MATH 591 Abridged notes

D: n-dim topological manifold (Hf, s.c, loc.Euc).

M : Hausdorff, Second countable, every $p \in M$ has a nbd homeomorphic to $\mathbb{R}^n$.

D: Locally finite ($p \in U \cap \mathcal{X}$ finitely) collection.

Collection $\mathcal{X}$ of sets $\subseteq M$, is locally finite if each $p \in M$ has a nbd U s.t U only intersects finitely many $C \in \mathcal{X}$.

D: Paracompact (open covers have locally finite refinements)

M is paracompact if every open cover admits a locally finite subcover (aka refinement)

Thm: Topological manifolds are paracompact.

D: Locally compact ($\exists$ precompact nbds $\forall p$)

M is locally compact if for every $p \in M$, and nbds U of p $\exists$ nbd $V \subseteq U$ s.t $\overline{V} \subset U$ is compact. (i.e $\exists$ precompact nbd)

Lemma: Top. mfl are locally compact.

D: Exhaustion

An exhaustion is a sequence of sets $K_1, \dots, K_n, \dots$ S.T $\bigcup K_n = M$ and $K_1 \subseteq K_2 \dots \subseteq K_n \subseteq K_{n+1} \subseteq \dots$.

Proposition: A 2^{nd} countable locally compact Hf space M admits an exhaustion by compact sets.

Proposition: M top mfl, path connected $\iff$ connected.

D: Differentiable manifold. (see Jonh Lee notes)

Transition maps $\longrightarrow$ Smooth atlas $\longrightarrow$ smooth structure (maximal atlas) $\longrightarrow$ Smooth manifold.

Thm: $\exists$ Top mfls. which do not admit any differentiable structure.

Thm: X Hf, 2^{nd} countable, $\sim$ open eq. rel. Then $X/\sim$ is 2^{nd} countable

Proposition X compact, Hf. $[x]$ are closed $\forall x \in X$. (i.e. $\sim$ is closed). Then $X/\sim$ is compact Hf.

D: Lie Group

A topological group G is called a Lie group if it has a differentiable structure S.T

$$\begin{aligned} G \times G &\longrightarrow G \quad (x \times y \longrightarrow xy) \\ G &\longrightarrow G \quad (x \longrightarrow x^{-1}) \end{aligned}$$

are smooth.

D: Topological group

A top. sp. with a group structure, S.T

$$\begin{aligned} G \times G &\longrightarrow G \\ G^{-1} &\longrightarrow G \end{aligned} \quad \text{are } \underline{\text{continuous}}.$$

D: Differentiable map between manifolds.

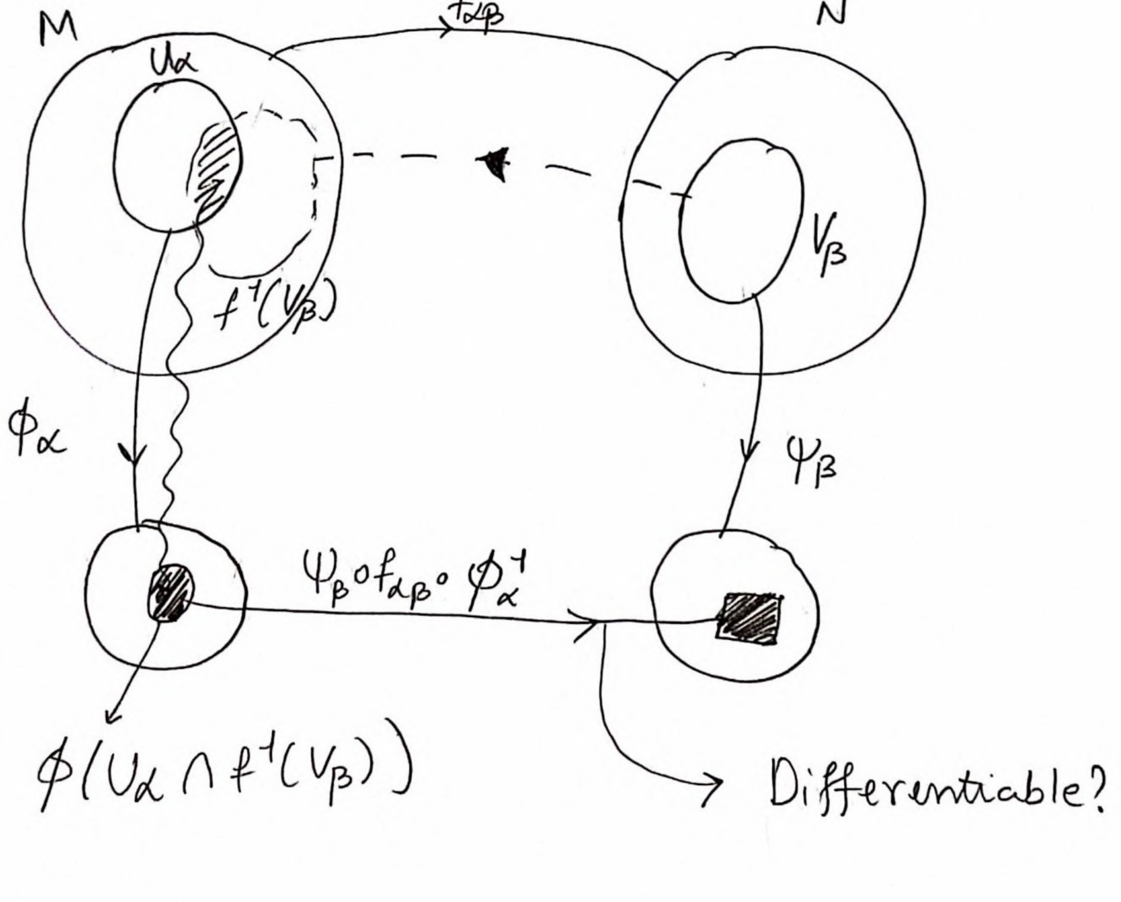
M, N differentiable manifolds. $f : M \longrightarrow N$ continuous.

① For every pair of charts (U_α, ϕ_α) of M and (V_β, ψ_β) of N

$\psi_\beta \circ f_{\alpha\beta} \circ \phi_\alpha^{-1} : \phi(U_\alpha \cap f^{-1}(V_\beta)) \longrightarrow \psi(V)$ is

differentiable.

or ② f is differentiable at every pt. of M.



D: Graph of $\sim$

$$\Gamma = \{ (x,y) \mid \in X \times X \text{ and } x \sim y \}$$

Thm 1: X top sp., Hf, $\sim$ open equivalence relation.
Then $X/\sim$ is Hf. iff Γ is closed in $X \times X$.

D: Action.

G group, X top. space.

$G \curvearrowright X$ is an action if $\exists$ map $\begin{array}{l} G \times X \longrightarrow X \\ g \cdot x \longrightarrow gx \end{array}$

S.T $\forall\, x \in X,\ 1 \cdot x = x$

$\forall g_1, g_2 \in G,\ (g_1 g_2) x = g_1 \cdot (g_2 \cdot x)$

D: Continuous action (If $G \times X \to X$ is cts).

D: Differentiable action (If $G \times X \longrightarrow X$ is differentiable)

D: Jointly diff. action (If G is Lie & $GX \to X$ is differentiable)

Thm: G compact, top. group; $G \curvearrowright X$ continuous action.
X compact Hf

$\Rightarrow X/G$ is Hausdorff.

D: Group action terminology

1) Orbit : $G \cdot p = \{ g \cdot p \mid g \in G \}$

2) Stabilizer of p or isotropy group

$$G_p = \{ g \in G \mid g \cdot p = p \}$$

3) Transitive

$\forall p, q \in M \quad \exists g \in G$ s.t $g \cdot p = q \Rightarrow$ only orbit $= M$.

4) Free

Only the identity fixes p. (ie. isotropy group $\forall p$ is trivial)

D: Homogeneous spaces (not definition but a general example)

$G \subset H$, G compact subgroup of $H \to$ Lie group. Then

H/G Mf (called Homogeneous.)

Thm : (Topological orbit stabilizer theorem)

$G \curvearrowright M$ transitively $\Rightarrow G/G_p \underset{\text{homeo}}{\cong} M$.

D: Grassmanian

V is a n-dim linear vector space. ; $k \leq n$, $k \in \mathbb{N}$.

$Gr_{k,n}$ is the set of k-dimensional linear subspaces.

It can be given a smooth structure & is called a Grassmanian manifold.

Notes of Transversal & find differentiable structures (Recipe)

M Lie group $G \curvearrowright M$. Want to endow M with a diff structure.

- Choose $p \in M$. G_p = stabilizer group of p. $G_p \subset G$ closed
- Show $G \curvearrowright M$ transitive. Then $G/G_p \cong M$.
- Find T_p, "transversal" to G_p in G. Maybe it is locally $\mathbb{R}^n$.
- Try coordinate charts $T_p \longleftrightarrow T_p \cdot p$ which would define local homeos from $T_p \cong \mathbb{R}^n \longleftrightarrow p$.
- If $q = g_0 \cdot p$ Then $G_q = g_0 \, G_p \, g_0^{-1}$ and Try $T_q = g_0 T_p \, g_0^{-1}$ for T_q.

D: Proper map: (Preimages of compact sets are compact)

D: Properly discontinuous action.

Γ discrete group. $\Gamma \curvearrowright \tilde{M}$ (diff mfl).

① $\Gamma \times \tilde{M} \to \tilde{M}$ is proper. (i.e. $\forall p \in \tilde{M}$, $\Gamma \cdot p \cap K$ is finite for K compact subset of $\tilde{M}$)

② $\Gamma \curvearrowright M$ is free (i.e. $\gamma \cdot p = p \Rightarrow \gamma = 1$)

Prop: $\Gamma \curvearrowright \tilde{M}$ properly discont. $\Rightarrow M = \tilde{M}/\Gamma$ is a top. mfl. if $\tilde{M}$ diff mfl. M has a diff. structure.

D: Tangent Vector & Tangent Space

> M diff mfl, $p \in M$.

Let (U, ϕ) & (V, ψ) be two smooth charts at p. ($p \in U \cap V$.)

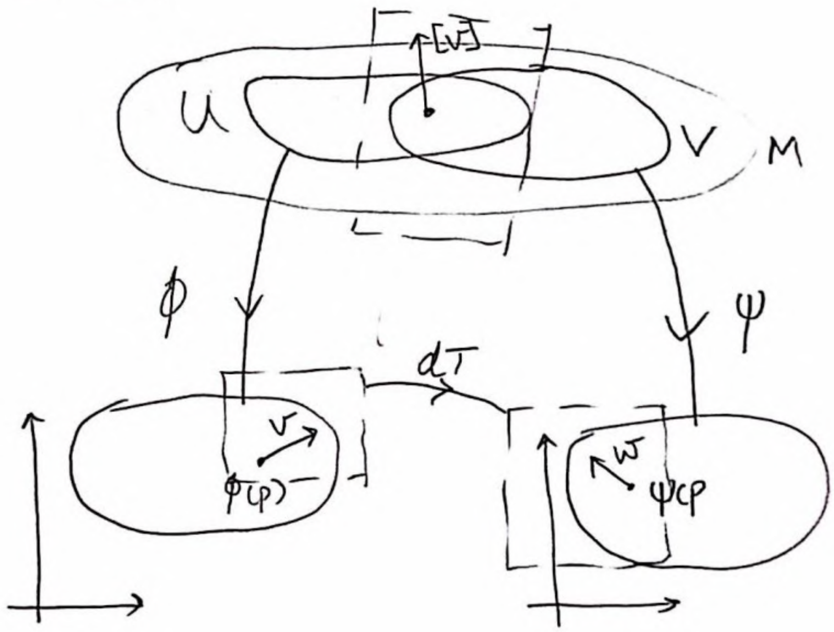
Let T be the transition map $\psi \circ \phi^{-1}$.

Then $v \sim w$ are two equivalent tangent vectors at $\phi(p)$ and $\psi(p)$ iff $dT(w) = v$.

Then the tangent vector at p in M is given by $[v]$

and $T_p M = \{ [v] \mid [v] \text{ eq. class} \}$ is the

Tangent space



D: <u>Section</u> ("basically a vector assigned to a base pt.")

Given a fiber bundle $E \xrightarrow{\pi} B$, E smooth,
$$\begin{array}{c} E \\ \uparrow \\ F \end{array}$$

$\sigma : B \to E$ is called a section if $\pi \circ \sigma = id_B$.

D: <u>Vector field</u>

$X : M \longrightarrow TM$ is a section of TM.

<u>Note</u>: On a manifold, map vector field basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ via charts to $e_1, \quad, e_n$ so that calculations become easy.

i.e. $\frac{\partial}{\partial x_i}(p) := D\phi_\alpha^{-1}{}_{\phi_\alpha(p)}(e_i)$

D: <u>Derivation</u>

$\delta : C^\infty(M) \longrightarrow \mathbb{R}$ is a derivation at p if $\forall f, g \in C^\infty(M)$,

$\delta(fg) = f\,\delta(g) + \delta(f)\,g$ & $\delta(f+g) = \delta f + \delta g$.

<u>Thm</u>: M smooth, X & Y are vector fields (smooth). Then $X \circ Y - Y \circ X$ is a derivation.

<u>Thm</u>: Every derivation δ of p defines a unique tangent vector, i.e. $\exists\, v \in T_pM$ s.t $\delta = \delta_v = \frac{d}{dt}\big|\, f(c(t))$

Cor: Every derivation $\Delta : C^\infty(M) \to C^\infty(M)$ defines a v-field.

D. Lie Bracket

$$[X,Y] = X \circ Y - Y \circ X.$$

Lemma: $v \in T_pM$, $\delta_v =$ direction derivative. δ_v is a derivation and if $\delta_v = \delta_w \Rightarrow v = w$.

Sard's Theorem (591 notes)

D: Full measure & Zero measure

$B \subset M$ has zero measure if $\forall\ \phi_\alpha, U_\alpha$, volume $(\phi_\alpha(B \cap U_\alpha))$ is zero in $\mathbb{R}^n$.

$A \subset M$ has full measure if $M-A$ has 0 measure.

Recall:

D: Regular value & Critical value (note, if $\phi^{-1}(c) = \emptyset$ it is a r.v.)

> $\phi : M \to N$; $c \in N$ is a regular value of ϕ if $\phi^{-1}(c)$ is a regular pt. (ie $\phi(\phi^{-1}(c))$) is a submersion.

> Critical value if it is not a regular value.

Thm: (Sard's Theorem)

$F : M \to N$ all C^∞. Then the set of critical values of F has zero measure in N.

Proof: (Induction)

$m = \dim M$, $n = \dim N$.

if $m = 0$, $F(M)$ is countable $\Rightarrow$ measure 0.

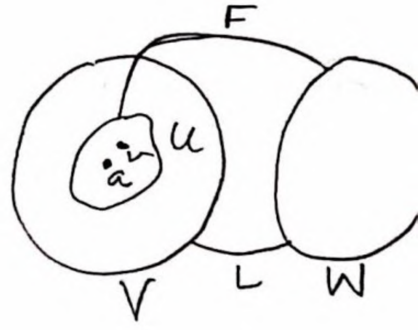
(Look at Alex Wright's proof).

D: Critical point (alternate definition) & Critical value

$f: M \to N$ all C^∞. $x \in M$ is a critical point of f if $df_x(T_x M) \neq T_{f(x)} N$. A critical value is the image of a critical point.

(Basically these are points where the Jacobian vanishes).
(i.e dim of Tangent space decreases).

Calculus Review for Math 591



- F differentiable at a if $\exists$ a linear map $L: V \longrightarrow W$ s.t

$$\lim_{v \to 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} = 0$$

$V, W \to$ fdVs

> L is unique.

> $L = DF(a)$ the total derivative of F at a. "best linear approx.

Chain rule: $D(G \circ F)(a) = DG(F(a)) \circ DF(a)$.

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D $U \subseteq \mathbb{R}^n$ & f is a real valued function.

$$\frac{\partial f}{\partial x^j}(a) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{e_j}) - f(\vec{a})}{h}$$

$\vec{e_j}$ → unit vector along j

In general $\left[\frac{\partial f^i}{\partial x^j}\right]$ is the Jacobian matrix

D: C^1: If F has continuous partial derivatives of order 1 at all pts. in U.

- C^k: $1 \to k^{th}$ partial derivatives exist & continuous.

C^∞: All partial derivatives exist a.k.a smooth.

D: Diffeomorphism

$F: U \to V$ is a diffeomorphism if it is smooth & bijective with a smooth inverse.

P: $F: U \to V$ diffeo (where $U \in \mathbb{R}^m$ & $V \in \mathbb{R}^n$), then

$m = n$ and $DF(a)^{-1} = D(F^{-1})(F(a))$.

D: Directional derivative along $\vec{v}$

$$D_{\vec{v}} f(a) = \frac{d}{dt}\Big|_{t=0} f(\vec{a} + t\vec{v}) = Df(a)\, v$$

Thm: Inverse Function Theorem

Suppose U and V are open subsets of $\mathbb{R}^n$, and $F: U \to V$ is a smooth function. If $DF(a)$ is invertible at some point $a \in U$, then $\exists$ connected nbds $U_0 \subseteq U$ of a and $V_0 \subseteq V$ of $F(a)$ s.t. $F|_{U_0}: U_0 \to V_0$ is a diffeo.

Implicit Function Theorem

Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$ be an open subset, and let $(x, y) = (x^1, \dots, x^n, y^1, \dots, y^k)$ denote the standard coordinates on U. Suppose $\phi : U \to \mathbb{R}^k$ is a smooth function, $(a, b) \in U$. and $c = \phi(a, b)$. If the $k \times k$ matrix $\left(\frac{\partial \phi^i}{\partial y^j}(a, b) \right)$ is non singular, then $\exists$ nbds $V_0 \subseteq \mathbb{R}^n$ of a and $W_0 \subseteq \mathbb{R}^k$ of b and a smooth function $F : V_0 \to W_0$ such that $\phi^{-1}(c) \cap (V_0 \times W_0)$ is the graph of F, i.e, $\phi(x, y) = c$ for $(x, y) \in V_0 \times W_0$ iff $y = F(x)$.

(Conditions under which a level set of a smooth fn. is locally a graph of a smooth fn.)

Jacobian is invertible.

Group Theory Review for Math 591

D: $GL(n,\mathbb{R})$ (also $GL(n,\mathbb{C})$) General linear group.

> Invertible $n\times n$ matrices over $\mathbb{R}$. (i.e. $\det(A)\neq 0$)
> Submanifold (open) of $M(n,\mathbb{R}) \to$ v-s of $\mathbb{R}^{n\times n}$.

D: $GL^+(n,\mathbb{R})$ (positive determinant with GL condition)

D: $O(n)$ Orthogonal group (columns & rows are orthogonal vecs)

$= \{A \in GL(n,\mathbb{R}) \mid A.A^t = I_n\}$ rotations & flips.

