compact}}{\underline{\text{pre compact}}}$ open sets, since M is locally compact.

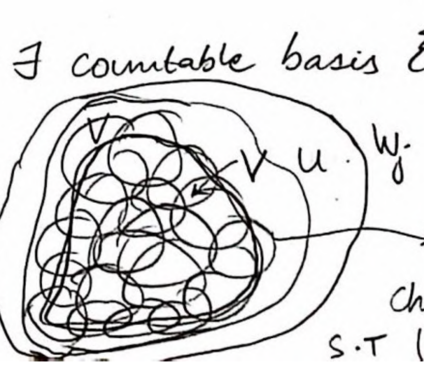
2 2nd countable ⇒ $\exists$ countably many precompact open sets $\{U_i\}_{i=1,\dots}$ S.T $\bigcup U_i = M$

Idea: $p \in M$ (2nd countable ⇒ $\exists$ countable basis $\{W_j\}$)

$p \in W_j$

$\exists\, p \in V$ S.T $\underset{\uparrow \text{compact}}{\overline{V}} \subset W_j$

(by local compactness)



$\in \{W_\ell\}$ Countable

Choose $W_\ell \subset W_j$

S.T $\bigcup W_\ell \supset \overline{V}$

Then can get $W_{\ell_1}, \dots, W_{\ell_k}$ which cover $\bar{V}$. Make the previous argument inside some nbd U which is precompact thus all $W_{\ell_i} \subset U$.

Then W_{ℓ_i} are precompact. $\overline{W_{\ell_i}} \subset \overline{U_i}$ compact.

3 Let's define the exhaustion by compact sets:

$K_1 := \bar{U}_1$

Recursively, if $K_1, \dots, K_k$ are defined. $K_1 \subset K_2 \subset \dots K_k$, K_i are compact.

~~K_m~~ sets $U_i \subset K_i$. K_k compact $\Rightarrow \exists$ sets $U_{k_1}, \dots, U_{k_m}$ s.t

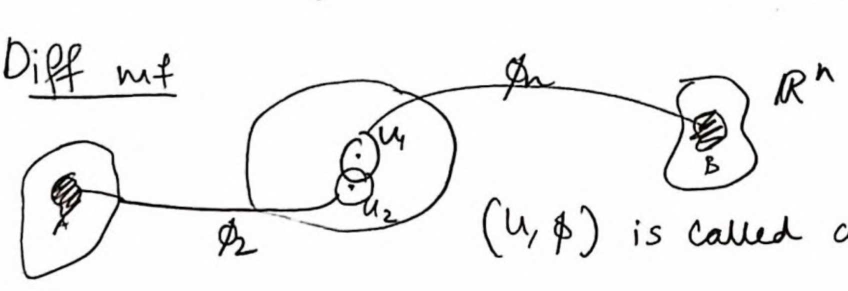
$K_k \subset U_{k_1} \cup \dots \cup U_{k_m}$.

~~Let $K^*_{k+1} := \bar{U}_{k_1} \cup \dots \cup \bar{U}_{k_m}$ compact (finite union of compact sets)~~

~~Set $K_{k+1} = K^*_{k+1} \cup \bar{U}_\ell$, with ℓ smallest not contained in K^*_{k+1}.~~

Let $K_k = \bigcup_{i=1}^{k} \bar{U}_i$

Diff mf



$\phi_1 \circ \phi_2^{-1}$

(U, ϕ) is called coordinate chart.

Call (U_1, ϕ_1) & (U_2, ϕ_2) <u>compatible</u> if $\phi_1 \circ \phi_2^{-1}$ is <u>differentiable</u>.

<u>Diff mf</u> → Compatible diff. coord. charts that cover the manifold.

Re proving the proposition

Prop: A second count loc. comp. Hf sp. admits an exhaustion by compact sets.

Pf:

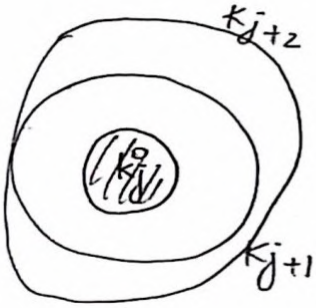
⋮ Pf in John Lee.//

Fact Top mfs M are paracompact. (i.e. open covers have loc. finite refinements).

Pf: (Lees' book).

Find an exhaustion by compact sets

$\cdots \subset K_j \subset K_{j+1} \subset \cdots \qquad \bigcup K_j = M$



Set $V_j = K_{j+1} - (\underbrace{Int(K_j)}_{K_j^o})$ Closed-compact

$W_j := \underset{Open}{Int(K_{j+2}) - K_{j-1}}$

$\Rightarrow \underset{compact}{V_j} \subset \underset{open}{W_j}$

⇒ Given an open cover $\mathcal{X} = \{\ldots\}$. Given $x \in M$, let $X_x \in \mathcal{X}$ be a set in $\mathcal{X}$ containing x.

$\mathcal{B}$ - basis countable. Find $B_x \in \mathcal{B}$ s.t $x \in B_x \subset X_x$.

V_j compact $\Rightarrow \exists$ finitely many B_{x_ℓ} which cover V_j

$\{B_x\}$ refinement of $\mathcal{X}$.

Can also arrange $B_{x_\ell} \subset W_j$.. B_{x_ℓ}'s do not touch K_{j-1} ⇒ loc. finite.

Deep Fact (100 years ago)

$\underset{\text{homeo}}{\phi}: \underset{\overset{\text{open}}{\cap}\,\mathbb{R}^n}{U} \longrightarrow \underset{\overset{\text{open}}{\cap}\,\mathbb{R}^\ell}{V} \Rightarrow n = \ell$ (Invariance of domain).

Special case

$\phi: (a,b) \xrightarrow[\text{homeo}]{} U \overset{\text{open}}{\subseteq} \mathbb{R}^\ell$

⇒ $\ell = 1$ proof: (a,b) can be disconnected by removing one pt.

Similar in $\mathbb{R}^2$ & $\mathbb{R}^3$.

Cor: dim is well defined for manifolds. (dim $M = \ell$ s.t. open nbds of pts. in M are homeo to $\mathbb{R}^\ell$). (M must be connected)

Convention: Assume dim. is constant for diff connected comp.

Prop: M is top Mf. M is conn ⇔ path connected.

⇐ obvious.

⇒ idea:

$X = \{y \mid \exists$ cont. path from p to $y\}$

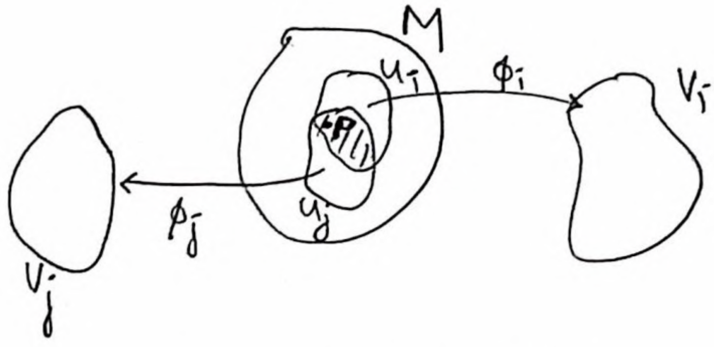
Claim: X is open & closed.

(Look at book?)

Def: Diff MF

$M^{\to \dim n}$ is a top mf. has a differentiable $\left(C^{k}\right)$ $\overset{k=1,\dots,\infty;\,\omega \to \text{real analytic}}{}$ if $\exists$ a cover by open sets U_i of M & homeos $\phi_i : U_i \to V_i \subset \mathbb{R}^n$

S.T



Note that ϕ_i and ϕ_j are defined on $U_i \cap U_j$.

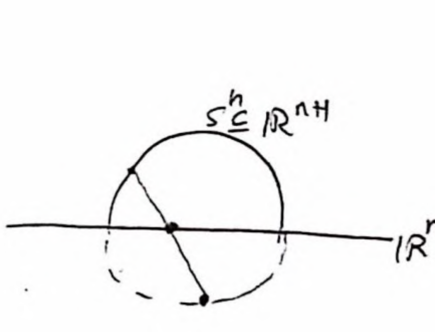
$$\underbrace{\phi_j(U_i \cap U_j)}_{\text{open}} \subset V_j \subset \mathbb{R}^n \qquad \phi_i \underbrace{(U_i \cap U_j)}_{\text{open}} \subset V_i \subset \mathbb{R}^n$$

$\Rightarrow$ Define: $\phi_j \xrightarrow[\phi_j \circ \phi_i^{-1}]{\substack{\text{Transition} \\ \text{map}}} \phi_i$

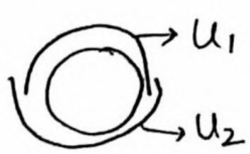
Want: $\phi_j \to \phi_i$ is C^k. diff.

This data is called an "atlas".

Ex: $\mathbb{R}^7$ atlas: $U_1 = \mathbb{R}^7$
$\phi_1 = \text{id}$



Ex: $S^1 \to$ circle =



map to $\mathbb{R}^2$ by Stereographic projection.

Thm: $\exists$ top mfl. which do not admit any differentiable structure.
(Deep)

In fact: PL manifolds (piecewise linear). are an example.

$$\begin{tikzcd} \text{Diff mf} \arrow[r, shift left] & \text{Top mf} \arrow[l, dashed, shift left, "?"] \end{tikzcd}$$
(Not invertible)

(and ∞ly many higher dims)

Milnor (late 50s) constructed exotic S^7.

ie: $M_1, \quad , M_{28}$ S.T

$M_i \underset{\text{homeo}}{\cong} M_j$ but $M_i \underset{\text{diffeo}}{\ncong} M_j$ unless $i=j$.

(differentiable structure)

> $(M, \Omega_1) \longrightarrow$ aka atlas.

(open cover, bunch of charts, transition maps)

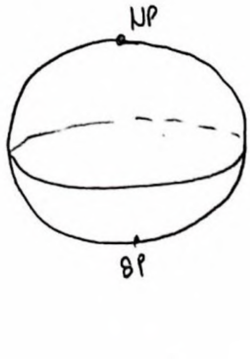
& (M, Ω_2) are "compatible" (or the same) if $\Omega_1 \cup \Omega_2$ is a differentiable structure.

> Maximum atlas = $\bigcup$ compatible atlases.

Lemma: Union of compatible atlases is an atlas.

Baubakian method $\leftarrow$ use max atlas | Hands on method
(very precise) | ↳ find your atlas & cook with it

Ex: S^2



$S^2 - \{NP\} \cup S^2 - \{SP\}$ is an atlas using stereographic projection.

Max atlas is $\cup$ of all open sets diffeo to $\mathbb{R}^2$

Construction

Products : $\left.\begin{matrix} M \\ N \end{matrix}\right\}$ diff mfs. $M\times N$ product top:

basis: $\{U\times V\}$ where U, V open in M, N resp.

$M\times N$ has diff. structure?

Say $(U_\alpha, \phi_\alpha)_\alpha$ is an atlas for M

$(V_\beta, \psi_\beta)_\beta$ . . . . N

Then $\{U_\alpha \times V_\beta\ ,\ (\phi_\alpha, \psi_\beta)\}_{\alpha,\beta}$ is an atlas for $M\times N$.

What to check?

① $\{U_\alpha \times U_\beta\}_{\alpha,\beta}$ is a cover for $M\times N$

② $(\phi_\alpha , \psi_\beta) \longrightarrow \mathbb{R}^m \times \mathbb{R}^n$

③ $(U_{\alpha_1}, U_{\beta_1}) \cap (U_{\alpha_2} \times U_{\beta_2})$ transition fns. are diff.

Quotients

① $\underset{\mathbb{C}}{\overset{\cap}{S^1}}$ $x \sim y$ if $xy^{-1} = e^{in\alpha 2\pi}$ $\alpha \in \mathbb{R}$, $n \in \mathbb{Z}$

$S^1/_\sim$

ex: ⓐ $\alpha = \frac{1}{3}$

$S^1/_\sim \underset{\text{homeo}}{\cong} S^1$

ⓑ $\alpha \notin \mathbb{Q}$. (irrational).

$[x]$ = equiv class $\sim$ of x is dense in S^1

$\Rightarrow S^1/_\sim$ is not Hausdorff. $\Rightarrow$ not mf.

> M manifold $\sim$ eq. rel. Assume $\sim$ is open i.e if $x \sim y$ $y \in \mathcal{U}$ open then $\exists$ V open nbd. of x s.t $y' \in \mathcal{U}$ is equivalent to $x' \in V$.

<u>Fact</u>: X 2^{nd} countable, $\sim$ open equivalence relation. Then

$X \xrightarrow{\pi} X/_\sim$. $x \overset{}{\underset{\pi}{\longmapsto}} [x]$ = equiv. class
endowed with quotient top.
(π is open map)

$\Rightarrow X/_\sim$ is 2^{nd} ctble

<u>Claim</u>: $\emptyset$ & $S^1/_\sim$ are the only open sets.

<u>P</u>: Suppose $\mathcal{U} \subset S^1/_\sim$ is open

$\Rightarrow \pi^{-1}(\mathcal{U}) := U$ open in S^1

Assume $\mathcal{U} \neq \emptyset$, then $V \neq \emptyset$

Take any $z \in S^1$; $[z] \cap V \neq \emptyset$

$\Rightarrow [z] \in \mathcal{U}$

$\Rightarrow \mathcal{U} = S^1/.$

> Cor: $e^{2\pi i[a,b]}$ $\pi : S^1 \longrightarrow S^1/N$

open map? Yes since every set in S^1 maps to S^1/N.

Def: Call N open if $\pi : X \longrightarrow X/N$ is open.

Equiv. if $U \subset X$ is open then $\underset{\pi^{-1}(\pi(U))}{\overset{\shortparallel}{\{x \in X \mid \exists y \in U \; x \sim y\}}}$ is open.

Prop: X is compact, Hf, & $[x]$ are closed $\forall x \in X$. (i.e. N is closed)

$\Rightarrow$ X/N is compact. & Hf.

Pf ① Hf (HW) Show b & $[a]$ can be sep. by taking $\cup$ nbds of a & $\cap$ nbds of b?

② Compact (trivial)

Prop: N is open, X is 2^{nd} ctbl Hf. $\Rightarrow X/N$ is 2^{nd} ctble.

Pf idea: Take a ctble basis of open sets for X, $\mathcal{B} = \{B_i\}_{i \in \mathbb{N}}$ then Consider $\pi(B_i)$ open, ctbl.

Aside: Look into Zariski top. on $\mathbb{R}^n$, $\mathbb{C}^n$ (Some alg. variety).

Closed sets = zero sets of polynomials. Fact: It is compact.

> G is a group. Suppose G has a topology. s.t. $G \times G \to G$, $x \times y \to x \cdot y$ cts. (↳ use product topology)

$x \mapsto x^{-1}$ cts.

Then G is a topological group.

Ex: 1) $\{1\}$

2) $\mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n, \mathbb{Z}, \mathbb{Z}^n$

3) any discrete top.

4) $GL(n,\mathbb{R})$; $GL(n,\mathbb{C})$. $\longrightarrow \{A \ n\times n \text{ mtx} \mid \det A \neq 0\}$

$\longrightarrow$ real entries $\longrightarrow$ complex entries.

5) $O(n)$ = orthogonal group.

$= \{\underset{\underset{GL(n,\mathbb{R})}{\cap}}{A} \ n\times n \mid A.A^t = I\}$ → rotations & flips.

$SO(n) = \{A \in O(n) \mid \det A = 1\}$ → rotations.

$SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \mid \det = 1\}$.

6) S^1 $(= SO(2))$

① $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$

$T^\infty = S^1 \times \cdots \times S^1 \times \cdots$ (product top) & (compact) (by Tychonoff)

② $\mathbb{Q}_p$ p-adic numbers. → is a topological field & group.

↑ completion of $\mathbb{Q}$ wrt $\| \ \|_p$ p-adic norm.

$SL(n,\mathbb{Q}_p)$
$GL(n,\mathbb{Q}_p)$

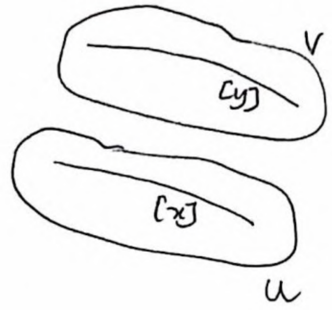
$a = p^k . a'$; $b = p^l . b'$

$\left\|\frac{a}{b}\right\| \underset{p}{=} p^{l-k}$ top. dim = 0 (Cantor set).

$p^n \xrightarrow[n\to\infty]{} 0$

Fixes from last time

X Hf, 2^{nd} ctbl compact, $\sim$ open eq. rel; $\Rightarrow X/_\sim$ is Hausdorff.
with $[x]$ compact



Problem was

$\exists\, z_1, z_2 \in U, V$ s.t $z_1 \sim z_2$

Fix:

Thm 1 : X top sp, Hf $\sim$ open eq. relation.

graph of $\sim = \Gamma := \{ (x,y) \mid \in X \times X \;\&\; x \sim y \}$

Then $X/_\sim$ is Hf iff Γ is closed in $X \times X$.

$\Rightarrow$: exercise

$\Leftarrow$: (only need this)

Suppose $x, y \in X$ & $[x] \neq [y]$. i.e. $(x,y) \notin \Gamma$

X Hf $\Rightarrow$ $X \times X$ is Hf under prod. top.

$\Rightarrow \exists\; U, V$ open in X s.t $x \in U$, $y \in V$.

and $U \times V \cap \Gamma = \phi$.

(Since $X \times X - \Gamma$ is open, $U^* \times V^*$ form a basis)

$\Rightarrow \pi(U) \cap \pi(V) = \phi$ & $\pi(U), \pi(V)$ are open.

Last time: G top group:

n torus $\rightsquigarrow T^n$

G Lie group: Ex: G Lie, $G = \mathbb{R}, S^1$; $G = \mathbb{R}^n, \overbrace{S^1 \times \dots \times S^1}^{n \text{ times}}$

$\hookrightarrow e^{it} \cdot e^{is} = e^{i(t+s)}$

$G = GL(n, \mathbb{R})$.

Note: S^2 this is not a Lie group

Reason: $\chi(S^2)$ = Euler char = 2

Thm: If a manifold M has a "non-vanishing" vector field then $\chi(M)=0$.

Note: $S^3 \subset$ Quaternion $(=\mathbb{R}^4)$ is a Lie group.

Note: $GL(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}$ (use inclusion chart).

Aside: $\exists$ exotic S^7 (with diff. structure) $= Sp(2)/Sp(1)$ (To be discussed).

Notion: G group, X space $(G \overset{\text{action}}{\curvearrowright} X)$

G acts on a space X if $\exists$ map $G\times X \to X$

$$(g\times x)\mapsto g.x.$$

S.T $\forall x$ $1.x = x$,

$\forall g_1 g_2 \in G$, $(g_1.g_2).x = g_1.(g_2.x)$

Ex: S^1 acts on S^1 $(g,x) = g\ x$ → complex product

Ex: $\mathbb{R}^n$ acts on $\mathbb{R}^n$ $(a,b)\mapsto a+b$.

$GL(n,\mathbb{R}) \curvearrowright \mathbb{R}^n$

Ψ

$A \cdot v = A \underset{\text{mult}}{\text{matrix}}\ v.$

$\overset{\text{top gr}}{G} \times \overset{\text{top manifold } (C', C^k, C^\infty)}{X}$

\> Call an action continuous cts if $G \times X \to X$ is cts.

\> The action is differentiable (C', C^k, C^∞).

\> if $G \times X \to X$ is cts. and for each $g \in G$, $x \to g$ is diff (C', C^k, C^∞)

\> if G is a Lie group. call action jointly differentiable (C', C^k, C^∞). if $G \times X \to X$ is diff(C', C^k, C^∞)

Thm: If G compact, top. group; and cont. action $G \curvearrowright X$, X compact Hf $\Rightarrow$ X/G is Hausdorff

Pf: Let $\sim$ be the orbit eq. rel.

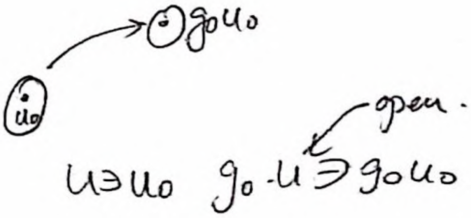
Check a) $\sim$ is open.

b)) graph $\sim = \Gamma$ is closed.

a) $U \subset X$ open.

$Y := \{y \in X \mid y \sim x\}$

$= \{g \cdot u \mid u \in U, g \in G\}$



b) Γ is closed; Γ = graph of $\sim$, $\phi: G \times X \to X \times X$

$(g, x) \mapsto (x, g \cdot x) \in \Gamma$; $\text{Im } \phi = \Gamma$. X comp. Hf $\Rightarrow$ ~~X~~X is comp. Hf

$G \times X$ compact

ϕ is cts. $\Rightarrow \phi(G \times X)$ is also compact thus closed.

(cts. image of compact in Hf is compact) ■

Ex: $X = S^n$, $G = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2 = \{1, \overset{\text{antipodal}}{A}\}$

G acts on S^n, $x \in S^n \xrightarrow{A} -x$

$S^n/\sim \; = \mathbb{RP}^n$

Ex: $S^{2n-1} \subset \mathbb{C}^n$. $e^{i\alpha} \in S^1 \curvearrowright \mathbb{C}^n$. $z = (z_1, \dots, z_n)$

$e^{i\alpha} \cdot z = (e^{i\alpha} z_1, \dots, e^{i\alpha} z_n)$

Check: This is a cts. action on S^{2n-1} by S^1. S^{2n-1}/S^1 is Hf $= \mathbb{CP}^n$ (or $\mathbb{CP}^{n-1}$?)

Ex: (General) H top. group $\overset{\text{Hf}}{}$: $G \subset H$ compact subgroup.

$G \curvearrowright H$ by $(g, h \mapsto g \cdot h)$. $\underline{\underline{H/G}}$ compact if H is compact.

$\hookrightarrow$ Important space. (Homogeneous spaces).

Ad: Homogeneous spaces are important ① you can calculate!

② "Systems with symmetry" are typically homogeneous.

③ $GL(n, \mathbb{R}) / GL(n, \mathbb{Z})$ ← not compact tho.

(Diophantine properties are studied using this)

<u>Ex</u> : $\mathbb{Z} \subset \mathbb{R}$. $\mathbb{R}/\mathbb{Z}$ is nice (compact) (Hf)

$\mathbb{Z} \curvearrowright \mathbb{R}$. $(z, a) \mapsto z+a$) . The graph of $\sim$ is closed.
↳ show this ⇒ Hf

<u>Note</u>: $\underset{\text{Subgroup}}{G} \subset H$ → Liegroup. H/G Hf ⇒ called Homogeneous.

> The quotient H/G sends each x to its orbit under G.

> G group $G \curvearrowright M$ ← want to understand M

<u>D:</u> Call the action <u>transitive</u> if M is one G-orbit.

ie. $\exists\, p \in M$ s.t $G\cdot p = \{gp \mid g \in G\} = M$.

<u>Ex:</u> $\mathbb{R}^{n+1} \supset S^n \curvearrowleft SO(n+1)$ is transitive.

$v \in S^n$

$$SO(n+1) \ni g = \begin{bmatrix} v & * \end{bmatrix}$$

↳ use to make O.N basis

$$g \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = v$$

<u>D:</u> $G \curvearrowright M \ni p$. G_p is the stabilizer of p in G = <u>isotropy group of p in G</u>

s.t $\{g \in G \mid g\cdot p\} = p$.

<u>Lemma:</u> G_p is a subgroup of P, Closed subgroup.

$$\left(\begin{array}{l} G\times M \to M \\ (g,p) \to g\cdot p \text{ is cts} \end{array}\right)$$

$$\begin{array}{ccc} g_n & \xrightarrow[n\to\infty]{\lim} & g \\ \cap & & \cap\,? \\ G_p & & G_p \end{array} \qquad g\cdot p = \lim_{n\to\infty} g_n p = p = \lim_{n\to\infty} p \Rightarrow \text{closed}.$$

<u>Future:</u> <u>(Thm)</u> G Lie group, $H \subset G$ closed subgroup then H has a Lie group structure.

HW

Ex: $SO(n+1) \curvearrowright S^n$ (dim of $\mathbb{Z} = 0$)

$$SO(n+1)_{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \,\middle|\, * \in SO(n) \right\}$$

$$S^n = SO(n+1)/SO(n) = G/G_p$$

☆

Lemma: $G \overset{\text{transitive}}{\curvearrowright} M \ni p$ (Top. orbit stabilizer thm).

G_p

Then $G/G_p \underset{\text{homeo}}{\cong} M$

Proof: $G \longmapsto M$ since G is transitive.
$g \mapsto g \cdot p$

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] & M \\ G/G_p \arrow[ur, "H"'] & \end{tikzcd}$$ ⓒⓣⓢ

$f(g \cdot x) = g \cdot \underbrace{x \cdot p}_{p} = f(g) = g \cdot p$

one-one: $F(g_1 G_p) = F(g_2 G_p) \Rightarrow g_1 \cdot p = g_2 \cdot p$

$\Rightarrow g_2^{-1} g_1 p = p$

$\Rightarrow g_2^{-1} g_1 \in G_p$.

Cts. inverse?

Ex: $GL(2, \mathbb{R}) \to \mathbb{R}^2$

$\underset{A}{\cup}$, $x \in \mathbb{R}^2$

$(A, x) \mapsto A.x$

Not transitive: $A \cdot 0 = 0$

Transitive on $\mathbb{R}^2 - \{0\}$

Transitive for any pt. equivilent to transitive $\forall$ pts.

$$0 \neq X = \underset{\substack{\pitchfork \\ GL(2,\mathbb{R})}}{A} \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}$$

$$A = \begin{pmatrix} x & y \end{pmatrix}$$

$\hookrightarrow$ pick y s-T x, y is a basis

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$P := G\ L(2,\mathbb{R}) = \left\{ \begin{pmatrix} 1 & c \\ 0 & d \end{pmatrix} \middle| \, c \in \mathbb{R}, d \neq 0 \right\}$$

$$GL(2,\mathbb{R}) / P \cong \mathbb{R}^2 - \{0\}$$

Ex: $G = GL(n+1) \circlearrowright \mathbb{R}P^n$ transitively. Want to find stabilizer of $l_1 \in \mathbb{R}P$

$$l_1 = \mathbb{R} \cdot \underset{\hookrightarrow \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}}{e_1} \longrightarrow \text{span of } e_1.$$

$$A \cdot e_1 = r \cdot e_1 \quad \text{where } r \in \mathbb{R} - \{0\}.$$

$$A = \begin{pmatrix} v & * \end{pmatrix} \cdot e_1 = \begin{pmatrix} v \end{pmatrix}$$

$$\Rightarrow G_{l_1} = \left\{ A \in GL(n+1,\mathbb{R}) \,\middle|\, A = \begin{pmatrix} r & \times \\ 0 & \boxed{\diagup} \\ \vdots & \end{pmatrix} \right. \longrightarrow \det(\bar{A}) = \frac{1}{r} \det A$$

$$\neq 0$$

$$\Rightarrow \bar{A} \in GL(n,\mathbb{R})$$

$$\Rightarrow \underset{\hookrightarrow \text{stabilizer.}}{\underline{G_{l_1}}} = \begin{pmatrix} * & \times \times \cdots \\ 0 & \\ 0 & B \\ \vdots & \end{pmatrix} ; \; B \in GL(n,\mathbb{R}).$$

> G/G_{l_1} is a diff mf.

G_{l_1} is a stabilizer.

Once we find the stabilizer. Finding the homeomorphism can be done by find elements in G transversal to the Isotropy group.

So we want $T \subset G$ "transversal" to G_{l_1}
(opposite)

& $\dim(T) = \mathbb{R}P^n = n$.

So take
$$T = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_1 & & & \\ a_2 & & \mathrm{id} & \\ \vdots & & & \\ a_n & & & \end{pmatrix} \cong \mathbb{R}^n$$

$$\Rightarrow T . l_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \cdots 0 \\ a_1 & & \\ a_2 & \mathrm{id} & \\ \vdots & & \end{pmatrix} \cdot l_1 \right\} = \mathbb{R}^n \to \text{chart to } \mathbb{R}P^n.$$

These charts give it a diff structure
to $\mathbb{R}^n$

<u>Recipe</u>

$G \curvearrowright M$ transitive. → Lie group. Want to endow M with a diff structure

Choose $p \in M$. G_p = stabilizer gp. of p.

$G_p \underset{\text{closed}}{\subset} G$. If you can find a "transversal" subspace of G maybe its locally $\mathbb{R}^n$, then try out coord. charts of the form

$T \longmapsto T.p$, $t \longmapsto t.p$.

<u>In the future</u>: G Lie, H closed subgroup. Then G/H is always a C^∞ mf.

Review ~~of~~ Grassmannian of k space?

Lecture

Ex: of a homogeneous space.

G Lie gp. $H \subset G$ closed subgroup.

$M = G/H$

$G \curvearrowright M$

$H = G_p$ for some $p \in M$.

Warning: G_p depends on p

Note: If G_p does not depend on p then?

