Digital Signal Processing. - Richi Radke.

Lec 1 : Signals.

> Review of signals, systems, continuous time signals, discrete time signals.

> Basic computations.

① Flipping $x[n] \longrightarrow x[-n]$ → mirrored by y axis

② Scaling $x[2n] \Rightarrow$ throw away odd values of n ⇒ samples are lost.

③ Shifting $x[n-n_0] \longrightarrow$ move right by n_0.

Order: Shift → Flip → Scale.

$x[n] \longrightarrow x[-2n+3] \Rightarrow$ $z[n] = x[n+3]$; $w[n] = z[-n]$; $y[n] = w[2n]$

Substituting back gives,

$\Rightarrow y[n] = w[2n] = z[-2n] = x[-2n+3]$.

> Properties of signals: Even, Odd, periodicity,

> Given any signal $x[n]$

$$\left.\begin{aligned} \text{Even}(x[n]) &= \frac{1}{2}\left(x[n] + x[-n]\right) \\ \text{Odd}(x[n]) &= \frac{1}{2}\left(x[n] - x[-n]\right) \end{aligned}\right\}$$ Decomposing a signal to even + odd

> Special signals: $\delta[n]$, $u[n]$,

> $u[n] = \delta[n] + \delta[n-1] + \ldots = \sum_{k=0}^{\infty} \delta[n-k]$

$\left\{ \text{Similar to } u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau \right\}$

> $\delta[n] = u[n] - u[n-1]$

$\left\{ \text{Similar to } \delta(t) = \frac{d}{dt} u(t) \right\}$

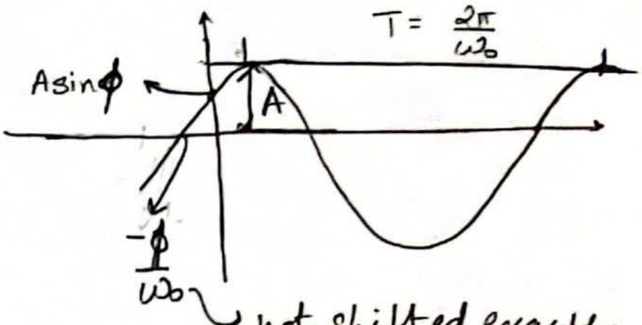
→ Making a signal out of deltas. aka. Sifting/sampling.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

> Reviews complex numbers... Euler $e^{j\theta} = \cos\theta + j\sin\theta$

> Reviews sinusoids.

$$x(t) = A\sin(\omega_0 t + \phi)$$



not shifted exactly by ϕ.

> What about signal Ce^{at} when c & a are complex?

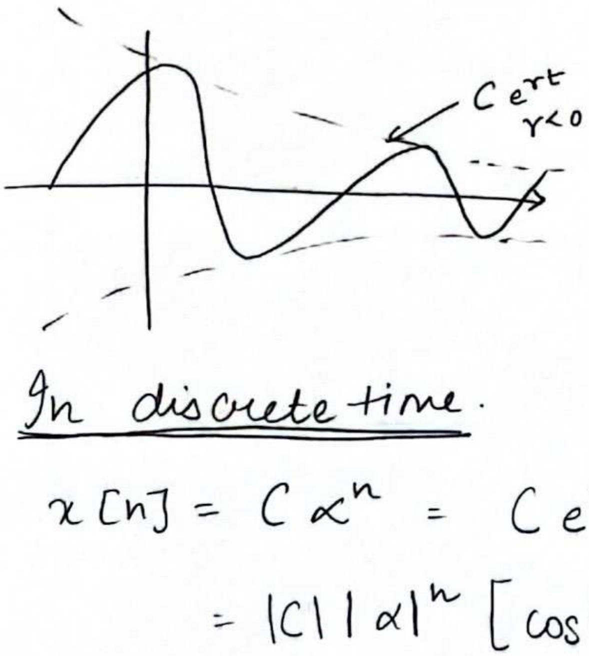
$C = 2+3j \quad a = -1-2j \Rightarrow 2+3j\; e^{(-1-2j)t}$

Convert to polar $\Rightarrow$ $x(t) = |C|\,e^{j\theta}\,e^{(r+j\omega_0)t}$

$$= |C|\,e^{rt}\cdot e^{j(\omega t+\theta)}$$

$$= |C|\,e^{rt}\left[\cos(\omega_0 t+\theta) + j\sin(\omega_0 t+\theta)\right]$$

↳ envelope of exponential since amplitude is exponential.