D: $SL(n,\mathbb{R})$ Special linear group

$= \{A \in GL(n,\mathbb{R}) \mid \det(A)=1\}$

D: $SO(n)$ Special orthogonal group

$= \{A \in O(n) \mid \det(A)=1\}$ rotations

$= O(n) \cap SL(n,\mathbb{R}) \subseteq GL(n,\mathbb{R})$

D: $U(n)$ Unitary group $\{A^*$ adjoint = conj transpose of $A\}$

$= \{A \in GL(n,\mathbb{R}) \mid A^*.A = I_n\}$

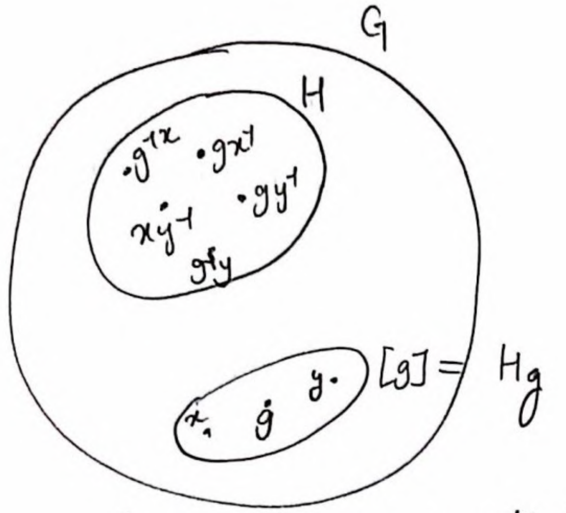
D: $SU(n)$ Special unitary group

$= \{A \in U(n) \mid \det(A) = 1\}$

Cosets, Equivalence classes and quotients.

Proposition: Let G be a group. H a subgroup & $g \in G$. Let $[g]$ be it's equivalence class under $\sim$. Then $[g] = Hg$ i.e. the right cosets. (Here $x \sim y$ if $xy^{-1} \in H$.)

Corollary: Let Ha & Hb be two right cosets. Then $Ha = Hb$ iff $ab^{-1} \in H$.



If $xy^{-1} \in H$, x pulls H onto $[x]$ by Hx.

Lemma: Quotient map coming from a continuous group action is open.

Note: p is a quotient map $\Leftrightarrow$ p maps saturated open sets to open sets. & p is continuous.

D:

Normal subgroup. $N \subseteq G$ (a group) is called a normal subgroup if it is a group and if $gng^{-1} \in N$ for any $n \in N$ & $g \in G$. ($N \triangleleft G$ notation)

General Review (for problem solving).

Jacobian

$F: \mathbb{R}^n \to \mathbb{R}^m$ is $F(\vec{x}) = \vec{y}$ where $\vec{x} \in \mathbb{R}^n$, $\vec{y} \in \mathbb{R}^m$

& $F(x_1, x_2, \ldots, x_n)$ s.t

$$J = \begin{bmatrix} \frac{\partial F_1(x_1 \ldots x_n)}{\partial x_1} & \frac{\partial F_1(\ldots)}{\partial x_2} & \cdots \\ \frac{\partial F_2(x_1 \ldots x_n)}{\partial x_1} & & \\ \vdots & & \end{bmatrix}$$

$y_1 = F_1(x_1, \ldots, x_n)$

$y_2 = F_2(x_1, \ldots x_n)$

$\vdots$

Smooth function (C^∞)

$\frac{\partial^p F^i}{(\partial x^j)^p}$ exists $\forall\ p \geq 0$ & $\forall\ i$. Here j runs from $1 \ldots n$ & i from $1 \ldots m$.

And $(\partial x^j)^p$ is all possible p^{th} order "polynormals" made from the x^js.

Smooth atlas

If $\psi \circ \phi^{-1}$ is smooth, injective and has non singular Jacobian at each point. ϕ & ψ are smoothly compatible & they belong to a smooth atlas.

Smooth Manifolds (John Lee)

Def Topological n-manifold

Let M be a topological space. It is an n-manifold if

> M is Hausdorff, second countable & locally Euclidean of dimension n.

Hf: $p, q \in M \Rightarrow \exists$ nbds U, V s.t $U \cap V = \phi$ & $U, V \subset M$.

S.C: $\exists$ a countable basis for M.

Locally Euc: $\forall p \in M \; \exists$ nbd U s.t U homeomorphic to an open subset of $\mathbb{R}^n$.

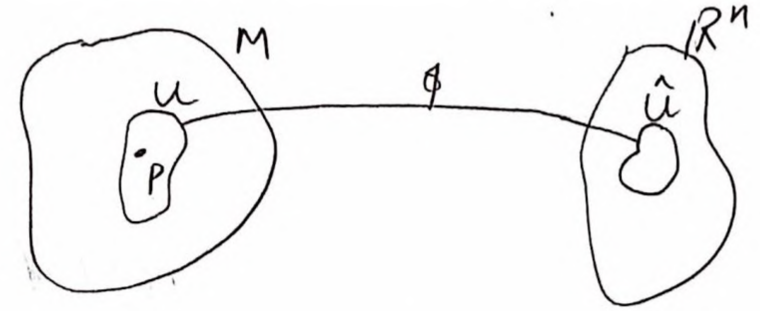
i.e. $\exists$ i) nbd U of p & $U \subseteq M$
ii) open $\hat{U} \subseteq \mathbb{R}^n$ and
iii) homeo $\phi: U \to \hat{U}$.

Thm: (Topological invariance of dimension) i.e. it is well defined.

> Hausdorf & S.C properties are inherited by subspace and finite products.
> Every open subset of an n-manifold is an n-manifold.

Def Coordinate Chart

M top n-man. A Coord. chart (U, ϕ) s.t U is an open subset of M and $\phi: U \to \hat{U}$ is a homeo from U to open subset $\hat{U} = \phi(U) \subseteq \mathbb{R}^n$.



In a top. man. every p is associated with a chart.

> If $\phi(p) = 0$; the chart is "centered at p".

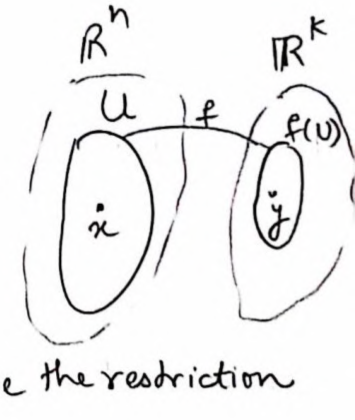
> Note, $\phi(p)$ is an n-dim vector. By subtracting $\phi(p)$, a new chart is obtained that is "centered at p".

> Given (U, ϕ) $\quad U \to$ coordinate domain/nbd

> if $\phi(U)$ is an open ball in $\mathbb{R}^n$, U is called a coordinate ball
" " " " cube " " " " " " cube

> ϕ is a coordinate map & is composed of n functions
$\phi(p) = (x^1(p), x^2(p), \dots x^n(p))$ called the local coordinates on U.

$\therefore (U, \phi) \equiv (U, (x^i))$



Ex: $U \subset_{\text{open}} \mathbb{R}^n$, $f : U \to \mathbb{R}^k$ cont. Graph of f is

$$\Gamma(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : x \in U \text{ \& } y = f(x)\}$$

Let $\pi_1 : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ be the proj. & $\phi : \Gamma(f) \to U$ be the restriction of π_1 onto $\Gamma(f)$. $\Rightarrow \phi(x, y) = x$. ϕ is cont. (due to restriction on cont. map)
Inverse $\phi^{-1}(x) = (x, f(x))$ is also cont. $\Rightarrow$ homeo! $\Rightarrow \Gamma(f)$ is n-manifold although it has dim. $(n+k)$. $(\Gamma(f), \phi)$ is a coord. chart called graph coord.

t3

Ex. Sphere $S^n \subseteq \mathbb{R}^{n+1}$. Let $U_i^+ = \{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : x^i > 0\}$ $\forall i = 1, \ldots, n+1$. That is, U_i^+ is the +ve half space along x_i^{th} coordinate. Similarly U_i^- is where $x^i < 0$.

Let $f : \mathbb{B}^n \to \mathbb{R}$ (unit ball in $\mathbb{R}^{n-1}$) be $f(u) = \sqrt{1 - |u|^2}$

then $\forall i$, $U_i^+ \cap S^n$ is the graph of the function

$$x^i = f(\underbrace{x^1, \ldots, \hat{x}^i, \ldots, x^{n+1}}_{\text{Squished open ball } \mathbb{B}^n \text{ onto } \mathbb{R}^n})$$ where $\hat{x}^i$ is omitted.

graph of f takes the shadow onto the surface of S^n. f itself only takes it to the missing $\hat{x}^{i\text{th}}$ coordinate.

Therefore, $\phi_i^{\pm} : U_i^{\pm} \cap S^n \longrightarrow \mathbb{B}^n$ given by

$\phi_i^{\pm}(x^1, \ldots, x^{n+1}) = (x^1, \ldots, \hat{x}^i, \ldots, x^{n+1})$ are

<u>graph coordinates</u> of S^n.

Graph coordinates are a projection from $\mathbb{R}^{n+k} \to \mathbb{R}^n$ restricted to an open subset of $\mathbb{R}^{n+k}$. It is the set & the map. See earlier example.

Each point on S^n is contained in atleast one of these $2n+2$ ($\pm$) charts $\Rightarrow$ S^n is a top. n-man.

Ex: Projective Spaces

> $\mathbb{RP}^n$ is the set of 1-dim linear subspace of $\mathbb{R}^{n+1}$. (ie. lines in $\mathbb{R}^{n+1}$ are the eq. classes).

$\pi: \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{RP}^n$ sends each pt. $x \in \mathbb{R}^{n+1} \setminus \{0\}$ to it's span.

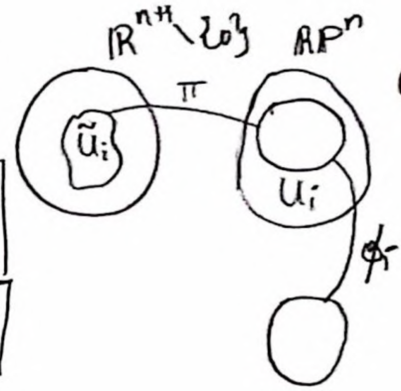
ie. $[x] = \pi(x)$ & $[x] \in \mathbb{RP}^n$.

If $\tilde{U}_i \subset \mathbb{R}^{n+1} \setminus \{0\}$ s.t. $\tilde{U}_i = \{x \mid x^i \neq 0 \ \forall i\}$. ($n=2 \Rightarrow \tilde{U}_1 = \mathbb{R}^3 - yz$ plane) Then $\tilde{U}_i$ is a saturated open subset.

$\underset{\llcorner \pi(\tilde{U}_i)}{U_i}$ is open & $\pi\big|_{\tilde{U}_i}$ is a quotient map (from Thm in Top).

Define $\phi_i [x^1 \dots x^{n+1}] = \left(\frac{x^1}{x^i}, \dots, \cancel{\frac{x^i}{x^i}}, \dots \frac{x^{n+1}}{x^i} \right)$

$\boxed{\text{Scaling sends all pts on line } [x] \text{ to single pt. of } YZ \text{ plane.}}$



ϕ_i is a homeo with inverse

$\phi_i^{-1}(u^1, \dots, u^n) = [u^1, \dots, u^{i-1}, 1, u^i, \dots, u^n]$

$\Rightarrow \phi([x]) = u$ means $(u, 1)$ is the pt. in $\mathbb{R}^{n+1}$ where the line $[x]$ intersects the plane $x^i = 1$.

Since $U_1 \dots U_{n+1}$ cover $\mathbb{RP}^n$, it is locally Euclidean of dim n.

"Idea: divide proj plane into open subsets cover. one per axis. Project all pts onto leftover plane for that axis. Since they all map to same plane $\mathbb{R}^n$ it is loc Euc."

Ex: If $M_1, \ldots, M_k$ are top mf of dimensions $n_1, \ldots, n_k$ then product space $M_1 \times \cdots \times M_k$ is a top. mf of dim. $n_1 + \cdots + n_k$ with charts of the form $(U_1 \times \cdots \times U_k, \phi_1, \ldots, \phi_k)$

Lemma: Every top. mf has a countable basis of precompact coordinate balls.
↳ closure is compact.

Prop: M top mf:

a) M is locally path connected ↗ i.e. has a basis of path conn. open sets

b) M is connected ⟺ path connected

c) components in M = path components in M.

d) M has countably many components, each of which is an open subset of M and a connected top. mf.

Prop: Every top mf is locally compact.
↳ every pt. has a nbd contained in a compact subset of X.

Pf: Trivial from Lemma.

D: Locally finite: M top. space. A collection $\mathcal{X}$ of subsets of M is locally finite if each $x \in M$ has a nbd that intersects atmost finitely many sets in $\mathcal{X}$.

Refinement: $\mathcal{V}$ & $\mathcal{U}$ are covers of M. If for each $U \in \mathcal{U}$ $\exists\ V \in \mathcal{V}$ s.t $V \subseteq U$ then $\mathcal{V}$ is a refinement of $\mathcal{U}$.

D: Paracompact: M is paracompact if every open cover of M admits an open, locally finite refinement.

Lemma: $\mathcal{X}$ is a locally finite collection of subsets of a top sp. M.

a) $\overline{\mathcal{X}} = \{\overline{X} : X \in \mathcal{X}\}$ is locally finite

b) $\overline{\bigcup_{X \in \mathcal{X}} X} = \bigcup_{X \in \mathcal{X}} \overline{X}$.

Thm (Manifolds are paracompact).

Given M top mf. $\mathcal{X}$ cover & $\mathcal{B}$ basis. $\exists$ a countable, locally finite open refinement of $\mathcal{X}$ consisting of elements of $\mathcal{B}$.

Prop: For connected spaces, paracompact $\Leftrightarrow$ 2^{nd} countability.

Prop: Fundamental group of a top. space is countable.

Smooth Structures

- > Diffeomorphism $\Rightarrow$ homeomorphism (since smooth $\Rightarrow$ continuous).

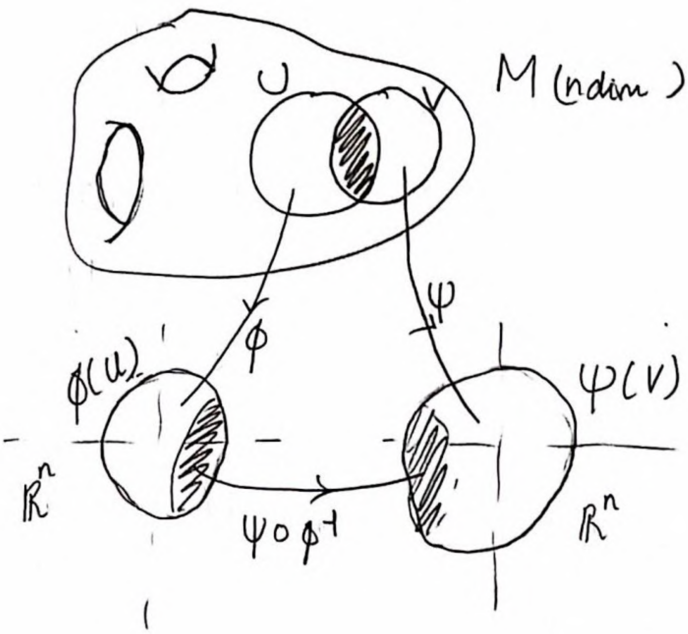
Transition map

Given M top mf, let (U,ϕ), (V,ψ) be two C-charts s.T $U \cap V \neq \phi$. Then the map

$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is called the transition map from ϕ to ψ. It is a homeomorphism.

Smoothly compatible

- If $U \cap V = \phi$ or the transition map $\psi \circ \phi^{-1}$ is a diffeomorphism, they are smoothly compatible.



- D: Smooth atlas: An atlas A whose any 2 charts are smoothly compatible. (collection of charts covering M)

> To show an atlas is smooth, only verify that $\psi \circ \phi^{-1}$ is smooth. It's inverse $\phi \circ \psi^{-1}$ is also smooth.

> To show (U, ϕ) & (V, ψ) are smoothly compatible, verify that $\psi \circ \phi^{-1}$ is smooth and injective with nonsingular Jacobian at each pt. Diffeo follows from Corollary C.36.

D: Maximal Atlas & complete atlas

A smooth atlas $\mathcal{A}$ is maximal if it is not properly contained in any larger smooth atlas.

i.e. any chart smoothly compatible with every chart in $\mathcal{A}$ is already in $\mathcal{A}$. (ie the smooth atlas is complete).

D: Smooth structure (A maximal smooth atlas).

(aka differentiable/C^∞ structure).

D: Smooth manifold A pair $(M, \mathcal{A})$ where M is a top mf and $\mathcal{A}$ is a smooth structure on M.

Prop: M top mf.

a) Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the smooth structure determined by $\mathcal{A}$. Call it $\bar{\mathcal{A}}$ (ie. $\mathcal{A}$ + all charts smoothly comp. with every chart in $\mathcal{A}$)

b) Two smooth atlases determine the same smooth structure iff their union is a smooth atlas.

Definitions with smooth

> M smooth mf, any $(U, \phi) \in$ Maximal smooth atlas is a smooth chart.

$U \to$ smooth coordinate domain/nbd

$\phi \to$ smooth coordinate map

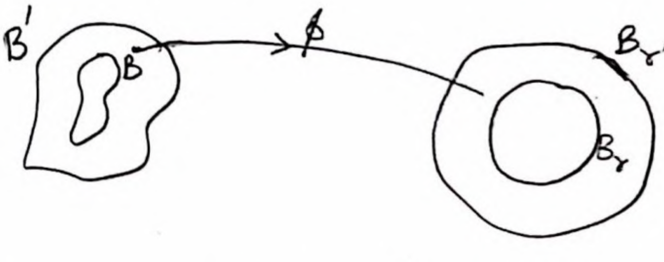
If $\phi(U) = $ ball/cube then U is a smooth coord. ball/cube.

D: Regular coordinate ball

$B \subseteq M$ is a rgb if $\exists$ a smooth c.b $B' \supseteq \bar{B}$ and a s.c. map $\phi : B' \to \mathbb{R}^n$ s.t for some $r, r' \in \mathbb{R}$ & $r < r'$

$$\phi(B) = B_r(0), \quad \phi(\bar{B}) = \bar{B}_r(0) \quad \& \quad \phi(B') = B_{r'}(0)$$

(B sits "nicely" inside B')



Prop: Every smooth manifold has a countable basis of regular coordinate balls.

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Ch 2: Smooth Maps

Def: Smooth map: $F: M \to N$ smooth if $\psi \circ F \circ \phi^{-1}$ is smooth from $\phi(\underbrace{U \cap F^{-1}(V)}_{\text{must be open in } M}) \to \psi(V)$.

Prop: Constant, identity, composition & inclusion maps are smooth.

Note: To show a map is smooth it is easiest to just write down the coordinate representation of it.

Prop: Compositions, finite products & restrictions to open submanifolds of diffeos are diffeos.

Ch 3 : Tangent Vectors.