Ex: Grassmannian of k-planes in n-space $(\mathbb{R}^n)$ $Gr_{k,n}$

Special case: $Gr_{1,n} = \mathbb{R}\mathbb{P}^{n-1}$ = homogeneous. $= GL(n,\mathbb{R}) \Big/ \begin{pmatrix} * & * \cdots * \\ 0 & \\ \vdots & \boxed{*} \\ 0 & \end{pmatrix}$

(These are all equal)

$$= \Big(\underbrace{SO(n-1)/SO(n)}_{S^{n-1}}\Big)\Big/\mathbb{Z}_2 \longleftarrow \overset{?}{=} SO(n)\Big/O(n-1)$$

detA $\begin{pmatrix} * & 0\,0 \cdots 0 \\ 0 & \\ \vdots & A \\ 0 & \end{pmatrix}$

General case:

If $e_1, \dots, e_n$ is a basis for $\mathbb{R}^n$.

$p := \text{Span}\langle e_1, \dots, e_k\rangle$ = k-dim subspace of $\mathbb{R}^n$.

$$\begin{matrix} GL(n,\mathbb{R}) & \curvearrowright & Gr_{k,n} \\ \Psi & & \Psi \\ A & & V \end{matrix}$$ → k dim subsp.

$A \cdot V = \{A \cdot v \mid v \in V\}$ prove this is an action.

Transitive?

$V = \langle v_1 \ \ v_k \rangle$

Let $A = (\overset{\curvearrowright \text{augment to a basis} \Rightarrow \text{Transitive.}}{v_1 \ , v_k, ???})$ $\Rightarrow$ A can take any form.

Thm $\Rightarrow$ $Gr_{k,n} = GL(n, \mathbb{R}) / G_p$

Warning

$\underbrace{G_p}_{\text{Isotropy.}} \ \cancel{\cong}\ \underbrace{G \cdot p}_{\text{Orbit}}$

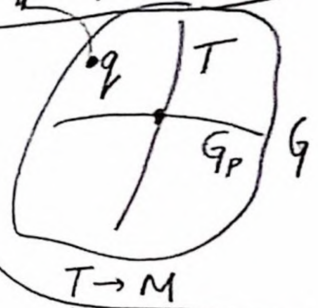
$Gr_p \ni A$ if $A \cdot V = V$

$\Rightarrow A \cdot e_i \in V \quad i = 1 \dots k$ $\Rightarrow$ A keeps the subspace unchanged.

$$A = \left(\begin{array}{c|c} k \times k & 0 \\ \hline (n-k) \times k \ 0 & 0 \end{array}\right) \rightarrow \text{"parabolic subgroup of } GL(n, \mathbb{R})\text{".}$$

Lemma! Prove it

$\boxed{q = g_0 \cdot p \Rightarrow G_q = g_0 G_p g_0^{-1}}$



$T \rightarrow M$
$t \rightarrow t \cdot p.$

Q: Why does $Gr_{k,n}$ have a diff. str.?

Find a transversal str. $T = \left(\begin{array}{c|c} \begin{matrix} 1 & \\ & \ddots \end{matrix} \ 0 & 0 \\ \hline n-k \times k & 0 \ \ddots \ 1 \end{array}\right)$

(did an example earlier?)

$T \cong \mathbb{R}^{\underbrace{(n-k) \cdot k}}$ → dim of Gr diff mfd

→ put here stuff that moves transversally to p.

hw: $SO(n) \curvearrowright Gr_{k,n}$ Transitive

for T_q try $T_q = g_0 T_p g_0^{-1}$

Properly Discontinuous Action.

Ex: $S^1 = \mathbb{R}/\mathbb{Z}$

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n\text{-times}} = \mathbb{R}^n/\mathbb{Z}^n$$

Bad (interesting) ex: $\mathbb{Z} \curvearrowright S^1$ by irrational rotation.

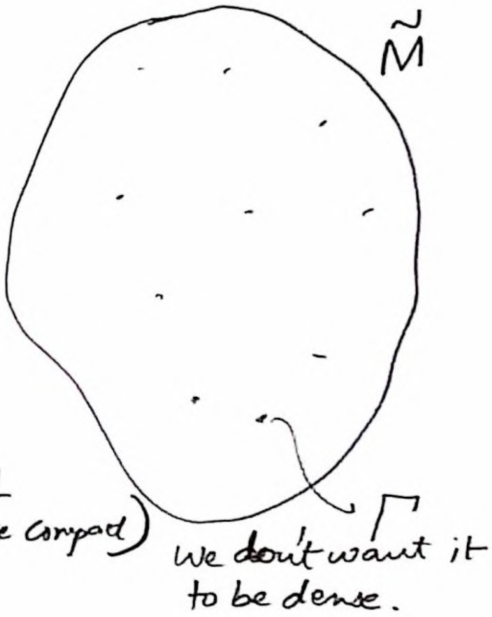
Γ discrete group (countable)

$\Gamma \curvearrowright \tilde{M}$ top mf (diff. mf)

Let $K \subset \tilde{M}$ compact set & $K \subseteq \tilde{M}$

Want ① $\forall p \in \tilde{M}$, $\Gamma \cdot p \cap K$ is finite.

i.e. $\Gamma \times \tilde{M} \longrightarrow \tilde{M}$ is proper ($\rightarrow$ pre images of compact are compact)



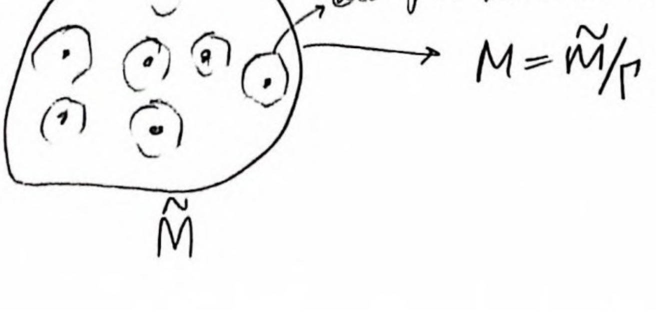
② If for some $p \in \tilde{M}$, $\gamma \in \Gamma$

$\gamma \cdot p = p \Rightarrow \gamma = 1$

(No. fixed pts)

Definition of free action

hw ① Suppose $\Gamma \curvearrowright \tilde{M}$ prop. disc. Then $M = \tilde{M}/\Gamma$ is a topological mf.



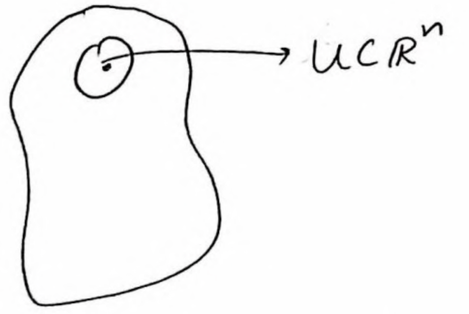
hw②: If $\tilde{M}$ is a diff (C^k) mfl. $\Gamma \curvearrowright \tilde{M}$ by diff (C^k) mfl.
Then $M = \tilde{M}/_{\Gamma}$ has a diff (C^k) structure.

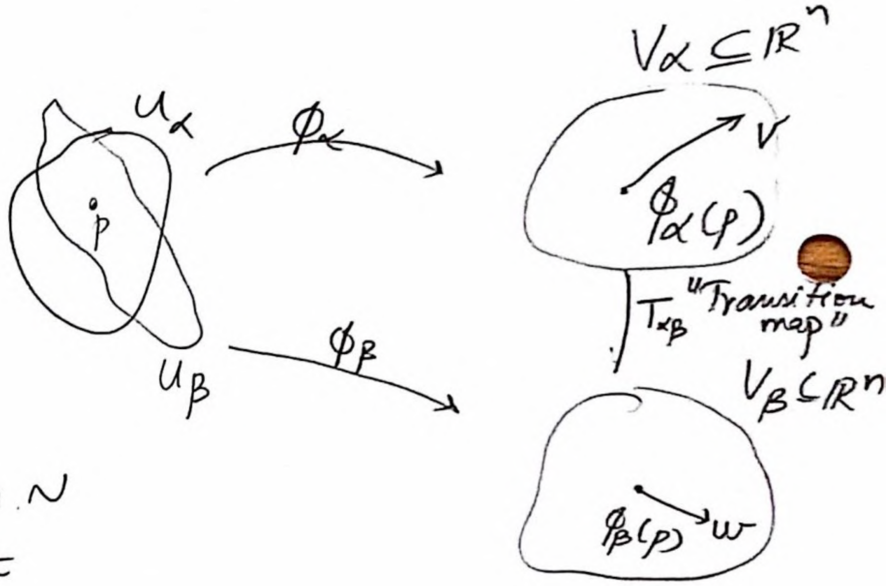
Ex: $SL(n, \mathbb{R}) / SL(n, \mathbb{Z})$ = space of lattices in $\mathbb{R}^n$ of vol. 1.
↓
shows up a lot in number theory.

Tangent vectors.

These two vectors are diff because of base pts.

M diff mfl.





Define an equivalen relation ~ between tangent vectors at $\phi_\alpha(p)$ & $\phi_\beta(p)$ by saying $v = w$ if

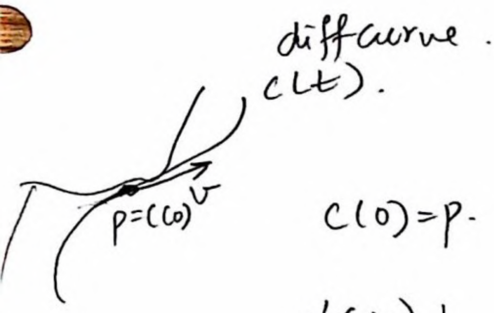
$$dT_{\alpha\beta}(w) = v.$$

deriv ↳ $\phi_\beta^{-1}\phi_\alpha$

$$T_p M = \{ [v] \mid [v] \text{ eq. class of } .. \}$$

↳ abstract tang. vector.

ice interpretation: in $\mathbb{R}^n$



$t \in (-\epsilon, \epsilon)$ some $\epsilon > 0$.

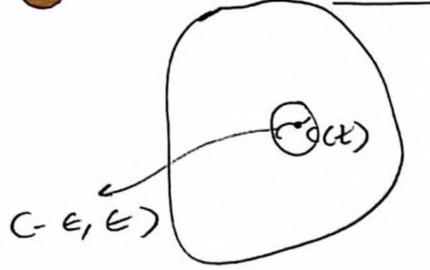
$c(0) = p$.

$c'(t)\Big|_{t=0} = c'(0)$.

another curve can have the same tang vector.

Can talk about two diff curves through p, C_1, C_2 as equivalent if $C_1'(0) = C_2'(0)$.

$p \in M$ diff mfl.



$[c]_p$ "deriv of c at p $c'(0)$".

$$M \xrightarrow[f]{\text{diff mf}} N$$

Call f differentiable at $p \in M$, then f is differentiable in charts.

$$df_p : T_pM \longrightarrow T_{f(p)}N$$

$\longrightarrow [f \circ c]_{f(p)} \in T_{f(p)}N$.

Homogeneous space

H closed subgroup of G Lie group. G/H & H/G are "homogeneous" spaces.

HW① F_k k frames in $\mathbb{R}^n$. Wanted G Lie $\curvearrowright F_k$ transitively.

H = Stabilizer & describe the transversal.

Def:

$$T_pM = \{ [(U_\alpha, \phi_\alpha, v \in \underset{\substack{\cong \\ \mathbb{R}^n}}{T_{\phi_\alpha(p)}\mathbb{R}^n})] \mid (U_\alpha, \phi_\alpha, v) \sim (U_\beta, \phi_\beta, w)$$

$$\text{if } dT_{\alpha,\beta}\Big|_{\phi_\beta(p)}(w) = v, \}$$

$$dT(y) = \underbrace{\left(\frac{\partial T_i}{\partial x_j}(y) \right)}_{} \rightarrow \text{matrix (Jacobian)}$$

$$T(y+\epsilon) = T(y) + L(\varepsilon) + E_y(\varepsilon).$$

Def (with curves):

$c:(-\epsilon,\epsilon) \to M$, $c(0)=p$ & differentiable at 0

$d:(-\delta,\delta) \to M$, $d(0)=p$ & " at 0.

Call $c \sim d$ if "$c'(0) = d'(0)$" if for one (& hence every) chart in the given atlas. $\phi_\alpha \circ c$ & $\phi_\alpha \circ d$ $\rightarrow (\phi_\alpha \circ c)'(0) = (\phi_\alpha \circ d)'(0)$.

well defined

$\Rightarrow T_pM = \{ [c] \mid c \text{ diff at } 0 \}$, Claim: T_pM has a V-space structure.

$A \in T_pM$, U_α ϕ_α cood ch. at p

$$A \longleftrightarrow v \in T_{\phi_\alpha(p)}\mathbb{R}^n$$

Similarly $B \in T_pM$ s.t $B \longleftrightarrow w$.

$v+w \in T_{\phi_\alpha(p)}\mathbb{R}^n$ & $v+w \longleftrightarrow A+B \in T_pM$.

The linearity is preserved using a diff chart because dT is linear.

> Suppose M, N diff mfl. $p \in M$. $f: M \to N$ diff. (at p).

D: $D_pf = (f_*)_p$ differential.

$Df_p: T_pM \longrightarrow T_{f(p)}N$ in terms of curves. $C: (-\epsilon, \epsilon) \to M$
$C(0) = p$, diff at 0.

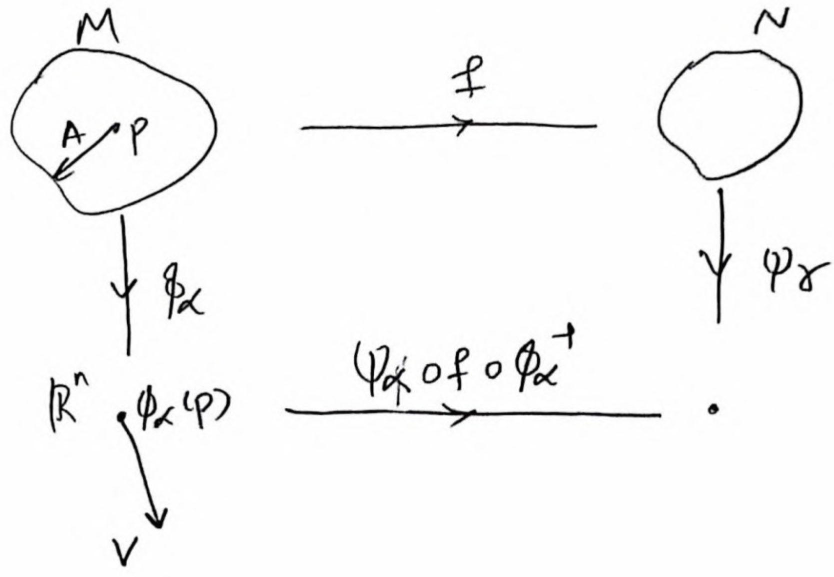
Then $(f \circ c): (-\epsilon, \epsilon) \longrightarrow N$ diff at 0. $(f \circ c)(0) = f(p)$.

$Df_p: [c] \longrightarrow [f \circ c]$. $\longrightarrow$ good for intuition.

In terms of charts, p U_α, ϕ_α a chart. $A \in T_pM$, then

$A \longleftrightarrow v \in T_{\phi_\alpha(p)}\mathbb{R}^n$

$Df_p(A) = ?$



$$Df_p(A) = \left[D_{\phi_\alpha(p)}(\psi_\alpha \circ f \circ \phi_\alpha^{-1})(v) \right]$$

$$\hookrightarrow \in T_{(\psi_\alpha f)(p)} \mathbb{R}^l$$

Thm: $f : \underset{\underset{p}{\cup}}{\overset{\overset{\mathbb{R}^n}{\cup}}{U}} \longrightarrow \overset{\overset{\mathbb{R}^l}{\cup}}{V}$ differentiable at p.

Supp. Df_p is onto then $f^{-1}(f(p))$ = level set is a manifold.

Review inverse

Ex: $SL(n,\mathbb{R}) = \{ A \in GL(n,\mathbb{R}) \mid \det(A) = 1 \}$

[illegible] det fn. is surj with onto deriv. $\Rightarrow$ $SL(n,\mathbb{R})$ is a manifold.

Lecture

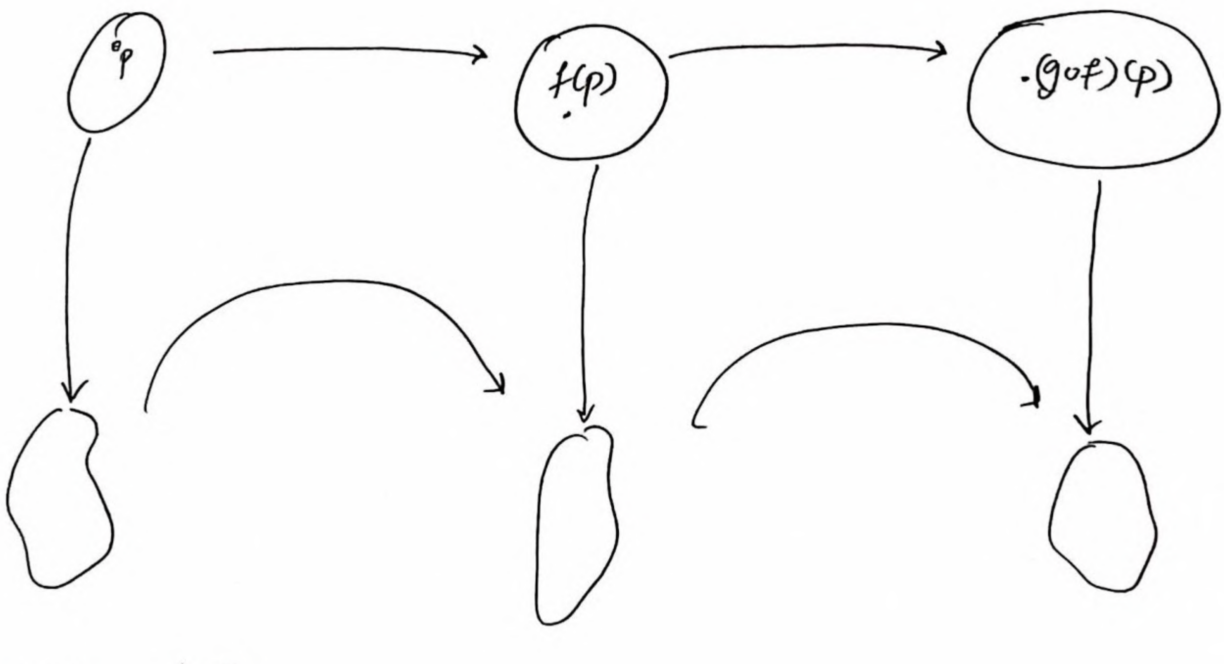
From now on all mfls are assumed to be diff.

(Aside: M has C^1 str. then it has a comptible C^∞ structure.)

> $f : M \longrightarrow N$ & $g\ N \longrightarrow O$

$p \in M$, $f(p) \in N$, $g(f(p)) \in O$. f, g diff $\Rightarrow$ $g \circ f$ diff.

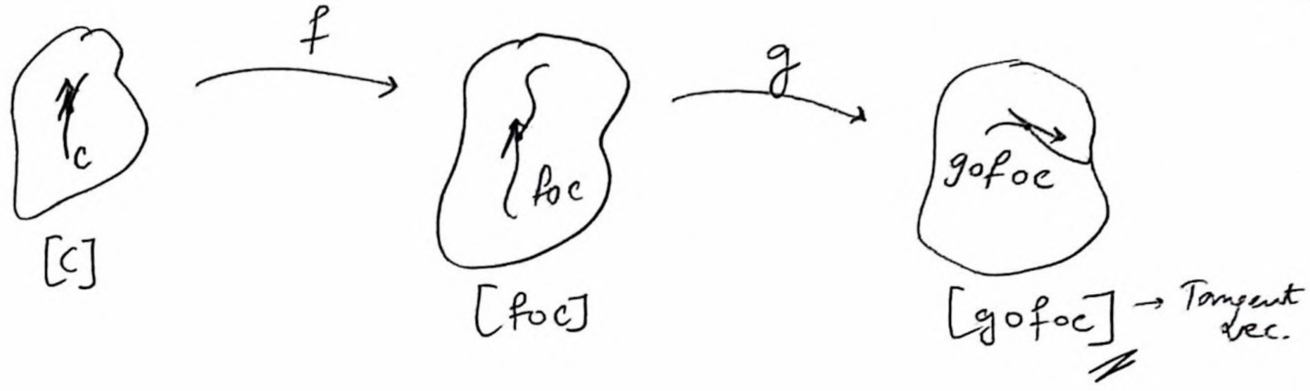
$$\begin{tikzcd}
D_p(g\circ f) : T_pM \arrow[rr] \arrow[dr, "Df_p"'] & \circlearrowright & T_{(g\circ f)(p)}O \\
& T_{f(p)}N \arrow[ur, "Dg_{f(p)}"'] &
\end{tikzcd}$$



Take ~~diff~~

Apply chain rule through chart maps.

With curves



What is a diffeomorphism?

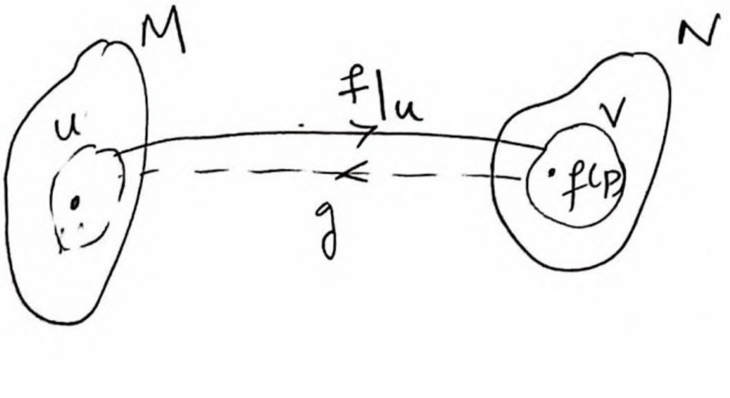
$M \xrightarrow{f} N$, f diff with diff inverse. i.e. $g \circ f = id_M$

$f \circ g = id_N$

Note: $D_p(g \circ f) = D_{f(p)}g \circ D_p f = D_p(id) = id$

Corr: $D_p f : T_pM \longrightarrow T_{f(p)}M$ has an inv. if f is a diffeo.

D: $f: M \to N$ is a local diffeo at $p \in M$ if $\exists$ nbds U of p & V of f(p) s.t. $f: U \to V$ is a diffeo.



g is the inv. of $f|_U$.

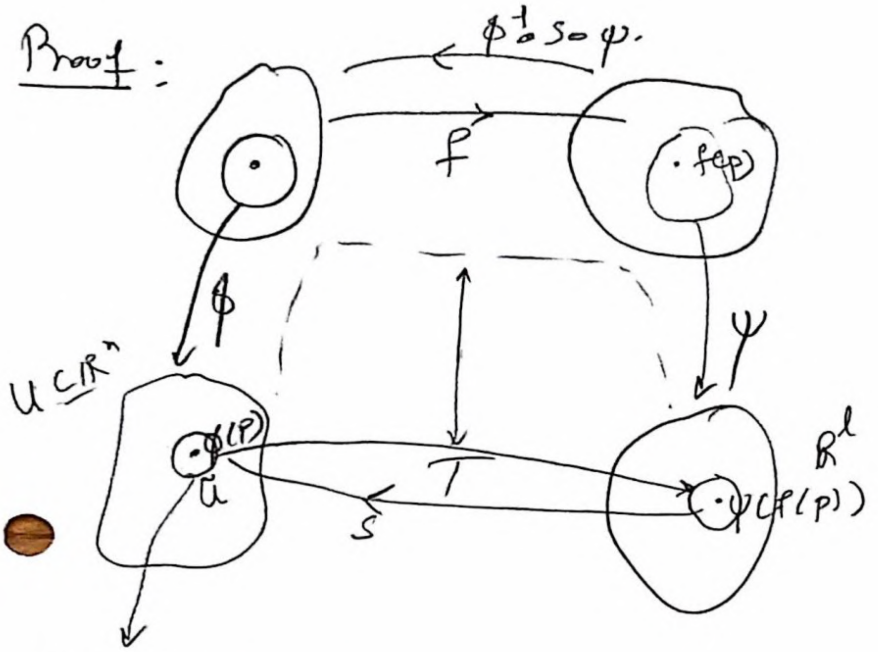
Inverse Function Theorem.

Thm: $f: M \to N$ is C^1 and suppose

$$Df_p : T_pM \longrightarrow T_pN$$ is investible (as a linear map)

then f is a local diffeo at p.

Proof:



$n = l$ since Df_p is investible.

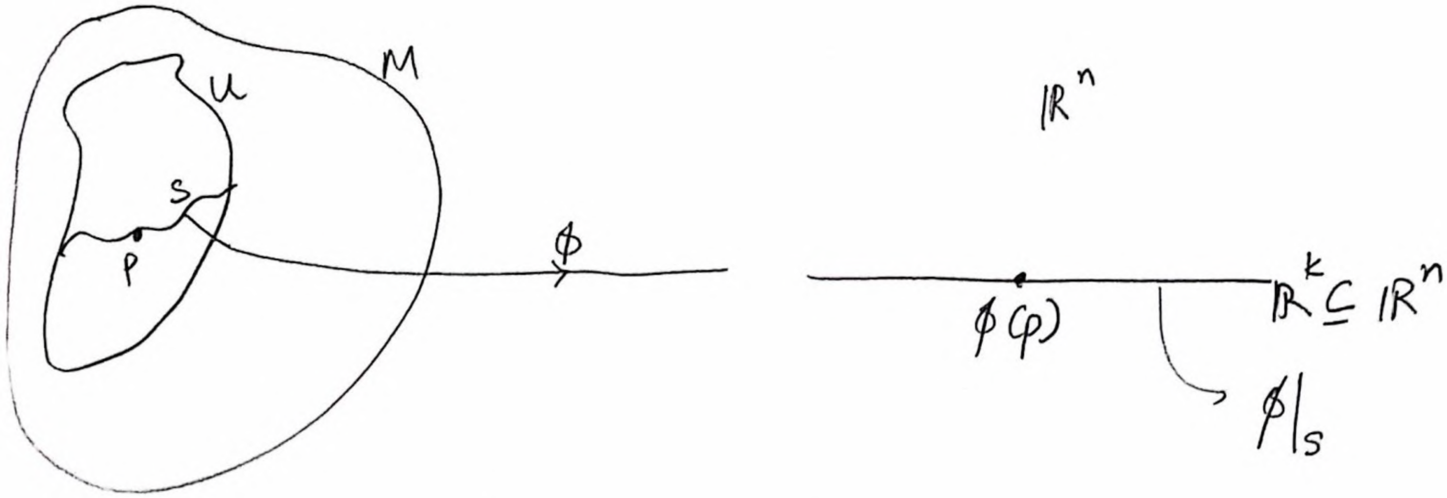
domain restricted since
$f(\phi(\tilde{U}))$ may not be in nbd $f(p)$.

Note that dT is investible at $\phi(p) \in \mathbb{R}^n$.

$\Rightarrow$ using inverse fn. thm from Real analysis (⇒ T has inverse) Call it S.

$\Rightarrow$ $\phi^{-1} \circ S \circ \psi$ is the inverse of f.

D: Suppose M C^1 mfl. $S \subset M$ is a $(C^1 C^k C^\omega)$ is an $\boxed{\text{embedded}}$ sub mfl of M if $\forall p \in S$ if $\exists$ nbd U & Chec chart $\phi : U \to \mathbb{R}^n$ s.t (U, ϕ) is a coord chart of p in M.



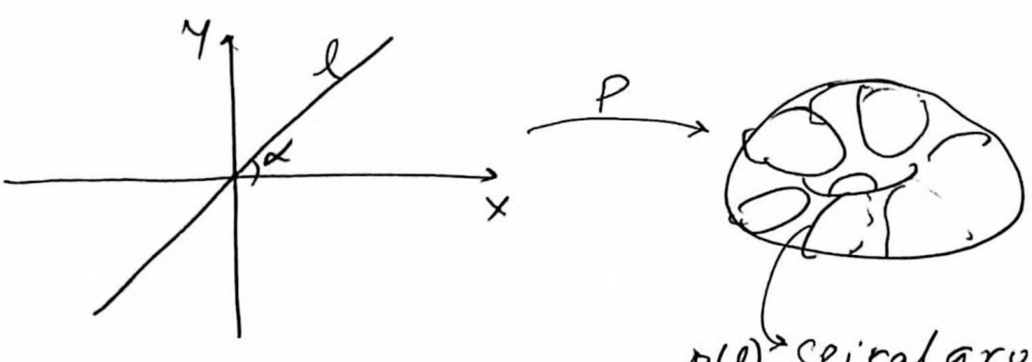
called an adapted chart $\leftarrow \phi|_S \rightarrow \mathbb{R}^k \subset \mathbb{R}^n$

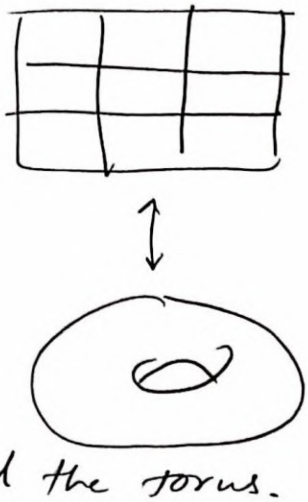
Also $\forall\, q \in U$ S.T $S = \{q\}$ $\rightarrow \phi(q) = (\underbrace{\psi_1 \dots \psi_k}_{k \text{ times.}}, 0, 0, 0 \dots \underset{n}{0})$

Note: S is a C^1 mfl by $\phi|_{S \cap U}$ as charts.
↳ some U in M.