In discrete time.

$$x[n] = C\alpha^n = Ce^{Bn} \quad \text{where } \alpha = e^B$$

$$= |C|\,|\alpha|^n\left[\cos(\omega_0 n + \phi) + j\sin(\omega_0 n + \phi)\right]$$

> Discrete time sinusoids: Frequency can NOT go faster & faster.

- Discrete signals cannot keep growing in frequency.

→ $e^{j\omega_0 n}$ ⇒ what is $e^{j(\omega_0+2\pi)n}$.

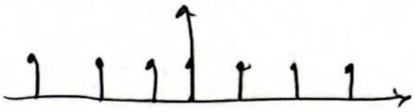
$$= \underset{1}{\cancel{e^{j2\pi n}}} \cdot e^{j\omega_0 n}$$

since n is an integer.

- There is only a 2π wide range of frequencies in discrete time!

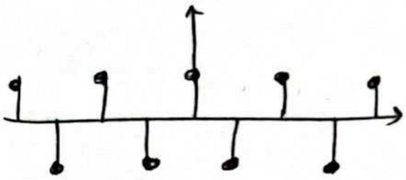
$\omega = 0$

$e^{0n} = 1$

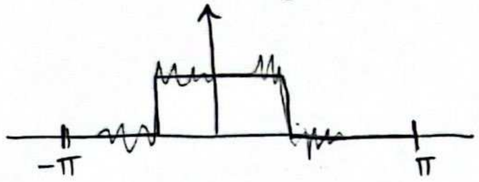


$\omega = \pi$

$e^{\pi n} = \begin{cases} 1 \\ -1 \end{cases}$



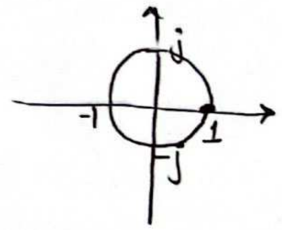
- Discrete time / Digital low pass filter.



- When is $e^{j\omega_0 n}$ periodic (discrete time)?

$e^{j\omega_0(n+N)} = e^{j\omega_0 n}$ for some integer N.

⇒ $e^{j\omega_0 N} = 1$, ⇒ $\omega_0 N = 2\pi K$ for some integer K.



⇒ $\omega_0 = \frac{2\pi K}{N}$

⇒ $\boxed{\text{Period } N = \frac{2\pi K}{\omega_0}}$

$\cos \frac{4\pi}{5} n$ can be periodic if K=2

⇒ N=5 ⇒ period is 5.

> Is $x[n] = \cos(7n)$ periodic? **No!**

$x(t) = \cos(7t)$ is periodic with $T_0 = \frac{2\pi}{7}$

Lec2 Systems.

> Difference equations, instead of differential equations.

$$y(t) + a y'(t) = b x(t)$$

$$y[n] + a y[n-1] = b x[n]$$

> Review of Serial, Parallel, Feedback systems.

> Review of system properties: Causality, Linearity, Time Varian
and Superposition.

> $x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow$ Sifting property.

> Impulse response review

$$H(x[n]) = H\Big(\sum_{k=-\infty}^{\infty} \overset{\text{constants!}}{x[k]}\,\delta[n-k]\Big)$$

By linearity, $\sum_{k=-\infty}^{\infty} x[k]\,H(\delta[n-k])$

By Time invariance, $= \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$.

$$\Rightarrow \boxed{H(x[n]) = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]}$$ Convolution.

$$y[n] = x[n] * h[n]$$

Lec 3 : Convolution

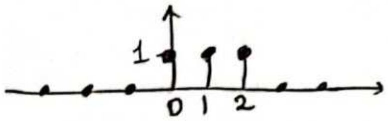
$$y[n] = x[n] * h[n]$$

$$= \sum_{K=-\infty}^{\infty} x[K]\, h[n-K]$$

Consider,

$$y[n] = x[n] - 2x[n-1] + 3x[n-2]$$

Q) What is the response to $x[n] =$



A) Could solve for each point $x = [\underline{1}\ \ 1\ \ 1]$ ↳origin.