$\vec{v}|_a$

> Geometric tangent vector $\vec{v_a}$ at $\vec{a}$ is a tangent vector at $\vec{a}$ with it's base at $\vec{a}$. The set of all vectors like this form the geometric tangent space at $\vec{a}$. (denoted by $\mathbb{R}^n_a$)

$\mathbb{R}^n_a = \{a\} \times \mathbb{R}^n = \{(a, v) \mid v \in \mathbb{R}^n\}$

> If $a \in S^{n-1}$. Then, the tangent space to S^{n-1} at a is a subspace of $\mathbb{R}^n_a$ s.t all vectors in it are orthogonal to $\frac{\vec{a}}{|\vec{a}|}$. This formulation does not generalize to arbitrary smooth manifolds.

why to $\mathbb{R}$

> Directional derivative along v at a is $D_{v|_a}(f) : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$

$$D_v\big|_a (f) = D_v(f(a)) = \frac{d}{dt}\Big|_{t=0} f(a+tv)$$

$$= v^i \frac{\partial f}{\partial x^i}(a) \qquad \text{where } v\big|_a = v^i e_i\big|_a$$

unit vector $x^i, y^i \ldots$

Summation Convention $v^i e_i = \sum_i v_i e_i$

Def: Derivation

If $a \in \mathbb{R}^n$, a map $w : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a derivation at a if it is linear over $\mathbb{R}$ & satisfies the product rule

$$w(fg) = f(a)\, wg + g(a)\, wf.$$

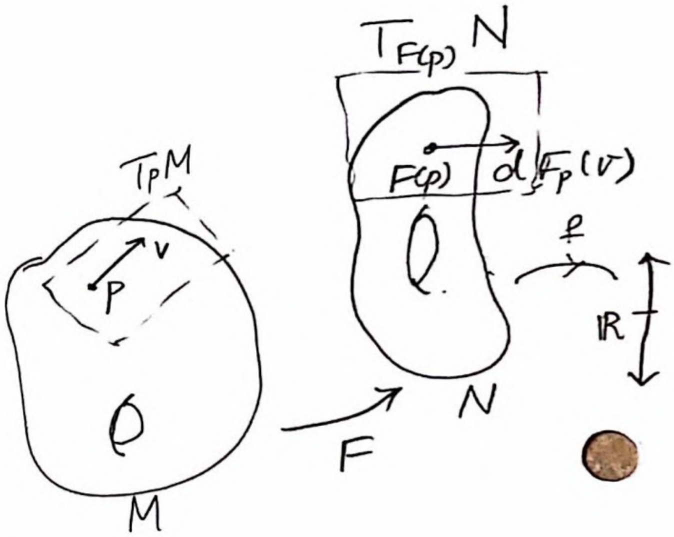
$T_a \mathbb{R}^n$ is the set of all derivations of $C^\infty(\mathbb{R}^n)$ at a.

> $T_a \mathbb{R}^n$ is isomorphic to the geometric tangent space $\mathbb{R}^n_a$.

<u>Cor</u>: The n derivations $\frac{\partial f}{\partial x^i}(a)$ form a basis for $T_a \mathbb{R}^n$.

> The above definition also apply to M.

<u>Def</u>: <u>Differential</u>.



M, N smooth manifolds,

$F: M \to N$ smooth map.

$p \in M$, define

$dF_p : T_pM \to T_{F(p)}N$,

as the "differential of F at" as follows.

Given $v \in T_pM$, let $dF_p(v)$ be the derivation at $F(p)$ that acts on $f \in C^\infty(N)$ by the rule

$$dF_p(v)(f) = v(f \circ F)$$

<u>Prop</u> (a) $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G \circ F(p)} P$.

(b) F diffeo $\Rightarrow dF_p$ iso. & $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Tangent vectors in terms of coordinates

- For smooth mfl M with smooth chart (U, ϕ), $\phi : U \to \hat{U} \subset \mathbb{R}^n$ is a diffeo & $d\phi_p$ is an iso b/w T_pM & $T_{\phi(p)}\mathbb{R}^n$.

Using this, the basis partial derivatives (i.e. basis derivations) are pulled back from $T_{\phi(p)}\mathbb{R}^n$ to define derivations at p.

i.e
$$\left.\frac{\partial}{\partial x^i}\right|_p = (d\phi_p)^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\phi(p)}\right) = d(\phi^{-1})_{\phi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\phi(p)}\right).$$

$$\Rightarrow \left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}\right|_{\phi(p)} (f \circ \phi^{-1}) = \frac{\partial \hat{f}}{\partial x^i}(\hat{p})$$

$\hat{f} \to$ coord rep. of f. $f \circ \phi^{-1}$.
$\hat{p} \to \phi(p)$.

Coordinate vectors (form a basis of T_pM) $\Rightarrow$ $v \in T_pM \Rightarrow v = \sum_i v^i \left.\frac{\partial}{\partial x^i}\right|_p$

$\Rightarrow dF_p = \left[\frac{\partial F^m}{\partial x^n}(p)\right]$ in terms of the coordinate vecs.

Tangent Bundle

- $TM = \dot{\bigcup}_{p \in M} T_pM$, $(p, v) \in TM$.

> TM is a manifold of dim $2n$.

Prop: If M can be covered by a single chart, $TM \approx M \times \mathbb{R}^n$.

Def: Global differential

$$dF : TM \to TN \quad s.t \quad dF\big|_{T_pM} = dF_p$$

Prop: F smooth $\Rightarrow$ dF smooth.

F diffeo $\Rightarrow$ dF diffeo & $(dF)^{-1} = d(F^{-1})$.

Cor: Computing differential using curves.

$F : M \to N$ smooth, $p \in M$ & $v \in T_pM$.

Let $\gamma : J \to M$ smooth curve s.t $\gamma(0) = p$ & $\gamma'(0) = v$

Then $dF_p(v) = (F \circ \gamma)'(0)$

$$= \frac{d}{dt}\bigg|_{t=0} (F \circ \gamma)(t) \quad //$$

Ch3 (Loring Tu) The Tangent Space.

Euclidean space

$\vec{p} \in \mathbb{R}^n$ & $\vec{v}$ a "tangent vector" at p. $f : \mathbb{R}^n \longrightarrow \mathbb{R}$

$f \in C^\infty$. Then the directional derivative of f at p along v is :

$$D_v f = \lim_{t \to 0} \frac{f(\vec{p} + t\vec{v}) - f(\vec{p})}{t} = \frac{d}{dt}\bigg|_{t=0} f(\vec{p} + t\vec{v})$$

$$= \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(\vec{p})$$

$\Rightarrow$ $D_v : C^\infty \longrightarrow \mathbb{R}$ & $D_v = \sum v^i \frac{\partial}{\partial x^i}\Big|_p$

It is a scalar value representing the instantaneous rate of change of f at $\vec{p}$ along $\vec{v}$. $D_v \longleftrightarrow v$ have an association & hence $D_v \in T_p$.

$$D_v(fg) = (D_v(f))g + f\, D_v(g)$$

"Derivation" at p is any linear map that satisfies

> $\mathcal{D}_p(\mathbb{R}^n)$ is the set of all derivations at p. It is a VS from linearity

$\exists$ isomorphism $\phi : T_p(\mathbb{R}^n) \longrightarrow \mathcal{D}_p(\mathbb{R}^n)$. $\Rightarrow$ Tangent vectors $\Longleftrightarrow$ derivations

$\Rightarrow$ The explicit isomorphism is $e_i \longleftrightarrow \dfrac{\partial}{\partial x^i}$

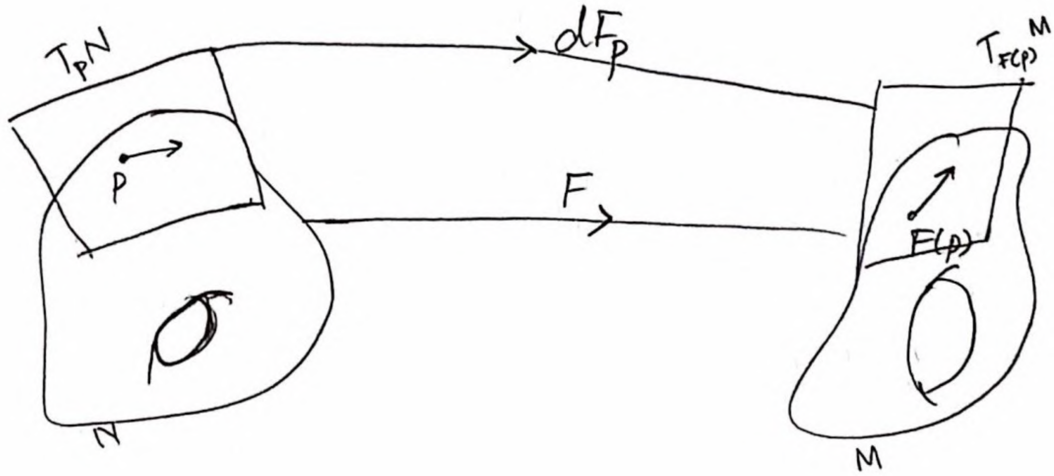
$$\vec{v} = \langle v^i \rangle = \sum_i v^i e_j = \sum v^i \frac{\partial}{\partial x^i}\Big|_p$$

Manifolds

> Tangent space to a manifold is the vector space of derivations at that point.

> A smooth map between manifolds "induces" a linear map, called its' "differential", of tangent spaces at corresponding points.

> In local coordinates, the differential is represented by the Jacobian matrix of partial derivatives of the map.



D: <u>Germ</u>: An equivalence class of C^∞ functions in a nbd of p in M. 2 fns. being equivalent if they agree on some (possibly) smaller nbd of p.

> The set of germs at p is $C_p^\infty(M)$. It forms an algebra over $\mathbb{R}$.

D: Derivation at p is a linear map $C_p^\infty(M) \to \mathbb{R}$

Such that $D(fg) = (D(f))g + f\, D(g)$

☆ [A tangent vector at p. is <u>defined</u> as a derivation at p.] ☆

> They form a <u>Tangent space</u> at p. $T_p(M)$.

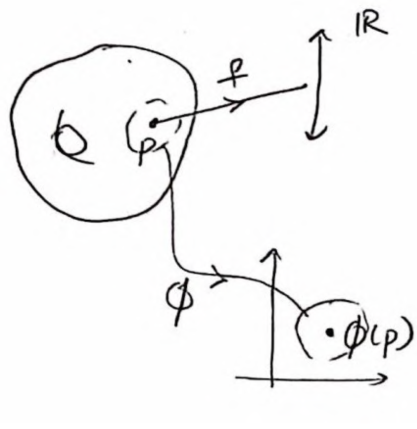
If $(U, \phi) = (U, x^1, x^2, \dots x^n)$ is a chart at $p \in M$,

Then $x^i = r^i \circ \phi : U \to \mathbb{R}$ be the coordinates of ϕ where r^i are the standard coordinates of $\mathbb{R}^n$.

D: <u>Partial derivatives on M</u>

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} (f \circ \phi^{-1})$$

$\hookrightarrow \mathbb{R}^n \to \mathbb{R}$



Differential of a map

$F: N \to M$ be a C^∞ map b/w mflds. F induces

dF_p or F_* the differential at p, as

$$dF_p : T_pN \longrightarrow T_{F(p)}M \text{ as follows}$$

If $X_p \in T_pN$ then $dF_p(X_p)$ is the tangent vector

$T_{F(p)}M$ defined by

$$\left(dF_p(X_p)\right)f = X_p(f \circ F) \in \mathbb{R} \text{ for } f \in C^\infty_{F(p)}(M).$$

↳ germ at $F(p)$.

It is independant of the germ or its function.

See Example 8.4 : Differential $\leftrightarrow$ Jacobian.

> $\phi: U \to \mathbb{R}^n \Rightarrow d\phi_p : T_pM \to T_{\phi(p)}\mathbb{R}^n$ is a VS isomorphism.

$$\boxed{d\phi_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \frac{\partial}{\partial r^i}\bigg|_{\phi(p)}}$$

T_pM has basis

$$\frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^n}\bigg|_p$$

Recall tangent vectors $\Leftrightarrow$ derivations ("directional derivatives")

Transition matrix

$(U, x^1, \dots, x^n)$ & $(V, y^1, \dots, y^n)$ are 2 charts at p.

$$\Rightarrow \frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i} \quad \text{on } U \cap V.$$

Local Expression for the Differential

$F: N \to M$ smooth, $p \in N$. $(U, x^1 \dots x^n)$ chart at p and $(V, y^1, \dots, y^m)$ chart at $F(p)$.

$$dF_p : T_pN \longrightarrow T_{F(p)}M.$$

Find charts x_j, y_i &
Find jacobian $\bar{\bar{J}} = \left[\frac{\partial y_i \circ F}{\partial x_j}\right]$
$\boxed{dF_p(\vec{v}) = \bar{\bar{J}}\vec{v}}$

$$\Rightarrow dF_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \sum_{k=1}^{m} a_j^k \left.\frac{\partial}{\partial y^k}\right|_{F(p)}, \quad j = 1 \dots m$$

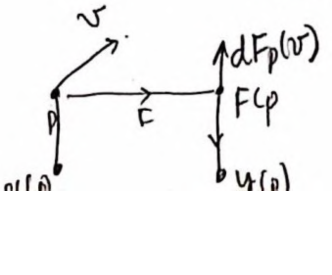
Where $a_j^i = \frac{\partial F^i}{\partial x_j}(p)$ where $F^i = y^i \circ F$.

i.e. $\left[\frac{\partial F^i}{\partial x^j}(p)\right]$ is the matrix that represents dF_p.

$$\Rightarrow dF_p(v) = \sum_{k=1}^{m} \frac{\partial (y^k \circ F)}{\partial x_j} \cdot \left.\frac{\partial v}{\partial y^k}\right|_{F(p)}$$

How much v moves when y^k changes

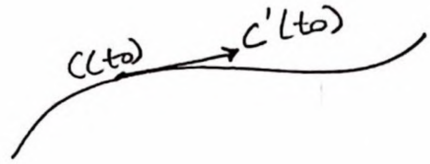
How much y^k moves when x



Curves on Manifolds

> Smooth curve $C : \overset{\circ}{\underset{\cap}{(a,b)}} \longrightarrow M$. (a smooth map)
s.T $c(0) = p$ for $p \in M$.

$$c'(t_0) = dc_{c(t_0)}\left(\frac{d}{dt}\Big|_{t_0} \right) \in T_{c(t_0)} M.$$ is the <u>velocity of c at $c(t_0)$</u>.



> $c'(t_0) \neq \frac{dc(t)}{dt}\Big|_{t=t_0}$. In fact, $c'(t) = \dot{c}(t) \frac{d}{dx}\Big|_{c(t)}$.

Because this would be in M & not $T_{c(t_0)}M$.

Basis vectors in the tangent space.

Velocity of a curve in local coordinates

$$c'(t) = \sum_{i=1}^{n} \dot{c}^i(t) \frac{\partial}{\partial x^i}\Big|_{c(t)} = \begin{bmatrix} \dot{c}^1(t) \\ \vdots \\ \dot{c}^n(t) \end{bmatrix}$$

$c^i(t)$ → components of $c(t)$ in M. <u>Note that M & T_pM have same dimensionality</u>

Prop: $X_p \in T_pM$ a tangent vector. $f \in C_p^\infty(M)$.

If $c : (-\epsilon, \epsilon) \to M$ s.T $c(0) = p$ and $c'(0) = X_p$. Then

$$X_p f = \frac{d}{dt}\Big|_{t=0} (f \circ c)$$

Intuition behind differential

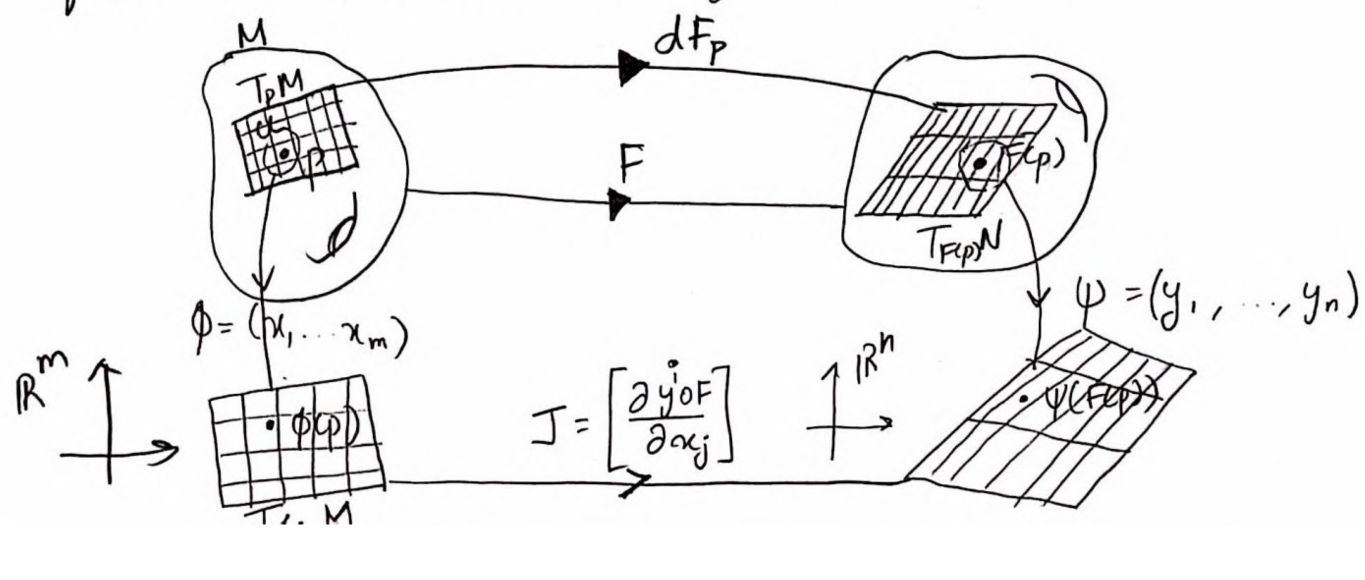
① > Given a manifold, at $\vec{p}$, the tangent space is a vector space that approximates the manifold.

② > Think of the image under a chart as a small enough nbd of the manifold itself.

① & ② => Given a general map b/w 2 manifolds. Near a pt. p, the space looks like a vector space in $\mathbb{R}^m$.

The Jacobian describes the linear transformation of this V.S to another one around $f(p)$ that approximates $f(p)$ in a nbd.

This Jacobian is the differential. It takes vectors in the tangent space to vectors in the image tangent space via a linear transformation.



$\Rightarrow$ If $\det(J) = 0$, tangent space collapses $\Leftrightarrow$ F cannot be inverted, which is the inverse function thm.

> It is much easier to work with dF_p since it is a linear map!

<u>Prop</u>: $\pi : M \to N$ smooth submersion. Then π is an open map. If π surjective it is a quotient map.