Ex: $\mathbb{R}^l \subset \mathbb{R}^n$, $S^l \subset S^n$, $\mathbb{RP}^l \subseteq \mathbb{RP}^n$

Ex (non): $\mathbb{R}^2 \xrightarrow{\pi} \mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1 = T^2$





$p(l)$ spiral around the torus.

if $\alpha \notin \mathbb{Q}\pi \Rightarrow p(l)$ is dense in T^2

$\pi(\ell)$ is not a submanifold of T^2

<u>Def</u>: $f: M \to N$ C^1 map is called an <u>immersion</u> if $\forall p \in M$, Df_p is injective.

<u>Cor</u>: $f(M)$ is "locally" a submanifold.

Submersions

i.e. $\forall x \in f^{-1}(y)$ we want $Df_x : T_xM \to T_yN$ is surj.

$M \xrightarrow[f]{} N$ C^1

s.T every value is a $\underbrace{\text{regular value}}$ $y \in N$.

Convention : If $y \notin f(M)$, then y is a reg value.

If $f(x) = y$, we want that

$Df_x : T_xM \to T_{f(x)}M$ is surjective.

Know : $\dim M \geq \dim N$

Reg. Value Thm

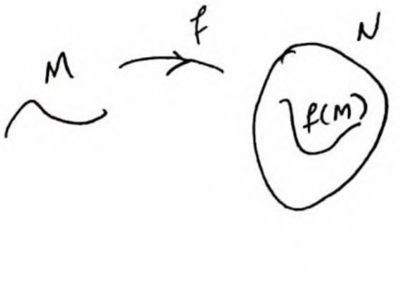
Suppose $f : M \to N$, and $q = f(p)$ is a reg. value, then $f^{-1}(q)$ is an embedded submfld of M.

Submanifolds → embedded : Defined last time

Submanifolds → immersed : ie $S \subseteq M$; $i : S \to M$ & $i(s) = S$. Call S an immersed submanifold if i is an immersion.

Recall : D : Immersion

$f : M \to N$ is an immersion if $\forall p \in M$, $Df_p : T_pM \to T_{f(p)}N$ is 1-1



if 

is also an immersion but not an immersed submfl.

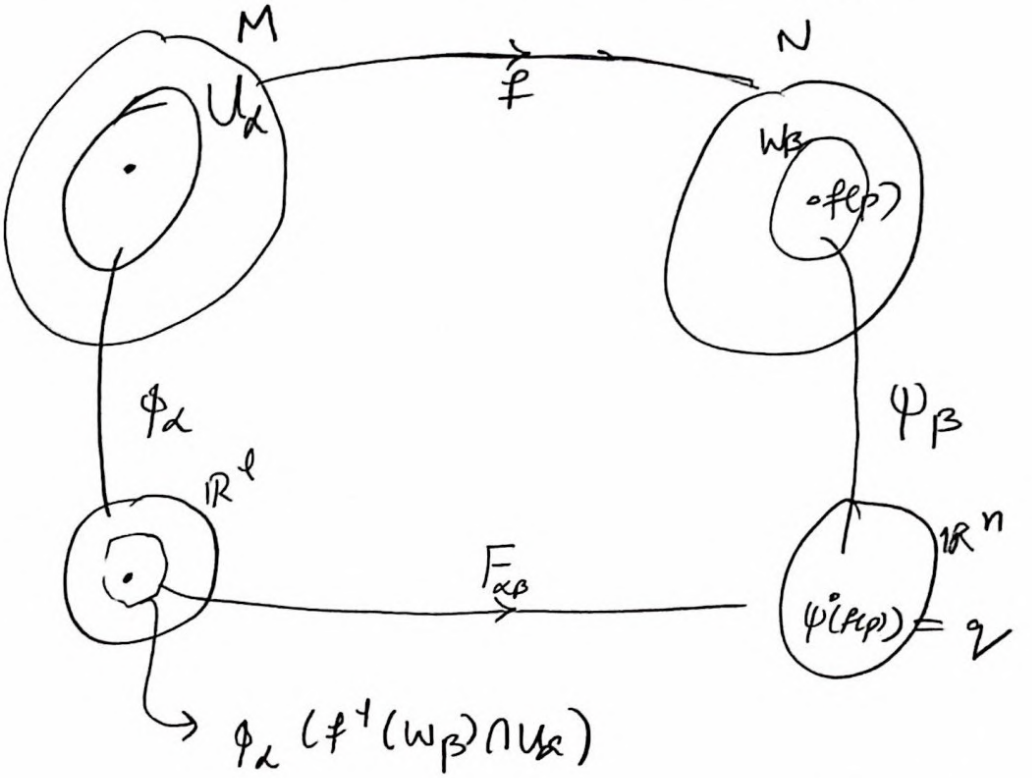
1st application: $SL(n,\mathbb{R}) \subset GL(n,\mathbb{R})$ is a sub. mfl.

det : $GL(n,\mathbb{R}) \to \mathbb{R}$ is surjective. to $\mathbb{R}$.

$\Rightarrow$ $\det^{-1}\{1\}$ is a submfl $= SL(n,\mathbb{R})$.

Proof (RVT) : (Idea)

Work with coord charts for M at $p \in M$. (U_α, ϕ_α).



Note: It suffices to check the claim for the "transfer" map

$$F_{\alpha\beta} : \underset{\text{open}}{\overset{\mathbb{R}^l}{\cup}} \longrightarrow \mathbb{R}^n$$

$F^{-1}(q)$ is a submfl of $\mathbb{R}^l$

$$\begin{tikzcd}
DF_{\phi_\alpha(p)} : T\mathbb{R}^l \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & T_q\mathbb{R}^n \arrow[d, phantom, "\cong" rotate=90] \\
\mathbb{R}^l \arrow[r] & \mathbb{R}^n
\end{tikzcd}$$

know $DF_{\phi_\alpha(p)}$ is surj.

Define an extended map $\mathbb{R}^l \longrightarrow \mathbb{R}^n \times \mathbb{R}^{l-n}$ (Recall $l \geq n$)

$G : \mathbb{R}^l \to \mathbb{R}^n \times \mathbb{R}^{l-n}$.

$$G(x_1, \dots, x_l) = (\underbrace{F(x_1 \dots x_l)}_{\mathbb{R}^n}, \underbrace{x_? \dots x_?}_{?})$$

$DF_{\phi_\alpha(p)} : \mathbb{R}^l \to \mathbb{R}^n$

$\ker DF_{\phi_\alpha(p)} \subseteq \mathbb{R}^l$

$\leftarrow$ lin subspace of dim $l-n$ because its surj.

Put $\ker DF_{\phi_\alpha(p)}$ into $\mathbb{R}^l$ as the last $l-n$ std. vectors using

Some linear map $\underset{\hookrightarrow \in GL(n,\mathbb{R})}{B}$. $B^{-1}(\ker DF) = \mathbb{R}^{l-n}$.

Now let $\bar{G} : DF_{U_\alpha(p)} \circ B : \mathbb{R}^l \to \mathbb{R}^n$, $\ker \bar{G} = \mathbb{R}^{l-n}$.

$\Rightarrow \bar{\bar{G}}(x_1 \dots x_l) = (\bar{G}(x_1. \ x_l), x_{l-n+1}, \dots, x_l)$

$\Rightarrow D\bar{\bar{G}}_{\phi(\alpha)} = (D\bar{G}_{\phi(\alpha)}, id_{\mathbb{R}^{l-n}})$ is invertible $\left(\begin{array}{c|c} * & * \\ \hline 0 & Id \end{array}\right)$

Now use IFT on $\bar{\bar{G}}$. $\Rightarrow$ $\bar{\bar{G}}$ is a local diffeo.

$$\underset{\text{level set}}{\underset{\parallel}{\bar{\bar{G}}^{-1}(q)}} \longrightarrow \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^{l-n} = \mathbb{R}^l.$$

Use $\bar{\bar{G}}$ to get adapted coords. for $\bar{\bar{G}}^{-1}(q)$, thus $\bar{\bar{G}}^{-1}(q)$ is a submfl. ■

<u>Application.</u>

$SO(n) = \{ A_{n\times n} \mid A \cdot A^T = Id \}$ is a submfl.

<u>Proof</u>: $\underset{GL(n,\mathbb{R})}{\underset{\cap}{A}} \rightarrow A \cdot A^T$ is a submersion from $GL(n,\mathbb{R}) \rightarrow$ symmetric matrices $= \mathbb{R}^l$.

Thus a Lie gp.

Proof (sketch) (redoing)

M, N C^1-mfl, $f: M \longrightarrow N$, q reg. value, i.e $\forall\ p \in f^{-1}(q)$, Df_p is surjective

Reg. Val. Thm

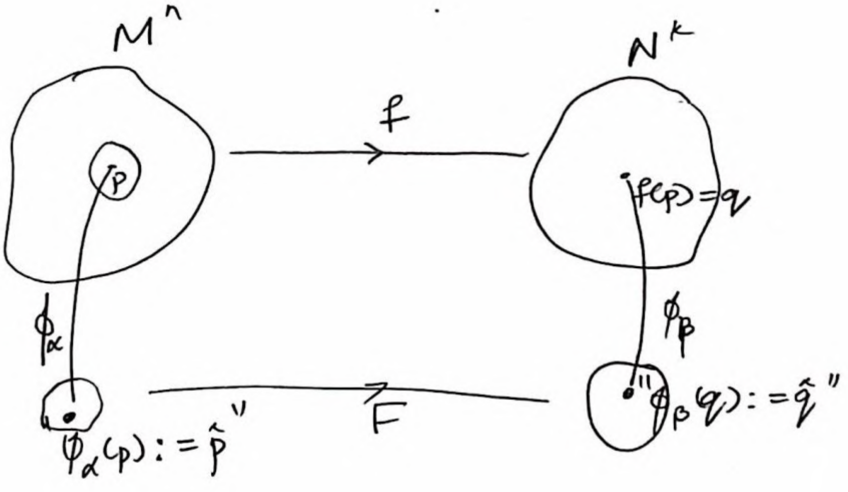
q reg. val. then $f^{-1}(q)$ is a submfl of M.

Proof

step ① $\dim M = n$, $\dim N = K$, use coord. charts to reduce to a problem about fns $F: \underset{\underset{\mathbb{R}^n}{\cap \text{open}}}{U} \longrightarrow \mathbb{R}^k$

step ② $F: U \to \mathbb{R}^n \Rightarrow \hat{q}$ is a reg. value of F

W.T.S $F^{-1}(\hat{q})$ is a submfl of $\mathbb{R}^n$



know, $DF_{\hat{p}}$ is surj, $F = (F_1, \dots, F_K)$

$$DF_p = \text{Jacobian} = \begin{pmatrix} \frac{\partial F_1(\hat{p})}{\partial x_1} & \cdots & \overline{\partial x_n} \\ \vdots & & \\ \underline{\partial F_k} & & \end{pmatrix}$$

DF_p surj => rank k => k lin ind col. vecs.

WLOG assume 1^{st} k

step ③

Idea : Somehow apply IFT

Want a map, $G: \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^{n-k}$

Expand F to G,

$$G(x_1 \ldots x_n) = (F(x_1 \ldots x_n), \underbrace{x_{k+1}, \ldots, x_n}_{\text{completion}})$$

$$DG_p = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_k} & \\ \vdots & & \vdots & 0 \\ \frac{\partial F_k}{\partial x_1} & \cdots & \frac{\partial F_k}{\partial x_k} & \\ & 0 & & I_{n-k} \end{pmatrix} \Rightarrow DG_p \text{ invertible}$$

$\overset{\Rightarrow}{\text{IFT}}$ G is a local diffeo.

Step 4

$$F^{-1}(q) = G^{-1}(q_1,\dots,q_k, * \cdots *) = G^{-1}(\{q\} \times \mathbb{R}^{n-k})$$

↑ locally a submfl. since G is diffeo. //

Ex: Canonical submersion.

$\pi : \mathbb{R}^n \to \mathbb{R}^k$ projection.

Thm: (Normal form) $\underset{C^1}{f} : M \to N$, q reg. value, $p \in f^{-1}(q)$.

then ∃ coord charts U, ϕ, & V, ψ around p & q resp.

$$\begin{tikzcd}
\mathbb{R}^n \supset \phi(U) \arrow[r, dashed, "\pi = F"] & \psi(V) \subset \mathbb{R}^k \\
U \arrow[u, "\phi"] \arrow[r, "f"'] & V \arrow[u, dash, "\psi"']
\end{tikzcd}$$

Engine of IVT: (Uses contraction mapping fixed pt. thm.)

↳ also used in existence & uniqueness of ODEs

↳ shows up in dynamical systems.

Application

$O(n)$ is an embedding submfl of $GL(n,\mathbb{R}) \underset{\text{open}}{\subset} \mathbb{R}^{n^2}$.

$F : GL(n,\mathbb{R}) \longrightarrow \text{sym}(n \times n)$ & use level set of I.

Calc. deriv. DF_g , $g \in GL(n,\mathbb{R})$, $g \in f^{-1}(I)$, i.e. $gg^T = I$.

$v \in \mathbb{R}^{n^2}$, $F(g+tv) = (g+tv)(g+tv)^T$

$$= (g+tv)(g^T + tv^T)$$

$$= gg^T + g\cdot(t\,v^T) + t\,v.\,g^T + t^2 \underset{\substack{t \\ \text{Something}}}{\underline{\qquad}}$$

$$= I + t(gv^T + vg^T)$$

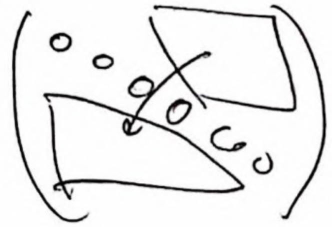
$$\Rightarrow DF_g = gv^T + vg^T$$

<u>Claim</u>: <u>Onto</u> : i.e. any symmetric matrix has this form for some $v \in \mathbb{R}^2$

Calc. $\ker(DF_g) = \{v \mid gv^T + vg^T = 0\}$

$\Rightarrow gv^T = -vg^T$ $\Rightarrow vg^T$ is skew symmetric.

$\Rightarrow$ $\text{Ker}(DF_g)$ has dim $\frac{n(n-1)}{2}$.



$\Rightarrow$ dim of Image $DF_g = n^2 - \frac{n(n-1)}{2}$

$$= \frac{n(n+1)}{2} \longrightarrow \text{dim. of symmetric mtcs.}$$

SO(n) comes from level set of I.

$\det(O(n)) = \pm 1$

$\Rightarrow SO(n) \supset$ conn. component of I in $O(n)$.

& $SO(n) \subset O(n)$ is clopen.

Closed since $\det^{-1}(1)$

$O(n) - SO(n) = \det^{-1}(-1)$ is closed $\Rightarrow$ open.

Show $SO(n)$ is connected.

& use Lemma :- L is a mfl, then any open ~~set~~ subset is a mfl.

Lecture

Cor: Submersions are open maps.

P: Local normal form. //

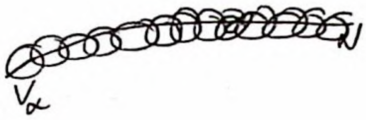
Ex (of submersions)

$M \times F \longrightarrow M$ (projections)

Fiber bundle

Def: A submersion $f: M \to N$ is a fiber bundle if f is surjective, & $\exists$ covering of N by open sets $\{V_\alpha\}$

S.T $f^{-1}(V_\alpha) \xrightarrow[\varphi_\alpha]{\text{diffeo}} V_\alpha \times F$ for some mfl F.



Where F is called the fiber.

Also need

$$\begin{tikzcd}
f^{-1}(V_\alpha) \arrow[rr, "\phi"] \arrow[dr, "f"'] \arrow[drr, phantom, "\circlearrowleft"] & & V_\alpha \times F \arrow[dl, dashed, "\text{proj to } V_\alpha"] \\
& N &
\end{tikzcd}$$

<u>Ex</u>: ① $N \times F \longrightarrow N$ {another example of products} → aka trivial fiber bundle.

② Möbius band,  $F = (-1,1)$ (or $\mathbb{R}$)

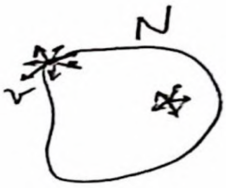
$\downarrow$

M & $N = S^1$

$F = (-1,1)$

Mobius band is not diffeo to $S^1 \times (-1,1)$

③ $M \subset \mathbb{R}^L$ embedded submfl



$$S(M) = \{ v \in T_pM,\ p \in M \ \& \ \overset{\text{Euclidean}}{\|v\|} = 1 \}$$

↳ unit tangent bundle of M.

$S(S^1) = S^1 \times \{-1,+1\}$

Fact $S(S^2)$ is not a trivial fiber bundle. (Hairy ball Thm.)

<u>HW</u> $S(S^3)$ is trivial. (Hint: S^3 is a group).

M (abstract) diff mfl.

Tangent bundle of M

as a set: $\dot{\bigsqcup\limits_{p\in M}} T_pM = TM$ ← disjoint union.

$$\begin{tikzcd} TM \arrow[d, "\pi"] \\ M \end{tikzcd}$$

Claim: This is a fiber bundle.

$p = \pi(v)$ = footpt. of v (proj. v onto its foot pt.)

Vector Bundle.:

A vector bundle of a diff mfl Z is a fiber bundle $V \xrightarrow[\pi]{} Z$.

V is another mfl with fiber $= V$, a vector space with S.T the local trivializations, i.e. covering of Z by open sets $\{Z_\alpha\}$

$\pi^{-1}(z_0)$ has the structure of a VS

$$\begin{tikzcd} \pi^{-1}(Z_\alpha) \arrow[r, equal, "\phi_\alpha"] \arrow[d, "\pi"'] & Z_\alpha \times V \arrow[dl, "\Omega", "\text{proj.}"'] \\ Z_\alpha & \end{tikzcd}$$

$\phi_\alpha \Big|_{\underbrace{(\pi^{-1}(z_0))}} = \{z_0\} \times V$

restricted V. space ↳ linear isomorphism

$$\begin{tikzcd} V \arrow[d, "\pi"'] & \pi^{-1}(Z_\alpha) \arrow[d, "\pi"] \arrow[rrr, bend left, "\phi_\alpha"] & \supset \pi^{-1}(z_0) \arrow[r] \arrow[d] & \{z_0\} \times V \arrow[r] & V_\alpha \times V \\ Z \supset & Z_\alpha & z_0 & & \end{tikzcd}$$

Collection of Vector spaces

$\phi_\alpha \big|_{\pi^{-1}(z_0)}$ is linear

Ex: $M \subset \mathbb{R}^L$ embedded submfl.

$TM = \{v$ based tang vector in $\mathbb{R}^L$ at some $p \in M$, tangent to $M\}$.

> M (absrract) diff mfl.

$$\begin{tikzcd} TM \arrow[d, "\pi"] \\ M \end{tikzcd}$$

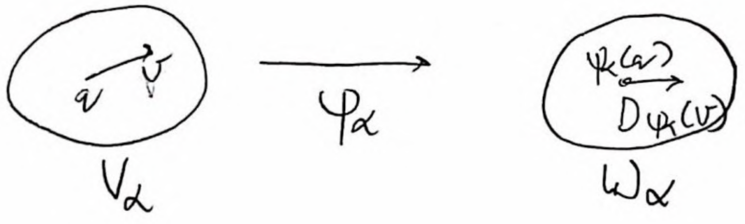
data: Collection of open sets for M. Just take an atlas for the diff. sdr. of M.

$\{V_\alpha\}$ $\quad \bigcup V_\alpha = M$.

$$\pi^{-1}(V_\alpha) = \bigsqcup_{p \in V_\alpha} T_pM \xrightarrow{D\psi_\alpha} W_\alpha \times \mathbb{R}^n$$

atlas $=$ $V_\alpha \xrightarrow[\psi_\alpha]{} W_\alpha \subset \mathbb{R}^n$ open.

$\{V_\alpha\ \psi_\alpha\}_\alpha$ $\quad \hookrightarrow$ charts for M.



a tangent vector in $\mathbb{R}^n$ at $\psi_\alpha(q)$, $q \in V_\alpha$.

Check $D\psi_\alpha \big|_{T_qM} : T_qM \xrightarrow[\text{linear}]{} \mathbb{R}^n \ (= T_{\psi_\alpha(q)}\mathbb{R}^n)$

linear sdructure on T_qM comes from the one on $T_{\psi_\alpha(q)}\mathbb{R}^n = \mathbb{R}^n$.

Topology of TM, = top. on $\pi^{-1}(V_\alpha)$ via $D\psi_\alpha$.

diff. sdr. also comes similarly via $D\psi$.

Lecture

Tangent bundle

M mfl. $\underset{\underset{\underset{T_pM}{\downarrow}}{(p,v)}}{\underset{\cup\!\!\!|}{TM}} = \bigsqcup_{p\in M} T_pM$. "foot pt. map".

Claim A: TM has a diff. str.

Claim B: $TM \underset{\text{foot pt. map}}{\xrightarrow{\pi}} M$ can be endowed with vector bundle structure.

Need a cover of M by open sets $\{U_\alpha\}$. $V \xrightarrow{\pi} M$

Recall v-bdl defn (I) $\pi^{-1}(U_\alpha)$ open & $\phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow[\text{diffeo}]{\text{local}} U_\alpha \times F$

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "\text{local diffeo}"] \arrow[d, "\pi"'] & U_\alpha \times F \arrow[dl, "\text{projection to } U_\alpha"] \\
M &
\end{tikzcd}$$

commutes

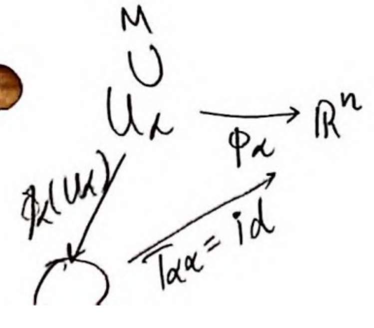
F a vector space (will be $\mathbb{R}^n$, $n = \dim M$)

& each $\pi^{-1}(p) \mid$ all p has a v.s structure.

(II) $\phi_\alpha\big|_{\pi^{-1}(p)} \longrightarrow F$ is linear.
$p\in M$

$\begin{matrix} TM \\ \downarrow \pi \\ M \end{matrix}$ $\pi^{-1}(p) = T_pM$ is a v-space. Pick a coord chart cover $\{U_\alpha\}$ of M.

Then $\phi_\alpha : U_\alpha \to$ open set in $\mathbb{R}^n$ are C^1 or C^∞. Check with coord charts.



$W \subset \mathbb{R}^n$ open

$TW \underset{\text{open}}{\subset} \mathbb{R}^{2n}$

$\Rightarrow TW$ smooth str.

$$\begin{tikzcd}
TU_\alpha = \dot{\bigsqcup}_{p\in U_\alpha} T_p U_\alpha \arrow[d, "D\phi_\alpha"'] \\
T\phi_\alpha(U_\alpha) \arrow[d, phantom, "\cap"] \\
\mathbb{R}^{2n}
\end{tikzcd}$$

<u>Claim 1</u>: TM has a top str. generated by the open sets in $\{TU_\alpha\}$.

$\Rightarrow$ <u>Claim 2</u>: $(TU_\alpha, D\phi_\alpha)$ are an atlas.

Take $\beta \neq \alpha \Rightarrow U_\beta$, TU_β.

$$\begin{tikzcd}
 & TU_\alpha \arrow[dl, "D\phi_\alpha"'] & TU_\beta \arrow[dr, "D\phi_\beta"] & \\
T(\phi_\alpha U_\alpha) \arrow[ur, bend left, "D(\phi_\alpha^{-1})"] \arrow[rrr, "\text{diffeo?}"] & & & T(\phi_\beta(U_\beta))
\end{tikzcd}$$

$D\phi_\beta \circ D_{\phi_\alpha}^{-1} = D(\phi_\beta \circ \phi_\alpha^{-1}) = D(\underset{T_{\alpha,\beta}}{\text{transition map}})$.

defined on $D\phi_\alpha(TU_\alpha \cap TU_\beta)$

well defined $\Rightarrow$ diffeo!

<u>Regularity</u>: If M has a C^1-mfl structure only, then this str. on TM is only C^0. $\Rightarrow$ M must be atleast C^2, to get TM with C^1 str.

<u>Note</u>: later we show that a manifold with C^1 str. has a compatible C^∞ str.

$\Downarrow$

<u>Worry</u>: TM is a diff mfl. T(TM), TTTM? Hence just ~~use~~ assume C^∞ M.

↳ shows up a lot

TM has C^∞ structure, given by $\pi^{-1}(U_\alpha)$, $D\phi_\alpha$. Also gives a vector bundle structure..

$TM \xrightarrow{\pi} M$. Take cover $\{U_\alpha\}$ as before. then $\pi^{-1}(U_\alpha) \xrightarrow[D\phi_\alpha]{} U_\alpha \times \mathbb{R}^n$

$\pi^{-1}(p) = T_pM \xrightarrow[D\phi_\alpha]{} T_{\phi_\alpha(p)}\mathbb{R}^n \; (= \mathbb{R}^n)$

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "D\phi_\alpha"'] \arrow[d] & T\phi_\alpha(U_\alpha) = \phi_\alpha(U_\alpha) \times \mathbb{R}^n \arrow[d, "\pi_{\mathbb{R}^n}"] \\
U_\alpha \arrow[r, shift right, "\phi_\alpha"'] & \phi_\alpha(U_\alpha) \arrow[l, shift right, "\phi_\alpha^{-1}"']
\end{tikzcd}$$

Thus get a fiber bundle. Last need to check $\phi_\alpha := D\phi_\alpha = T(\phi_\alpha, U_\alpha)$ is linear map on fibers T_pM, $p \in U_\alpha$ .. $D\phi_\alpha\Big|_{T_pM} = (D\phi_\alpha)_p$ linear..

Constructions.

> V, W are vector bundles. HW5#1? Define $\underset{\pi_1^{-1}(p)}{\underset{\parallel}{V_p}} \oplus \underset{\pi_2^{-1}(p)}{\underset{\parallel}{W_p}}$ & show all the stuf done befor

$$\begin{tikzcd}
V \arrow[d, "\pi_1"'] & W \arrow[d, "\pi_2"] \\
M & M
\end{tikzcd}$$

> V vspace, $V^* =$ dual. Given a v-bundle $\begin{tikzcd} V \arrow[d, "\pi"'] \\ M \end{tikzcd}$ want $\begin{tikzcd} V^* \arrow[d, "\pi^*"] \\ M \end{tikzcd}$

$\pi^{*-1}(p) = (\pi^{-1}(p))^*$

For V, U_α cover of M ..., $\pi^{-1}(U_\alpha) \xrightarrow[\phi_\alpha]{} U_\alpha \times V$

$(\pi^*)^{-1}(U_\alpha) = \bigsqcup_{p \in U_\alpha} (\pi^{-1}(p))^* \to$

$\phi_\alpha\Big|_{\pi^{-1}(p)} \quad \pi^{-1}(p) \xrightarrow[\text{linear}]{} \{p\} \times V$

$\pi^{*-1}(p) = (\pi^{-1}(p))^* \longrightarrow V^*$

$\pi^{-1}(p) \xrightarrow[L_p \,=\, \text{isomorphism}]{} V$ $\Rightarrow$ $V^* \xrightarrow{L_p^*} \pi^{-1}(p)$

we want $\pi^{-1}(p)^* \longrightarrow V^*$ so use $(L_p^{-1})^*$

> Another construction

> V, W vector bundles. $V \overset{\text{tensor product?}}{\otimes} W$. $\pi_1^{-1}(p) \otimes \pi_2^{-1}(p)$

> $\underline{V \otimes V \otimes V \cdots \otimes V}$ k times (k fold tensor product)

$\cup$

$\Lambda^k V$ exterior product.

k-alternating linear forms. (determinant is an example)

$\Lambda^k V$ x $\Lambda^k TM \to$ diff. k-forms $\rightsquigarrow$ de Rham cohomology.

Multilinear form (review it) on $V_1 \dots V_k$.

> $V_1 \oplus V_2 \oplus \cdots \oplus V_k \xrightarrow[\lambda]{} \mathbb{R}$ s.t $\lambda(v_1 \dots v_k)$ is linear in each coord. v_i ∀ i.

> λ is alternating $\lambda(v_1 \dots v_i \dots v_j \dots v_k) = -\lambda(v_1 \dots v_j \dots v_i \dots v_k)$

Lecture

Fiber bundle

$$\begin{tikzcd} \text{fiber } F \arrow[r] & E \text{ total space} \arrow[d, "\pi"] \\ & B \text{ base space} \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

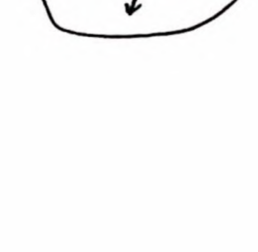
E is (cts, diff, C^∞)

$\sigma : B \to E$ is called a *section* if $\pi \circ \sigma = id_B$.

Ex:

$$\begin{tikzcd} TM = E \text{ tangent bundle} \arrow[d, "\pi = \text{foot pt. map}"] \\ M \arrow[u, bend left, "\sigma \text{ section}"] \end{tikzcd}$$

D: A vector field $X : M \longrightarrow TM$ is a section (cts, C^1, C^k, C^∞) of TM.