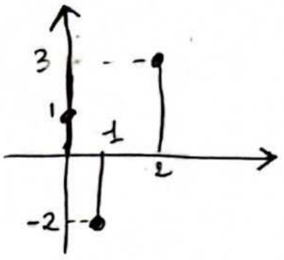
$y = [\underline{1}\ \ -1\ \ 2\ \ 1\ \ 3]$

2) What is impulse response?

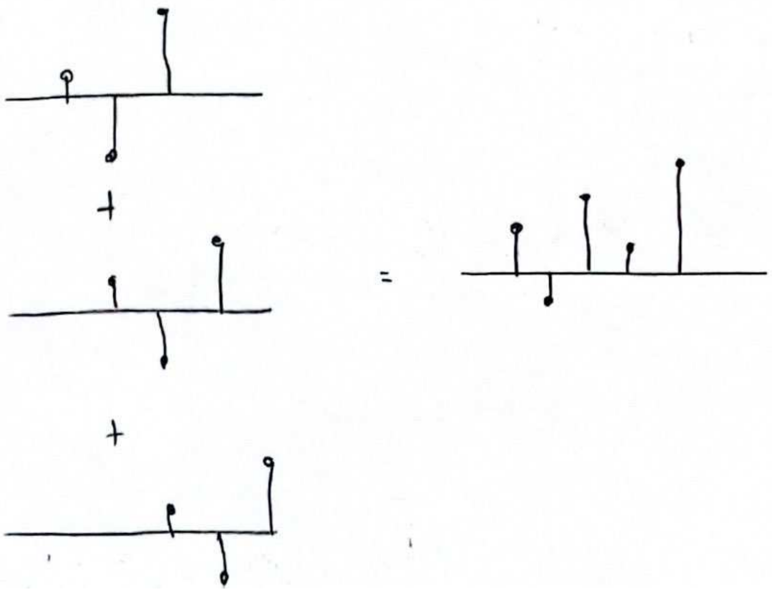
$\delta[n] = [\ \underline{1}\ ]$

$$\left.\begin{aligned} h[0] &= 1 \\ h[1] &= -2 \\ h[2] &= 3 \end{aligned}\right\} \text{Coefficients!}$$