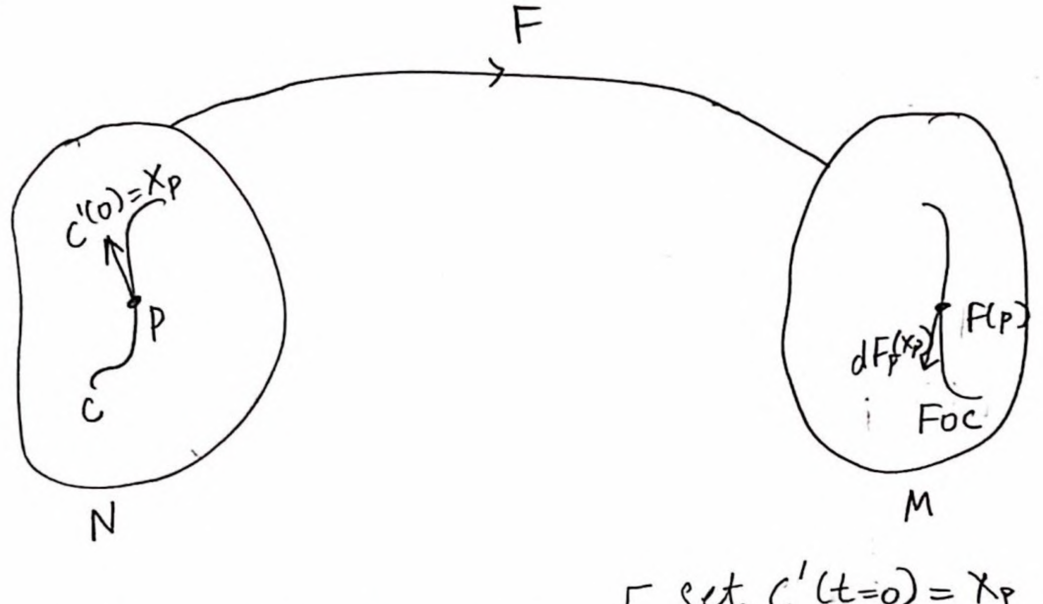
Computing differential using curves

$F: N \longrightarrow M$, $p \in N$, $X_p \in T_pN$. c is a smooth curve in N

$c(0) = p$ and velocity $X_p = c'(0)$. Then

$$dF_p(X_p) = \frac{d}{dt}\Big|_{t=0} (F \circ c)(t).$$

i.e. dF_p is the velocity vector of the image curve $F \circ c$ at $F(p)$



Example 8.19 (is a good reference)

$$\left[\text{set } c'(t=0) = X_p \text{, then } \quad \frac{d}{dt}\Big|_{t=0} F(c(t)) = F(c'(0)) \right.$$

Note,

$$c'(t) = \begin{bmatrix} \dot{c}^1(t) \\ \vdots \\ \dot{c}^n(t) \end{bmatrix} \quad \text{with basis } \frac{\partial}{\partial x^i}\Big|_{c(t)}.$$

$c(t) \in M$ & $c'(t) \in T_pM$ have same dimension.

Ch 4: Submersions, Immersions and Embeddings. (John Lee)

D: Rank of a map

$F: M \longrightarrow N$ smooth map with M, N smooth manifolds.

then rank of F at $p \in M$ is the rank of the linear map

$dF_p : T_pM \longrightarrow T_{F(p)}N$. It is the rank of the Jacobian matrix of F in any smooth chart. F is constant rank if the rank is independant of p.

D: Full rank

If the rank $F = \min\{\dim M, \dim N\}$, then F has full rank.

D: Submersion (dF is surjective)

$F: M \to N$ smooth, is called a submersion if it's differential is surjective at each pt. (i.e. rank $F = \dim N$)

D: Immersion (dF is injective)

$F: M \longrightarrow N$ is an immersion if dF_p is injective $\forall p$.

(i.e. rank $F = \dim M$)

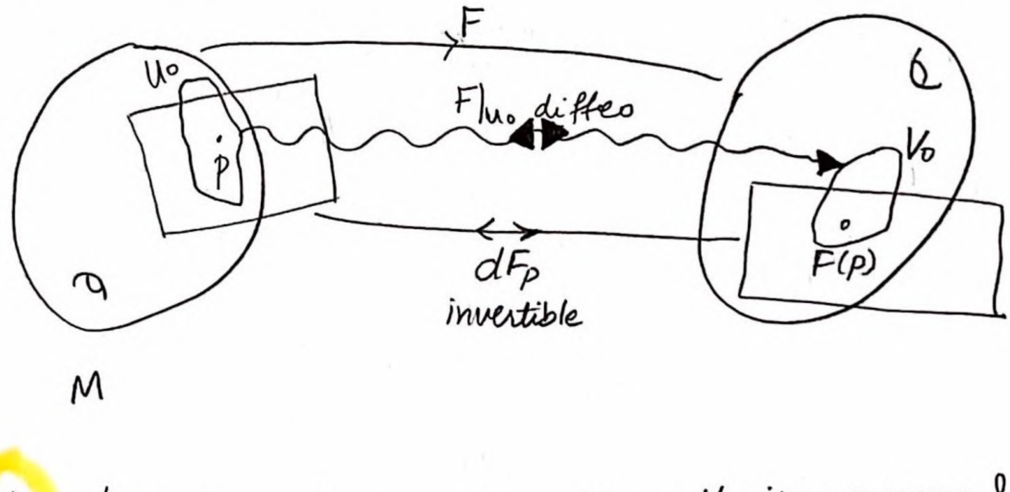
Diffeo $\Rightarrow$ Local diffeo ; Bijective local diffeo $\Rightarrow$ diffeo.

D: Local Diffeomorphism

M, N smooth. $F: M \longrightarrow N$ is a local diffeomorphism if every pt. $p \in M$ has nbd U s.t $F(U)$ is open in N and $F|_U : U \longrightarrow F(U)$ is a diffeomorphism.

Thm (Inverse function theorem for Manifolds)

M, N smooth; $F: M \longrightarrow N$ smooth map. $p \in M$ s.t dF_p is invertible, then there are connected nbds U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \longrightarrow V_0$ is a diffeomorphism.



★ Prop a) Local diffeo $\Leftrightarrow$ smooth immersion & submersion

b) $\dim M = \dim N$ and if F is either a smooth immersion or a smooth submersion $\Rightarrow$ local diffeo.

(homeomorphic immersion)

D: Embedding: $F: M \to N$ a smooth immersion,

s.t it is as a homeomorphism onto its image

$F(M) \subseteq N$ in the subspace topology.

(ie, a topological embedding that is also an immersion).

Prop: $F: M \to N$ injective smooth immersion. F is a smooth embedding if any of the following hold:

a) F is an open or closed map.

b) F is a proper map

c) M is compact

d) M has no bdry & $\dim M = \dim N$.

Ex: The prototype of an immersion is the inclusion of $\mathbb{R}^n$ in a higher dimensional $\mathbb{R}^m$

$$i(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0)$$

The prototype of a submersion is the projection of $\mathbb{R}^n$ onto a lower dim $\mathbb{R}^m$

$$\pi(x^1 \dots x^n) = (x^1 \dots x^m) \qquad m < n.$$

Thm: (Rank Theorem)

t 226

M^m, N^n smooth, $F : M \to N$ smooth with constant rank r. $\forall p \in M \ \exists$ smooth charts (U, ϕ) for M at p & (V, ψ) for N at $F(p)$ s.t $F(U) \subseteq V$ in which F has coord. representation

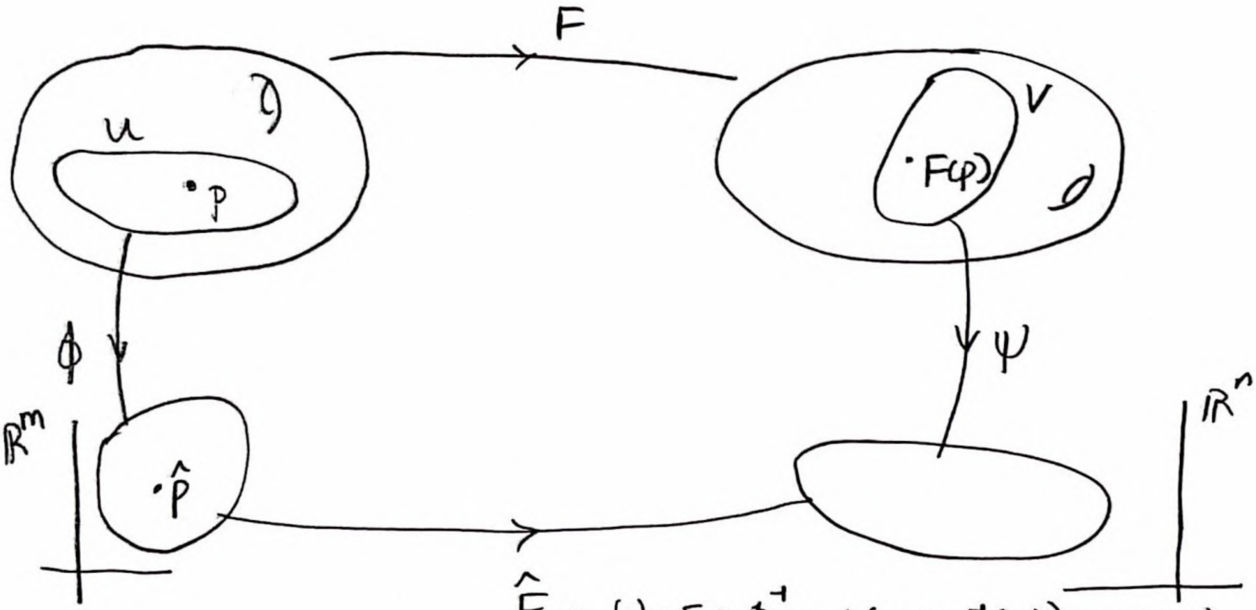
$$\hat{F}(x^1, \dots, x^r, x^{r+1}, \dots, x^m) = (x^1, \dots, x^r, 0, 0, \dots, 0)$$

If F is a smooth submersion

$$\hat{F}(x^1, \dots, x^n, x^{n+1}, \dots, x^m) = (x^1, \dots, x^n)$$

If F is a smooth immersion

$$\hat{F}(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0)$$



$\hat{F} = \psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(V)$.

Cor: F has constant rank $\Longleftrightarrow$ $\hat{F}$ is linear (i.e $\hat{F}$ behaves like dF_p) $\forall p$

$\#$

Thm: Global Rank Theorem (Follows from Cor.)

M, N smooth; $F: M \to N$ smooth with constant rank. Then,

a) F surjective $\Rightarrow$ Smooth submersion.

b) F injective $\Rightarrow$ Smooth immersion.

c) F bijective $\Rightarrow$ diffeo.

Embeddings (contd.)

> A smooth curve $\gamma: J \to M$ in a smooth manifold is an immersion iff $\gamma'(t) \neq 0$.

Thm: (Local Embedding Thm)

$F: M \to N$, F smooth immersion $\Leftrightarrow$ $p \in M$ has nbd U s.t $\underbrace{F|_U : U \to N \text{ is a smooth embedding}}_{\text{Local embedding.}}$

Def: Section (right inverse of a cont. map)

$\pi: M \to N$ continuous $\Rightarrow$ $\sigma: N \to M$ a section s.t $\pi \circ \sigma = id_N$.

Def: Local section.

$\pi: M \to N$, $U \subseteq N$ & $\sigma: U \to M$ s.t $\pi \circ \sigma = id_U$.

Prop: Smooth submersion $\Rightarrow$ open map.

$\Rightarrow$ Surjective smooth submersion $\Rightarrow$ quotient map.

Ch 5: Submanifolds

D. Submanifold (Smooth)

Smooth manifolds that are subsets of smooth manifolds

D: Embedded submanifold (aka regular submanifold)

M manifold. $S \subseteq M$ is called an embedded submanifold if it is a manifold with the subspace topology, & it is endowed with a smooth structure w.r.t which the inclusion map $S \hookrightarrow M$ is a smooth embedding.

(Subspace s.t inclusion map is a smooth embedding).

> $\dim M - \dim S =$ codimension of S
> M : ambient mfl.

D: Open submanifold: (Embedded submfl of co-dim 0)

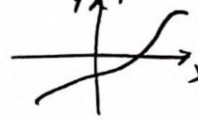
$S \subseteq M$ open subset with subspace topology with charts obtained from restricting those in M.

Prop: M, N smooth. $\forall\, p \in N$, the subset $M \times \{p\}$ (slice of $M \times N$) is an embedded submanifold of $M \times N$ diffeo to M.

Prop: (Graphs as submfls) (Graphs are embedded submfl of product space).

M, N smooth. $U \subseteq M$ open and $f : U \to N$ smooth.
dim(m) dim(n)

Let $\Gamma(f) = \{(x,y) \in M \times N : x \in U, y = f(x)\}$. $\to$ "Graph"



Then $\Gamma(f)$ is an embedded m-dimensional submanifold of $M \times N$.

D: Properly embedded

$S \subseteq M$ embedded submfl is properly embedded if the inclusion map $S \hookrightarrow M$ is proper.
{inverse image of compact sets are compact}

Prop: Properly embedded $\Leftrightarrow$ S is a closed subset of M.

Cor: Compact embedded submfl $\Rightarrow$ properly embedded.

Level Sets & Submanifolds

D: Level set: $\phi : M \to N$ & $c \in N$. $\phi^{-1}(c)$ is the level set of ϕ.

Thm: M smooth mfl. $K \subseteq M$ closed subset. $\exists$ a smooth non-negative fn $f : M \to \mathbb{R}$ ST K is the zero set of f.
(i.e the level set of $0 \Leftrightarrow f^{-1}(0) = K$).

* Thm (Constant-rank Level Set Theorem):

M, N smooth mfl. $\phi: M \to N$ smooth with constant rank r.

Each level set of ϕ is a properly embedded submfl of codimension r in M.

Cor (Submersion Level set Theorem) (Similar to RNT from 420)

M, N smooth mfl. $\phi: M \to N$ smooth submersion.

Each level set of ϕ is a properly embedded submfl of codimension $= \dim N$.

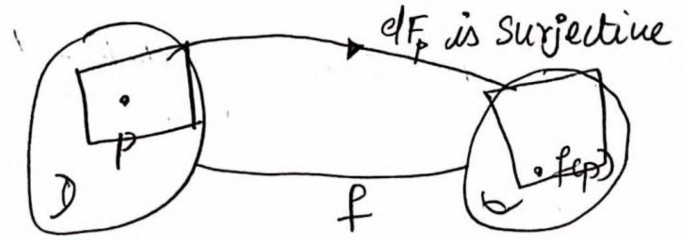
D: Regular point (& Critical point)

$\phi: M \to N$ smooth, $p \in M$ is a regular point of ϕ if $d\phi_p : T_pM \longrightarrow T_{\phi(p)}N$ is surjective. (otherwise p is a critical pt.)

$\Rightarrow$ (every pt. is regular iff ϕ is a submersion)

(every pt. is critical if $\dim M < \dim N$)

(set of regular pts. of ϕ is an open subset of M)



D: Regular Value (all pts. of level set are regular pts.)

$\phi : M \to N$ all smooth. $c \in N$ is a regular value of ϕ if it's level set $\phi^{-1}(c)$ is regular at all pts.

↳ called a regular level set → contains only regular pts

D: Critical Value (not a regular value)

Note: if $\phi^{-1}(c) = \emptyset \Rightarrow c$ is a regular value.

→ Level sets are graphs.

Thm (Regular Level Set Theorem)

Every regular level set of a smooth map b/w smooth mfl is a properly embedded submanifold whose codimension is equal to the dimension of the co-domain.

Prop: M m-dim mfl. $S \subseteq M$ subset. Then S is an embedded k-submanifold of M iff every pt. of S has a nbd U in M s.t. $U \cap S$ is a level set of a smooth submersion $\phi : U \to \mathbb{R}^{m-k}$.

Note: $T_p M = \ker [J(F, p)]$ if $M = F^{-1}(p)$ level set.

Since level sets are annihilated, the Jacobian takes the Tangent space to $\vec{0}$.

Ch 6 Sard's Theorem

D: Measure zero set

$A \subseteq \mathbb{R}^n$ has measure 0 if for any $\delta > 0$, A can be covered by a countable collection of open "rectangles", the sum of whose "volumes" is less than δ.

Prop: $A \subseteq \mathbb{R}^{n-1}$ & $f: A \to \mathbb{R}$ cont. $\Gamma(f)$ graph has measure zero in $\mathbb{R}^n$.

Prop: $A \subseteq \mathbb{R}^n$ measure zero. $F: A \to \mathbb{R}^n$ smooth $\Rightarrow$ $F(A)$ meas. 0.

Prop: $A \subseteq M$ meas 0 $\Rightarrow$ $M \setminus A$ is dense in M.

Thm (Sard's Theorem)

$F: M \to N$ (all smooth), the set of critical values of F has measure zero in N.
↳ not reg. value.

Cor: $F: M \to N$ & $\dim(M) < \dim(N)$, $F(M)$ has measure 0.

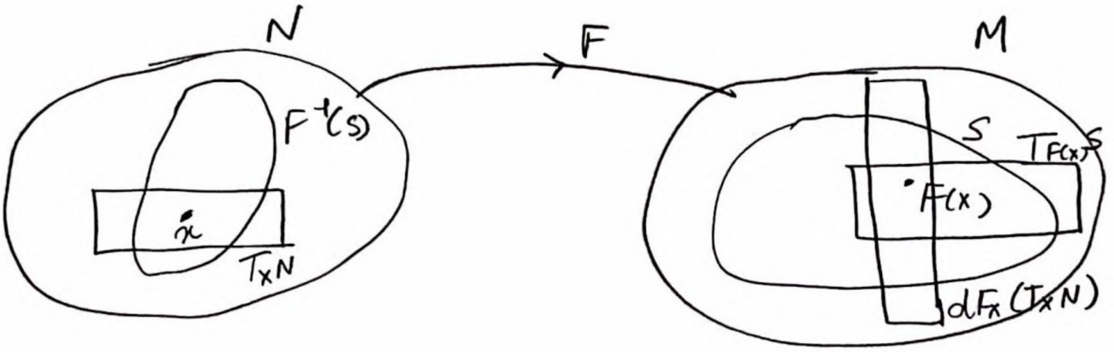
Transversality

D: "Intersect transversally"

$S, S' \subseteq M$ embedded submfls are said to intersect transversally if $\forall\ p \in S \cap S'$, T_pS & T_pS' together span T_pM.

D: Transverse map

$F : N \longrightarrow M$ smooth & $S \subseteq M$ embedded sub-mfl. F is "transverse to S" if $\forall\ x \in F^{-1}(S)$, the spaces $T_{F(x)}S$ & $dF_x(T_xN)$ together span $T_{F(x)}M$.



Thm: a) $F^{-1}(S)$ is an embedded submfl of N with codim equal to co-dim of S in M.

Def: Normal Subgroup H: $H \subseteq G$ a subgroup is normal if $H = gHg^{-1}\ \forall g \in G$.

Ch 7: Lie Groups.

D: Lie group

Smooth manifold G that is also a group S.T

$m: G \times G \to G$ & $i: G \to G$ given by

$m(g,h) = gh$ & $i(g) = g^{-1}$ are smooth.