Ex: $M = \mathbb{R}^n$, special v-fields $e_1, \dots, e_n$

$\frac{\partial}{\partial x_i} = e_i$; e_1 is placed at every pt.

$e_2 = \frac{\partial}{\partial x_2}$ ↑ ↑ ↑ ↑ ...

$X : \mathbb{R}^n \longrightarrow T\mathbb{R}^n$ is any v-field, we can write it as

~~$X(p) = a_1 \frac{\partial}{\partial x_1}(p)$~~

Then $X(p) = a_1(p) \frac{\partial}{\partial x_1}(p) + \cdots + a_n(p) \frac{\partial}{\partial x_n}(p)$

$a_i(p) =$ i'th coefficient functions.

Ex: $X(x_1, x_2) = e^{x_1 + x_2} \cos x_1 \frac{\partial}{\partial x_1} + e^{e^{x_1^2 + x_2 \cos x_1}} \frac{\partial}{\partial x_2}$

X cts $\Leftrightarrow$ all $a_i(p)$ are cts.
diff $\Leftrightarrow$ " " " diff
C^1 C^1
C^k " " " C^k
C^∞ C^∞

What does diff. mean?

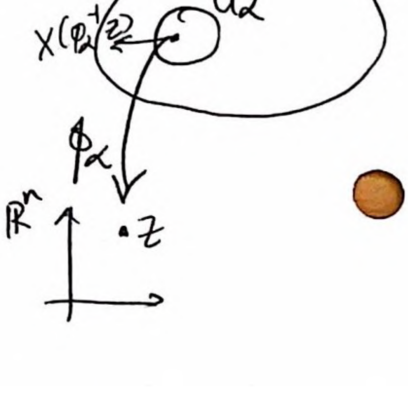
> $M \xrightarrow{\sigma} TM$ is a diff. map. from M to TM with their diff structures.

How to check in "practice"? TM C^∞ mfl. $M \xrightarrow{X} TM$.

Take a chart (U_α, ϕ_α) for M.

$D\phi_\alpha = TU_\alpha \rightarrow \underbrace{\cancel{T\mathbb{R}^n}} \rightarrow T(\phi_\alpha(U_\alpha)) \subset T\mathbb{R}^n$

WTS: $z \longrightarrow D_{\phi_\alpha}(X(\phi_\alpha^{-1} z))$ is C^∞?



Fact = hw : $M \xrightarrow{f\ C^\infty} N$

$TM \xrightarrow{Df} TN$

$Df((p,v)) = (f(p), Df_p(v))$ is C^∞ if f is C^∞.

Check (use coord. charts) (V_β, Ψ_β) is chart at $f(p)$

$$D\psi_\beta \circ Df \circ D\phi_\alpha^{-1} : TU_\alpha \longrightarrow T_{\phi_\beta}(V_\beta)$$

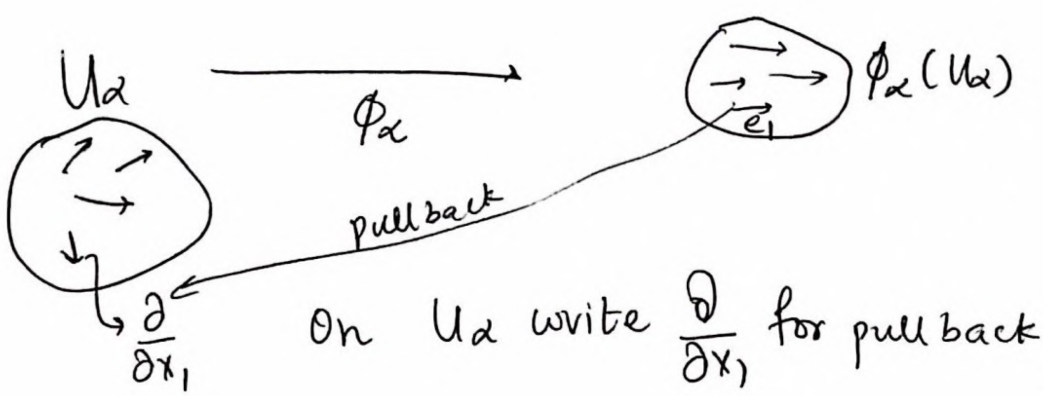
$$\shortparallel$$

$$D(\psi_\beta \circ f \circ \phi_\alpha^{-1})$$

Q: If $f: M \to N$ is C^k, What is $Df: TM \to TN$?
it is C^{k-1}.

Back to vector fields

To check smoothness (or do any calculation) write X in a chart (U_α, ϕ_α). $e_1, \dots, e_n$ on $\phi_\alpha(U_\alpha)$.



On U_α write $\frac{\partial}{\partial x_i}$ for pullback of e_i on $\phi_\alpha(U_\alpha)$

$\to \frac{\partial}{\ }(p) := D(\phi_\alpha^{-1})(e_i)$

$$X\Big|_{U_\alpha} = \sum_{i=1}^{n} a_i(p) \cdot \frac{\partial}{\partial x_i}(p)$$

$\uparrow$ i^{th} coord. fn. for the chart (U_α, ϕ_α).

Now, the computations can be done on M & X as though on $\mathbb{R}^n$.

Warning : $\frac{\partial}{\partial x_i}$ is a vector field.

There will also be "dx_i" which are cotangent fields or differential 1 forms. dx_i are local sections to T^*M.

$$(TM)^* =: T^*M = \Lambda^1 M$$

$\hookrightarrow$ alternating 1 forms.

$$dx_i \in (TU_\alpha)^* = T^*U_\alpha$$

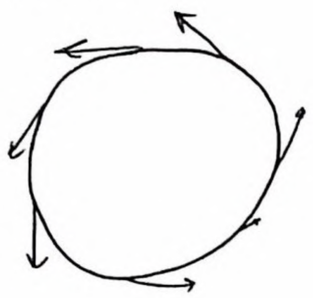
D: dx_i : At every pt. $p \in U_\alpha$,

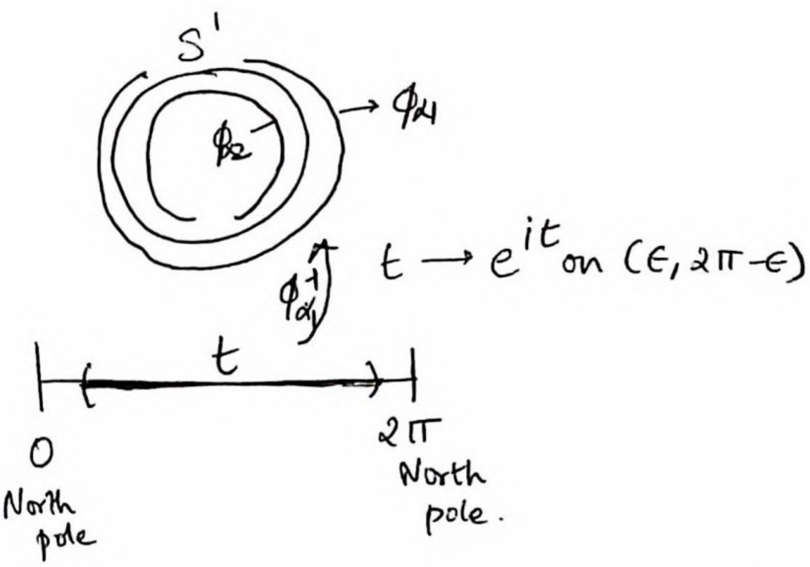
$\{dx_i(p)\}_{i=1\ldots n}$ is the dual basis to $\{\frac{\partial}{\partial x_j}(p)\}_{j=1\ldots n}$.

V $v_1 \ldots v_n$ a basis

V^* $w_1 \ldots w_n$ a dual basis $\Rightarrow w_j(v_i) = \delta_{ij}$.

Ex: $M = S^1$.





$\phi_\alpha^{-1}: t \mapsto e^{it}$

$X(\phi_\alpha^{-1}(t)) = \sin t \frac{\partial}{\partial t}$

$\phi_{\alpha_2}^{-1}: t \to e^{it}$ on $[0, \frac{\pi}{2}-\epsilon) \cup (\frac{\pi}{2}+\varepsilon, 2\pi]$

Note: σ_1, σ_2 sections

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ M \end{tikzcd}$$

Vector bundle. $(f_1\sigma_1 + f_2\sigma_2)_p$, $f_1, f_2 : M \to \mathbb{R}$.

⤷ $f_1(p)\sigma_1(p) + f_2(p)\sigma_2(p)$.

Note: C^k-sections of a v-bundle form a module over $C^k(M)$.
↳ of ∞ dim.

Derivations

$\delta : \overset{\to C^\infty \text{ fns on } M}{C^\infty(M)} \to \mathbb{R}$ is a map ~~[illegible]~~ called a derivation at p if

$\forall f, g \in C^\infty(M)$, the product rule works:

$\partial(f \cdot g) = f(p) \cdot \delta(g(p)) + \delta f(p) \cdot g(p)$. & $\delta(f+g) = \delta f + \delta g$.

115ex : $M = \mathbb{R}$ (or (a,b)) $p \in (a,b)$.

$\delta(f) = f'(p)$

215ex : $f \xmapsto[\delta]{}$ directional deriv of f at p.

gen-ex : M mfd

$(p,v) \in T_pM$

$f \mapsto$ "directional deriv of f at p in direction of v"

$:= (Df_p)(v)$.

Call W & V isomorphic as V-bundles if $\exists\, \Phi : V \to W$ & $\exists\, \phi : M \to N$ diffeo.

$$\begin{tikzcd}
V \arrow[r, shift left, "\Phi"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft \ \text{commutes}"] & W \arrow[l, shift left, "\Psi"] \arrow[d, "\tau"] \\
M \arrow[r, shift right, "\phi"'] & N \arrow[l, shift right, "\psi"']
\end{tikzcd}$$

S.T Φ takes fibers to fibers $\Phi(\pi^{-1}(p)) = \tau^{-1}(\phi(p))$

and $\forall\, p \in M$

$$\Phi\big|_{\pi^{-1}(p)} : \pi^{-1}(p) \to \tau^{-1}(\phi(p))$$

$\nearrow \qquad \nearrow$

V space is linear.

data so far says that, (Φ, ϕ) are a V-bdle map.

Call (Φ, ϕ) a vbdle isomorphism if $\exists$ ~~(Φ, ϕ)~~ (Ψ, ψ) vbdle. map which are inverses. of $(\phi$

HW1

$$\begin{tikzcd}
TS^3 \arrow[r, phantom, "\cong"] \arrow[d] & \mathbb{R}^3 \times S^3 \arrow[d] \\
S^3 \arrow[r, phantom, "\equiv", "id."'] & S^3
\end{tikzcd}$$

D: Trivial bundle.

$$\begin{tikzcd} V \arrow[d, "\pi"'] \\ M \end{tikzcd}$$

Call a trivial bundle if it is isomorphic (as a v-bundle) to $\mathbb{R}^l \times M$. $\mathbb{R}^l \cong \pi^{-1}(p)$, $p \in M$.

\> $$\begin{tikzcd} \mathbb{R}^l \times M \arrow[d, shift left, "\tau"] \\ p \in M \arrow[u, shift left, "\sigma_i"] \end{tikzcd}$$

$\sigma_i(p) = (p, e_i)$ → any vector in $\mathbb{R}^l$.

If $\{e_1, \ldots, e_l\}$ are a basis for $\mathbb{R}^l$. Then $\{\sigma_i(p)\}$ are lin. ind. & in fact a basis for $\tau^{-1}(p)$ $\forall p \in M$.

Thus get l lin. ind. sections of $\begin{tikzcd}[row sep=small] \mathbb{R}^l \times M \arrow[d] \\ M \end{tikzcd}$. (This is for trivial bundles)

\> Conversely, if $\begin{tikzcd} V \arrow[d, "\pi"] \\ M \end{tikzcd}$ is a v-bdle & $\exists$ l sections $\sigma_1 \ldots \sigma_l$ of π & if l is the dim. of $\pi^{-1}(p)$ then V is isomorphic to a trivial bundle. Thus V is trivial.

Ex: $v \in T_pM$. $\delta(f) =$

$v \in [c(t)]$.

$\xrightarrow{}$ diff. curve through $p = c(0)$

$$\delta(f) = \frac{d}{dt}\Big|_{t=0} f(c(t)) =$$ directional deriv of f at p in "direction" of v.

$T_pM \rightsquigarrow$ derivations.

> Can also look at a smooth v. field: X at M

$$\Delta : C^\infty(M) \longrightarrow C^\infty(M)$$

~~$\delta_{X(p)}(f)$~~ $\qquad (\Delta f)(p) : \delta_{X(p)}(f)$

Then $\Delta(f \cdot g) = f \cdot \Delta g + \Delta f \cdot g$

↑ true pointwise.

Ex: $X = y\frac{\partial}{\partial x}$ & $Y = x \cdot \frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

$$(X \circ Y) = \left(y \cdot \frac{\partial}{\partial x}\right)\left(x \cdot \frac{\partial}{\partial y}\right) = y\left(\frac{\partial}{\partial x} x\right)\frac{\partial}{\partial y} + \underbrace{yx \frac{\partial}{\partial x}\frac{\partial}{\partial y}}_{???} = y\frac{\partial}{\partial y} + yx\frac{\partial^2}{\partial x \partial y}$$

$$\left(x\frac{\partial}{\partial y}\right)\left(y\frac{\partial}{\partial x}\right) = x\left(\frac{\partial}{\partial y} y\right)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\frac{\partial}{\partial x} = x\frac{\partial}{\partial x} + xy\frac{\partial^2}{\partial y \partial x}$$

$Y, X : C^\infty(M) \longrightarrow C^\infty(M)$; $X \circ Y : C^\infty(M) \longrightarrow C^\infty(M)$.

> $X \circ Y - Y \circ X = y \frac{\partial}{\partial y} - x \frac{\partial}{\partial x}$ → vector field!!

Thm: M smooth mfl. X & Y are smooth vector fields

Then $X \circ Y - Y \circ X$ is a derivation.

Pf: (Check the rule).

$$\underset{\uparrow}{(X \circ Y - Y \circ X)}(fg) = [X,Y](f \cdot g) = [X,Y](f) \cdot g + f \cdot [X,Y](g)$$

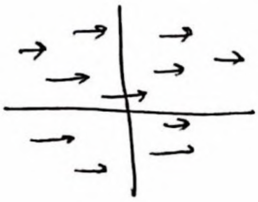
Thm: Every derivation δ of p defines a unique tangent vector, i.e. $\exists\, v \in T_pM$ s.t $\delta = \delta_v \left(\text{recall} = \frac{d}{dt}\Big|_{t=0} f(\alpha(t)) \right)$

Corr: Every derivation $\Delta : C^\infty M \to C^\infty M$ defines a v-field

~~Ex~~

D: Lie bracket of X & Y is $[X, Y] = X \circ Y - Y \circ X$.

EX: $\mathbb{R}^2$, $X = \frac{\partial}{\partial x_1}$ $Y = \frac{\partial}{\partial x_2}$,



$$[X, Y] = \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial}{\partial x_1}\right) = 0$$

(on C^2 fns. since order can be changed

If the vector fields don't commute,

don't go back!

Lecture

Derivations

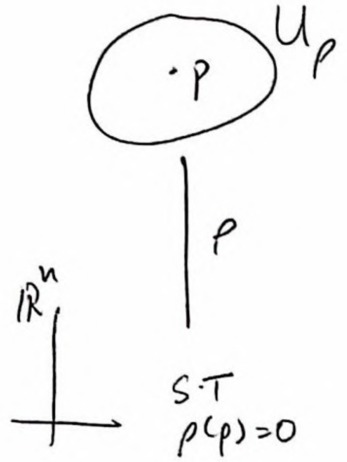
at $p \in M$. $\delta : C^\infty(M) \to \mathbb{R}$ linear.

s.t $\delta(fg) = \delta(f)\cdot g + f\cdot \delta(g)$.

Lemma : $v \in T_pM$, δ_v = directional deriv, then δ_v is a derivation.

Moreover, if $\delta_v = \delta_w$, then $v = w$ $(v, w \in T_pM)$.

Idea : Take a coord chart (U_ρ, ρ) around p.



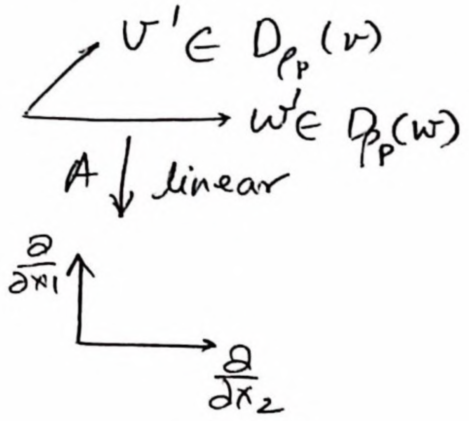
S.T $\rho(p) = 0$

Want

$D\rho_p(v) = \frac{\partial}{\partial x_1}$

$D\rho_p(w) = \frac{\partial}{\partial x_2}$

Can do unless $w = a\cdot v$.



Then replace ρ with $A\circ\rho$.

Look at $x_1 : \rho(U_\alpha) \to \mathbb{R}$

$\alpha : x = (x_1, \ldots, x_n) \mapsto x_1$

$\frac{\partial}{\partial x_1}\alpha = 1$; $\frac{\partial}{\partial x_2}\alpha = 0$ $\Rightarrow \frac{\partial}{\partial x_1} \neq \frac{\partial}{\partial x_2} \Rightarrow \delta_v, \delta_w$ are diff.

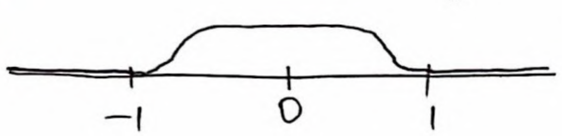
Problem

$$\left.\begin{array}{l} x_1 \circ \varphi \\ x_2 \circ \varphi \end{array}\right\} \text{ are only defined in a nbd of } p.$$

$$\begin{array}{ccc} \overline{g}|_{U_p} & \overset{?}{=} & g \\ C^\infty(M) & & C^\infty(U_p) \\ \rotatebox{90}{$\in$} & & \\ f & \longmapsto & f|_{U_p} \end{array}$$

$\Rightarrow$ Need Bump functions. (Comes up a lot).

On $\mathbb{R}$ want a fn $\overset{\psi(x)}{}$ that looks like



& want $\psi(x) \in C^\infty(\mathbb{R})$.

Cannot do in $C^\omega(\mathbb{R})$ (real analytic)

(Since C^ω fn. $= 0$ in an interval $\Rightarrow 0$ everywhere).

$$\psi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & \text{on } (-1,1) \\ 0 & \text{otherwise.} \end{cases}$$

on $\mathbb{R}^n$ Want $\overline{\psi} : \mathbb{R}^n \longrightarrow \mathbb{R}$

$\overline{\psi} \equiv 0$ outside $B_1(0)$

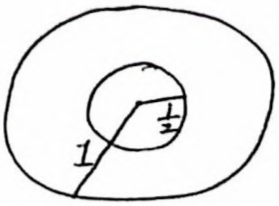
$\overline{\psi}$ is C^∞ on $\mathbb{R}^\infty$ $\neq 0$ on $B_1(0)$

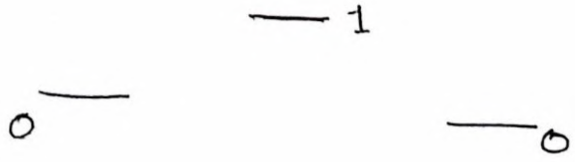
$\omega(x) = \psi(|x|^2)$

Really want $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$

s.t $\psi \equiv 1$ on $B_{1/2}(0)$

& $\psi \equiv 0$ outside $B_1(0)$





Pick (maybe?)

$\phi = \overline{\psi}(x) \cdot \psi(\frac{1}{2|x|})$

// end of Lemma.

$\Rightarrow \quad \delta_v = \delta_w \iff v = w$ in T_pM.

$\Rightarrow \exists$ n linearly independant derivations.

Since $T_pM \underset{\text{embeds into}}{\hookrightarrow} \{\text{derivs. at } p\}$

Lemma: $\{$derivs at $p\}$ are n-dim v-space.

P: a $\supset$ n-dim v-space $= \{\delta_v \mid v \in T_pM\}$.

b $f \in C^\infty(M) \xleftrightarrow{\text{bump fns}} C^\infty(U_p)$

just look locally

Can work locally around 0 in $\mathbb{R}^n$.

$I_p = \{ f \ C^\infty \text{ near } p \mid f(p) = 0 \} \subset \{ f \text{ defined \& smooth near } p \}$.

$I_p^2 = \{ \sum f_i g_i \mid f_i, g_i \in I_pM \}$

Let $f, g \in T_pM$: $\delta(f \cdot g) = \delta(f) g + f(p) \cdot \delta(g)$

$= \delta(f) \cdot 0 + 0 \cdot \delta(g) = 0.$

Thus δ vanishes on I_p^2.

$\Rightarrow$ derivations embed into $\left(I_p / I_p^2 \right)^{*}$ dual

(Sub)**Lemma**: $\dim(I_p / I_p^2) = n$.

⌈**Corollary**: derivations at $p = \{ \delta_v \mid v \in T_pM \}$

P: dims are finite & equal. ⌋

~~Pf~~ **Lemma**: Suppose ~~[illegible]~~ $f : \overset{\overset{\circ}{\cap}}{U} \longrightarrow \mathbb{R}$ & C^∞.
open in $\mathbb{R}^n$

Then $\exists$ C^∞ fns on U (on $\mathbb{R}^n$) s.t. $f(x) = f(0) + \sum_i x_i f_i$.

Pf: By fund. thm. of calc.

Note $f_i(0) = \frac{\partial f}{\partial x_i}(0)$.

$$f(x) = f(0) + \int_0^1 \frac{d}{dt} f(tx)\, dt$$

$$= f(0) + \int_0^1 \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(tx)\, dt$$

$$= f(0) + \sum_{i=1}^n x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx)\, dt}_{\longrightarrow\ f_i(x) \in C^\infty}$$

Lemma: f C^∞ (near 0) in $\mathbb{R}^n$, then $\exists$ C^∞ fns f_{ij} on $\mathbb{R}^n$ (near 0)

s.t $f(x) = f(0) + \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(0) + \sum_{i,j=1}^{n} x_i x_j f_{ij}(x)$.

Pf: Apply last lemma to $f_i(x)$. //

$\hookrightarrow$ any derivation is zero here $\Rightarrow$ done!

$\Rightarrow$ only linear term survives.

Dealing with germs of functions.

$p \in M \xleftarrow[C^\infty]{}$ mfl. Suppose f is C^∞ defined on an open nbd U_f of p. (same for g . . . . U_g of p). Say f & g define the same germ if $f|_{U_f \cap U_g} = g|_{U_f \cap U_g}$.

Note: $f_1, f_2 \rightsquigarrow$ same germ. $[f]$

$h_1, h_2 \rightarrow$ same germ $[h]$

$\Rightarrow f_1 + h_1$, $f_2 + h_2$ define same germ.

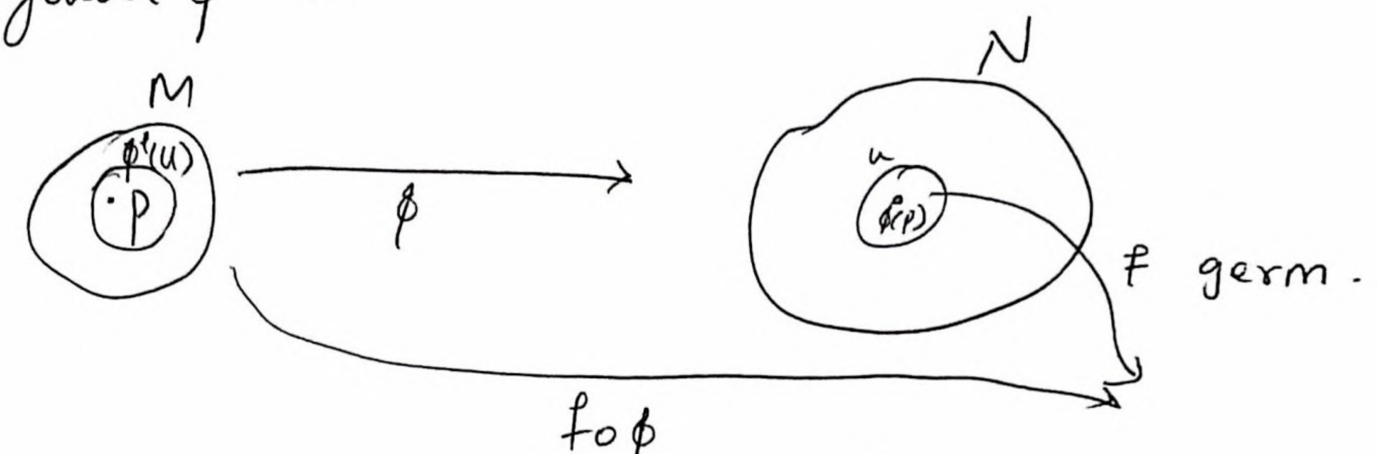
Really f_1, f_2 are $\sim$ & define $[\ ]$.

Note: Have globally defined fns & locally defined fns. on nbds & germs at p.

Note: I_p, I_p^2 make sense for germs.

$$\dim\left(I_p^{\text{germ}} / I_p^{2\,\text{germ}} \right) = n$$ & is a v-space.

- Given $\phi : M \longrightarrow N$, C^∞



is a germ on $\phi^{-1}(U)$.

$$\phi^*: I_{\phi(p)} / I_{\phi(p)}^2 \longrightarrow I_p / I_p^2$$

- $D\phi = \phi_* : T_pM \longrightarrow T_{\phi(p)}N$

<u>Duality:</u> $T_pM \underset{\text{dual}}{\longleftrightarrow} I_p / I_p^2 \cong T_pM^*$

Thus can think of/identify $T_pM^* = \underset{\substack{\uparrow \\ \text{"cotangent space" at } p.}}{I_p / I_p^2}$

$$\phi^* = T^*N \to T^*M$$

$$\left(\text{dual}\right) \; D\phi : TM \to TN$$

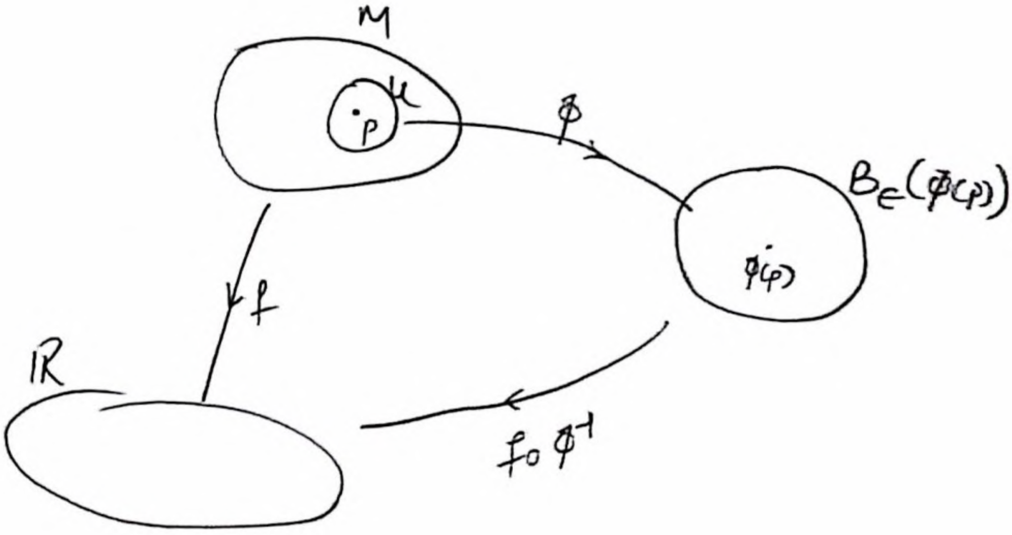
Recall, $V \xrightarrow{L} W$

- $\Rightarrow \underline{\underline{W^* \xrightarrow{L^*} V^*}}$

Lecture (Monday)

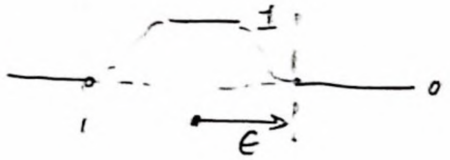
Last time

$C^\infty(M)$, M C^∞
$\overset{\Psi}{f}$



Want $f \mapsto \bar{f}$ which is 0 outside $\phi^{-1}(B_\epsilon(\phi(p)))$.

For this take a bump function $\rho : \mathbb{R}^n \to \mathbb{R}$ s.t $\rho \equiv 0$ outside $B_\epsilon(\phi(p))$ and $\rho \equiv 1$ inside $B_{\epsilon/2}(\phi(p))$.



Back to Immersions etc.