$h[3] = 0$

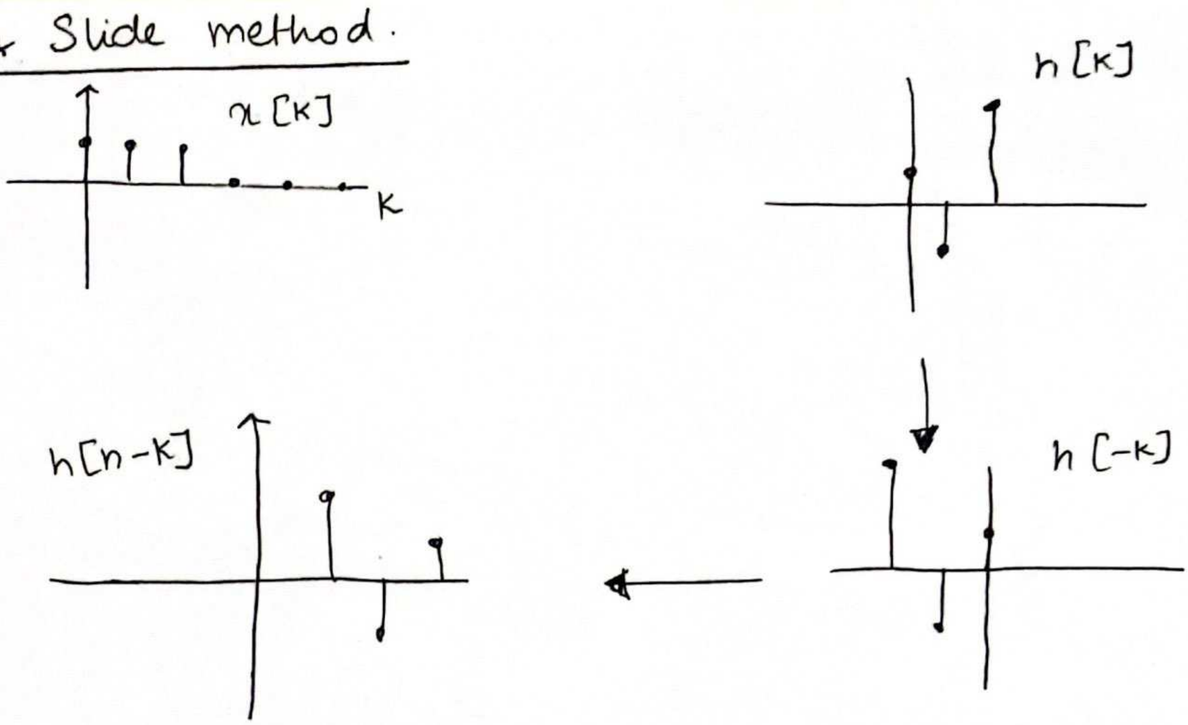


$$y[n] = x[0]\, h[n] + x[1]\, h[n-1] + x[2]\, h[n-2] + \dots$$



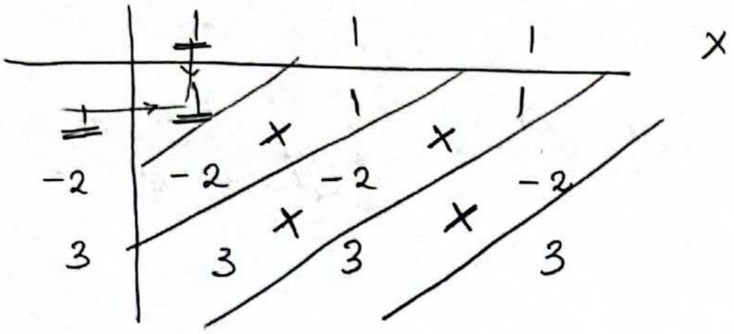
$$\begin{aligned} & [\underline{1}\ \ -2\ \ 3] \\ + & [\underline{0}\ \ 1\ \ -2\ \ 3] \\ + & [\underline{0}\ \ 0\ \ 1\ \ -2\ \ 3] \\ = & [\underline{1}\ \ -1\ \ 2\ \ 1\ \ 3] \end{aligned}$$

3) Flip & Slide method.



→ Slide, multiply & add.

4) Convolution array method.



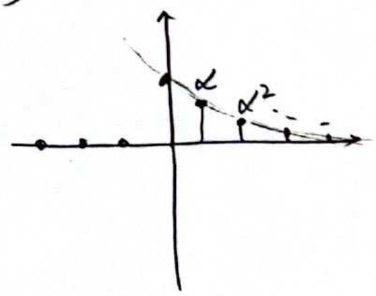
	$\underline{1}$	1	1
$\underline{1}$	$\underline{1}$	1	1
-2	-2	-2	-2
3	3	3	3

$[\underline{1}\ \ -1\ \ 2\ \ 1\ \ 3]$

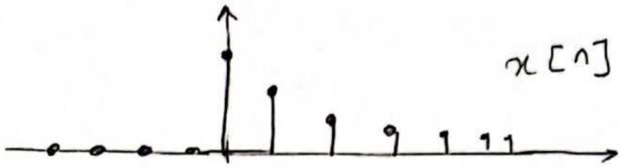
› Cannot use this technique for infinitely long signals.

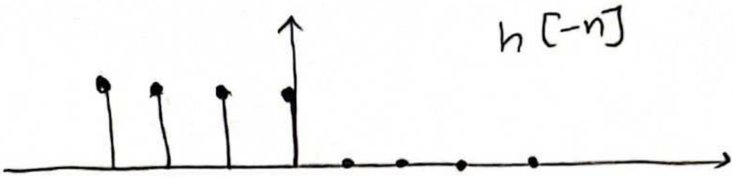
Eg: $x[n] = \alpha^n u[n]$ $\quad \alpha \in [0,1)$

$h[n] = u[n]$



> Use flip + shift : Can flip either!