D: Translations by $g \in G$ (Left & right)

$L_g \quad R_g$

$L_g : G \to G \mid L_g(h) = gh$ is the left translation map

L_g: is a diffeomorphism of G.

Ex: $GL(n,\mathbb{R})$, $GL(n,\mathbb{C})$, $\mathbb{R}^n$, $\mathbb{C}^n$, S^1, $\mathbb{T}^n$.

D: Lie group homomorphism

G, H Lie groups, $F: G \to H$ smooth & also a group homomorphism $\Rightarrow$ it is a Lie group homomorphism.

D: Lie group isomorphism

Lie group homomorphism that is also a diffeomorphism.

Ex: $\det: GL(n,\mathbb{R}) \to \mathbb{R}^* = \mathbb{R} - \{0\}$ is a L-G-Homo

Ex: conjugation by g: $C_g : G \to G$, $C_g(h) = ghg^{-1}$ is a L-G-Iso.

*Thm: Every LGH has constant rank.

Thm: LGH is a LGI iff it is bijective.

D: Lie Subgroup

A subgroup of a Lie group G that is also a Lie group and an immersed submanifold of G.

Prop: Embedded subgroups are Lie subgroups.

D: Subgroup generated by a subset

G group & $S \subseteq G$. Then the subgroup generated by S is the smallest subgroup containing S.

Prop: G Lie group & $W \subseteq G$ a nbd of e.

a) W generates an open subgroup of G.

b) W connected $\Rightarrow$ generates a connected open subgroup of G.

c) G connected $\Rightarrow$ W generates G.

Prop: $F: G \to H$ LGH. Ker(F) is a properly embedded Lie subgroup of G, whose co-dim = rank of F.

Thm: G Lie group, $H \subseteq G$ Lie subgroup. H is closed in G iff it is embedded

Group Actions

G group & M a set,

Left Action
$$\begin{cases} G \curvearrowright M : G \times M \to M & \text{s.T} \quad \theta_g(p) = g \cdot p \\ g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p \ \text{(associative)} & \theta : G \times M \to M \\ e \cdot p = p \quad \text{(identity)} & \end{cases}$$

Smooth action if $G \curvearrowright M : (g, p) \to g \cdot p$ is smooth & M smooth mfl.

* Smooth actions $\theta_g : M \to M$ are diffeomorphisms. *

Defs

orbit of p : $G \cdot p = \{ g \cdot p : g \in G \}$

isotropy group or stabilizer of p : $G_p = \{ g \in G : g \cdot p = p \}$

Transitive action : $\forall p, q \in M, \exists g \in G$ s.T $g \cdot p = q$
(i.e M is the only orbit)

Free action : only the identity fixes p $\forall p \in M$.
(i.e. stabilizer is trivial)

$g \cdot p = p \Rightarrow g = e.$

Ch 21: Quotients by Group Actions

› Let the orbits of G in M be the equivalence classes of Group Action, denote the quotient space obtained as M/G called the <u>orbit space</u>.

left action

Lemma: A continuous action $G \curvearrowright M$ gives a quotient map $\pi : M \to M/G$ that is open.

D: <u>Proper action</u>

$G \curvearrowright M$ continuous action is a proper action if the map $G \times M \longrightarrow M \times M$ given by $(g, p) \to (g \cdot p, p)$ is a proper map (i.e inverse image of compact sets are compact).

Cor: Every continuous action by a compact Lie group on a mfl is proper.

Thm: Smooth, free, proper group actions yield smooth manifolds as orbit spaces.

Thm: $G \curvearrowright M \Rightarrow G/G_p \underset{\text{homeo}}{\cong} M$. , $\underset{\text{subgroup}}{G_p \text{ compact}} \Rightarrow M$ is homogeneous

Lie Algebras (Loring Tu)

Lie groups are special manifolds because multiplication is smooth.

Recall, for any $g \in G$, left translation $l_g : G \to G$ by g is a diffeomorphism with inverse $l_{g^{-1}}$.

Therefore $l_g(\overset{\text{identity}}{e}) = g$ & induces an isomorphism of tangent spaces.

$$l_{g*} = (l_g)_{*,e} : T_e(G) \to T_g(G).$$

Therefore, for Lie groups it suffices to identify the tangent space at e.

Parallelizable

$\exists \{V_i\}$ s.t. $\forall p \in M$, $\{V_i(p)\}$ is a basis for T_pM.

set of v-fields. $\Leftrightarrow$ Trivial bundle.

- Every Lie group is parallelizable.

Ch 8: Vector Fields

t26f

D: Vector Field

M smooth; X a vector field is a section of the map $\pi : TM \to M$.

> Smooth vector field has smooth π.

D: Support

Set of all $p \in M$ s.t $X_p \neq 0$.

X has compact support $\Rightarrow$ support of X is compact set.

> Component functions : $X_p = X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$.

Lie bracket

$[X, Y] f = (X \circ Y) f - (Y \circ X) f$. It is also a vector field.

Tangent Bundle

D: Trivial bundle

$\pi : M \times F \longrightarrow M$ is a submersion, aka a trivial bundle.

D: Fiber bundle

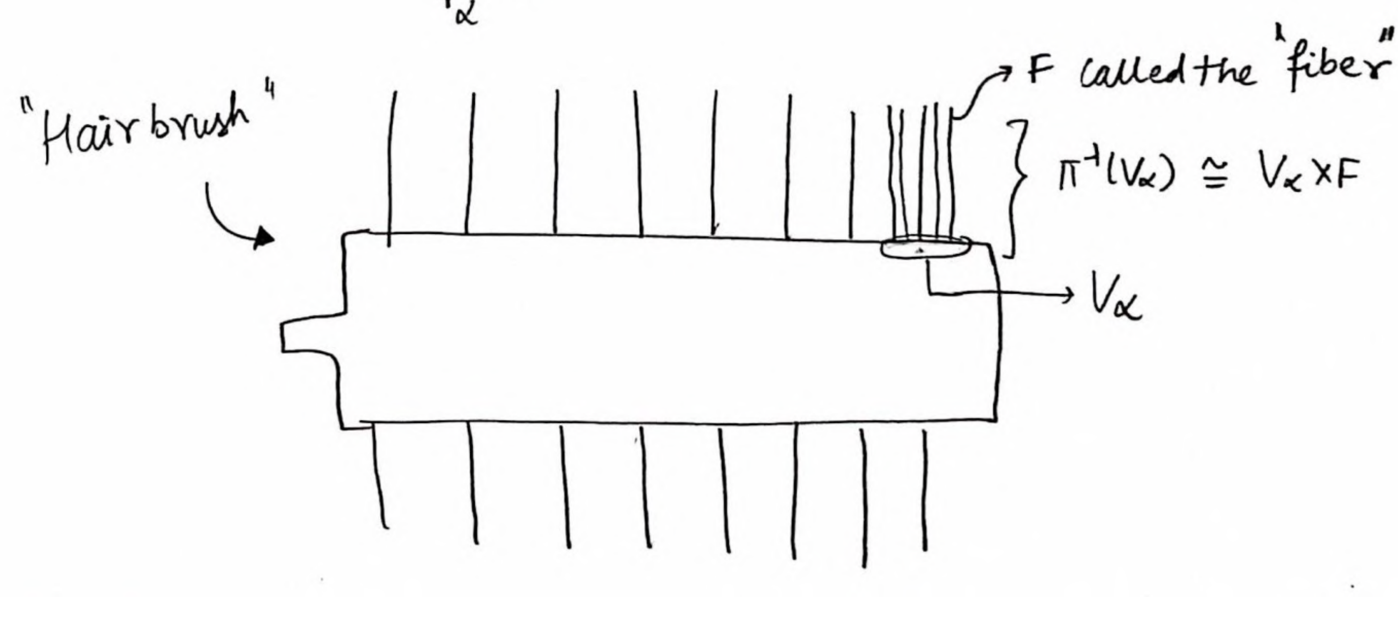
Given $\pi : M \longrightarrow N$, π is a <u>fiber bundle</u> if

i) π is surjective

ii) N is covered by open sets $\{V_\alpha\}$ S.T $\pi^{-1}(V_\alpha)$ is diffeo by ϕ_α to $V_\alpha \times F$ for some fixed manifold F. These are called <u>local</u> <u>trivializations</u>.

iii) For each V_α, the following diagram commutes.

$$\begin{tikzcd} \pi^{-1}(V_\alpha) \arrow[rr, leftrightarrow, "\phi_\alpha"] \arrow[dr, "\pi"'] & & V_\alpha \times F \arrow[dl, dashed] \\ & V_\alpha & \end{tikzcd}$$



D: Tangent bundle

$$TM = \bigsqcup_{p \in M} T_p M.$$

D: Vector bundle

$\pi : M \to N$ a fiber bundle with fiber F a vector space such that $\pi^{-1}(z_0)$ is intrinsically a vector space and for any local trivialization (U_α, ϕ_α), induces a linear map $\phi_\alpha : \pi^{-1}(z_0) \longrightarrow z_0 \times F \quad \forall z_0 \in U_\alpha$.

Equivalently if $(U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta)$ are 2 trivializations, then

$$\begin{tikzcd} \pi^{-1}(U_\alpha \times U_\beta) \arrow[r, bend left, "\phi_\alpha"] \arrow[r, bend right, "\phi_\beta"'] & (U_\alpha \cap U_\beta) \times F \end{tikzcd} \quad \text{has } \phi_\beta^{-1} \circ \phi_\alpha .$$

Tangent Bundle (Loring Tu)

D: Disjoint union.

$$\bigsqcup_{i \in I} A_i = \bigcup_{i \in I} (\{i\} \times A_i)$$

D: Tangent Bundle

$$TM = \bigsqcup_{p \in M} T_pM$$

Topology on TM

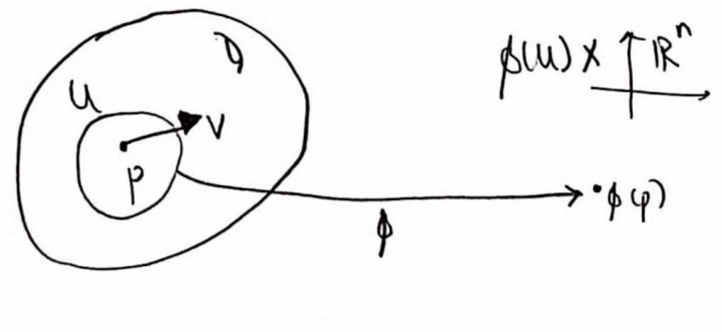
For any $v \in T_pM$, $\exists$ a natural map $\pi(v) = p$ $\quad \pi : TM \to M$

π is the projection to base point.

Also $v = \sum_{i}^{n} c^i \frac{\partial}{\partial x^i}\Big|_p$. Let (U, ϕ) be a chart at p.

Let $\bar{x}^i = x^i \circ \pi$ & define map $\tilde{\phi} : TU \to \phi(U) \times \mathbb{R}^n$ by

$$v \to (x^1(p), \ldots, x^n(p), c^1(v), \ldots, c^n(v)) = (\bar{x}^1, \ldots, \bar{x}^n, c^1, \ldots, c^n)(v)$$



"Every vector & basepoint is mapped to a product of their images under the chart."

Then $\tilde{\phi}$ has an inverse since it is a bijection. The topology on TU is inherited from $\phi(U) \times \mathbb{R}^n$, i.e. A is open in TU if $\tilde{\phi}(A)$ is open in $\phi(U) \times \mathbb{R}^n$.

subset of $\mathbb{R}^{2n}$.

$\tilde{\phi}$ can also be described as $\tilde{\phi} = (\phi \circ \pi, \phi_*)$

differential

$$\mathcal{B} = \bigcup_{\alpha} \{A \mid A \text{ open in } T(U_\alpha),\ U_\alpha \text{ is a coordinate open set in } M\}$$

Then $\mathcal{B}$ is a basis for TM.

<u>Manifold structure on the Tangent Bundle.</u>

WTS, if $\{(U_\alpha, \phi_\alpha)\}$ is a smooth atlas for M, then $\{(TU_\alpha, \tilde{\phi}_\alpha)\}$ is a smooth atlas for TM.

i.e. $\tilde{\phi}_\alpha$ and $\tilde{\phi}_\beta$ are smoothly compatible on $(TU_\alpha) \cap (TU_\beta)$.

Vector Bundles (John Lee)

D: Vector bundle of rank k over M

dim of each fiber.

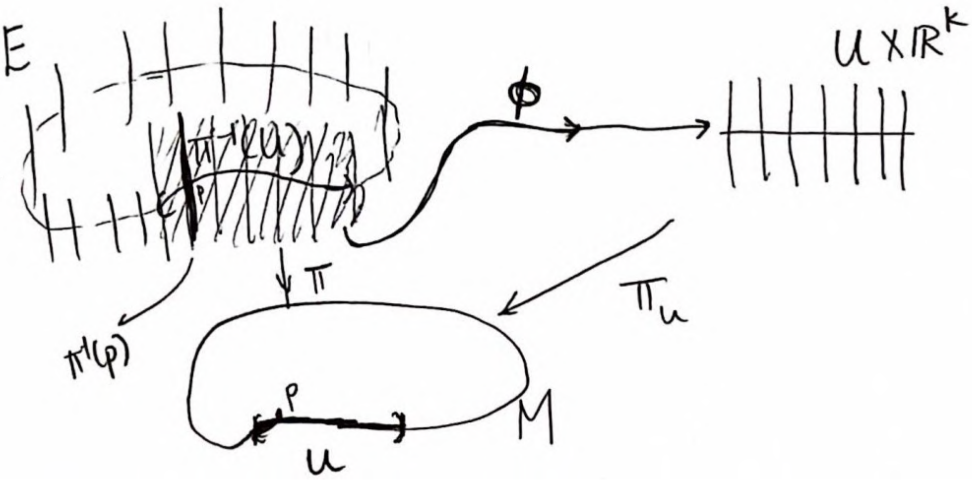
A topological space E together with a surjective continuous map $\pi: E \to M$ satisfying:

i) For each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ over p is endowed with the structure of a k-dim vector space.

ii) For each $p \in M$, $\exists$ nbd U of p in M & a homeo $\phi: \pi^{-1}(U) \to U \times \mathbb{R}^k$ (called a local trivialization of E over U) satisfying the following conditions:

- $\pi_U \circ \phi = \pi$ (where $\pi_U : U \times \mathbb{R}^k \to U$ is the proj)
- for each $q \in U$, the restriction of ϕ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$

(is a trivial bundle ← E if $\exists$ local trivialization on all of M.)



Lemma Let $\pi : E \to M$ be a smooth vector bundle of rank k over M. Suppose $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$ are two smooth local trivializations of E with $U \cap V \neq \phi$. There exists a smooth map $\tau : U \cap V \to GL(k, \mathbb{R})$ such that the composition $\phi \circ \psi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ has the form

$$\phi \circ \psi^{-1}(p, v) = (p, \tau(p) v),$$

where $\tau(p) v$ denotes the usual action of the $k \times k$ matrix $\tau(p)$ on the vector $v \in \mathbb{R}^k$.

$$\begin{tikzcd}
(U \cap V) \times \mathbb{R}^k \arrow[rr, bend left, "\phi \circ \psi^{-1}(p, v) = (p, \tau(p) v)"] \arrow[dr, "\pi_1"'] & \pi^{-1}(U \cap V) \arrow[l, "\psi"'] \arrow[r, "\phi"] \arrow[d, "\pi"] & (U \cap V) \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\
& p \in U \cap V \arrow[d, "\tau"] & \\
& GL(k, \mathbb{R}) &
\end{tikzcd}$$

$\tau \longrightarrow$ called the <u>Transition Function</u>
(it takes one fiber to another b/w trivializations)

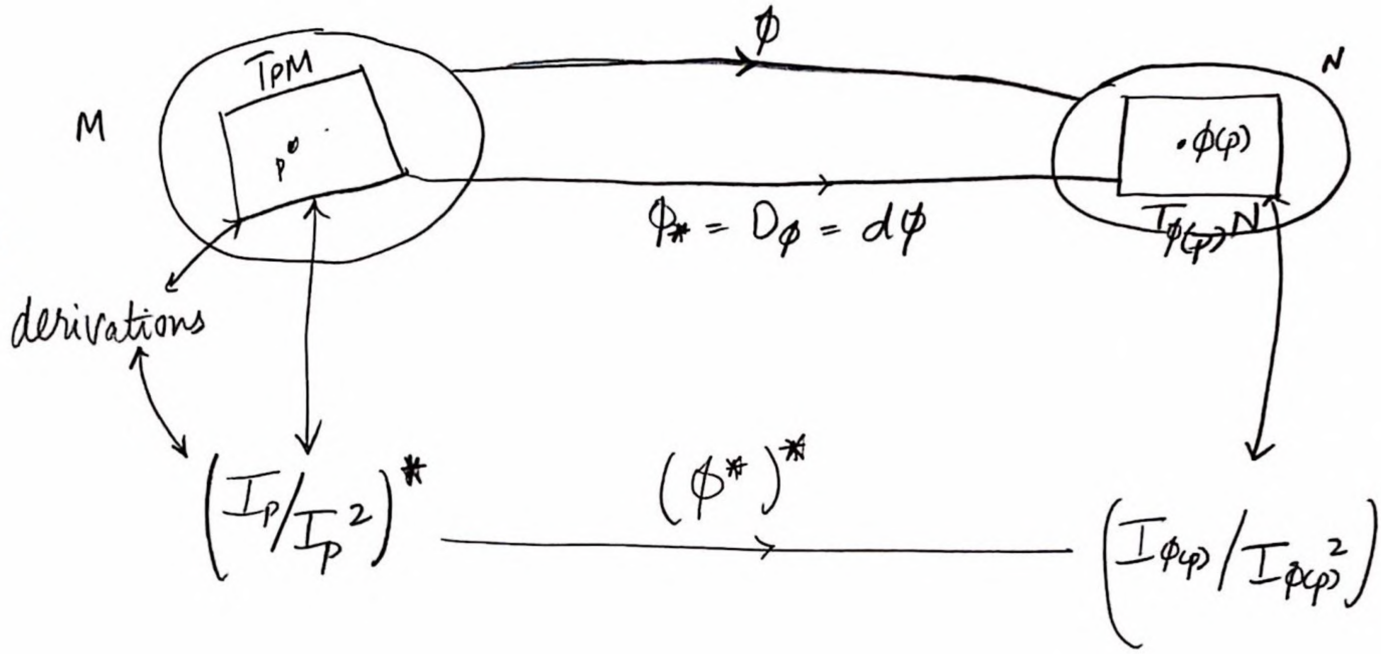
* **Lemma** (Chart Lemma) Existence of Transition Fn. $\Rightarrow$ E is a v.b over M.

Co-tangent bundle

$$I_p = \{ f \in C^\infty \text{ near } p \mid f(p) = 0 \}$$

$$I_p^2 = \{ \sum f_i g_i \mid f_i g_i \in I_p \}$$

> Derivations δ embed into $\left(I_p / I_p^2 \right)^*$.



and,

$$\left(I_p / I_p^2 \right) \xleftarrow{\ \phi^*\ } \left(I_{\phi(p)} / I_{\phi(p)}^2 \right)$$

$$\Rightarrow T_p^*M = \left(I_p / I_p^2 \right) \Rightarrow \begin{array}{|l} \phi^* : T^*N \to T^*M \\ \phi_* : T_*M \to T_*N \end{array}$$

Ex: $SL(n,\mathbb{R})$ is a mfl.