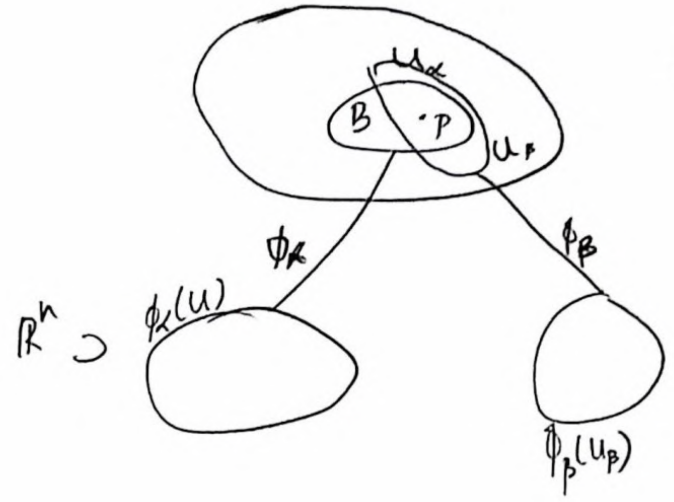
M, N are two manifolds (C^∞-mfl). $f : M \to \underset{\overset{\Psi}{q}}{N}$.
q is a reg. value of f if $df_p \mid T_pM \to T_qN$ is surjective for all $p \in f^{-1}(q)$.

Note: If $q \notin Im(f)$, then it is a regular value.

Measure: $A \subset \mathbb{R}^n$, volume $vol(A) = \int_A 1\,dx \in [0,\infty]$. (If you know Lebesgue measure, A is a measure set.) for Riemann need more assumptions.

Call A has <u>meas 0</u> if vol $A=0$

M a C'-mfl.



Consider $B \subset U_\alpha$. B has meas 0

if $\phi_\alpha(B)$ has meas. 0

$$\Rightarrow 0 = \int_{\phi_\alpha(B)} 1 \, dx$$

Suppose U_β, ϕ_β is another coord. chart & $B \subset U_\beta$.

Main Q: Is meas $=0$ well defined on C' mfls?

Q1: If $\phi_\alpha(B)$ is meas. (i.e. Riemann / Lebesgue), is $\phi_\beta(B)$ measurable?

Q2: If vol $\phi_\alpha(B) = 0$, is vol $\phi_\beta(B) = 0$?

$$\phi_\beta(B) = \underbrace{\phi_\beta \circ \phi_\alpha^{-1}}_{\text{transition map } C'}\left(\phi_\alpha(B)\right) \Rightarrow \underline{Q1} \checkmark$$

Fact. if ψ is C', ψ of a measurable is still measurable.

$$\text{vol}\left(\phi_\beta(B)\right) = \int_{\phi_\beta(B)} 1 \, dx = \int_{x \in \phi_\alpha(B)} \det\left(\overset{\to \text{Jacobian.}}{D}\left(\phi_\beta \circ \phi_\alpha^{-1}\right)\right)\Big|_x \, dx.$$

change of variables.

If $\phi_\alpha(B)$ has 0 measure then so does

Def $B \subset M$ has 0 measure (volume) if $\forall \phi_\alpha, U_\alpha$

$$\text{Vol}\ \phi_\alpha(B \cap U_\alpha) = 0.$$

$A \subset M$ has full measure if $M - A$ has 0 measure.

↳ (Also well defined since 0 meas is)

Sard's Theorem

$f: M \to N$, $M\ N$ C^∞. Then the set of regular values
↳ C^∞

has full measure. (needs C^∞, does not work for C^1).

Application.

> Sometimes makes calculation unnecessary.

Ex: $SL(n, \mathbb{R})$ is a manifold.

$$\| \\ \det^{-1}(1)$$, enough to show 1 is a reg. value.

$$\det(\lambda A) = \lambda^n \det(A).$$

Claim: 1 is not a reg value $\Rightarrow$ neither is λ^n.

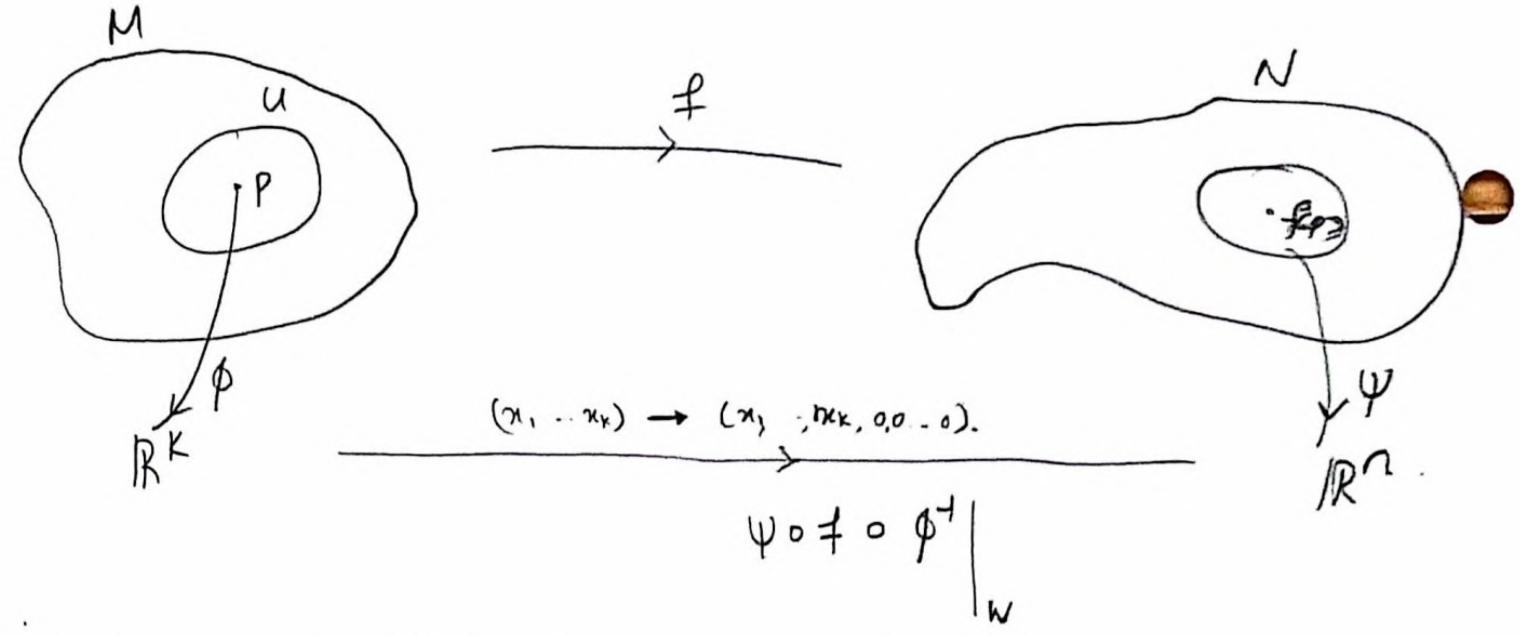
But λ^n does not have measure 0, $\Rightarrow$ 1 is a reg. value !

Immersions - Normal form

① Simplest immersion: $\mathbb{R}^k \subset \mathbb{R}^n$, $k \leq n$.

$$(x_1 \dots x_k) \to (x_1, \dots x_k, 0 \dots, 0)$$

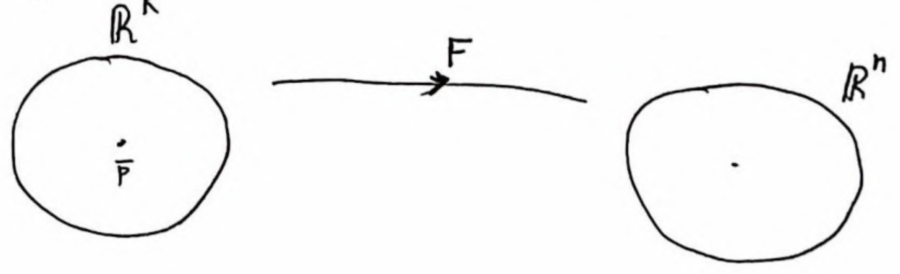
Normal Form $f: M \to N$ an immersion, i.e. $Df_p : T_pM \to T_{f(p)}N$. is injective. Then $\exists$ charts (U, ϕ) at p & (V, ψ) at $f(p)$,



$\Rightarrow$ Get a map locally investible & use IFT.

$$\begin{tikzcd}
M \arrow[r] & N \\
\times & \\
\mathbb{R}^{n-k} \arrow[uur, dashed, "\text{local diff.}"'] &
\end{tikzcd}$$

Good enough to work on given charts (U, ϕ) & (V, ψ)



Df_p is injective. $Df_p = \underset{n\times k}{A} = \begin{pmatrix} A(e_1) & A(e_2) & \cdots & A(e_k) \end{pmatrix}$

rank $A = k$

$$(id, —)^T \xleftarrow[0]{\overbrace{0((A_1, A_k)^{-1}, —)^T}^{\hat{A}}} \overset{=}{} \begin{pmatrix} \underbrace{A_1 \dots A_k}_{\text{lin. ind.}}, & — \end{pmatrix}^T$$

$\Rightarrow \hat{A} \circ F : \mathbb{R}^K \to \mathbb{R}^n$. extend to $\mathbb{R}^n \to \mathbb{R}^n$. $\Rightarrow \hat{\hat{A}} \cdot f(x_1 \dots x_n) = (\hat{A} \circ F(x_1 \dots x_k), x_{k+1} \dots x_n)$

$\hat{\hat{\vec{A}}} \overset{?}{\equiv} \begin{pmatrix} id & ? \\ 0 & id \end{pmatrix}$. Get local diffeo G , $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} \xrightarrow[G]{} \mathbb{R}^n$.

Use G to define a coord. chart in $\mathbb{R}^n$.

By const. in this chart F looks like $(x_1 .\ x_k) \longrightarrow (x_1, .. x_k, 0 ... 0)$ $/\!/$

Lecture (Wednesday)

Vector Fields

Going from v-fields to curves:

Define: X is a v-field on M (C^∞-mfd). $c : (a,b) \to M$ is a solution curve for X. if $\forall_{t_0 \in (a,b)}$ $\left.\frac{d}{dt}\right|_{t=t_0} c(t) = X(c(t_0))$

In coords: $\frac{\partial}{\partial x_i}$ std v fields pulled back from $\hat{e}_i$.

Then under chart U,

$$X|_U = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i} \qquad X \ C^\infty \Leftrightarrow a_i \text{ are } C^\infty$$

Write curve $c(t) = (c_1(t), .. , c_n(t))$ in the coords.

$$\dot{c}(t_0) = (c_1'(t_0), .. , c_n'(t_0)) \underset{/\!/}{=} \sum_{i=1}^{n} c_i'(t_0) \frac{\partial}{\partial x_i}(c(t_0))$$

$$X(c(t_0)) = \dot{c}(t_0) \ ;$$

$\Rightarrow a_i(c(t_0)) = c_i'(t_0)$ → system of ODEs.

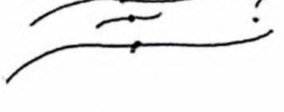
⇒ can solve uniquely if a_i's are Lipschitz functions. ← c' will do.

↓

Vague. (For what t do we get a soln.)

↓

depends on base pt.



<u>More precisely</u>

X some C^∞ v field, $p \in \mathbb{R}^n$, then $\exists\, \epsilon > 0, \exists\, \delta > 0$ s.t

$\forall q \in B_\epsilon(p)$, $\exists$ soln. to the ODE on the interval $(-\delta, \epsilon)$.

(Existence & Uniqueness of local solns.).

<u>Def</u>: Call X; vf, on M <u>complete</u> if solution curves exist through any pt. ∀ time.

<u>Lemma</u>: M compact C^∞. X C^1 v-field ⇒ X is complete.

<u>Proof</u>: True for short time on nbd. an nbd U_p.

$p \in U_p$, $(-\epsilon(p), \epsilon(p))$. Then ∃ finitely many $p_1, \ldots, p_\ell$ s.t

$\bigcup^{\ell} U_{p_i} = M$.

⋮

//

<u>Def</u>: $\Phi_t : \mathbb{R} \times M \to M$ s.t $\Phi(t,p)$ = a soln. curve for X fixing p.

$\phi_t(p) := \Phi(t,p)$ → soln-curve through p & ϕ_t is the <u>global flow</u> determined by X

MID Term

Charts, T_pM, V-bundles, TM, TM*, Examples (Basic & counter) & constructions. Lie stuff, Products, Group actions, reg. value stuff, Sards Thm (SLNR is a manifold type), Level sets, HW could show up again. (50mins). {until Before flow stuff}

> X v-field ⟷ ϕ_t flow
↳ curves tangent to vector field.

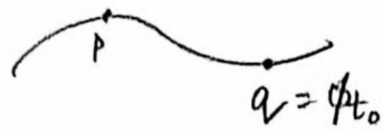
$$\left.\frac{d}{dt}\right|_{t=0} \phi_t(x) = X(\phi_{t_0}(x)).$$

Thm: X $C^1 \Rightarrow \exists$ unique soln. on small intervals (i.e. ~~global~~ local flow).

> X complete ⇒ ϕ_t is defined for all time.

Thm: M compact ⇒ any v-field is complete.

> X v-field



$\phi_s(q) = \phi_{t_0+s}(p)$
from uniqueness of ODE.

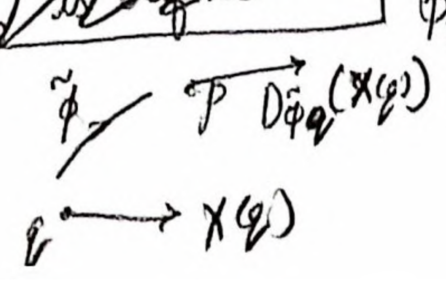
$\Rightarrow$ $\phi_{s+t_0}(p) = \phi_s(\phi_{t_0}(p))$. i.e. $\boxed{\phi_{t_0+s} = \phi_s \circ \phi_{t_0}}$

⇝ gives $\underbrace{\text{action of } \mathbb{R}}_{\text{called "flow"}}$ on M if X is complete.

~~Prooise Suppose ϕ_{s+t_0} is defined~~

Suppose X v-field. $\tilde{\phi}: M \to M'$ a diffeomorphism
$\overset{Y \text{ v-field in } M'}{}$

$\tilde{\phi}_*(x)(p) = D\tilde{\phi}_q(X(q))$ where $p = \tilde{\phi}(q)$.

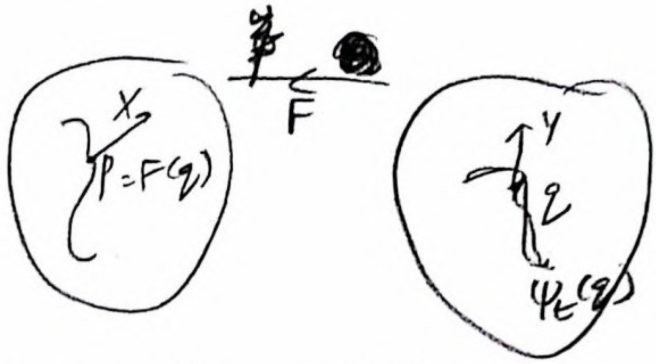


& X, Y are $\tilde{\phi}$-related. (still makes sense for local diffeo).

> Also say X & Y are $C^{?}$ conjugate if ϕ is $C^{?}$.

ϕ_t = flow of X

ψ_t = flow of Y



Q: If $\tilde{\phi}_*(X) = Y$,

Let $F = \tilde{\phi}$. $F_*(Y) = X$

$$\frac{d}{dt}\Big|_{t=0} \overbrace{F(\psi_t(q))}^{\text{curve in } M.} = DF_{\psi_t(q)}(Y(\psi_t(q)))$$

$$= X(F(\psi_t(q))).$$

→ $\boxed{\phi_t = F \circ \psi_t \circ F^{-1}}$ Conjugacy of Flows.

More generally, $M \times F = M$ $$\begin{tikzcd} M \arrow[loop right, "\phi_t"] \arrow[d, "\pi"] \\ N \arrow[loop right, "\psi_t"] \end{tikzcd}$$ $\Rightarrow (\psi_t \circ \pi)(x) = (\pi \circ \phi_t)(x)$

> Suppose X, Y are v-fields in M.

L: $[X, Y]$ is a v-field. Recall X, Y v-fields $\leftrightarrow$ Derivations

Lie
racket. $X : C^\infty \circlearrowleft$ $\quad X \circ Y - Y \circ X : C^\infty(M) \circlearrowleft$

$Y : \sim$

Recall 1-1 correspondence b/w v-fields & derivations.

- $f, g \in C^\infty(M)$ linear, $(X \circ Y)(f \cdot g) = X(Y(f \cdot g))$

$$= X(Y(f) \cdot g + f \cdot Y(g))$$

$= X\circ Y(f)\,g + Y(f)\cdot X(g) + X(f)\cdot Y(g) + f\cdot X\circ Y(g).$

$(Y\circ X)(f\,g) = Y\circ X(f)g + X(f)\,Y(g) + Y(f)\cdot X(g) + f\cdot Y\circ X(g)$

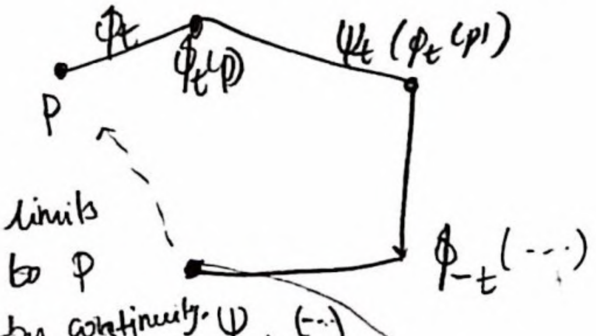
$\Rightarrow [X,Y] = (X\circ Y)(f)\cdot g + f\cdot(X\circ Y)(g)$
$\quad - (Y\circ X)(f)\cdot g - f\cdot(Y\circ X)(g)$

$= [X,Y](f)\cdot g + f[X,Y](g)$ ⇒ is a deriv
⇒ a v-field.

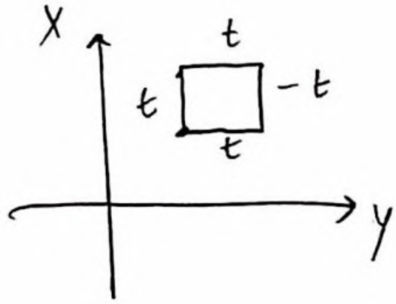
<u>Message</u> $[X,Y]$ measures how much X & Y don't commute.

> In terms of v-fields rather than derivations ??

Suppose φ_t – (local) flow of X
ψ_s – flow of Y



EX.
$X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y}$



→ This end pt. is a curve $\frac{d}{dt}\Big|_{t=0}(c(t^2)) = [X,Y](p)$ //

(Look at Spivak intro to D-G: vol 1 - page 150)

Vector Fields (contd.)

M mfl, X, Y C^∞ v-fields.

$[X,Y] = X \circ Y - Y \circ X$ defines a vector field since we showed it's a derivation.

Special situation, suppose X, Y don't vanish on M (or some $U \subset M$ open).

Assume $[X,Y] = 0$. Then $\exists$ coordinates on a chart s.t $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y}$

(i.e. X, Y commute).

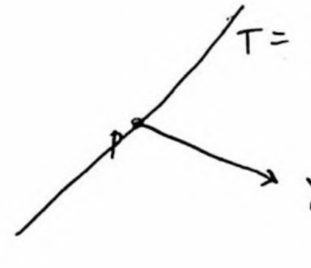
More generally, $X_1 \ldots X_k$ vector fields that don't vanish on U. Then $\exists$ C-char such that $x_1 = \frac{\partial}{\partial x_1}, \ldots, x_k = \frac{\partial}{\partial x_k}$ if $[X_i, X_j] = 0$ $\forall$ i, j (Need $x_1(p), \ldots, x_k(p)$ are lin. independant).

Proof (Idea)

① K = 1 X ⇝ gives flow $\phi(t)$ (local flow)

$\bar{U} \subset U$

Pick transversal $T \subset \bar{U}$ submfl of dim n−1.



$T = \{(z_1, \ldots, z_n) \mid z_u = 0\}$
$\hookrightarrow \in \text{span } X(p)$

Can make coords. $y_1, \ldots, y_n$ s.t
$T = \{(0, y_2, \ldots, y_n)\}$

⇒

$T = \{(0, y_2, \ldots, y_n)\}$ Use the flow & apply to T

$\Phi = (t, y_2, \ldots, y_n) \mapsto \phi_t((0, y_2, \ldots, y_n))$

Check $D\Phi(0,0,\ldots,0)$ invertible., Then Φ is a local diffeo & is the chart needed.

$D\Phi_{(0,\ldots,0)} = \begin{pmatrix} 1 & - \\ & \ddots & 0 \\ 0 & & 1 \end{pmatrix}$ is invertible since X(p) is transversal to T.

② $\underline{k=2}$: X, Y

wlog in suitable coords. $X = \frac{\partial}{\partial x}$

$\underline{\text{Cheat}}$: suppose $n = 2$

$$X = \frac{\partial}{\partial x}, \quad Y = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$$

& $[X, Y] = 0$

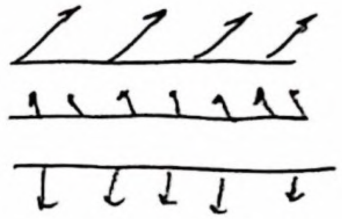
$$\overset{\shortparallel}{}$$

$$\frac{\partial}{\partial x}\left(a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}\right) - \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x}$$

$$= \frac{\partial a}{\partial x}\frac{\partial}{\partial x} + \frac{\partial b}{\partial x}\frac{\partial}{\partial y} = 0. \Rightarrow \frac{\partial a}{\partial x} = 0 \ \& \ \frac{\partial b}{\partial x} = 0.$$

Thus $a(x,y) = a(y)$ & $b(x,y) = b(y)$.

$\Rightarrow Y = a(y)\frac{\partial}{\partial x} + b(y)\frac{\partial}{\partial y}$ $\Rightarrow Y =$



Let ψ_t = local flow for Y.

$\Psi(x,t) = \psi_t(x,0)$ & $\Psi : \underset{(\text{local})}{\mathbb{R}^2} \longrightarrow M$

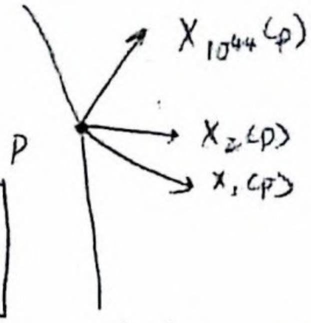
If $\dim M > 2$, pick a submfl (local) through p, that is transversal to both x, y (say Z).
$(\pitchfork)$

$$\Psi(s,t,\underset{\uparrow}{z}) = \psi_t \circ \phi_s(z).$$

③ k, dim M arbitrary

$$\boxed{(t_1, \quad t_{10^{44}}, z) \mapsto \phi^{10^{44}}_{t_{10^{44}}} \circ \cdots \circ \phi^{1}_{t_1}(z))}$$

(Flows commute)



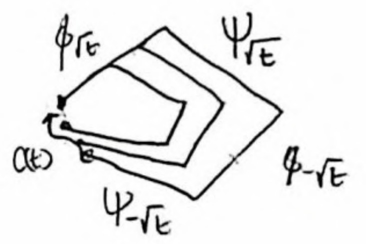
Z transversal. //

<u>Cor</u>: If X, Y commute ⇔ $[X,Y]=0$ then they flows (local) commute.

<u>Notes on proof</u> ↺

<u>k=2</u>

Any v-fields X, Y, C^∞.
ϕ_t ψ_t flows



<u>Thm</u>: $c(t)$ is differentiable (C^{k-1}) & $c'(0) = [X,Y](p)$. ← Geometric interpretation.

If $[X,Y]=0 \Rightarrow c(t)=p \quad \forall t$.

Spivak vol 1

<u>~~Fabinis~~ Thm & Lie gps. (next t)</u>

Frobenius

Lec (Monday)

> $F: M \to N, C^\infty$, X v-field on M, Y v-field on N. X & Y are F-related

if $dF_p(X(p)) = Y(F(p)) \in T_{F(p)}N$.

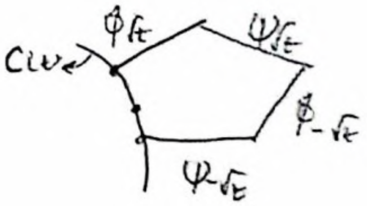
> Suppose X_1 is F related to Y_1

& X_2 is F related to Y_2

$\Rightarrow [X_1, X_2]$ " " " " $[Y_1, Y_2]$

$X_1 \leftrightarrow \phi_{1,s}$ & $X_2 \leftrightarrow \phi_{2,t}$ are 2 flows.

Why is $\sqrt{t}$ used in $[\underset{\phi_t}{X} \circ \underset{\psi_s}{Y}](p)$



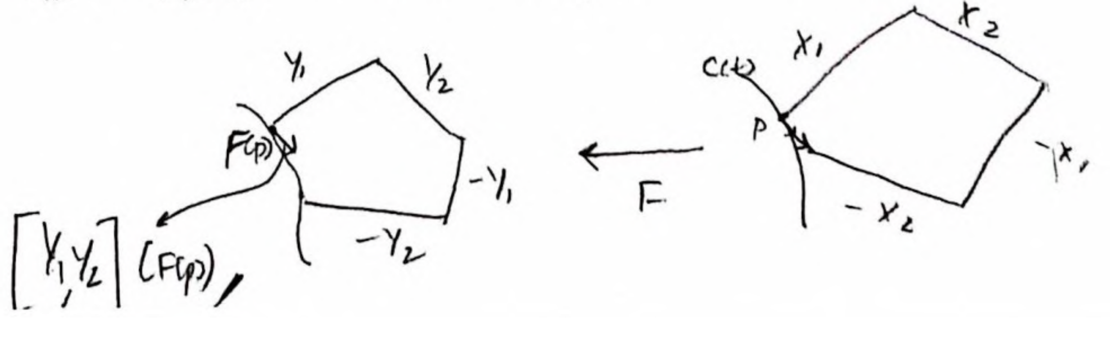
$G(s,t) = \psi_{-s} \circ \phi_{-t} \circ \psi_s \circ \phi_t(p)$

$$\frac{\partial G(s,t)}{\partial s} \quad \frac{\partial G}{\partial t}(p) = 0 \ \& \ \frac{\partial}{\partial s}(G(s,0)) = \frac{\partial}{\partial s}\Big|_{s=c}(p) = 0.$$

$\Rightarrow$ higher order terms come into the picture $\Rightarrow \sqrt{t}$ is used.

$F(\phi_{1,s}(p))$ is a soln-curve to the ODE for Y_1. Thus,

$F \circ \phi_{1,s} = \psi_{1,s}$ & $F \circ \phi_{2,s} = \psi_{2,s}$.



Distributions (D)

(K- plane fields)

$$T_pM \rightsquigarrow Gr_k(T_pM)$$

$$\parallel$$

$$\{k\text{-dim v-subspaces of } T_pM\}$$

Make a fiber bundle out of TM, $Gr_k(M) = \bigsqcup_{p\in M} Gr_k(T_pM)$.

Topologize using local product structure of TM.
(make it a C^∞ mfd)

$$\begin{tikzcd} Gr_{k,n} \arrow[r] & Gr_k(M) \arrow[d, "\pi"] \\ n = \dim M & M \arrow[u, bend left, "D"] \end{tikzcd}$$

D is by def. is a smooth section of this construction.
(i.e distribution).

> $T_pM \supset D(p) \to$ k-dim subspace s.T $\overset{\nearrow \text{span}}{\langle v_1 \ldots v_k \rangle} = D(p)$
> & $v_1, \ldots v_k \in T_pM$.

> Doing this $\forall p$ gives locally, $v_1(q), \ldots, v_k(q)$ lin-ind. $\forall q \in \underset{U}{\overset{\shortparallel}{\text{Nbd}(p)}}$
> $v_i(q) \in T_qM$.

> Want v_i are smooth v-f's on U.

Thus smooth distributions D of rank k are given by the following data:

1) $\forall p \in M$, $D(p) \subset T_pM$ is a k-dim subspace

2) To define smoothness, suffices to do it locally, i.e. $\forall p \in M$, $\exists$ nbd U of p & smooth v-fields V_i on U s.t

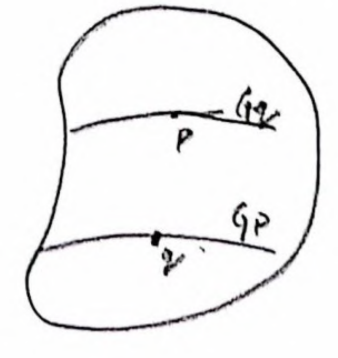
① $\forall i$, $V_i(q) \in D(q)$

② $V_1(q), \ldots, V_k(q)$ are lin. ind.

Ex:

① $\mathbb{R}^n = M$ & $D(p) = \mathbb{R}^k$

② Suppose G is a Lie group that acts on some mfd M. Suppose $\forall p \in M$, G_p is discrete (e.g. $G_p = \{1\}$).



$D(p) = T_p(Gp)$.

③ V v.f. on $\mathbb{R}^n$ (nonvanishing), $D(p) = V(p)^\perp$.

③a $M = \mathbb{R}^n - \{0\}$. $V(p) =$ radial v-f $v(p) = p$.

$D(p) =$ planes orthogonal to $v(p)$.

④ $\left\{\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\right\}$ = Heisenberg group (Lie group).

$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ → Tangent vectors at I.

$\bar{A}_1, \bar{A}_2$ are the left inv. vector fields.

$D = \mathbb{R} \cdot \bar{A}_1 + \mathbb{R} \cdot \bar{A}_2$ is an exciting

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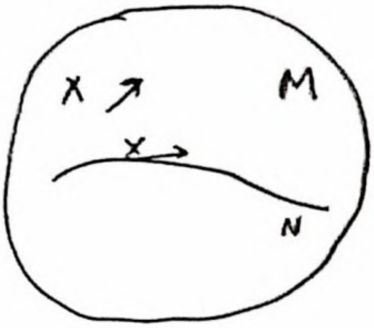
HW7

P3 corrected.