Show 1 is a reg. value of det.

Assume 1 is not a regular value.

$\det(\lambda A) = \lambda^n \det(A)$.

Claim: 1 not regular value $\Rightarrow$ λ^n also not reg. value.

1 not regular $\Rightarrow$ $D_A(\det(A))$ is not surjective.

$\Rightarrow$ $D_A(\det(\lambda A)) = D_A(\lambda^n \det(A))$

$= \lambda^n D_A(\det(A))$

is also not surj.

But λ^n does not have zero measure.

$\Rightarrow$ 1 is a regular value. (Sard's Thm).

Ch 8 : Vector Fields (Full version)

D: Vector Field :

def ① > $X : M \longrightarrow TM$ a continuous map s.T $\pi \circ X = Id_M$.
↳ proj to basept. map.

def ② > It is a section of the map $\pi : TM \longrightarrow M$.

D: Component functions of X

$X_p = X(p)$ where $p \in U$ and let $(U, (x^i))$ be a chart at p.

Then $X_p = X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$ & each $X^i : U \rightarrow \mathbb{R}$ is a component function of X in that chart.

Prop: $X|_U$ is smooth iff each of it's component fn. in U is smooth.

Prop: X, Y smooth v.f & $f, g \in C^\infty(M)$, then $fX + gY$ is a smooth v-f.

Frames

> The ordered k-tuple $(X_1, \dots, X_k)$ of v-f defined on some $A \subseteq M$ is <u>linearly independant</u> if $(X_1|_p, \dots, X_k|_p)$ is linearly independant at all $p \in A$.

> $(X_1, \dots, X_k)$ spans the tangent bundle if

$(X_1|_p, \dots, X_k|_p)$ spans T_pM $\forall p \in M$.

D: Local Frame (LI basis for TM)

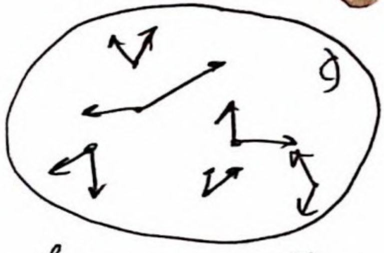
An ordered n-tuple of vector fields $(E_1, \dots, E_n)$ defined on an open subset $U \subseteq M$ that is linearly ind & spans TM

> If $U = M$, it is called a global frame.

> If each E_i is smooth it is called a smooth frame.

Eg: Coordinate frame (i.e pull back of $\frac{\partial}{\partial x^i}$)

> If the frame is orthonormal at each p, it is called an orthonormal frame. (in $\mathbb{R}^n$)



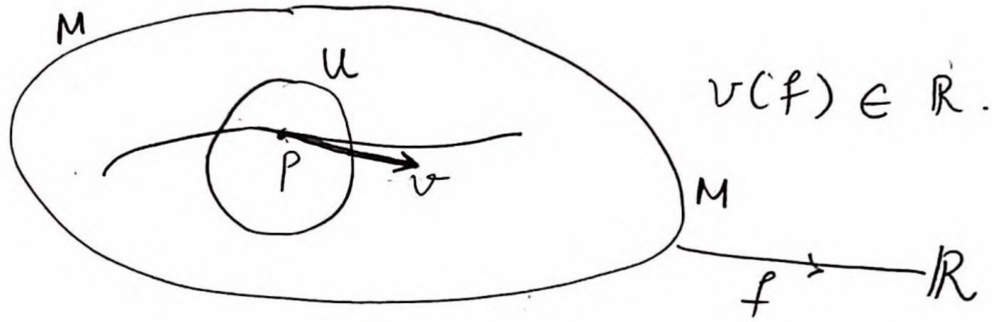
> Gram Schmidt for othornormal frame generation from given X_j in $T\mathbb{R}^n$.

D: Parallelizable

A smooth manifold is parallelizable if it admits a smooth global frame.

Intution behind "Tangent vetors act on functions"

- Tangent vetors can be naturally identified by directional derivatives along them, so they are in 1-1 correspondance with the operator $\frac{\partial}{\partial \vec{v}}$. Therefore, given a function $f \in C^\infty(M)$ the tangent vector represents how much the function is changing along a curve that defines that tangent vector.



$v(f) \in \mathbb{R}$.

$\therefore \quad Xf : U \to \mathbb{R}$

and $(Xf)(p) = X_p f$.

D: Derivation:

$X : C^\infty(M) \to C^\infty(M)$ is a derivation if it is linear over $\mathbb{R}$ and satifies $X(fg) = f\,Xg + g\,Xf$.

Prop: A map $D : C^\infty(M) \to C^\infty(M)$ is a derivation iff $Df = Xf$ for some smooth vector field X on M.

Vector Fields and smooth maps

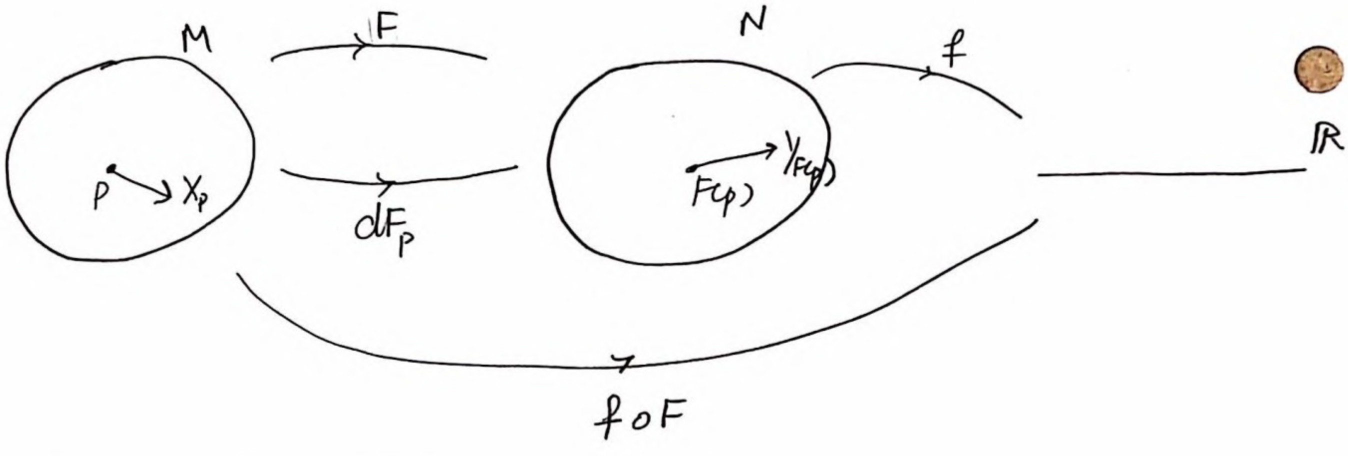
D: F-related vector fields

$F: M \to N$ smooth. X is a vector field on M and suppose $\exists$ vector field Y on N s.t. $\forall p \in M$, $dF_p(X_p) = Y_{F(p)}$.

Then X & Y are F-related.

Prop: X & Y are F-related iff $\forall f \in C^\infty(N)$,

$$X(f \circ F) = (Yf) \circ F$$



Prop: Diffeomorphism guarantees an F-relation (uniquely).

D: Push forward

$F_* X$ is the v-f Y in N that is F-related to X by diffeo F.

$$(F_* X)_q = dF_{F^{-1}(q)}\left(X_{F^{-1}(q)}\right).$$

Prop : (Restricting Vector Fields to Submanifolds)

M smooth, $S \subseteq M$ immersed submfl. $i : S \hookrightarrow M$ is the inclusion map. If $Y \in \mathfrak{X}(M)$ is, tangent to S, then
(↳ set of all v-f)

$\exists$ a unique smooth vector field on S, that is i related to Y. (call it $Y|_S$).

Lie Brackets

> f is a real valued function from $p \to f(p)$. Xf is also a real valued function from $p \to Xf(p)$ (think $\frac{d}{dx}f$). So we can compose vector fields (remember they are operators).

> XY or YX is not a derivation but $XY(f) - YX(f)$ is and is called the Lie bracket

$$[X, Y] f = XYf - YXf$$

and it is a Vector field itself

In coordinates, it simplies after cancellation to

$$[X, Y] = (XY^j - YX^j) \frac{\partial}{\partial x^j}$$

Properties of the Lie bracket

i) Bilinearity

$$[aX+bY, Z] = a[X,Z] + b[Y,Z]$$

$$[X, aY+bZ] = a[X,Y] + b[X,Z]$$

ii) Anti-symmetry

$$[X,Y] = -[Y,X]$$

iii) Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

iv) $f,g \in C^\infty(M)$

$$[fX, gY] = fg[X,Y] + (fXg)Y - (gYf)X.$$

Prop: $X_i \underset{F\text{rel}}{\longleftrightarrow} Y_i$ for $i=1,2$ then $[X_1, X_2] \underset{F\text{-rel}}{\longleftrightarrow} [Y_1, Y_2]$

Prop: $F_*[X_1, X_2] = [F_* X_1, F_* X_2]$

Prop: $S \subseteq M$ immersed submfl. If X, Y are tangent to S, $[X,Y]$ is also tangent to S.

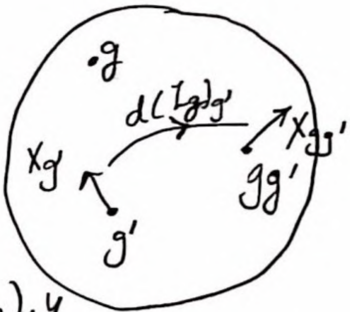
Lie Algebra

D: Left invariant vector field

G Lie group. $L_g(h) = gh$ is a smooth transitive action from $G \to G$. $X \in \mathfrak{X}(G)$ is left invariant if it is invariant under all left translations L_g; i.e. it is L_g-related to itself for all $g \in G$.

$$\Rightarrow \quad d(L_g)_{g'}(X_{g'}) = X_{gg'}$$

$$\text{or} \quad (L_g)_* X = X \quad \forall g \in G.$$



Note, $(L_g)_*(aX + bY) = a(L_g)_* X + b(L_g)_* Y$,

$\Rightarrow$ the set of all smooth left invariant vector fields is a linear subspace of $\mathfrak{X}G$.
↳ set of all smooth vector fields.

Prop: X, Y are smooth LIVF (left inv. v-f), then $[X, Y]$ is also LIVF.

(A V-S with a bracket) ($\mathfrak{X}(M)$ is a Lie algebra)

Lie algebra ($\mathfrak{g}$) (over $\mathbb{R}$)

A real vector space endowed with a map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denoted by $[X, Y]$ that is bilinear, alternating & satisfies the Jacobi identity. (In $\mathfrak{X}(M)$ each v-f is a vector)

> A linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called a Lie sub-algebra if it is closed under brackets.

> A Lie algebra homomorphism $A: \mathfrak{g} \to \mathfrak{h}$ if it preserves brackets $A[X, Y] = [AX, AY]$.

> If A is invertible it is called a Lie algebra isomorphism.

Eg: $\mathfrak{X}(M)$, smooth LIVF is a Lie subalgebra of $\mathfrak{X}(M)$ & called the Lie Algebra of G.

D: Lie Algebra of G ($Lie(G)$).

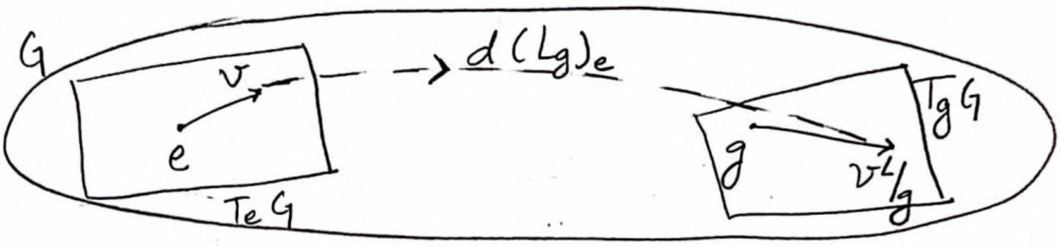
The set of smooth LIVF under the Lie bracket.

Thm: The evaluation map $\mathcal{E}: Lie(G) \to T_e G$ given by $\mathcal{E}(X) = X_e$ is a v-s isomorphism. Thus $Lie(G)$ is finite dim & $\dim(Lie(G)) = \dim(G)$.

⌈> This maps every SLIVF to a tangent vector at the identity isomorphically.⌋ (→ nontrivial).

> If $v \in T_e G$, then it's associated SLIVF is given by

$$v^L\big|_g = d(L_g)_e(v)$$



Cor: Every Lie group admits a left-invariant smooth global frame, and therefore every Lie group is parallelizable.

Ch 9 Integral Curves & Flows

D: Integral Curve

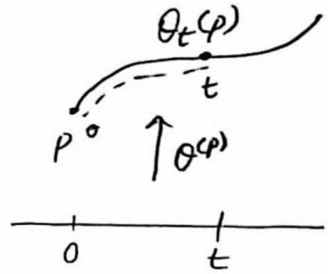
V is a v-f on M. $\gamma : J \to M$ is an integral curve on M if $\gamma'(t) = V_{\gamma(t)}$ $\forall\, t \in J$.

Prop: If X & Y are F-related ; an integral curve γ of X is taken to $F \circ \gamma$ an integral curve of Y.

Flows

M C^∞ mfl. $V \in \underbrace{\mathfrak{X}(M)}_{C^\infty \text{ v-fs}}$; for $p \in M$, V has a unique integral curve starting at p & say defined $\forall\, t \in \mathbb{R}$. Call this curve $\theta^{(p)} : \mathbb{R} \to M$. For each t, define a map $\theta_t : M \to M$ by sending p to the pt. obtained by flowing for time t along $\theta^{(p)}$.

$$\theta_t(p) = \theta^{(p)}(t).$$



Note $\theta_t \circ \theta_s(p) = \theta_{t+s}(p)$

"The manifold "slides" along the curves to generate a flow".

Also $\theta_0(p) = \theta^{(p)}(0) = p$.

$\Rightarrow$ We have a group action of $\mathbb{R}$ on M given by

$\theta : \mathbb{R} \times M \longrightarrow M$, called the "global flow on M."

Since θ is a continuous left action,

$\theta_t : M \longrightarrow M$ is a homeomorphism & if the flow is smooth, it is a diffeomorphism.

D> Complete v-f

A smooth v-f is complete if it generates a global flow.

> If $\exists$ an integral curve that is not defined $\forall \mathbb{R}$, it is not complete.

Ch 19 Distributions and Foliations

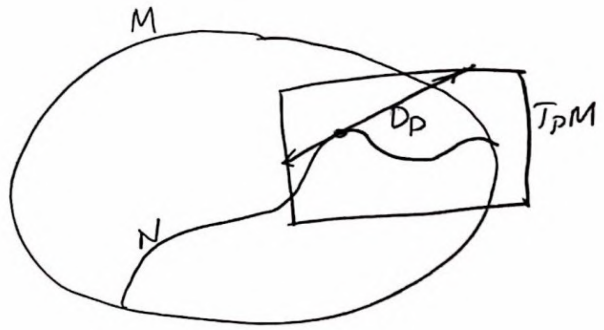
D: Distribution

> A distribution on M of rank k is a rank-k sub-bundle of TM. (Generalization of vector fields).

i.e. at each $p \in M$, $\exists$ a linear subspace $D_p \subseteq T_pM$ of dim k.

D: Integral manifold

$D \subseteq TM$ a smooth distribution. A non-empty immersed submanifold $N \subseteq M$ is called an "integral manifold of D" if $T_pN = D_p$ at each $p \in N$.



Exponential Map

D: One parameter subgroup of G : It is a LGH
$\gamma: \mathbb{R} \to G$ with $\mathbb{R}$ as a Lie group under addition.

Thm : One parameter subgroups are precisely the maximal integral curves of LIVF starting at the identity.

D: One parameter subgroup determined by X

The maximal integral curve given by the LIVF X.

> LIVF are uniquely determined by their values at 'e' the identity $\Rightarrow$ every one parameter subgroup is uniquely determined by its initial velocity in T_eG

$$\left\{ \begin{array}{c} \text{one parameter subgroups of } G \\ \{\text{ie. max int curves from } e\} \end{array} \right\} \leftrightarrow \mathfrak{g} \leftrightarrow \text{LIVF } G \leftrightarrow T_eG$$

Prop: $A \in \mathfrak{gl}(n, \mathbb{R})$, let $e^A = \sum \frac{A^k}{k!}$; then $e^A \in GL(n, \mathbb{R})$ & the one parameter subgroup of $GL(n, \mathbb{R})$ generated by A is $\gamma(t) = e^{tA}$.

Ch 12 : Tensors

D: Tensors

Real valued multi-linear functions of one or more variables.

eg: covectors (i.e dual vectors), inner products, determinants.

$L(V_1 \ldots V_k; W)$ is the space of all multilinear maps from

$V_1 \times \cdots \times V_k \longrightarrow W$ & it is a vector space.

D: Tensor product of co-vectors. (or in general)

V vs, $\omega, \eta \in V^*$. Def: $\omega \otimes \eta : V \times V \longrightarrow \mathbb{R}$ by

$$\boxed{\omega \otimes \eta\,(v_1, v_2) = \omega(v_1)\,\eta(v_2).}$$

> It is a bilinear fn. $\in L(V_1, V_2; \mathbb{R})$

if $e^1, e^2 \in V^*$ basis elts, then

$$e^1 \otimes e^2((w_1, w_2), (v_1, v_2)) = w_1 \cdot v_2$$

In general, Let $F \in L(V_1, \ldots, V_k : \mathbb{R})$ & $G \in L(W_1, \ldots, W_l; \mathbb{R})$

$F \otimes G : V_1 \times \cdots \times V_k \times W_1 \times \cdots \times W_l : \mathbb{R}$ as

$\in L(V_1 \ldots V_k W_1 \ldots W_l; \mathbb{R})$

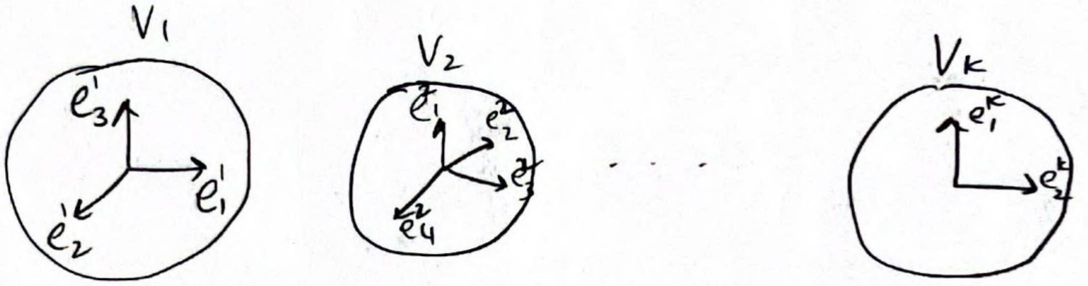
$$\boxed{F \otimes G(v_1, \ldots v_k, w_1, \ldots, w_l) = F(v_1, \ldots v_k)\,G(w_1, \ldots, w_k).}$$

Prop: A basis for a space of multilinear functions can be formed by taking all possible tensor products of basis co-vecs.