$N < M$ immersed submfd. $p \in N$, X vf on M, s.t $\forall q \in N$, $X(q) \in T_q N$



Q: ϕ_t = (local) flow of X on M.

Claim: $\forall p \in N$, $\phi_t(p) \in N$. → wrong.

Corrected version

$\phi_t(p) \in N$ $\forall$ small enough t.

Hint Q3

Any v.f that is non vanishing can be mapped to $\frac{\partial}{\partial x}$

Hint Q2 (Sard's Thm)

$V = y - x$, $F : M \times N \longrightarrow \mathbb{R}^n$.

$(x, y) \longrightarrow (y - x) = v$ & then use Sard's Thm.

Lec (Wed)

Examples of distributions

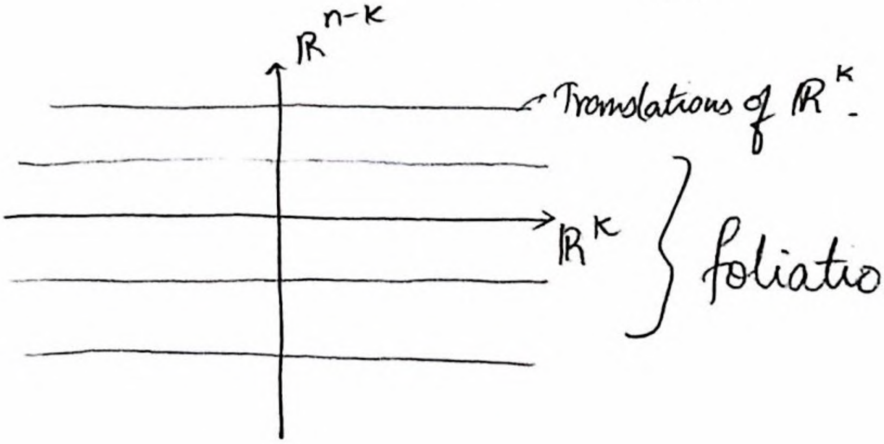
① Integrable

Say V is a distribution (smooth).

Call V integrable if $\forall p \in M$ $\exists$ coord-chart (U, ϕ) s.t

$$D\phi(V(x)) \subset T_{\phi(x)}\mathbb{R}^n$$
$$\| \; \{(x_1, \ldots, x_k, 0, \ldots, 0) \mid x_i \in \mathbb{R}\} = \mathbb{R}^k \subset \mathbb{R}^n.$$

Example : $V(p) = \langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k} \rangle$



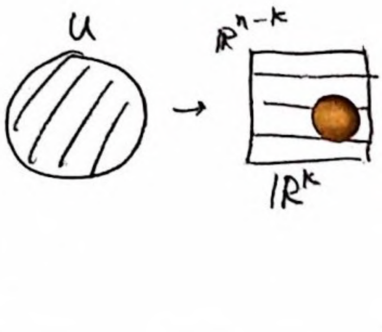
} foliation. (each subspace is called a leaf)

Def : M C^∞ mfl. $\mathcal{F}$ = partition of M. i.e. $\forall x \in M$,

F_x = immersed submfl of M s.t $F_x \cap F_y \neq \phi \Leftrightarrow F_x = F_y$.

(i.e. F defines an equivalence relation).

and s.t $\forall p \in M$ $\exists$ coord. chart (U, ϕ) s.t

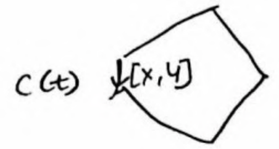


Frobenius theorem

Given a foliation $\mathcal{F}$, $\mathcal{F}_x$ is called the leaf of $\mathcal{F}$ through x. Then can define a distribution $V(p) := T_p\tilde{\mathcal{F}}_p$ of dim k & smooth

Consider X, Y v-fields on M s.t $\forall p \in M$, $X(p), Y(p) \in V(p) = T_p \mathcal{F}_p$.

By HW7 (3c), $[X,Y](p) \in V(p)$.



Thm (Frobenius)

If $V(p)$ = tang. dist to a foliation $\mathcal{F}$, i.e $V(p) = T_p\mathcal{F}_p$

then any two v-fields X, Y s.t $\forall p$, $X(p), Y(p) \in V(p)$

then $[X,Y](p) \in V(p)$.

Def: Given any smooth k-dim distr. V on a C^∞ mfl M.

Call V integrable if any two v-fields X, Y s.t $\forall p$ $X(p), Y(p) \in V(p)$ then $[X,Y](p) \in V(p)$.

The 2 defns. of integrable are equivalent // . (John Lee p490

Proof : $\Leftarrow$: just did.

$\Rightarrow$: Book?

Ex: $\mathbb{R}^3$, $V(p) = p + \mathbb{R}^2 \Rightarrow$ (x y planes) $\Rightarrow$ Integrable.

Now quotient by $\mathbb{Z}^3$. $(\mathbb{R}^3/\mathbb{Z}^3 = T^3)$.

$\overline{V(p)} = D\pi(V(p))$ where $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3/\mathbb{Z}^3$.

for $\bar{p} = \pi(p)$.

$\Rightarrow$ get 2 tori foliating T^3.

Mess it up a little,

$V(p) =$ angle (irrational) with $\mathbb{R}^2$.

$= \mathbb{R}v_1 + \mathbb{R}v_2$ where v_1 & v_2 are irrational w.r.t $\mathbb{Z}^2$.

Push down to T^3 (can do - check if well defined).

T^3 is foliated by "planes" densely.

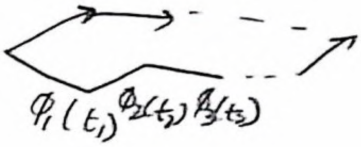
Frob Thm (special case)

Proof of $\Rightarrow$: Special case, suppose $\forall p$ $\exists$ U nbd & Vf on U $x_1 \dots x_k$ on U s.t. $\langle X_1(q), \dots, X_k(q) \rangle$ $\forall q \in U = U(q)$.

& $\forall i,j$ $[X_i, X_j] = 0$. Then Frobenius holds.

By last friday, the local flows ϕ_i associated with X_i commute.

$(t_1, .. , t_k) \xrightarrow[\text{immersion}]{}$



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Lec (Friday)

Frobenius (contd.)

Ex Heis $= \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ 0 & & 1 \end{pmatrix} \right\}$

$$V(1) = \Big\langle \underset{A \uparrow}{\overset{X(1)}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}}, \underset{\nearrow B}{\overset{Y(1)}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}} \Big\rangle$$

left invariant vector fields on Heis

$$DL_g(V(1)) = V(g) = \langle X(g), Y(g) \rangle$$

Then $[X, Y] = \underbrace{Z}$. left inv. vf. determined by $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$AB - BA = Z$$

Last time: Frobenius is true if V is spanned by k lin. ind. vf $X_1 \ldots X_k$ which commute.

$\nearrow \Phi_1(t) \qquad \uparrow \Phi_k(t)$

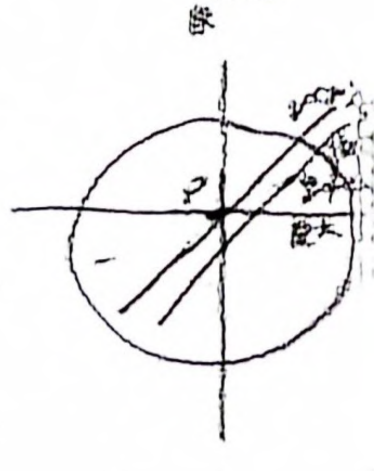
To prove Frobenius in general find k lin-ind commuting v-fields.

① This is a local problem $\Rightarrow$ work in $\mathbb{R}^n$.

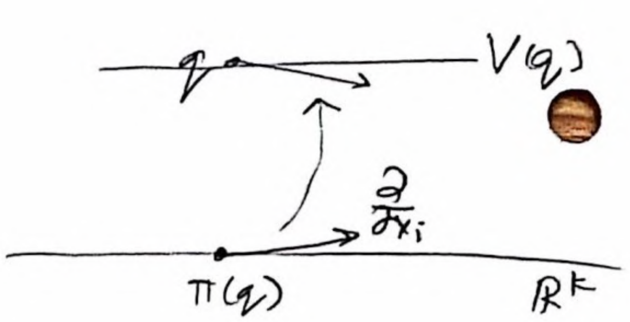
② wlog $V(p) \underset{\text{transversal}}{\pitchfork} \mathbb{R}^{n-k}$ & $\forall q \in$ nbd of p $V(q) \pitchfork \mathbb{R}^{n-k}$

$$\Rightarrow \exists\ \Pi : \mathbb{R}^n \to \mathbb{R}^k$$

s.t. $D\Pi : V(q) \to \mathbb{R}^k$ is 1-1 & onto.



$X_i(q) = (D\pi)^{-1}\left(\frac{\partial}{\partial x_i}(\pi(q))\right)$



$Y_i = \frac{\partial}{\partial x_i} \Rightarrow Y_i$ commute

X_i & Y_i are π-related

& hence X_i commute.

(~~since~~ Recall that $[X_i, X_j]$ is π related $[Y_i, Y_j] = 0$) //

> G Lie group (C^∞). Say $\exists$ {left inv. v-fields V on G}

i.e. $V(g) = DL_g(V(1))$.

this is a vector space

since $(V_1 + V_2)_g = DL_g(V_1(1) + V_2(1))$

$\dim\{\ldots\} = \dim G$.

Called the Lie Algebra of G.

<u>Note</u>: Could just as well discuss right inv. v f. $\xleftrightarrow[1-1]{} T_1 G$.

Sometimes define $\underbrace{\text{Lie } G}_{\text{Lie Alg.}} = T_1 G$

<u>Additional structure</u>: X, Y are left inv. v-fields.

$[X, Y](g) = DL_g([X, Y](1))$ | <u>Note</u>: $D\phi([X, Y]) = [D\phi(X), D\phi(Y)]$.

Apply this to $\phi = L_g$. gives $[X_i$

Denote Lie Alg of G by $\mathfrak{g}$. $\mathfrak{g}$ is endowed with

$$[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}.$$
$$(X,Y) \to [X,Y]$$

> It is bilinear, $[a_1X_1 + a_2X_2, Y] = a_1[X_1,Y] + a_2[X_2,Y]$.

> $[X,Y] = -[Y,X]$ → obviously $(XY-YX)$.

> Jacobi identity,

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

These are the 3 main properties of Lie Algebras

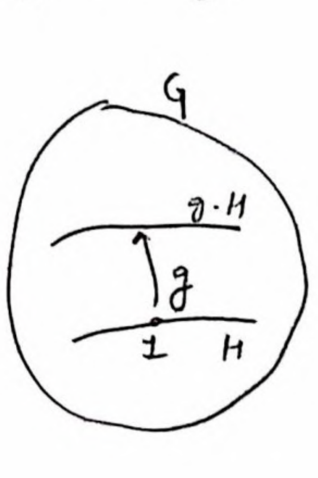
G Lie group. $H \subset G$ Lie subgroup. i.e. an immersed submfl.

Look at right cosets by $H \to g \cdot H$ they are leaves ⇒ get a foliation.
(partition)

$$V(g) = T_g(g\cdot H) \text{ distribution. Note } DL_g(V(1)) = V(g).$$

Look at left inv. vf tangent to $V(g)$

ie tangent to $g \cdot H$.



$\Rightarrow \mathfrak{h} < \mathfrak{g} \Rightarrow X, Y \in \mathfrak{h}$

integrable

$\Rightarrow [X,Y] \in \mathfrak{h}$.

<u>Thm</u> H a Lie subgroup of G. Then $\mathfrak{h} < \mathfrak{g}$ is a Lie sub algebra i.e $[\mathfrak{h}, \mathfrak{h}] < \mathfrak{h}$.

Conversely, If $\mathfrak{h} < \mathfrak{g}$ is a V-subspace <u>AND</u> $[\mathfrak{h}, \mathfrak{h}] < \mathfrak{h}$ then $\exists\ H \subset G$ a Lie subgroup.

S.T $\mathfrak{h} = T_g(g \cdot H)$ i.e $\mathfrak{h} =$ left inv vf coming from H.

<u>Proof of converse</u> : $\mathfrak{h}$ is integrable $\Rightarrow$ use Frobenius

$H = \tilde{\mathcal{F}}(1)$.

↳ leafs of a foliation ($\mathcal{F}$ is left inv. under G)

Lec (Monday)

Cleaning up part of Frobenius proof.

$$\begin{tikzcd} \Pi: \mathbb{R}^n \arrow[r] & \mathbb{R}^k \\ & \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_k} \arrow[u] \end{tikzcd}$$

. V is a k dim. distribution

Find $Y_1, \ldots, Y_k$ tangent to V s.t $D\pi(Y_i) = \frac{\partial}{\partial x_i}$

Y_i, $\frac{\partial}{\partial x_i}$ are π-related. $[Y_i, Y_j]$ is tangent to V since V is involutive & $D_\pi([Y_i\, Y_j]) = \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$

$\Rightarrow [Y_i, Y_j] = 0.$

Lie Groups (contd.)

G Lie group. $\mathfrak{g} = T_1 G$ Lie algebra. $H < G$ Lie subgroup.

ie $H =: \mathfrak{h} < \mathfrak{g}$. Why?

ay $V(1) \in T_1 H \Rightarrow V$ LIVF on $G \Rightarrow V$ LIFV for H.

ef: Lie sub-alg.

Given a Lie algebra $\mathfrak{g}$, then $\mathfrak{h} < \mathfrak{g}$ is a Lie sub-alg. if $\mathfrak{h}$ is a v-space &. $\forall$ $H_1, H_2 \in \mathfrak{h} : [H_1, H_2] \in \mathfrak{h}$.

Thm: There is a 1-1 correspondance b/w connected Lie subgroups of G & Lie sub algebras of $\mathfrak{G}$.

Proof: $\Rightarrow$ just discussed.

$\Leftarrow$ main application of Frobenius.

Given $\mathfrak{h} < \mathfrak{g}$ a Lie sub-algebra.

$\hookrightarrow$ Collection of LIVF or $\mathfrak{h} \subset T_1 G$.

Define a distribution $V(g) = DL_g(\mathfrak{h})$, (i.e. evaluate the LIVF in $\mathfrak{h}$ at g)

Claim : V is integrable (or involutive)
ie closed under brackets.

Take X_1, X_2 vf tangent to V.

WTS $[X_1, X_2] \in V$.

Let $Z_1, \ldots, Z_k$ $(k = \dim \mathfrak{h})$ be a basis of $\mathfrak{h}$.

Thus, Z_i are LIVF & $[Z_i, Z_j] \in \mathfrak{h}$ since $\mathfrak{h}$ is a Lie sub alg. of $\mathfrak{g}$.

$\Rightarrow X_1 = \sum_{i=1}^{k} a_i Z_i$ where $a_i : G \to \mathbb{R}$ &

$X_2 = \sum_{j=1}^{k} b_j Z_j$ where $b_j : G \to \mathbb{R}$.

$[\sum a_i Z_i, \sum b_j Z_j] = ?$

Lemma: $[aF, bG] = aF(bG) - bG(aF)$

$$= \overset{\text{directional deriv.}}{\overline{aF(b) \cdot G}} + ab\, FG - b\, G(a) \cdot F - ba\, GF$$

$$= aF(b)G + ab[F, G] - bG(a)F.$$

Apply thus to $\left[\sum a_i Z_i, \sum b_j Z_j\right]$

$$= \sum \text{some fns of } \underbrace{Z_i \,\&\, Z_j}_{\in V} + \sum \text{some fns.} \underbrace{[Z_i, Z_j]}_{\in V} \quad \&$$

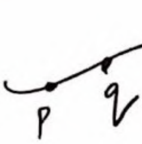
$$\Rightarrow [\quad] \in V \quad // \quad \text{use Frobenius \& done}$$

Thus V is integrable. ▨

Let $\mathcal{F}$ be the foliation determined by V. This exists atleast locally in a nbd of every pt. of G.

Thm: $\exists$ Foliation globally. M smooth mfl. V is an integrable distr. on M.

Proof $\mathcal{F}(p)$, Have local charts : $\mathcal{F}_{loc}(p)$ $\forall p \in M$



$q \in \mathcal{F}_{loc}(p)$, look at $\mathcal{F}_{loc}(q)$ & use $\mathcal{F}_{loc}(p) \cup \mathcal{F}_{loc}(q)$.

Need the following Lemma

<u>L:</u> Given $q \in \tilde{F}_{loc}(p)$, $\exists$ nbd U of q s.t

$$F_{loc}(p) \cap U = F_{loc}(q) \cap U.$$

<u>P:</u> both are tangent to U. ⌟

Keep building up with that construction.

$$F_{loc}(p) = F^1(p)$$

$$\bigcup_{q \in F_{loc}(p)} F_{loc}(q) = F^2(p)$$

$$\bigcup_{q \in F^n(p)} F_{loc}(q) = F^{n+1}(p)$$

$$\Rightarrow \quad \bigcup_{n=1}^{\infty} (F^n(p)) = \tilde{F}(p) \text{ global leaf of } \tilde{F} \text{ through } p.$$ ∎

<u>Back to Lie groups</u>

$V \rightsquigarrow$ global foliation $\tilde{F}$. global leaves $\tilde{F}(g)$.

<u>Note:</u> $\tilde{F}$ is left-invariant. $\Rightarrow \tilde{F}(gp) - L_g(\tilde{F}(p))$

Now set $H = \tilde{\mathcal{F}}(1)$. Suppose $h \in H$.

$h \cdot H = L_h H = L_h(\tilde{\mathcal{F}}(1)) = \tilde{\mathcal{F}}(h \cdot 1) = \tilde{\mathcal{F}}(h) = \tilde{\mathcal{F}}(1).$

$\Rightarrow \quad \underline{h \cdot H = H}$

Suppose $h \in H$, is $h^{-1} \in H$?

$h^{-1}h = 1 \; ; \; h^{-1} = h^{-1} \cdot 1 \in h^{-1}\tilde{\mathcal{F}}(h) = \tilde{\mathcal{F}}(1) = H.$ //

$\Rightarrow$ H is a subgroup.

<u>Note</u>: Construction shows that global leaf is path conn.

<u>Ex</u>: $\mathbb{Z} \subset \mathbb{R}$, $\text{Lie}(\mathbb{Z}) = \{0\} = \text{Lie}(\{0\})$

<u>Ex</u>: $\mathbb{Z} \times \mathbb{R} \subset \mathbb{R}^2$, $\text{Lie}(\mathbb{Z} \times \mathbb{R}) = \{0\} \times \mathbb{R} \subset \mathbb{R}^2$.

<u>Facts</u>: Know given a Lie group, can attach a Lie alg. of G.

<u>Q</u>: Given any fin dim Lie alg. $\mathfrak{g}$, does $\exists$ a Lie group G with Lie alg $\mathfrak{g}$? & how many?

<u>A</u>: ① Yes true. Trick is to embed the Lie Alg into Lie Alg of $GL(n,\mathbb{R})$ & then use Thm.

② No. $\text{Lie}(\mathbb{R}) = \mathbb{R}$, $\text{Lie}(S^1) = \mathbb{R}$ ③ Yes sort of true.

Lec (Wednesday)

Lie groups contd.

Lie groups $\longleftrightarrow$ Lie algebras.

Thm: G Lie gp. H connected Lie subgroup. $\Leftrightarrow$ $\exists\, \mathfrak{h} \subset \mathfrak{g}$

S.T $\mathfrak{h}$ is the Lie subalgebra.

> $G_1 \overset{}{\underset{\phi}{\longleftrightarrow}} G_2$, Lie gps & ϕ is a regular homomorphism
$$\left(\begin{aligned} \phi(ab) &= \phi(a)\phi(b) \\ \phi(a^{-1}) &= \phi(a)^{-1} \end{aligned} \right)$$

Trouble: $\exists\, \phi : \mathbb{R} \to \mathbb{R}$ homomorphism which are not diff.

Since $\exists$ transcendental nos. (Galois theory ?)

Assume ϕ is C^∞, then ϕ is called a Lie group homomorphism (LGH)

Remark: Good enough to assume ϕ is measurable (continuous?).
↳ (w.r.t charts.)

$$\phi : G_1 \longrightarrow G_2$$

$$\begin{tikzcd} D\phi_1 : T_1 G_1 \arrow[r] \arrow[d, phantom, "\parallel"] & T_1 G_2 \arrow[d, phantom, "\parallel"] \\ \mathfrak{g}_1 \arrow[r] & \mathfrak{g}_2 \end{tikzcd}$$

> $X_1, X_2 \in \mathfrak{g}_2$. Think of them as left invariant vector fields.

$\phi_*(X_1)$ is a vf on G_2, so is $\phi_*(X_2)$.

pullback?

⋮ [not sure about this]

End result,

$$\left.\begin{array}{l} \phi_*(X_2) \longleftrightarrow X_2 \\ \phi_*(X_1) \longleftrightarrow X_1 \\ \phi_*[X_1, X_2] \longleftrightarrow [X_1, X_2] \end{array}\right\} \text{all are } \phi\text{-related.}$$

Thus; $D\phi_1 : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a linear map

& $D\phi_1([X_1, X_2]) = [D\phi(X_1), D\phi(X_2)]$

↳ Lie alg. Homomorphism.

> **Thm:** Given Lie alg. Homo, & Lie group Homo. $\exists$ a 1-1 correspondance b/w them.

Proof (Idea)

$\Rightarrow$ OK. (just discussed above (not really tho)).

$\Leftarrow$ Trick: $\mathfrak{g}_1 \xrightarrow{\Psi} \mathfrak{g}_2$

graph $\psi = \{(X, \psi(X)) \mid X \in \mathfrak{g}_1\}$

↑ Lie subalg. of $\mathfrak{g}_1 \times \mathfrak{g}_2$. Why?

$$[(X_1, \psi(X_1)), (X_2, \psi(X_2))]$$

Note,

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [Y_1, Y_2])$$ → Check?

$$= ([X_1, X_2], [\psi(X_1), \psi(X_2)])$$

$$= ([X_1, X_2], \psi([X_1, X_2]))$$ ⇒ it is a Lie subalg.

⇒ $\exists$ a connected Lie subgroup $H \subset G_1 \times G_2$

(By main result last time)

<u>Note</u>: Lie $(G_1 \times G_2) = \mathfrak{g}_1 \times \mathfrak{g}_2$

<u>Claim</u>: H = graph of homomorphism, Ψ.

i.e. $\Psi(g_1) = g_2$ if $(g_1, g_2) \in H$.

Check: ① welldefined ② Homomorphism.

Exponential Map

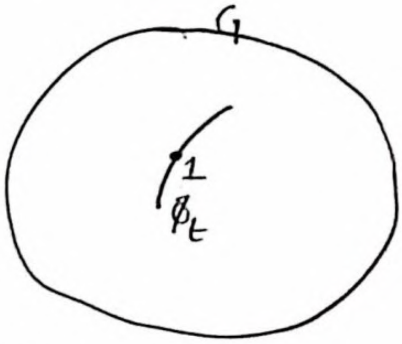
G Lie group. $X \in \mathfrak{g}$ Lie algebra. $\{tX,\ t \in \mathbb{R}\}$

is a Lie subalg. $[sX, tX] = 0.$

Thus $\exists$ a subgroup of G corresponding to $X \in \mathfrak{g}$.

Down to earth: $X \in \mathfrak{g}$ a(LIVF) $\overset{\text{gives}}{\rightsquigarrow}$ ϕ_t - local flow on G.

Say ϕ_t is local at $1 \in G$.



$$g_t := \phi_t(1)\ ;\ g_t \cdot g_s = \phi_t \cdot \phi_s(1).$$
$$= \phi_{t+s}(1)$$
$$= g_{t+s} \Rightarrow \text{homomorphism.}$$

Claim: ϕ_t is global ; i e defined for all time.

Proof : option ① appeal to subgroup arg.

or option② The local flow of X through g is simply $L_g \cdot$ (local flow at 1) $= L_g - (\phi_t(1)) = g \cdot g_t$.

Thus if the local flow at 1 is defined on $(-\epsilon\ \epsilon)$ it is defined at g. Thus ϕ_t is global.

Eg (prime) - $G = GL(n,R)$.

$\mathfrak{g} = \mathfrak{g}L(n,\mathbb{R}) = M_{n,n}$

$A \in M_{n,n}$, what is ϕ_t : corresponding 1parameter subgroup.

$e^{tX} = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!}$ power series. (converges).

$\| \sum \frac{(tX)^n}{n!} \| \leq \sum \frac{|t|^n \, \|X\|^n}{n!}$ ✓

Operator norm : $A \in M_{n,n}$; $\|A\| = \max\limits_{X \in \mathbb{R}^n} |Av|$ s-t $|v| = 1$

$\underline{1}$: $\|AB\| \leq \|A\| \, \|B\|$.

Finally note, $\frac{d}{dt}(e^{tX}) = X.e^{tX}$.

Thus e^{tX} solves the local flow ODE.

Also, $e^{tX} \in GL(n,\mathbb{R}) \Rightarrow e^{-tX}$ is defined

$\Rightarrow e^{tX} . e^{-tX} = e^{0} = I$.

Lec (Friday)

Q: What are the continuous homomorphisms from $\phi : S^1 \rightarrow \mathbb{R}$.

$\phi(S^1)$ compact in $\mathbb{R} \Rightarrow$ bounded.

$a \in \phi(s) \Rightarrow a + a + \quad \in \phi(s)$

$\Rightarrow$ cannot be bounded unless $a = 0$

$\Rightarrow \phi(S^1) = 0$ is the only homomorphism!

But, $\text{Lie } S^1 = \mathbb{R}$; $\text{Lie } \mathbb{R} = \mathbb{R}$

$$\begin{tikzcd}\mathbb{R} \arrow[r, bend right, "\Phi = \mathrm{id}"'] & \mathbb{R}\end{tikzcd}$$

Lie alg. homomorphism $\longrightarrow$ $\Phi = \mathrm{id}$

Then which $\phi : S^1 \rightarrow \mathbb{R}$ corresponds to Φ? Does not exist!

$\Rightarrow$ Homo Lie alg donot lead to Homo Lie groups.

But almost true,

$$\begin{tikzcd}\mathfrak{g}_1 \arrow[r, "\Phi"] \arrow[d] & \mathfrak{g}_2 \arrow[d] \\ G_1 \arrow[r, dashed, "?"'] & G_2\end{tikzcd}$$

where $\mathfrak{g}_1$ & $\mathfrak{g}$ are FDim Lie Alg.

Look at $\tilde{G}_1$ (the universal cover of G_1). There is a homomorphism from $\tilde{G}_1 \longrightarrow G_2$. ($\tilde{G}_1 \xrightarrow[\pi]{} G_1$).

Fact: G Lie group $\Rightarrow$ $\tilde{G}$ is also a Lie group.

orrecting proof from Wednesday)

$\mathfrak{g}_1 \xrightarrow[\Phi]{} \mathfrak{g}_2$, graph $\Phi =: \mathfrak{h}$,

Then $\exists\, H \subset G_1 \times G_2$

$\hookrightarrow$ subgroup corresponding to $\mathfrak{h}$.

dim H = dim $\mathfrak{h}$ = dim G_1 (i.e dim of domain since its a graph).

$$\begin{tikzcd} H \arrow[dr, "\pi_1|_H"'] & \subset & G_1 \times G_2 \arrow[dl, "\pi_1"] \arrow[d, "\pi_2"] \\ & G_1 & G_2 \end{tikzcd}$$

<u>Claim</u>: $\pi_1|_H$ is a local diffeo

<u>P</u>: $D(\pi_1|_H) = \pi_1|_{\mathfrak{h}} \subset \mathfrak{g}_1 \times \mathfrak{g}_2$

where $(\pi_1(X, Y) = X)$

Q : $\pi_1|_H$ = 1-1 but not true since $H \cap \pi^{-1}(1)$ = discrete //

> Suppose G Lie & smooth. Suppose $G \curvearrowright M$ smoothly.

$\mathfrak{g}$ Lie alg. Then $\mathfrak{g} \xrightarrow[\text{Lie alg homo}]{}$ smooth v-fields on M.

Partitions of Unity

M compact, $\{U_\alpha\}_{\alpha \in A}$ is an open cover. Then $\exists$ a finite subcover $\{V_\beta\}_{\beta=1,\dots,l}$ i.e, $\forall \beta \exists \alpha$ s.t $V_\beta \subset U_\alpha$

& $\exists$ smooth functions $\phi_\beta : V_\beta \to [0,1]$ s.t $\forall x \in M$,

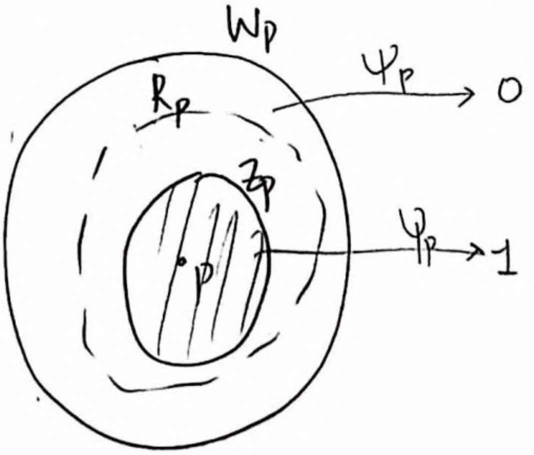
$$\sum_\beta \phi_\beta(x) = 1 .$$

Proof: $\forall$ $p \in M$, find nbd W_p & smooth bump function Ψ_p .

$\Psi_p : W_p \longrightarrow [0,1]$ s.t $\Psi_p \equiv 1$ on a nbd Z_p s.t $p \in Z_p \subset W_p$

& $\exists R_p$ s.t $Z_p \subset R_p \subset W_p$

& $\Psi_p \equiv 0$ outside R_p.



Wlog $W_p \subset V_\beta$. M compact $\Rightarrow$ $W_{p_1} \dots W_{p_l}$ cover l. Then $\exists$ finitely many Ψ_{p_i} & use $\dfrac{\Psi_{p_i}}{\sum_{i=0}^{l} \Psi_{p_i}}$ //

Lec (Monday)

HW

P3 Use the chart at each pt. & slow down v-f in each chart.

Look into partitions of unity to deal with overlapping regions.

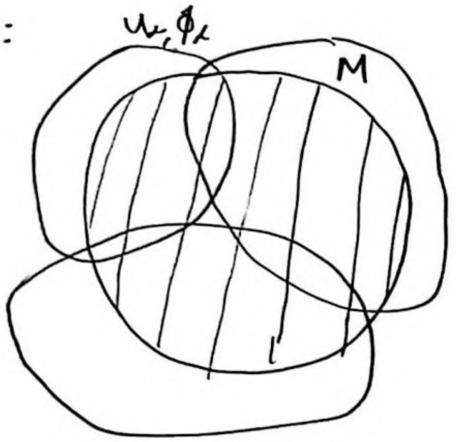
[If not, just do the compact case]

P5 Use the inverse fn. & make it into a diffeo.

Embedding of manifolds into $\mathbb{R}^N$

Thm : M compact ⇒ $\exists\ f: M \to \mathbb{R}^N$ an embedding.

Pf :



$\{U_\alpha, \phi_\alpha\}$ - Coord charts.

$$M \longrightarrow \prod_\alpha \mathbb{R}^n = \mathbb{R}^{N = n \cdot \#\{\alpha\}}$$

Say $x \in M$ & $x \in U_{\alpha_1}, \dots U_{\alpha_j}$

$$x \longmapsto \phi_{\alpha_1}(x) \times \dots \times \phi_{\alpha_j}(x)$$

If $x \notin U_{\alpha_p}$ Then set it 0 on those

Bad embedding since not continuous

> To make it smooth use (partitions of unity) tampering with partitions of 1 of $\{U_\alpha\} \to \{\tau_\alpha\}$

$\{\tau_\alpha\}_\alpha$ s.t $\sum \tau_\alpha = 1$ get smooth $\tau_\alpha : M \to [0,1]$

$$\Rightarrow \quad x \longmapsto \dots \times \tau_{\alpha_i} \phi_{\alpha_i}(x) \times \dots$$

> Also $\tau_{\alpha_i} \equiv 1$ on $V_{\alpha_i} \subset U_{\alpha_i}$.

> Need to massage it more to make it rigorous.

> What if M not compact?

use $\underset{\uparrow \text{infinite product}}{\prod_{\alpha_i}} \mathbb{R}^n \xrightarrow{\text{proj into}}$ large dim $\mathbb{R}^N$ & check such an $\mathbb{R}^N$ exists.

Differential Forms

Exterior product

Say $\mathbb{R}^n$ or V

$\Lambda^n V =$

$\{$multilinear alternating functionals$\}$

$\lambda : \underbrace{V \times \dots \times V}_{k \text{ times}} \to \mathbb{R}$ S.T λ is linear in each $v_i \in V$ $\forall i$.

$$\lambda(\dots, v_i, \dots, v_j, \dots) = -\lambda(\dots v_j, \dots v_i, \dots).$$

Thm : $\dim \Lambda^n \mathbb{R}^n = 1$.

Proof (Idea): $\dim \geq 1$ because $\underset{(n \times n)}{\det(v_1, \dots, v_n)}$.

Difficult part is to show det exists (& uniqueness up to scalar multiples is easy).

Geometric interpretation : $|\det|$ is volumes. (atleast over $\mathbb{R}$)

Ex ② $\Lambda^1 V = V^*$ since 'its' a linear functional of 1 variable.

Ex ③ $\Lambda^2 \mathbb{R}^n$. $e_1, \ldots, e_n$ std. basis for $\mathbb{R}^n$

$\lambda \in \Lambda^2 \mathbb{R}^n$

$\lambda(v,w)$ where $v = \sum \alpha_i e_i$ $w = \sum \beta_j e_j$

$$\Rightarrow \lambda(v,w) = \lambda\Big(\sum_i \alpha_i e_i , \sum_j \beta_j e_j\Big)$$

$$= \sum_{ij} \alpha_i \beta_j \, \lambda(e_i, e_j)$$

$\downarrow$ using alternating prop.

$$= \sum_{i<j} (\alpha_i \beta_j - \alpha_j \beta_i) \, \lambda(e_i, e_j) \quad \left(\begin{matrix}\text{note}\\ \lambda(e_i, e_i) = 0\end{matrix}\right)$$

Define $e_i \wedge e_j$ as element of $\Lambda^2 \mathbb{R}^n$ which assigns value

$$e_i \wedge e_j \left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right) = x_i y_j - x_j y_i$$ → notice this is alternating. (check)

↳ These are basis elements of $\Lambda^2 \mathbb{R}^n$.

Ex ④ $\Lambda^k \mathbb{R}^n$ has basis $e_{i_1} \wedge \ldots \wedge e_{i_k}$ where $i_1 < i_2 < \ldots < i_k$.

Meaning

Ex: $\mathbb{R}^3$. say $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$

$$e_1 \wedge e_2\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right) = 1$$ → area of square spanned by $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$.

$$e_1 \wedge e_2\left(\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right) = 0$$

$\lambda \in \Lambda^k \mathbb{R}^n, \to |\lambda|$ "measures" the k-dim area of a parallelopiped w.r.t a particular fixed k-dim subspace.

Lec (Wednesday)

Clean up from last time

K-form measures area of ~~an intersection~~ of a projection.

> V fdvs, $\dim(V) = n$. $\dim \Lambda^n V = 1$.

D: Two n-forms α, β "have the same orientation" if $\beta = c \cdot \alpha$; $c > 0$. $(\alpha, \beta \neq 0)$. Otherwise they have the opposite orientation.

> M n-dim mfd. $\Lambda^k M = \Lambda^k(TM)$

$$\begin{tikzcd} (\Lambda^k M)_p = \Lambda^k T_p M & \Lambda^k M \arrow[d, "\pi"] \\ & p \end{tikzcd}$$ foot point map.

D: A k-form α is a smooth section of $\begin{tikzcd} \Lambda^k M \arrow[r, "\pi"] & M \arrow[l, dashed, bend left, "\alpha"] \end{tikzcd}$.

<u>Ex</u>: ① $\mathbb{R}^n$, $\Lambda^1 \mathbb{R}^n \Rightarrow \alpha = dx_{10^{44}}$

$\Lambda^2 \mathbb{R}^n \to dx \wedge dy$.

$\Lambda^n \mathbb{R}^n \to \det$.

> <u>D</u>: Any section $\tau : M \to \Lambda^n M$ such that $\tau(p) \neq 0$. Any such thing is called an orientation of M.

Given an orientation τ & and another one σ. Say that σ & τ define the <u>same orientation</u> on M if $\exists f : M \to (0, \infty)$ & smooth s.t $\sigma = f \cdot \tau$.

> Also if σ is an orientation so is $(-\sigma)$.

> Orientation may not exist !! Ex: Möbius band

> Observe the following, S^n is orientable.

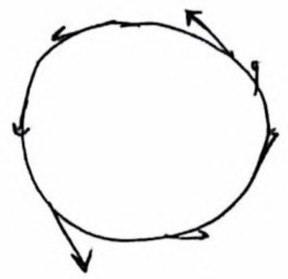
<u>D</u>: Call M orientable if has an orientation.

> Also an oriented mfl is a mfl with a given orientation σ.

> (M, σ) orientation $\Rightarrow$ $(M, -\sigma)$ also is given.

> <u>Ex</u>: $\mathbb{P}^n = S^n / \mathbb{Z}_2$. $x \to -x = A(x)$. Q: Does A preserve orientation.

>



Any Lie group is orientable.

> **Prop:** G Lie group $\Rightarrow$ it is orientable.

Pf: $\dim G = n$.

$\Lambda^n G$

pick $\sigma(1) \in \Lambda_1^n G$ & make it left-inv.

It is a G-inv. orientation (even nicer).

> Recall that $\mathbb{R}P^3 \equiv SO(3) \curvearrowright$ group $\Rightarrow \mathbb{R}P^3$ is orientable.

$$\underset{SU(2)}{\uparrow} \text{ double cover}$$

> If a quotient map leaves some given orientation on a manifold invariant, the quotient space has an orientation.

> Back to basic Multi-linear algebra.

Wedge product

Suppose $\alpha \in \Lambda^k V$, $\beta \in \Lambda^l V$. $(\alpha \wedge \beta)$ is a $k+l$ alternating form.

$$(\alpha \wedge \beta)(v_1 \dots v_{k+l}) = \frac{1}{k!\,l!} \sum_{\sigma \in S(k+l)} (-1)^\sigma \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}$$

Where $\sigma \in S(k+l)$ are permutations of $\{1, .., k+l\}$

s.t $\sigma(1) < \sigma(2) < \dots < \sigma(k)$ & $\sigma(k+1) < \sigma(k+2) < \dots < \sigma(k+l)$

Lec (Friday)

Book uses $\Lambda^n T^* M$ instead of $\Lambda^n M$.

So $(\Lambda^k M)_p = \Lambda^k (T_p^* M) = \{$alt k-multi linear maps from $T_pM \times \dots \times T_pM \to \mathbb{R}\}$.

> Ex: $(dx_1 \wedge dx_2)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$

$$= dx_1\left(\frac{\partial}{\partial x_i}\right) \cdot dx_2\left(\frac{\partial}{\partial x_j}\right) - dx_1\left(\frac{\partial}{\partial x_j}\right) \cdot dx_2\left(\frac{\partial}{\partial x_i}\right)$$

$$= \begin{cases} 0 & \text{if } \{i,j\} \neq \{1,2\} \\ 1 & \text{if } i=1, j=2 \\ -1 & \text{if } i=2, j=1 \end{cases}$$

Recall,

> α k-form is smooth if either X_1 X_k are smooth v-fields &

$p \longmapsto \alpha(X_1(p) \dots X_k(p))$ is smooth.

> OR, if α can be written in terms of smooth coordinates.

$$\alpha = \sum \alpha_i \, dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

↳ fns. that are smooth : $M \to \mathbb{R}$.

Ex:

$$dx_1 \wedge dx_2 \left(\sum a_i \overset{\frac{\partial}{\partial x_i}}{\delta x_i}, \sum b_j \overset{\frac{\partial}{\partial x_j}}{\delta x_j} \right)$$

$$= dx_1 \wedge dx_2 \left(a_1 \delta x_1 , b_2 \delta x_2 \right) + dx_1 \wedge dx_2 \left(a_2 \delta x_2 , b_1 \delta x_1 \right)$$

$$= a_1 b_2 - a_2 b_1$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Recall

$\alpha \in \Lambda^k V^*$

$\beta \in \Lambda^l V^*$

$$(\alpha \wedge \beta)(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l})$$

$$= \frac{1}{k!\,l!} \sum_{\sigma \in S(k+l)} (-1)^\sigma \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})$$

where σ is a permutation in $S(k+l)$ if $\sigma(i) < \sigma(j)$ for $1 \le i \le j \le k$

& for $k+1 \le i \le j \le k+l$

Ex: Suppose $k = l = 1$

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1) \cdot \beta(v_2) - \alpha(v_2) \beta(v_1).$$

$$= \begin{vmatrix} \alpha(v_1) & \beta(v_1) \\ \alpha(v_2) & \beta(v_2) \end{vmatrix}$$

<u>Prop</u>: $\alpha \in \Lambda^k V^*$, $\beta \in \Lambda^l V^*$; then $\alpha \wedge \beta \in \Lambda^{k+l} V^*$

<u>Future</u>: This leads to wedge product of forms which leads to Poincaré duality.

<u>Important</u>: Wedge product is a multiplicative operation.

<u>Q</u>: $\dim V = n$, $k+l=n$ & $\alpha \in \Lambda^k V^*$ & $\beta \in \Lambda^l V^*$, then

$$\alpha \wedge \beta \in \Lambda^{k+l} V^* = \Lambda^n V^* \cong \mathbb{R}.$$

↳ just the determinant.

> M dim n, + smooth which is oriented.

$$\begin{tikzcd} \Lambda^n M \arrow[d, "\pi"] \\ M \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

s-t σ never vanishes

<u>Note</u>: A section like σ is called a "volume form".

> If M is oriented, $\Lambda^n M$ is $\underset{\text{iso}}{\cong} \{f : M \to \mathbb{R}\}$ smooth.

> $\tau \in \Lambda^n M \Rightarrow \tau(p) = f(p) \cdot \sigma(p) \Rightarrow f$ is uniquely determined

$\Rightarrow \Lambda^n M$ gives a particular fn. from $M \to \mathbb{R}$.

Similarly f gives $f \cdot \sigma$. Also note that the space of $C^\infty M$ is $= \Lambda^0 M$. $\Rightarrow \Lambda^0 M \leftrightarrow \Lambda^n M$.

Properties (more)

① $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$.

Since V_k is switched with $V_{k+1} \cdots V_{k+l}$

than V_{k-1} is swiched with - - - .

② $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$ associative.

Eg: $(dx_1 \wedge dx_2 \wedge dx_3) \wedge (dx_4 \wedge dx_5)$

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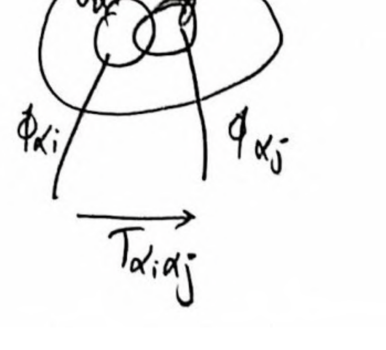
⋮

> An aside about orientability ; given M, suppose have coord charts (U_α, ϕ_α).

Note $\mathbb{R}^n$ is orientable since we have $dx_1 \ldots dx_n \neq \vec{0}$ i.e. never the zero map.

On volume form U_{α_i} can take $dx_1^{\alpha_i} \wedge \ldots \wedge dx_n^{\alpha_i}$

$$\begin{tikzcd} dx_1^{\alpha_i} \wedge \ldots \wedge dx_n^{\alpha_i} \arrow[d, "\text{Transition map - } T_{\alpha_i\alpha_j}"] \\ dy_1^{\alpha_j} \wedge \ldots \wedge dy_n^{\alpha_j} \end{tikzcd}$$



Want

$dy_1^{\alpha_j} \wedge \dots \wedge dy_n^{\alpha_j}$ pull back by $T_{\alpha_i \alpha_j}$

then you get $g\left(dx_1^{\alpha_i} \wedge \dots \wedge dx_n^{\alpha_i}\right)$ S.T $g > 0$ on U_{α_i}.

Need to do

① Pulling back

$\phi : M \to N$

$\phi^* \beta \in \Lambda^k M \qquad \beta \in \Lambda^k N$

$(\phi^* \beta)(v_1, \dots, v_n) = \beta(D\phi(v_1) \dots D\phi(v_n))$

② Put these forms together. (will use part of 1)

Lec (Monday)

HW9 8)

Show that locally we can go from p to p_1. Show it in $\mathbb{R}^n$ using a vector field to take you from p to p_1.

then use bump functions to make it a global flow.

Thm: M smooth, then TFAE

ⓐ M is orientable

ⓑ M has an atlas of c-charts (U_α, ϕ_α) S.T

S.T $DT_{\alpha,\beta}$ where defined.

__Lemma__ (about $\mathbb{R}^n$)

Suppose V fdvs dim n, $\mu \in \Lambda^n V^*$ &

$\{e_1, \dots, e_n\}$ is a basis of V. Suppose A is an $n \times n$ matrix $\{a_{ij}\}$

$f_i = \sum_{j=1}^{n} a_{ij} e_j$. Then $\mu(f_1, \dots, f_n) = \det A \, \mu(e_1, \dots, e_n)$.

vectors where A takes e_i

__Pf__: $\mu(f_1, \dots, f_n) = \mu\left(\sum_{j=1}^{n} a_{1j} e_j \dots \sum_{j=1}^{n} a_{nj} e_j\right)$

$$= \sum a_{1j_1} \dots a_{nj_n} \, \mu(e_{j_1}, \dots, e_{j_n})$$

$$= \sum_{\sigma \in \Sigma_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \, \mu(e_{\sigma(1)}, \dots, e_{\sigma(n)})$$

perm. gp. $\longrightarrow$

$$= \sum (-1)^\sigma \prod_{j=1}^{n} a_{i\sigma(i)} \, \mu(e_1, \dots, e_n)$$

$$= \det A \, \mu(e_1 \dots e_n)$$

__On Forms__

W, V - v-s. $F: W \to V$ linear then $\Lambda^k V^* \xrightarrow[\text{forms (pull back)}]{F^*} \Lambda^k W^*$

$(F^*\alpha)(w_1 \dots w_k) := \alpha(F(w_1), \dots, F(w_k))$

Lemma: Suppose now $\dim V = \dim W = n$.

$F: W \to V$ linear; $e_1, \dots, e_n$ basis of V. $f_1, \dots f_n$ a basis of W.

$\epsilon_1, \dots, \epsilon_n$ dual basis of V^*; $\phi_1, \dots, \phi_n$ dual basis of W^*.

Let A be the matrix of F w.r.t these bases.

$$F(f_i) = \sum_{j=1}^{n} a_{ij} e_j .$$

Then $F^*(\underbrace{\epsilon_1 \wedge \dots \wedge \epsilon_n}_{\Lambda^n V^*}) \in \Lambda^n W^*$

$$\Rightarrow \quad F^*(\epsilon_1 \wedge \dots \wedge \epsilon_n) = \det A\,(\phi_1 \wedge \dots \wedge \phi_n).$$

Pf: (Outline)

A: Repeat the argument from the last lemma.

B: $V \underset{L}{\cong} W$ L - linear isomorphism.

Pull back everything by L s.T you do the calc. on one v-sp.

Then use previous lemma.

Or evaluate $(F^*(\epsilon_1 \wedge \dots \wedge \epsilon_n))(f_1 \dots f_n) = (\epsilon_1 \wedge \dots \wedge \epsilon_n)(\underset{\uparrow}{F(f_1)} \dots, F(f_n))$

$$= \det A \cdot \underbrace{(\epsilon_1 \wedge \dots \wedge \epsilon_n)(e_1, \dots, e_n)}$$

$$\begin{tikzcd} & \mathbb{R}^n \arrow[d, phantom, "\cup"] & \\ & \hat{U} \arrow[r, "T"] & \hat{V} \subset \mathbb{R}^n \\ \phi_\alpha(U_\alpha) \arrow[ur, "\cong"] & T_{\alpha\beta} \arrow[u] & \phi_\beta(U_\beta) \arrow[u, bend left, "\cup"] \end{tikzcd}$$

> **Def**: $\underset{p}{M}, \underset{\phi(p)}{N}$ mfls. $\Phi : M \to N$ smooth. $\underset{\alpha}{\Lambda^k N} \; (= \Lambda^k T^* N)$

$$\underset{\Lambda^k_p M}{(\phi^* \alpha)_p} = (D\phi)_p^* \underset{\Lambda^k T^*_{\phi(p)} N}{\alpha_{\phi(p)}}$$

> $T : \hat{U} \to \hat{V}$. T^*. $x_i = x_i(y_1 \cdots y_n)$

diffeo, $\hat{U}$ open in $\mathbb{R}^n$; $y_1 \cdots y_n$ coords in $\mathbb{R}^n$; $x_1 \cdots x_n$ coords. in $\mathbb{R}^n$

$$\boxed{T^*(dx_1 \wedge \cdots \wedge dx_n)_p = \det(DT)_p \, (dy_1 \wedge \cdots \wedge dy_n)_p}$$

$(dx_i)_q$ are dual to $\left(\frac{\partial}{\partial x_i}\right)_q$.

Rec (Wed)

[illegible] on HW9 : ~~Good enough to do for H⊂G [illegible] compact connected.~~

↳ Q3 all of it ~~[illegible]~~ ~~a) H compact connected. Hint:~~

Skip this problem. do it on HW10.

Hint: Why is $GL(n, \mathbb{C})$ connected? For P2d

Note: $\mathbb{C}^* = \mathbb{C} - \{0\}$ is connected.

$$GL(n, \mathbb{C}) = \mathbb{C}^* \times SL(n, \mathbb{C})$$

↳ is this connected.

$$SL(n, \mathbb{C}) = SU(n) \cdot \begin{pmatrix} \diagdown & \\ 0 & \diagdown \end{pmatrix}$$

→ connected since it is $\mathbb{C}^k$.

Show this is connected (rotation group)

2d (maybe use this instead)

$$GL(1, \mathbb{C}) = \mathbb{C}^* - \{0\} = \{a + ib \mid \underset{a, b \neq 0}{a, b \in \mathbb{R}}\}$$

$$= \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 \neq 0 \right\}$$

& $\det(\downarrow) > 0$.

General comment/principle

Say $\Gamma \curvearrowright M$. M/Γ is a quotient space.

eg. ① $\mathbb{R}P^n = S^n/\mathbb{Z}_2$

② $\mathbb{C}P^n = S^{2n-1}/S^1$

If Γ leaves a "structure" on M invariant, it induces this kind of "structure" on M/Γ.

↳ Orientation for ex.

Ex: Suppose M is $\mathbb{C}$-diff, Γ acts by $\mathbb{C}$-diff maps,

$\Rightarrow M/_\Gamma$ is $\mathbb{C}$-diff.

Ex: If M has a Riemannian metric (i.e. $\exists < >_p$ on T_pM)

& Γ acts by isometries. i.e. $\gamma \in \Gamma$ &

$$\downarrow$$

$$\langle (D\gamma)_p(v), (D\gamma)_p(w) \rangle_{\gamma p} = \langle v, w \rangle_p$$

Then $M/_\Gamma$ inherits a Riemannian metric.

Thm (Last time): M C^∞ mfl, TFAE,

a) M is orientable (i.e. $\exists$ nonvanishing section of $\Lambda^n M$,)

$$\begin{tikzcd} & \Lambda^T T^* M \arrow[dl, leftrightarrow] \\ \Lambda^n M \arrow[d] & \\ M & \end{tikzcd}$$

b) $\exists$ charts (U_α, ϕ_α) of M s.t the transition maps have positive Jacobian. i.e. $\det(D_x T_{\alpha\beta}) > 0 \ \forall x \in \dots$

Pf cont. (Look at Faye's notes)

Use pullback of forms by $T_{\alpha\beta}^*$ & use part. of 1. $\not\!\!/$

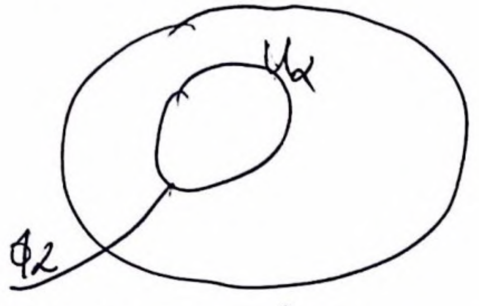
Why bother?

$f: M \to \mathbb{R}$ smooth. What is $\int_M f$? What is the difficulty on a manifold?

> In $\mathbb{R}^n$ there is a notion of a measure i.e. ▣.

> $\int_M f\tau$ ← vol. form (ie n-form that does not vanish).

How to do it?



$V_\alpha = \phi_\alpha(U_\alpha)$ & f is $\equiv$ outside U_α.

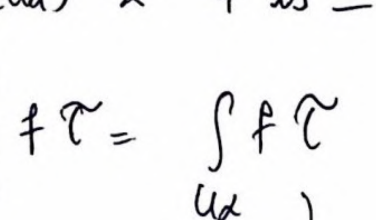
$\Rightarrow \int_M f\tau = \int_{U_\alpha} f\tau$

$\Rightarrow \int (f \circ \phi_\alpha^{-1}) \cdot \phi_\alpha^*(\tau)$

Is it well defined? What if we pick some other U_β -

Recall from vector calculus $\int_A h \, dx_1 \ldots dx_n = \int_B (h \circ T) \det(\mathrm{Jac}\, T) \, dy_1 \ldots dy_n$

↓ But this is how forms transform.



$$\int_W f = \int_{\phi_\alpha(W)} (f \circ \phi_\alpha^{-1}) \, \phi_\alpha^* \tau \quad \int_{\phi_\beta(W)} (f \circ \phi_\beta^{-1}) \, \phi_\beta^* \tau$$

$$\phi_\alpha(W) \xrightarrow[T_{\alpha\beta}]{} \phi_\beta(W)$$

Lec(Friday)

$M = G$ a Lie gp. Take $X_1, \dots, X_n$ a Lie $\overset{\mathfrak{g}}{\text{gp}}$ basis i.e. a basis of LIVF.

$\eta_1, \dots, \eta_n$ dual basis at $1 \in G$. Make η_i left inv.

Then $\forall\, q \in G$

$$\eta_i(X_i) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$\hookrightarrow$ Dual basis everywhere.

$\eta_1 \wedge \dots \wedge \eta_n$ left inv.

<u>Prop</u>: G Lie, then $\exists$ a LI vol. $\overset{(\nu_L)}{\text{form}}$ & it is unique upto scalar mult.

Also $\exists!$ (uptoscalar) right inv. volume form. (ν_R)

<u>Q</u>: When is $\nu_L = \nu_R$? Not always.

$$G = \left\{ \begin{matrix} a & b \\ 0 & \frac{1}{a} \end{matrix} \;\middle|\; a \neq 0, b \right\} \Rightarrow \mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ 0 & -A \end{pmatrix} \right\}$$

affine gp. Here $\nu_L \neq \nu_R$.

<u>But</u> for abelian, nilpotent eg. Heisenberg, $SL(n,\mathbb{R})$ ... have $\nu_L = \nu_R$.

Such groups are called unimodular.

> Compact groups are unimodular.

Also, if $\exists\, \Gamma \subset G$ discrete s.t G/Γ is compact, then G is unimodular

II. M Riemannian mfl, oriented. $\Rightarrow \langle\ \rangle$ induces a vol. form.

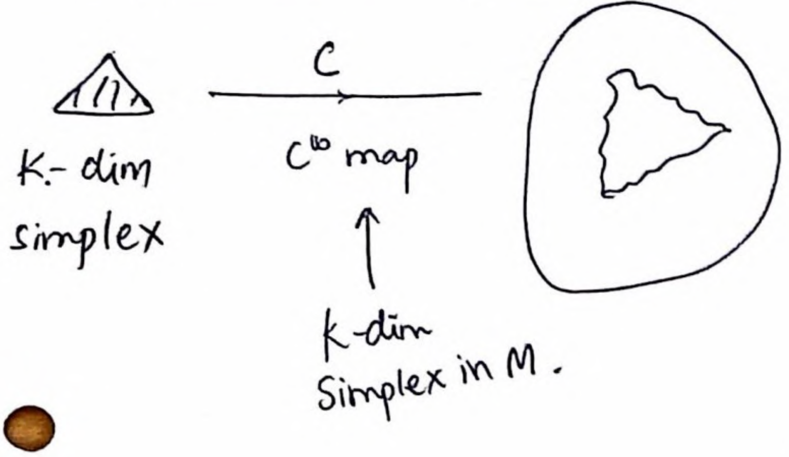
III. Suppose M has a (special) vol. form. Suppose $\Gamma \curvearrowright M$ prop. disc. M/Γ is also a mfl. If ν is Γ invariant, then ν descends to M/Γ.

$\Pi : M \to M/\Gamma$ is a submersion $\Rightarrow$ get a local diffeo.

E M has a 2-form α s.t $\underbrace{\alpha\wedge\alpha\wedge\cdots\wedge\alpha}_{n \text{ times}} \neq 0$ (does not vanish anywhere).

Let $\nu = \alpha\wedge\alpha\wedge\cdots\wedge\alpha$, this α is called a symplectic form.

More general integrals



α k-form on M

$C^*(\alpha)$ = k-form on Δ

$$\int_\Delta C^*(\alpha) =: \int_C \alpha$$

(Similar to $\int_{\text{line}} f$ line integral).

Exterior derivatives

α k-form on M

$$\rightsquigarrow$$

$d\alpha$ (k+1) forms on M

α is 0-form. i.e. a function. $f: M \to \mathbb{R}$.

then df 1-form.

if $v \in T_pM$, then $df(v)$ is the directional deriv. of f along v.

Want

Denote $\Lambda^k M$ - $\{$k-forms on M$\}$

$(= \Lambda^k(T^*M))\}$

for all k, $d: \Lambda^k M \longrightarrow \Lambda^{k+1} M$.

0) $\Lambda^0 M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n M \xrightarrow{d} 0$. → linear maps over $\mathbb{R}$.

Want: 1) d is d on Λ^0

2) $d^2 = 0$.

3) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$

↳ super Leibniz rule

Thm: $\exists !$ d satisfying props 0 − 3.