Let $V_1, \ldots, V_k$ be v-s with V_j having dim n_j.
Let $\{\epsilon_j^1, \ldots, \epsilon_j^{n_j}\}$ be the dual basis for V_j^*.

Then $\{\epsilon_1^{i_1} \otimes \cdots \otimes \epsilon_k^{i_k} : 1 \le i_1 \le n_1, \ldots, 1 \le i_k \le n_k\}$

is a basis for $L(V_1, \ldots, V_k; \mathbb{R})$.



$\epsilon_1^1 \otimes \epsilon_3^2 \otimes \cdots \otimes \epsilon_2^k$ is a basis elt.

D: Covariant tensor:

V fdvs; $k \in \mathbb{N}^+$; a covariant k-tensor on V is an element of the k-fold tensor product $V^* \otimes \cdots \otimes V^*$.

i.e $\alpha : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$. k is the rank of α.

> The set of all covariant k-tensors forms a V-S denoted by

$$T^k(V^*) = V^* \otimes \cdots \otimes V^*.$$

Ex: 1) Every co-vector is a covariant 1-tensor $\Rightarrow T^1(V^*) = V^*$.

2) Every inner product is a covariant 2-tensor.

3) det is a covariant n-tensor on $\mathbb{R}^n$.

D: Contravariant tensors on V of rank k.

Elements of the space $T^k(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}$

It can be identified with the space of multilinear functionals of k covectors (i.e dual space)

$$T^k(V) \cong \{ \alpha : V^* \times \cdots \times V^* \longrightarrow \mathbb{R} \}.$$

D: Symmetric tensors (i.e. Symmetric covariant k-tensor).

$$\alpha(v_1, \ldots, v_i, \ldots v_j, \ldots, v_k) = \alpha(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

D: Alternating tensors

$$\alpha(\ldots v_i \ldots v_j \ldots) = -\alpha(\ldots v_j \ldots v_i \ldots)$$

D: <u>Symmetric Product</u> of tensors

$$\alpha\beta\,(v_1, \dots, v_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \alpha\left(v_{\sigma(1)}, \dots, v_{\sigma(k)}\right)\beta\left(v_{\sigma(k+1)} \dots v_{\sigma(k+l)}\right)$$

<u>Note</u>: 1) $\alpha\beta = \beta\alpha$

2) $(a\alpha + b\beta)\gamma = a\alpha\gamma + b\beta\gamma = \gamma(a\alpha + b\beta)$.

3) if α, β are covectors.

$$\alpha\beta = \frac{1}{2}\left(\alpha \otimes \beta + \beta \otimes \alpha\right).$$

<u>Note</u>: Alternating covectors are also called exterior forms, multi-covectors or k-covectors.

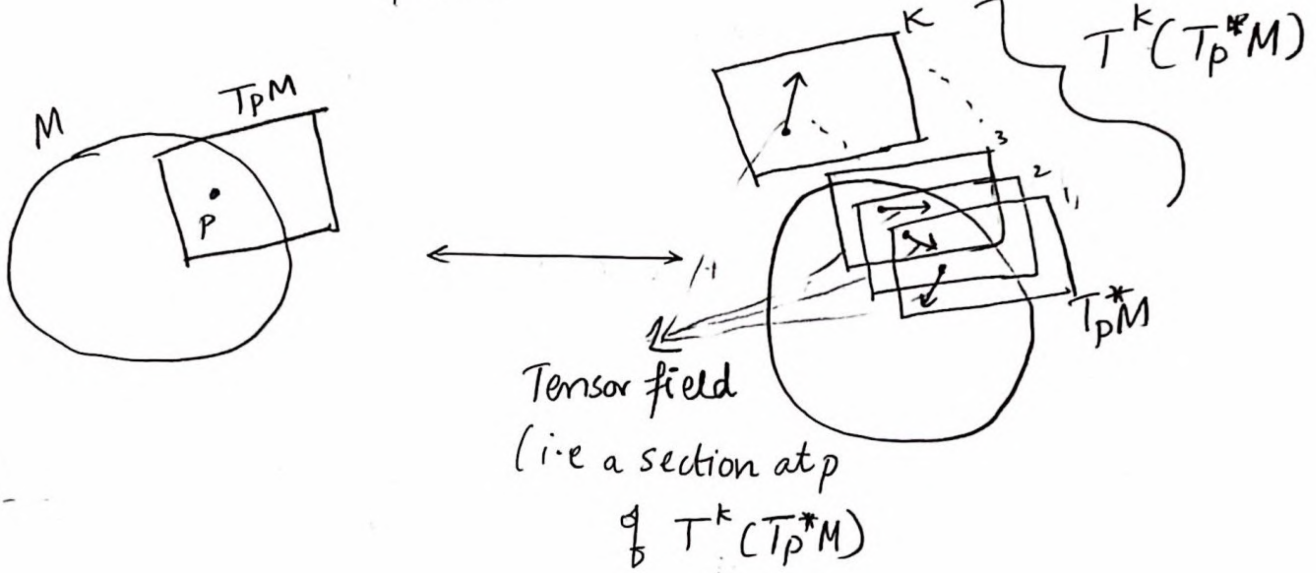
The subspace of all alternating covariant k-tensors on V is denoted by $\Lambda^k(V^*) \subseteq T^k(V^*)$.

Back to Manifolds

D: Tensor bundle on M

1) Bundle of covariant K-tensors on M.

$$T^k T^* M = \bigsqcup_{p \in M} T^k(T_p^* M).$$



> A section of a tensor bundle over M is called a tensor field on M

2) Bundle of contravariant K-tensors on M.

$$T^k TM = \bigsqcup_{p \in M} T^k(T_p M)$$

$$\Rightarrow \quad T^1 TM = TM \quad \& \quad T^1 T^* M = T^* M$$

Alternating tensor fields are called differential forms!

Partitions of Unity

"Taking locally smooth objects/functions to global ones: without them necessarily agreeing on overlaps"

D: Support of f: f is a real valued / vector valued fn., then $\text{supp}(f)$ = the closure of the set of pts. where $f \neq 0$.

$$\text{Supp} f = \overline{\{p \in M : f(p) \neq 0\}}.$$

Compact support $\Rightarrow$ $\text{supp} f$ is compact.

D: Partition of unity subordinate to $\mathcal{X}$ (Always exists on a smooth manifold)

M top sp; $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ be an arbitrary open cover of M. A partition of unity sub-ordinate to $\mathcal{X}$ is an indexed family $(\psi_\alpha)_{\alpha \in A}$ of continuous fns: $\psi_\alpha : M \to \mathbb{R}$ s.t.:

i) $0 \leq \psi_\alpha(x) \leq 1 \;\forall\; \alpha \in A$ & $x \in M$

ii) $\text{Supp}\, \psi_\alpha \subseteq X_\alpha$ for each $\alpha \in A$.

iii) $(\text{Supp}\, \psi_\alpha)_{\alpha \in A}$ is a locally finite family, i.e. every pt. intersec[ts] this family in finitely many α.

iv) $\sum_{\alpha \in A} \psi_\alpha(x) = 1 \;\forall\; x \in M$.

Ch14 Differential forms

> Recall, alternating covariant k-tensors are aka k-covectors.

& the v-s of k-covectors is given by $\Lambda^k(V^*)$.

Lemma $\alpha \in \Lambda^k(V^*)$, V fdvs. TFAE.

a) α is alternating

b) $\alpha(v_1, \ldots, v_k) = 0$ if $v_1 \ldots v_k$ are lin-dep.

c) $\alpha(v_1 \ldots w, \ldots w, \ldots v_k) = 0$.

D: Alternation (constructing alternating k-covectors).

$\mathrm{Alt}: T^k(V^*) \to \Lambda^k(V^*)$ is a projection given by

$$(\mathrm{Alt}\,\alpha)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, \alpha(v_{\sigma(1)} \cdots v_{\sigma(k)}).$$

Eg: $(\mathrm{Alt}\,\beta)(v, w) = \frac{1}{2}(\beta(v, w) - \beta(w, v))$

Note, $(\mathrm{Alt}\,\alpha) = \alpha \iff \alpha$ is alternating.

D: Multi-index of length k.

$I = (i_1, \ldots, i_k)$ where $i_j \in \mathbb{N}$

$I_\sigma = (i_{\sigma(1)}, \ldots, i_{\sigma(k)})$ where $\sigma \in S_k$ is a perm.

D: Elementary alternating tensor $(\epsilon^I) \to$ generalizing determinant.

For $I = (i_1, \ldots, i_k)$, define $\epsilon^I = \epsilon^{i_1, \ldots, i_k}$ a cov-k-tensor

$\hookrightarrow (\epsilon^1, \ldots \epsilon^n)$ is a basis of V^*

$$\epsilon^I(v_1, \ldots, v_k) = \det \begin{pmatrix} \epsilon^{i_1}(v_1) & \cdots & \epsilon^{i_1}(v_k) \\ \vdots & & \\ \epsilon^{i_k}(v_1) & \cdots & \epsilon^{i_k}(v_k) \end{pmatrix}$$

$$= \det \begin{pmatrix} v_1^{i_1} & \cdots & v_k^{i_1} \\ \vdots & & \\ v_1^{i_k} & \cdots & v_k^{i_k} \end{pmatrix}$$

$$= \det \left(\underbrace{[v_1 \; v_2 \; \cdots \; v_k]_{k \times k}}_{\text{First } k \text{ rows of } [v_1 \ldots v_k]} \right)$$

ϵ^I is an alternating k-tensor & is called the elementary alt k-tensor

Eg: Let e^1, e^2, e^3 basis of $(\mathbb{R}^3)^*$.

$$e^{13}(v,w) = \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix} = v_1 w_3 - v_3 w_1$$

Note; if I, J are 2 multiindeces,

$$\delta^I_J = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_k} \\ \vdots & & \\ \delta^{i_k}_{j_1} & \cdots & \delta^{i_k}_{j_k} \end{pmatrix}$$

$$\epsilon^I(\underbrace{E_{j_1}, \ldots, E_{j_k}}_{\text{basis of } V}) = \delta^I_J$$

Prop: (A basis for $\Lambda^k(V^*)$).

V n-dim vs & (ϵ^i) a basis for V^*. then for $k \leq n$,

$\mathcal{E} = \{ \epsilon^I : I$ is an increasing multi-index of length $k \}$

is a basis for $\Lambda^k(V^*)$.

$$\Rightarrow \dim \Lambda^k(V^*) = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

if $k > n$, $\dim \Lambda^k(V^*) = 0$

Note, $\Lambda^n(V^*)$ is 1-dim & is spanned by the determinant of $\mathbb{V} = [v_1 \ldots v_n]$

Prop: V n-dim vs, $\omega \in \Lambda^n(V^*)$. If $T: V \to V$ is any linear map, and $v_1, \ldots, v_n$ are arbitrary vectors in V, then

$$\omega(Tv_1, \ldots, Tv_n) = (\det T)\, \omega(v_1, \ldots, v_n)$$

Wedge Product (Exterior product)

V fdvs, $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, then

$$\omega \wedge \eta = \frac{(k+l)!}{k!\,l!} \mathrm{Alt}(\omega \otimes \eta)$$

Lemma: V n-dim v-s, $(\epsilon^1, \ldots \epsilon^n)$ basis of V^*.

$I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_l)$,

$$\epsilon^I \wedge \epsilon^J = \epsilon^{IJ} \text{ where } IJ = (i_1, \ldots, i_k, j_1, \ldots j_l).$$

Properties

a) Bilinearity: $(a\omega + a'\omega') \wedge \eta = a(\omega \wedge \eta) + a'(\omega' \wedge \eta)$

b) Associativity: $\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi$

c) Anticommutativity: $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$.

d) (ϵ^i) basis for V^* & $I = (i_1, \ldots, i_k)$

$$\Rightarrow \quad \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = \epsilon^I$$

*e) $$\omega^1 \wedge \ldots \wedge \omega^k (v_1, \ldots, v_k) = \det(\underbrace{\omega^j(v_i)}_{\text{matrix } a_{ij}})$$

Differential Forms on Manifolds

Recall, $T^k T^* M$ is the bundle of covariant k-tensors on M.

$\Lambda^k T^* M$ is the subset of alternating cov-k-tensors.

$$\Lambda^k T^* M = \bigsqcup_{p \in M} \Lambda^k (T_p^* M).$$

A section of $\Lambda^k T^* M$ is called a <u>differential k-form</u>.

It is a continuous tensor field whose value at each pt. is an alternating tensor. Denote the v-s of smooth k-forms by

$$\Omega^k(M) = \Gamma(\Lambda^k T^* M).$$

> Wedge product of 2 differential forms.

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$$

$\Rightarrow$ $\omega \wedge \eta$ is a $(k+l)$ form.

<u>Notation:</u> $$\sum_I{}' \alpha_I \epsilon^I = \sum_{\{I : i_1 < \cdots < i_k\}} \alpha_I \epsilon^I$$

In a smooth chart, a K-form ω can be written locally as

$$\omega = \sum_I{}' \omega_I \, dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_I{}' \omega_I \, dx^I$$

$$\Rightarrow \quad dx^{i_1} \wedge \cdots \wedge dx^{i_k}\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right) = \delta^I_J .$$

$\Rightarrow$ $\omega_I = \omega\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right)$ give the component fns. of ω

Eg: $\omega = (\sin(xy))\, dy \wedge dz$ & $\eta = dx \wedge dy + dx \wedge dz + dy \wedge dz$

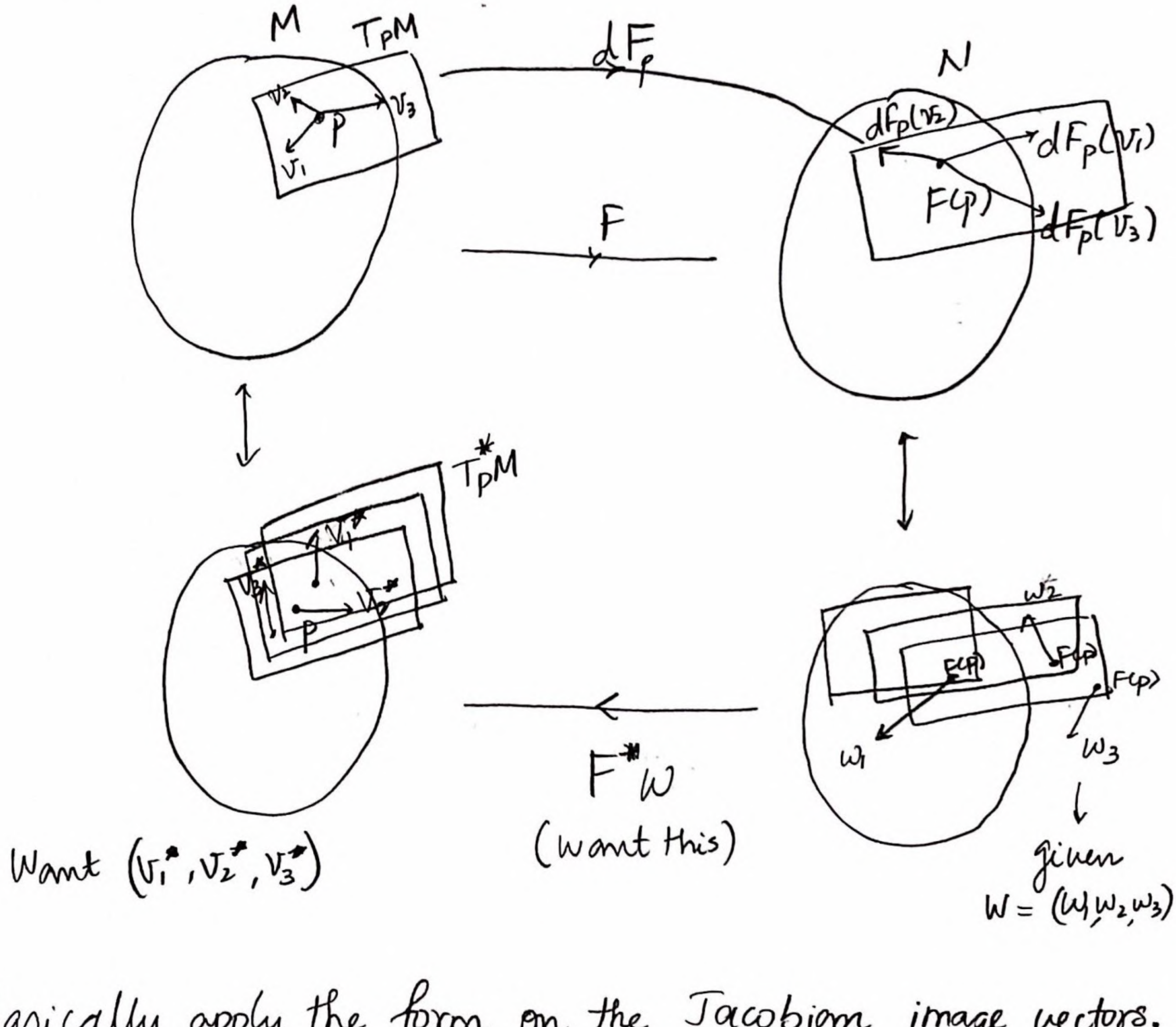
are 2 forms in $\mathbb{R}^3$.

Pullback of forms

$F : M \to N$ smooth, ω is a d-form on N.

$F^* \omega$ is a d-form of M defined by

$$(F^*\omega)_p (v_1, \ldots, v_k) = \omega_{F(p)} (dF_p(v_1), \ldots, dF_p(v_k)).$$



Basically apply the form on the Jacobian image vectors.

Lemma : $F : M \to N$ smooth.

a) $F^* : \Omega^k(N) \to \Omega^k(M)$ is linear over $\mathbb{R}$.

b) $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$

c) In a smooth chart,

$$F^*\left(\sum_I{}' \omega_I \, dy^{i_1} \wedge \dots \wedge dy^{i_k}\right) = \sum_I{}' (\omega_I \circ F)\, d(y^{i_1} \circ F) \wedge \dots \wedge d(y^{i_k} \circ F).$$

Eg: $F : \mathbb{R}^2 \to \mathbb{R}^3$ by $F(u,v) = (u, v, u^2 - v^2)$.