Note: $d(d x_i) = d^2(x_i) = 0$ → is the trick. (Look at Spivak).

Poincaré Lemma: On $\mathbb{R}^n$ if α has $d\alpha = 0$, (α is "closed" form).

Then $\exists \beta$ s.t $\alpha = d\beta$.

If $\alpha = d\beta$, then α is closed. $d(\alpha) = d(d(\beta)) = 0$.

Note: $\mathrm{Im}\, d|_{\Lambda^{k-1}} \subset \ker(d_{\Lambda^k})$

$$H^k_{\text{deRham}} M := \ker(d_{\Lambda^k}) / \mathrm{Im}(d)|_{\Lambda^{k-1}}$$

↑ f.dim.

What is $H^1(S^1)$?

Exterior derivative

$d := d^k : \Omega^k M \longrightarrow \Omega^{k+1} M$ where $\Omega^k M := \Lambda^k (T^* M)$

$\llcorner$ smooth k forms on M.

Properties ⓪ d_K is linear.

① $\Omega^0 M = C^\infty(M)$

$\overset{\Psi}{f}$

"df" := $df = df(v) =$ directional deriv of f at v.; $v \in T_pM$.

② $d^2 = 0$ i.e $d_{K+1} \circ d_K = 0$

③ Leibnitz rule,

$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ if $\alpha \in \Omega^k$

Prop: $\exists !$ d satisfying ① - ③

① $M = \mathbb{R}$

$\Omega^0 = C^\infty(\mathbb{R})$

$\overset{\Psi}{f}$ $\qquad df = df.$

Aside $A \in \Omega^0(M)$ & $B \in \Omega^l(M) \Rightarrow A \wedge B := A.B$ (convention).

$\alpha \in \Omega^1(\mathbb{R})$

$\parallel$

$f \cdot dx = f \wedge dx$ $\quad f \in \Omega^0(\mathbb{R})$

$\qquad\qquad\qquad\qquad \llcorner C^\infty(\mathbb{R})$

$d(f \wedge dx) = \underbrace{df \wedge dx}_{0 \text{ since}} + f \wedge \underbrace{d(dx)}_{0 \text{ by ② \& also by } \Omega^2 \mathbb{R} = 0.} = 0$

② Ex : $\mathbb{R}^2$ Ω^0, Ω^2 are $\{0\}$.

Ω^1: $f(x,y)\,dx = f(x,y) \wedge dx$

$$d(f \wedge dx) = df \wedge dx + (-1)^0 \cancel{f \wedge d(dx)}_{\to 0}$$

$$= df \wedge dx$$

$$= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right) \wedge dx$$

$$= \frac{\partial f}{\partial y} dy \wedge dx .$$

Can be generalized to $\mathbb{R}^n$.

> General case

M smooth mfd i.e. d is "local", figure out by coord. charts.

If α is a form on a mfd $\alpha \in \Omega^k M$.

But is $d(\alpha|_U) \overset{?}{=} (d\alpha)|_U$ where U is a chart.

Use bump functions. Let $U \subsetneq V$ $\Rightarrow \exists\, \phi$ bump fn. s.T

$\phi|_U \equiv 1$ & $\phi|_{M-V} \equiv 0$ & $\phi \geq 0$.

$$\phi \cdot \alpha = \begin{cases} \alpha|_U & \text{on } U \\ \text{- - -} & \\ 0 & \text{outside.} \end{cases}$$

$$d(\phi\alpha) = d(\phi\wedge\alpha) = d\phi\wedge\alpha + \phi\wedge d\alpha$$

$$\text{on } \mathcal{U} \;= \phi\wedge d\alpha = \underline{\underline{d\alpha}} \quad (\text{since } \phi\equiv 1 \text{ on } \mathcal{U}).$$

Suppose $F: M\to N$, Smooth. $\alpha\in\Omega^k N$ & $F^*\alpha\in\Omega^k M$.

<u>Is $d(F^*\alpha) = F^*(d\alpha)$. Yes (Lemma).</u>

<u>Fact</u> $\dim M = n$

$$\begin{tikzcd} 0 \arrow[r, "d"'] & \Omega^0 M \arrow[r, "d"'] & \Omega^1 M \arrow[r, "d"'] & \cdots \arrow[r, "d"'] & \Omega^n M \arrow[r, "d"'] & 0 \end{tikzcd} := \text{chain complex}$$

$$d_{k+1}\circ d_k = 0$$

$$\Rightarrow \operatorname{Im} d_k \subset \operatorname{Ker} d_{k+1}$$

$$\boxed{H^k M := \operatorname{Ker} d_{k+1} / \operatorname{Im} d_k} \quad \text{De-Rham Cohomology.}$$

① $M = \mathbb{R}$,
$$\begin{tikzcd} 0 \arrow[r] & \Omega^0(\mathbb{R}) \arrow[r, "d"'] \arrow[d, equal] & \Omega^1(\mathbb{R}) \arrow[r] \arrow[d, hook] & 0 \\ & C^\infty(\mathbb{R}) \arrow[r, "d"'] & \{f\cdot dx \mid f\in C^\infty(\mathbb{R})\} \arrow[r] & 0 \\ & g \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto, "d"'] & dg = g'\,dx. \end{tikzcd}$$

$$\operatorname{Im} d = \{f\cdot dx \mid f = g'\} \;\Rightarrow\; f \text{ any } C^\infty(\mathbb{R}) \text{ fn. S.T } g(x) = \int_0^x f(t)\,dt.$$

$\Rightarrow H^1(\mathbb{R}) = 0$ ~~$\& H^0(\mathbb{R}) = \operatorname{Ker} d$~~

$H^0(\mathbb{R}) = \ker d = \{g \mid g'. dx = 0\} = \{g \mid g' = 0\} = \{g = \text{const}\} = \mathbb{R}.$

Prop: M C^∞ mfd. $H^0(M) = \mathbb{R}^?$

Pf:- ① M connected.

$H^0 M = \{f \in \Omega^0 \mid df = 0\} = \{\text{const. fns.}\} = \mathbb{R}.$

② M has 2 conn. comp.

$\Rightarrow H^0 M = \mathbb{R}^2$

* in general $H^0 M = \mathbb{R}^{\#\text{conn. comp.}}$

It therefore connects analysis (d) to topology (connected comps.)

③ $M = \mathbb{R}^n$ (Poincaré Lemma)

$H^0(\mathbb{R}^n) = \mathbb{R}$

$1 \le k \le n$, $H^k(\mathbb{R}^n) = \{0\}$

~~Def^n: [illegible] α Ω^k~~

Def^n: $\alpha \in \Omega^k(M)$

Call α closed if $d\alpha = 0$

Call α exact if $\exists \beta \in \Omega^{k-1}M$ s.t. $\alpha = d\beta$.

Then,

$H^K(M) = \{\text{closed K-forms}\} / \{\text{exact k-forms}\}$.

Pf (Idea) of Poincaré Lemma: (True in general for any star shaped open region in $\mathbb{R}^n$.

$\exists\, p_0 \in A$ s.t if $p \in A$ then $\overline{p_0 p} \in A$.)

We'll construct a map $I_k : \Omega^k A \to \Omega^{k-1} A$ s.t

$$\begin{tikzcd} \Omega^{k-1} \arrow[r, bend left, "d_{k-1}"] & \Omega^k \arrow[r, bend left, "d_k"] \arrow[l, bend left, "I_k"] & \Omega^{k+1} \arrow[l, bend left, "I_{k-1}"] \end{tikzcd}$$

$$\underbrace{\underset{k-1}{d} \circ I_k + I_{k+1} \circ d_k = \text{id}}$$

I is a "Chain Homotopy".

Then if $\alpha \in \Omega^k A$ & α is closed $\Rightarrow d\alpha = 0$.

$\Rightarrow \alpha = d \circ I(\alpha) + I \circ d(\alpha) = d(I(\alpha)) + 0 = d\underbrace{(I(\alpha))}_{k-1 \text{ form}}$.

But what is I? How to define it?

HW10 (Assume unimodular problems are connected).

$H^0_{dR}(S^1) = \mathbb{R}^{\#\text{conn. comp}} = \mathbb{R}$.

What is $H^1(S^1)$?

Why was $H^1(\mathbb{R}) = 0$?

α closed 1-form i.e all 1-forms, then $\alpha = d\beta$ where $\beta = 0$-form $=$ fu. on

$\Rightarrow \beta(x) = \int_0^x f(t)\,dt$ where $\alpha = f \cdot dx$.

Now S^1, $S^1 = [0,1]/_{0\sim 1}$.

$\alpha = 1$-form & $\alpha = f\,dx$.

dx makes sense on $S^1 = \mathbb{R}/\mathbb{Z}$ since dx is invariant when $x \mapsto x+a$

Moreover if $\Omega^1(S^1) \ni \alpha$, $\alpha = f \cdot dx$; $f : \underset{[0,1]/_{0\sim 1}}{\underset{\shortparallel}{S^1}} \to \mathbb{R}$ smooth

$\alpha = d\beta$ & $\beta : S^1 \to \mathbb{R}$. $\beta(x) = \int_0^x f(t)\,dt$. But is $\beta(0) = \beta(1)$?

Wlog $0 = \beta(0) = \beta(1) = \int_0^1 f(t)\,dt$. Thus $\alpha = \text{exact} = d\beta$ iff $\int_0^1 f(t)\,dt = 0$.

Thus, given α closed, can subtract $A \cdot dt$ from α where $A = \int_0^1 \alpha$.

then $\alpha - A\,dt = $ exact. Thus, $\exists\, \beta \in \Omega^0(S^1)$ S.T $\alpha - A\,dt = d\beta$.

$\Rightarrow [\alpha] = [\underset{\hookrightarrow \in \mathbb{R}}{A}\,dt] \Rightarrow H^1(S^1) = \mathbb{R}$ //

Moral

The way that coord. charts are put together to give you a mfl. determines the cohomology $H^*_{dR} M$.

Poincaré Lemma

$A \subset \mathbb{R}^n$ open, star shaped. Then any closed k-form on A is exact.

Proof (motivation)

In dim 1, $f(x) = \int_0^x \alpha \, dt$ & $df = \alpha$.

In general for star shaped domain, the idea is to integrate radially.

Recall, chain homotopy

$$0 \xrightarrow{d} \Omega^0 M \xrightarrow{d} \Omega^1 M \xrightarrow{d} \Omega^2 M \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n M \xrightarrow{d} 0$$

$$\begin{tikzcd}
0 \arrow[r] & \Omega^0 M \arrow[r] & \Omega^1 M \arrow[r] & \cdots \arrow[r, "d"] & \Omega^{k-1} M \arrow[r, "d"] & \Omega^k M \arrow[r, "d"] \arrow[l, bend left, "I_k"] & \Omega^{k+1} M \arrow[r, "d"] \arrow[l, bend left, "I_{k+1}"] & \cdots
\end{tikzcd}$$

Want $d_{k-1} \circ I_k + I_{k+1} \circ d_k = \mathrm{id}$ & I_k linear.

Consequence: If $d\alpha = 0$ $\quad \alpha \in \Omega^k M \Rightarrow \alpha = I d\alpha + d(I\alpha)$

$$\begin{aligned} &= I(0) + d(I\alpha) \\ &= d(I\alpha). \end{aligned}$$

Suppose, $I : \Omega^{l} \to \Omega^{l-1}$ (l & k are switched).

$$\omega = \sum_{i_1 < i_2 < \cdots < i_l} \omega_{i_1,\dots,i_l}\, dx_{i_1} \wedge \dots \wedge dx_{i_l}$$. Wlog $p_0 = 0$.
(Since $dF^* = F^* d$)

not there

$$\text{Set } (I\omega)(x) := \sum_{i_1 < i_2 < \cdots < i_l} \sum_{\alpha=1}^{l} (-1)^{\alpha-1} \left(\int_0^1 t^{l-1} \omega_{i_1 \cdots i_l}(tx)\,dt \right) x_{i_\alpha}\, dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_\alpha}} \wedge \cdots \wedge dx$$

$\hookrightarrow \in \Omega^{l-1}(A)$.

Proof that $d \circ I + I \circ d = id$.

$$d(I\omega) = l \sum_{i_1 < \cdots < i_l} \left(\int_0^1 t^{l-1} \omega_{i_1 \cdots i_l}(tx)\,dt \right) + \sum_{i_1 < \cdots < i_l} \sum_{\alpha=1}^{l} \sum_{j=1}^{n} (-1)^{\alpha-1} \int_0^1 t^{l} \frac{\partial}{\partial x_j}(\omega_{i_1 \cdots i_l}(tx))\,dt$$

$dx_{i_1} \wedge \cdots \wedge dx_{i_l}$

$x_{i_\alpha} dx_j \wedge dx_{i_1} \wedge \cdots$
$\cdots \wedge \widehat{dx_{i_\alpha}} \wedge \cdots \wedge dx$

Comes from $\frac{\partial}{\partial x_{i_\alpha}}$ which produces a
dx_{i_α} that is switched in to
the $\widehat{dx_{i_\alpha}}$ position that kills $(-1)^{\alpha-1}$.

$$d\omega = \sum_{i_1 < \cdots < i_l} \sum_{j=1}^{l} \frac{\partial}{\partial x_j}(\omega_{i_1 \cdots i_l})\, dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_l}$$

Then,
$$I(d\omega) = \sum_{i_1 < \cdots < i_l} \sum_{j=1}^{n} \left(\int_0^1 t^{l} \frac{\partial}{\partial x_j}(\omega_{i_1} \cdots \omega_{i_l})(tx)\,dt \right) x_j\, dx_{i_1} \wedge \cdots \wedge dx_{i_l}$$

$$- \sum_{i_1 < \cdots < i_l} \sum_{j=1}^{n} \sum_{\alpha=1}^{l} (-1)^{\alpha-1} \left(\int_0^1 t^{l} \frac{\partial}{\partial x_j}(\omega_{i_1 \cdots i_l})(tx)\,dt \right) x_{i_\alpha} \cdot dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_\alpha}} \wedge \cdots \wedge dx$$

$d(I\omega) + I(d\omega) \Rightarrow$ The second terms cancel out.

Lec (Friday)

Stokes Theorem

Manifolds with boundary

$$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geqslant 0\}$$

M top sp. (paracompact) ~~s.t every pt $p \in M$ has a nbd~~ and an open cover of M by $\{U_\alpha\}_{\alpha \in I}$ & maps $\phi_\alpha : U_\alpha \longrightarrow \mathbb{H}^n$ s.t transition maps $T_{\alpha\beta}$ are C^∞.

① $\mathbb{H}^n$

② $\mathbb{R}^n$

③ 

④ 

Lemma: $\partial M = \{x \in M \text{ s.t } \phi_\alpha(x) \in \delta\mathbb{H}^n\}$ where $\delta \mathbb{H}^n = \{x \in \mathbb{H}^n \text{ s.t } x_n = 0\}$

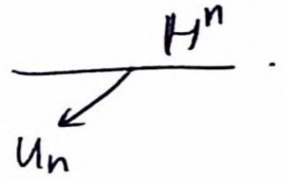
& ∂M is well defined. (independant of atlas).

> $M, \partial M$ is a mfl with bdry. Suppose M is oriented then ∂M is also oriented.

> Use outward normal u_n & tangent vectors $v_1 \dots v_{n-1}$ in $\mathbb{H}^n$ & give them an orientation. Pull this back to $M, \partial M$ to give the bdry an orientation. Lemma: It is well define

Outward normal $\neq$ perpendicular.

$u_n = (u_1, \dots, u_{n-1}, u_n)$ where $u_n < 0$



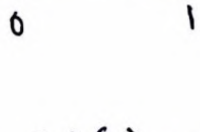
Stokes thm

M oriented with ∂M with induced orientation.

$\omega \in \Omega^{n-1} M$ & $d\omega \in \Omega^n M$.

Then

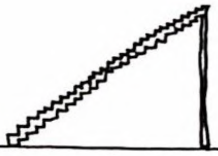
$$\boxed{\int_M d\omega = \int_{\partial M} \omega}$$

Ex: $M = [0,1]$ ; $\omega \in \Omega^0([0,1]) \to C^\infty$ fn.

$$\int_{[0,1]} d\omega = \int_0^1 \omega'(t)\, dt = \omega(1) - \omega(0) = \int_{\partial([0,1])} \omega = \int_{\{1\}} \omega \overset{\text{from orientat}}{-} \int_{\{0\}} \omega$$

Cor: Integrals of exact n-forms over manifolds w/o bdry are 0.

$$\int_M d\omega = 0$$



Monday lecture (Look at Faye's notes)

Def: A manifold M is called closed if M is compact & has no bdy $\delta M = \emptyset$.

P4: Check $M \times [0,1]$ is oriented.

β vol-form on M & dt form on $[0,1] \Rightarrow \beta \wedge dt$ is a vol-form on $M \times [0,1]$.

Lec Wed: C is a singular K-chain on M, $C = \sum \alpha_i c_i$. C_i Singular k-cubes. $\alpha \in \Omega^{k-1}(M)$.
Pf of Stokes for Singular k-chains $\int_C d\alpha = \int_{\partial c} \alpha$. Good enough to check on a singular k-cube $c: [0,1]^K \to M$. Good enough to check on special form α. $\int_C d\alpha := \int_{[0,1]^k} c^*(d\alpha) =$ LHS.
(Look at Faye's notes).

Lec (Friday): Degree & $H^n_{dR}(M)$

Thm: If M compact, oriented, then $H^n_{dR}(M) = \mathbb{R}$.

D: M, N dim = n, oriented compact. $f: M \to N$ orientation preserving, v Volform on N,

$$\int_M f^* v = \deg f \int_N v \xleftarrow{\text{holds}}$$

deg f is the # S.T.

Brouwer's fixed pt. thm: D^n closed ball in $\mathbb{R}^n$, $f: D^n \circlearrowleft$ cts. then f has a fixed pt.

MATH591 Final Review

D: <u>Locally compact</u>: Every pt. has nbd contained in a compact set

D: <u>Locally finite</u>: $p \in U$ intersects $\mathcal{X}$ finitely.

D: <u>Paracompact</u>: Every open cover admits an open locally finite refinement.

P: Connected $\Rightarrow$ (Paracompact $\Longleftrightarrow$ 2nd countable).

D: <u>Smoothly compatible</u>: Transition maps are diffeo. Enough to check $\psi \circ \phi^{-1}$ is smooth, injective & has non singular Jacobian at all

P: Constant, identity, composition & inclusion maps are smooth.

D: <u>Directional derivative</u>: $D_{\vec{v}}\big|_{\vec{a}}(f) = \frac{d}{dt}\Big|_{t=0} f(\vec{a}+t\vec{v}) = v^i \frac{\partial f(\vec{a})}{\partial x^i} = \nabla f \cdot \vec{v}$

D: <u>Differential</u>: $dF_p(v) = v(f \circ F) = \text{Jacobian}(F)\big|_p(v) = \left[\frac{\partial F^m}{\partial x^n}(p)\right]$

P: F diffeo $\Rightarrow$ dF_p iso & $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

C: Computing differentials with curves: $dF_p(v) = (F \circ \gamma)'(0) = \frac{d}{dt}\Big|_{t=0} (F \circ \gamma)($
S.T $\gamma(0) = p$ & $\gamma'(0) = v$.

D: Local diffeo: at p $\exists$ nbd U s.t $F(U)$ open in N & $F|_U$ is a diffeo.

Thm: IFT: dF_p invertible $\Rightarrow$ Local diffeo(F at p).

P: i) Local diffeo $\Leftrightarrow$ smooth immersion & submersion.

ii) dim M = dim N & F is either immersion or submersion $\Rightarrow$ local diffe

D: Embedding: Homeomorphic immersion.

P: F injective smooth immersion. F is a smooth embedding if

a) F is an open or closed map

b) F proper map (i.e. preimage of compact sets are compact).

c) M compact

d) M has no bdry & dim M = dim N.

L: A smooth curve is an immersion iff $\gamma'(t) \neq 0$.

D: Section: $\pi : M \to N \Rightarrow \sigma$ section: $N \to M$ s.t $\pi \circ \sigma = Id_N$.

Prop: smooth submersion $\Rightarrow$ open map. Surj. smooth submersion $\Rightarrow$ quotient map

D: Embedded submanifold: {Subspace s.t Inclusion map is a smooth embedding}

P: Graphs $\Gamma(f) = \{(x,y) \in M \times N : x \in U$ & $y \in f(U), y = f(x)\}$ are embedded submfls of product spaces.

D: Properly embedded: Inclusion map is proper.

P: Properly embedded $\Leftrightarrow$ S is a closed subset of M.

Cor: Compact embedded submfl $\Rightarrow$ properly embedded.

Thm (Constant-rank Level set Thm)

$\phi: M \to N$ constant rank r. Each level set of ϕ is a properly embedded submfl of co-dim r in M.

Cor (Submersion Level set Thm)

$\phi: M \to N$ smooth submersion. Each level set of ϕ is a properly embedded submfl of co-dim = dim N.

D: Normal subgroup: $H \subseteq G$ subgroup & $H = gHg^{-1}$ $\forall g \in G$.

D: Regular pt & critical pt

Regular pt ⇒ $d\phi_p$ is surjective. (critical pt otherwise).

D: Regular value: All pts. of level set are regular pts.

Thm: Regular Level Set Theorem : Every regular level set is a properly embedded submfl with co-dim = dim N. If $M = F^{-1}(p)$

$T_pM = \text{Ker}\,[J(F,p)]$.

Thm: (Sards' Thm) : The set of critical values of F has measure

Cor: $F: M \to N$ & $\dim(M) < \dim(N) \Rightarrow F(M)$ has measure 0.

D: Transversal intersection : $S, S' \subseteq M$ embedded submfls, intersect transversally if $\forall p \in S \cap S'$, T_pS & T_pS' span T_pM.

D: Transverse to S : $S \subseteq N$ embeded. $F: M \to N$. F is transverse to S if $T_{F(x)}S \oplus dF_x(T_xM)$ span $T_{F(x)}N$.

Thm: Every LGH has constant rank.

P: $F: G \to H$ LGHomo. $\mathrm{Ker}(F)$ is a properly embedded Lie subgroup with co-dim = rank = F.

Thm: G Liegroup, H Lie-subgroup. H closed in $G \Leftrightarrow$ it is embedded

Defs (of Actions)

Orbit: $G \cdot p$; $p \in M$; Isotropy group (Stabilizer) $G_p = \{g \in G : g \cdot p = p\}$

Transitive: $\forall p, q \in M$ $\exists g \in G$ s.t $gp = q$

Free: Isotropy is trivial.

D: Orbit space: Space of orbits as equivalence classes M/G.

L: $\pi: M \to M/G$ is open.

D: Proper action: $G \curvearrowright M$: $G \times M \to M \times M$ by $(g,p) \to (gp, p)$ is prop

Cor: Cont. action by a compact Lie group is proper.

Thm: $G/G_p \overset{\text{homeo}}{\cong} M$ where G_p is the isotropy group.

Thm: Every Liegroup is parallelizable $\to \exists \{V_i\}$ s.t $\{V_i(p)\}$ is a basis for T

D: Fiber bundle:

$$\begin{tikzcd}
\pi^{-1}(V_\alpha) \arrow[rr, leftrightarrow, "\phi_\alpha", "\text{diffeo}"'] \arrow[dr, "\pi"'] & & V_\alpha \times F \arrow[dl, dashed] \\
& V_\alpha &
\end{tikzcd}$$

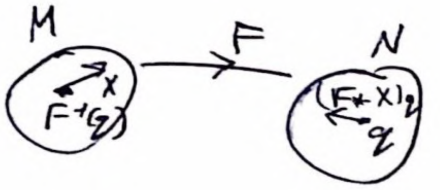
D: Local frame (LI basis for TM). $\overset{\exists}{\Leftarrow}$ Parallelizable.

D: F-related: $dF_p(X_p) = Y_p$ $\forall p \in M$.

F3

L: a) $[fX, gY] = fg[X,Y] + (fXg)Y - (gYf)X$.

b) $F_*[X_1, X_2] = [F_* X_1, F_* X_2]$



where $\hookrightarrow$ Push Forward.

$$(F_* X)_q = dF_{F^{-1}(q)}\left(X_{F^{-1}(q)}\right)$$

D: Lie Algebra

LAH: $A[X,Y] = [AX, AY]$

- Space of all LIVF of M i.e $\mathfrak{X}(G)$ with [ ] is the Lie Algebra.

$$\boxed{\mathfrak{g} \underset{\text{iso}}{\cong} T_eG}$$

D: Integral curve

$\gamma(t)$ s.t $\gamma'(t) = V_{\gamma(t)}$ for some V-f V.

D: Complete vector field

A smooth v-f with a global flow.

D: Distribution

D_p is a linear subspace of T_pM of dim k. $\bigsqcup_{p\in M} D_p$ is a distributio

D: Involutive: Given distribution D, If $X, Y \in D$ then $[X,Y] \in D$.

D: Integrable: Given D, if $\exists\ N \subseteq M$ s.t $T_pN = D_p\ \forall\ p \in N$ $\rightarrow$ If each pt of M is contained in an integral mfl of D, D is integrable.

N is called

Prop: $\omega(Tv_1, \dots, Tv_n) = (\det T)\,\omega(v_1, \dots, v_n)$

$$\boxed{\omega^1 \wedge \dots \wedge \omega^k (v_1, \dots, v_k) = \det(\omega^j(v_i))}$$

$$\boxed{\underset{(k)}{\omega} \wedge \underset{(l)}{\eta} = (-1)^{kl}\, \eta \wedge \omega}$$

$$(F^*\omega)_p (v_1, \dots, v_k) = \omega_{F(p)}(dF_p(v_1), \dots, dF_p(v_k))$$

$$F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$$

Eg: $F^*(w)$ = replace the function into the form. $F(u,v) = (u, v, u^2-v^2$

$w = y\,dx \wedge dz + x\,dy \wedge dz \Rightarrow F^*(w) = v\,du + d(u^2-v^2) + u\,dv \wedge d(u^2-v^2)$.

$$\boxed{d(\omega_j\, dx^j) = \sum_{i,j} \frac{\partial \omega_j}{\partial x^i}\, dx^i \wedge dx^j}$$

$$\boxed{d(\overset{k\text{ form}}{\omega} \wedge \overset{l\text{ form}}{\eta}) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta}$$

Basically substitute $\boxed{F \text{ into } \omega}$

D: If (E_i) & $(\tilde{E}_j)$ define the same orientations, then the transition mtx B_{ij} given by $E_i = B_i^j E_j$ has +ve determinant.

D: Atlas (U_α, ϕ_α) is consistently oriented if T-maps have pos Jac det.

$$\boxed{(\mathcal{L}_V W)_p = \lim_{t\to 0} \frac{d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) - W_p}{t}}$$

$$\boxed{\mathcal{L}_V W = [V, W]}$$

$$\boxed{\mathcal{L}_V (fW) = (Vf)W + f\,\mathcal{L}_V W}$$

$$\boxed{(\mathcal{L}_V A)_p = \lim_{t\to 0} \frac{d(\theta_t)_p^*(A_{\theta_t(p)}) - A_p}{t}}$$

$$\mathcal{L}_V f = Vf = \frac{d}{dt}\Big|_{t=0}(\overset{\text{flow of } V}{\theta_t^* f}) = \frac{d}{dt}\Big|_{t=0}(f\circ\theta_t)$$

$$\mathcal{L}_V(fA) = (\mathcal{L}_V f)A + f\,\mathcal{L}_V A$$

$$\mathcal{L}_V(df) = d(\mathcal{L}_V f). \quad \& \quad \mathcal{L}_V(d\omega) = d(\mathcal{L}_V \omega).$$

$$\mathcal{L}_V(\omega\wedge\eta) = (\mathcal{L}_V\omega)\wedge\eta + \omega\wedge(\mathcal{L}_V\eta)$$

$$H^k_{dR}(S^n \times I^n) = \begin{cases} \mathbb{R} & k>0 \text{ or } k=n \\ 0 & \text{otherwise} \end{cases} \quad \Bigg| \quad H^k_{dR}(\mathbb{R}\setminus\{0\}) = \begin{cases} \mathbb{R}^2 & n>1, k>0 \\ \mathbb{R} & n>1, k=0 \\ & n \\ 0 & \text{otherwise} \end{cases}$$

Thm: Global rank Thm

$F: M \to N$ smooth & constant rank $\begin{pmatrix} F \text{ surj} \Rightarrow \text{sub} \\ F \text{ inj} \Rightarrow \text{imm} \\ F \text{ bij} \Rightarrow \text{diffeo} \end{pmatrix}$

Thm: Graphs as submanifolds

M^m, N^n & $U \subseteq M$ open, $f: U \to N$ $C^\infty \Rightarrow \Gamma(f) = \{(x, y) \in M \times N, x \in U, y = f(x)\}$

is an embedded m dimensional mfl of $M \times N$.

Given a small enough nbd, the flow looks locally like $\frac{\partial}{\partial x}$.

$\gamma(t) = e^{tA}$ is the integral curve starting at e along V-f A

Since $\gamma'(t)\Big|_{t=0} = Ae^{tA}\Big|_{t=0} = A$. & $\gamma(0) = Id$. $\Rightarrow ge^{tA}$ is the i-c, starting from. from g.

Prop 15-21

$\mathbb{CP}^n \cong S^{2n+1}/S^1$, $\mathbb{RP}^n = S^n/f$ → orientable when n is odd
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ ↳antipodal

> Riemannian metric is the pullback of Eucl. metric by charts that are then passed thru a POU.