$\omega = y\,dx \wedge dz + x\,dy \wedge dz$ on $\mathbb{R}^3$

$$\begin{aligned} F^*(\omega) &= v\,du \wedge d(u^2 - v^2) + u\,dv \wedge d(u^2 - v^2) \\ &= v\,du \wedge (2u\,du - 2v\,dv) + u\,dv \wedge (2u\,du - 2v\,dv) \\ &= -2v^2\,du \wedge dv + 2u^2\,dv \wedge du \\ &= 2(u^2 + v^2)\,dv \wedge du \end{aligned}$$

(t63)

* Prop (Pull back from Top-degree forms)

$F : \overset{\dim n}{M} \longrightarrow \overset{\dim n}{N}$; (x^i) & (y^j) are smooth coordinates on open subsets $U \subseteq M$ & $V \subseteq N$, & u is a continuous real valued function on V, then the following holds on $U \cap F^{-1}(V)$:

$$F^*(u\, dy^1 \wedge \dots \wedge dy^n) = (u \circ F)(\det DF)\, dx^1 \wedge \dots \wedge dx^n.$$

↳ Jacobian of F.

Exterior Derivative

Motivation: given a 1-form ω; if ω is closed, i.e $\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} =$ then $\exists$ a smooth function f s.t $\omega = df$.

D: Exterior derivative in $\mathbb{R}^n$

If $\omega = \sum_J' \omega_J\, dx^J$

$$d(\omega) = \sum_J' d\omega_J \wedge dx^J$$

↳ differential of ω_J

If ω is a 1-form,

$$d(\omega_j\, dx^j) = \sum_{i,j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j = \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge d$$

Prop (Properties of d in $\mathbb{R}^n$)

a) d is linear over $\mathbb{R}$.

b) $\omega \to k$-form $\eta \to l$-form, $\Rightarrow d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

c) $d \circ d = 0$

d) $F^*(d\omega) = d(F^*\omega)$.

Def: (Exterior derivative on a manifold (Smooth))

$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ s.t

i) d is linear over $\mathbb{R}$

ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

iii) $d \circ d = 0$

iv) $f \in \Omega^0(M) = C^\infty(M)$, $df(X) = Xf$.

Def: Closed & exact forms

$\omega \in \Omega^k(M)$ is closed if $d\omega = 0$

$\omega \in \Omega^k(M)$ is exact if $\exists\, \eta \in \Omega^{k-1}(M)$ s.t $\omega = d\eta$.

Note exact $\Rightarrow$ closed.

Ch 15 : Orientation

On vector spaces

D: Consistently oriented

Two ordered bases $(E_1, \ldots E_n)$ & $(\tilde{E}_1, \ldots, \tilde{E}_n)$ are consistently oriented if the transition matrix B_i^j defined by:

$E_i = B_i^J \tilde{E}_j$ has positive determinant.

D: Orientation for V

Defined to be an equivalence class of an ordered basis.

If $(E_1, \ldots, E_n)$ is an ordered basis, the orientation it determines is given by $[E_1, \ldots, E_n]$ and the opposite orientation is given by $-[E_1, \ldots, E_n]$.

Given an orientation $[E_1, \ldots, E_n]$, a basis is positively oriented if it $\in$ [ ] & negatively oriented otherwise.

Prop: (Connection b/w orientations on alternating tensors).

V n dim vs. Each nonzero $\omega \in \Lambda^n(V^*)$ determines an orientation O_ω of V as follows:

If $n \geq 1$, O_ω is the set of ordered bases $(E_1, \dots, E_n)$ s.t $\omega(E_1, \dots, E_n) > 0$.

If $n = 0$, O_ω is $+1$ if $\omega > 0$ & -1 if $\omega < 0$.

> If V is an oriented v-s (i.e a vs with a specified orientation), ω above is a positively oriented n-covector if it determines the same orientation.

On Manifolds

Prop: M smooth n-manifold. Let ω be an n-form on M s.t $\forall p \in M$ $\omega|_p \neq 0$. Then ω determines a unique orientation of M for which ω is positively oriented.

τ & ω give the same orientation on M if $\exists$ $f: M \to (0, \infty)$ s.t $\tau = f\omega$. ω is itself called the orientation form on M

Using coordinate charts

D: Consistently oriented atlas

A smooth atlas $\{(U_\alpha, \phi_\alpha)\}$ is consistently oriented if $\forall \alpha, \beta$ the transition maps $\phi_\beta \circ \phi_\alpha^{-1}$ have positive Jac.-det. everywhere on $\phi_\alpha(U_\alpha \cap U_\beta)$.

Prop: A consistently oriented atlas determines an orientation by each chart having a positive orientation.

Prop: M connected, orientable, smooth. Then M has exactly 2 orientations. If 2 orientations agree at p, they are equal.

D: Orientation preserving map

M, N oriented mfls smooth. $F: M \to N$ a local diffeo.

if $\forall p \in M$, dF_p takes an oriented basis of T_pM to an oriented basis of $T_{F(p)}N$ it is called orientation preserving.

P: (Pull back orientation $F^*\theta$)

F local diffeo: $M \to N$, ω is an orientation 2-form of $N \Rightarrow F^*(\omega)$ is an orientation form on M.

Prop: Every parallelizable smooth manifold is orientable.

Interior multiplication (v into ω)

V fdvs, $v \in V$, $i_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$

is the interior multiplication by v given by

$$v \lrcorner \omega = i_v\omega(w_1, \ldots, w_{k-1}) = \omega(v, w_1, \ldots, w_{k-1})$$

Prop: (Orientation of Hypersurfaces)

M oriented smooth n-manifold, S is an immersed hypersurface in M. Let N be a vector field on S that is nowhere tangent to S. Then S has a unique orientation s.t $\forall\, p \in S$; $(E_1, \ldots, E_{n-1})$ is an oriented basis for T_pS iff $(N_p, E_1, \ldots, E_{n-1})$ is an oriented basis for T_pM.

If ω is an oriented form for M, then $i_S^*(N \lrcorner \omega)$ is an
↳ pull back of inclusion
orientation form for S, w.r.t this orientation.

Prop: M oriented smooth mfl. $S \subseteq M$ is a regular level set of $f : M \to \mathbb{R}$. Then S is orientable.

Ch 13 - Riemannian Metrics

D: Riemannian metric

> A smooth symmetric covariant 2-tensor field on M that is pos def at each pt.

D: Riemannian manifold (M, g).

> A manifold with a Riemannian metric. g

$\forall p \in M$, g_p is an inner product on T_pM.

$$g_p(v, w) = \langle v, w \rangle_g \quad \text{where } v, w \in T_pM.$$

In a smooth local coordinate (x^i),

$$g = g_{ij}\, dx^i \otimes dx^j$$

$\hookrightarrow$ symmetric pos def matrix of smooth fns.

$$\Rightarrow \quad g = g_{ij}\, dx^i\, dx^j$$

Eg: Euclidean metric $(\bar{g})$: $g_{ij} = \delta_{ij}$

$$\Rightarrow \bar{g}_p(\vec{v}, \vec{w}) = g_{ij} v^i w^j = \delta_{ij} v^i w^j = \sum v^i w^i = \vec{v} \cdot \vec{w}$$

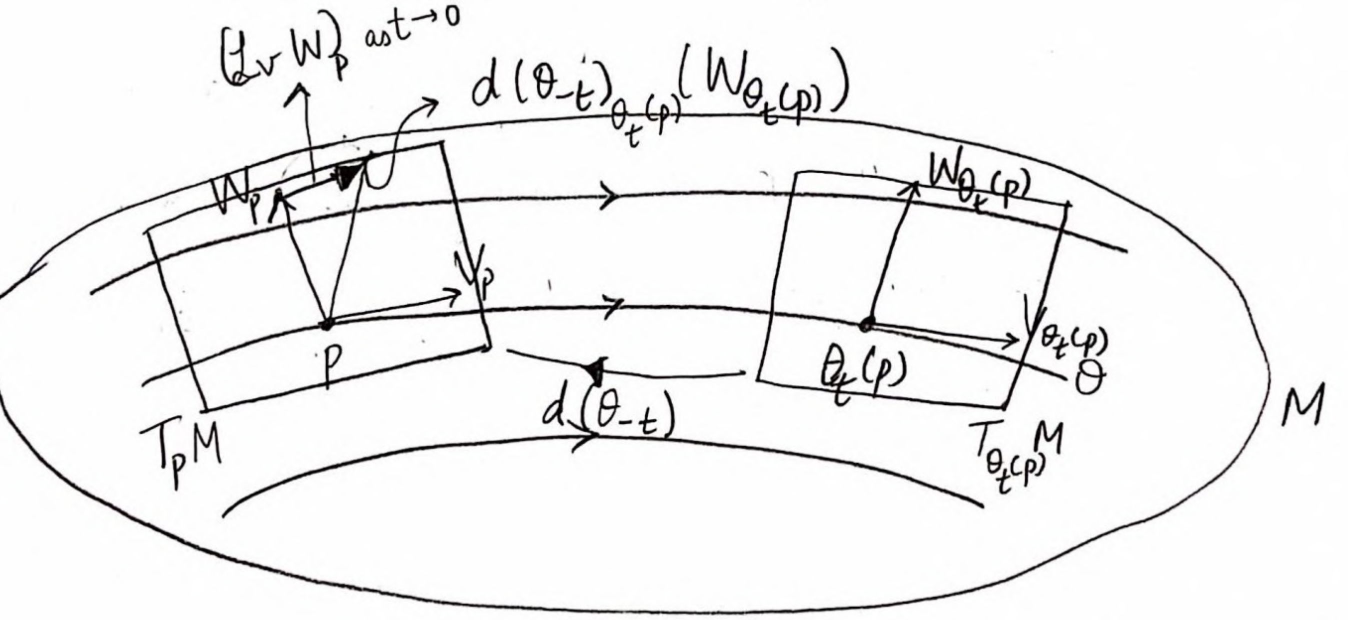
Lie Derivatives

D: Lie derivative of W w.r.t V

Suppose M is a smooth manifold. Let V & W be 2 smooth vector fields on M. Let θ be the flow of V. Then, the Lie derivative of W w.r.t V is:

$$(\mathcal{L}_V W)_p = \left.\frac{d}{dt}\right|_{t=0} d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right)$$

$$= \lim_{t\to 0} \frac{d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right) - W_p}{t}$$



"$\mathcal{L}_V W$ is the directional derivative of W along the flow of V".

Theorem: $\mathcal{L}_V(W) = [V,W]$ (which gives a geometrical interpretation to $[V,W]$).

Cor: a) $\mathcal{L}_V W = -\mathcal{L}_W V$

b) $\mathcal{L}_V [W,X] = [\mathcal{L}_V W, X] + [W, \mathcal{L}_V X]$

c) $\mathcal{L}_{[V,W]} X = \mathcal{L}_V \mathcal{L}_W X - \mathcal{L}_W \mathcal{L}_V X$.

d) $g \in C^\infty(M) \Rightarrow \mathcal{L}_V(gW) = (Vg)W + g\mathcal{L}_V W$.

e) If $F: M \to N$ is a diffeo, $F_*(\mathcal{L}_V X) = \mathcal{L}_{F_* V} F_* X$.

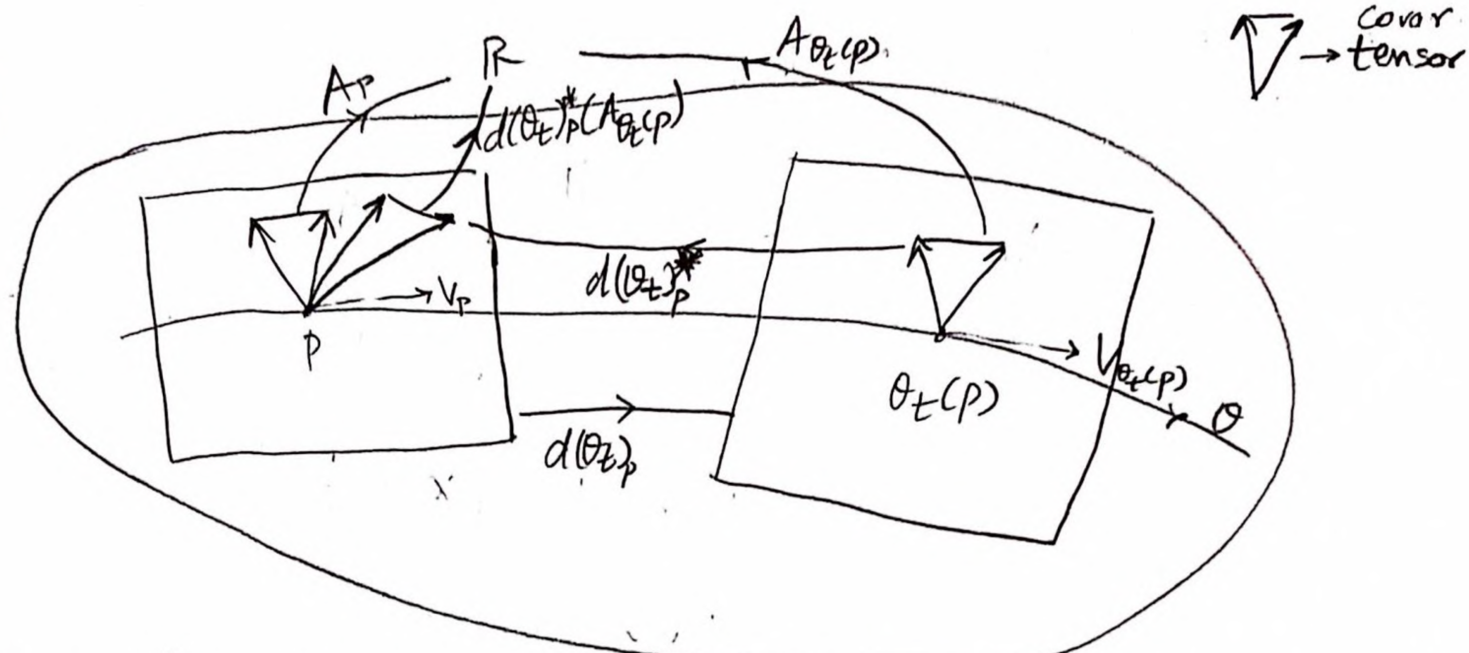
Lie derivatives of Tensor Fields

D: Lie derivative of a Tensor field

M smooth mfl, A smooth covariant tensor field on M, V smooth v-field on M with flow θ.

$$(\mathcal{L}_V A)_p = \left.\frac{d}{dt}\right|_{t=0} (\theta_t^* A)_p = \lim_{t\to 0} \frac{d(\theta_t)_p^*(A_{\theta_t(p)}) - A_p}{t}$$

Recall, $\theta_t(p)$ is a diffeo from nbd of p to nbd of $\theta_t(p)$ & hence $d(\theta_t)_p^*$ pulls back tensors from $\theta_t(p)$ to p.



<u>Prop</u>: $f \in C^\infty(M)$ i.e a 0-tensorfield on M. A, B are smooth covariant tensor fields.

a) $\mathcal{L}_V f = Vf$

b) $\mathcal{L}_V (fA) = (\mathcal{L}_V f)A + f\,\mathcal{L}_V A$

c) $\mathcal{L}_V (A \otimes B) = (\mathcal{L}_V A) \otimes B + A \otimes \mathcal{L}_V B$

<u>Cor</u>: Let $X_1 \dots X_k$ be smooth vector fields & A is a k-tensor. Then,

$$(\mathcal{L}_V A)(X_1, \dots, X_k) = V(A(X_1, \dots, X_k)) - A([V, X_1], X_2, \dots, X_k) - \dots - A(X_1, \dots, X_{k-1}, [V, X_k]).$$

<u>Cor</u>: $\mathcal{L}_V (df) = d(\mathcal{L}_V f)$.

Lie derivatives of Differential Forms

> Extends naturally from Tensors.

<u>Prop</u>: M smooth manifold, $V \in \mathfrak{X}(M)$, $\omega, \eta \in \Omega^*(M)$

$$\mathcal{L}_V(\omega \wedge \eta) = (\mathcal{L}_V \omega) \wedge \eta + \omega \wedge (\mathcal{L}_V \eta).$$

<u>Prop</u>: $\mathcal{L}_V(d\omega) = d(\mathcal{L}_V \omega)$.

Useful equations

If $f \in C^\infty(M) = 0$-form & X is a v-f that generates flow ϕ_t.

$(Xf)_x$ = dir. der of f at x.

pull back

$$Xf = \frac{d}{dt}\Big|_{t=0} (\phi_t^* f) = \frac{d}{dt}\Big|_{t=0} (f \circ \phi_t)$$

Ch17 de Rham Cohomology

D: de Rham Cohomology group in degree p. (p^th deRham group)

Recall, $d : \Omega^p(M) \to \Omega^{p+1}(M)$.

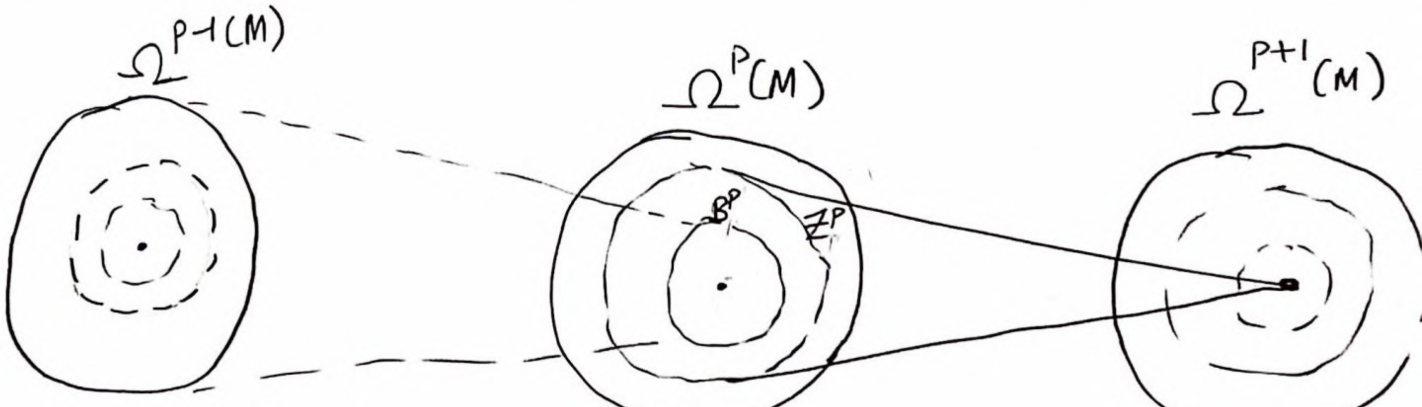
Let

$$\mathcal{Z}^p(M) = \operatorname{Ker}\left(d : \Omega^p(M) \to \Omega^{p+1}(M)\right) = \{\text{closed } p\text{-forms on } M\}$$

$$\mathcal{B}^p(M) = \operatorname{Im}\left(d : \Omega^{p-1}(M) \to \Omega^p(M)\right) = \{\text{exact } p\text{-forms on } M\}.$$

$$H^p_{dR}(M) = \mathcal{Z}^p(M) / \mathcal{B}^p(M) \qquad \text{(i.e. quotient)}$$

which is a real vector space & hence a group under +.



We know exact ⇒ closed. Want to know closed ⇒ exact?

If so $H^p_{dR}(M) = 0$.

· Poincare Lemma ⇒ every star shaped open $U \subseteq \mathbb{R}^n$ has $H^1_{dR}(U) = 0$.