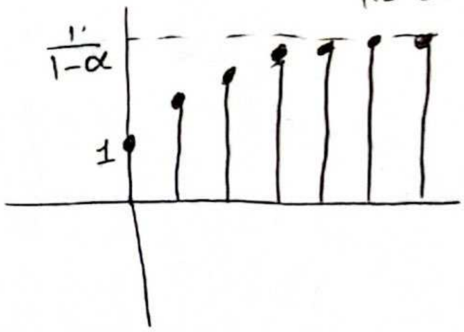
$$y[-1] = 0$$
$$y[0] = 1$$
$$y[1] = 1+\alpha$$
$$y[2] = 1+\alpha+\alpha^2$$

$$y[n] = \begin{cases} 0 & n<0 \\ \sum_{K=0}^{n} \alpha^{k} \end{cases}$$

$$= \left(\sum_{K=0}^{n} \alpha^{K}\right) u[n] = \sum_{k=0}^{\infty} \alpha^{k} - \underbrace{\sum_{K=n+1}^{\infty} \alpha^{k}}_{\alpha^{n+1}\sum_{K=0}^{\infty}\alpha^{K}}$$

$$= \frac{1}{1-\alpha} - \frac{\alpha^{n+1}}{1-\alpha}$$

$$\boxed{y[n] = \frac{1-\alpha^{n+1}}{1-\alpha} u[n]}$$



> Use convolution sum

$$y[n] = \sum_{K=-\infty}^{\infty} \alpha^{K} u[K] u[n-K]$$

$$= \left[\sum_{K=0}^{\overset{\text{from } u[n-K]}{n}} \alpha^{K}\right] u[n] = \frac{1-\alpha^{n+1}}{1-\alpha} u[n].$$

↳ from u[K]

↳ keep this to make sure no negative values

Properties of LTI systems

) Entirely defined by impulse response (or some other inputs).

) Commutative $x[n] * h[n] = h[n] * x[n]$

3) Distributive $x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n]$

) Associative $x[n]*(h_1[n]*h_2[n]) = (x[n]*h_1[n])*h_2[n]$

Note, $x[n]*h_1[n]*h_2[n] = x[n]*h_2[n]*h_1[n]$.

5) Causality : For causal system $h[k]=0$ for $k<0$.

5) Step response : $h[n] = \underbrace{s[n]}_{\text{step response.}} - s[n-1]$

$$\Rightarrow s[n] = \sum_{k=-\infty}^{\infty} h[k]$$

Difference Equations.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$y[n] = \underbrace{y_h[n]}_{\text{homogeneous soln.}} + \underbrace{y_p[n]}_{\text{particular soln.}}$$

$$y[n] = \sum b_k x[n-k] \rightarrow \text{FIR.}$$

Lec 4 Fourier Series

> Assume we have a periodic signal $x(t)$.

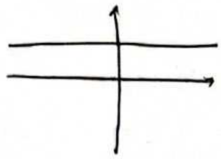
$x(t+T) = x(t)$. ; $x(t)$ could be sinusoidal.

> $e^{jk\omega_0 t}$ also has a period T.

> Any combination $\sum_{k=-\infty}^{\infty} \underset{\text{could be any complex \#.}}{a_k} e^{jk\omega_0 t}$ is periodic with period T.

>

$k=0 \Rightarrow a_0$.



$k=1 \Rightarrow a_1 . (\cos\omega_0 t + j\sin\omega_0 t)$

$k=-1 \Rightarrow a_{-1} (\cos\omega_0 t - j\sin\omega_0 t)$

Fourier Series

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$$

How to find a_k?

multiply

> $x(t) e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}$

> Integrate from 0 to T

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t} dt$$

$$\int_0^T e^{j(K-n)\omega_0 t}\,dt = \int_0^{+} \cos(K-n)\,\omega_0 t\,dt + j\int_0^T \sin(K-n)\omega_0 t)\,dt.$$

Say $K = n$

$$\Rightarrow \int_0^1 1\,dt = T \qquad \int_0^T 0\,dt = 0$$

$\Rightarrow K = n \Rightarrow \int LHS = T.$

Say $K \neq n.$

$$\int_0^T \underbrace{\cos(\text{integer})\,\omega_0 t\,dt}_{\text{oscillates equal no. of times in one T.}} + j\sin\ldots$$

$\Rightarrow = 0.$

$$\Rightarrow \int_0^T x(t)\,e^{-jn\omega_0 t}\,dt = a_n T$$

$$\Rightarrow \boxed{a_K = \frac{1}{T}\int_0^T x(t)\,e^{-jK\omega_0 t}\,dt}$$

→ a_K could be complex even for real $x(t)$.

$$x(t) = \sum_{K=-\infty}^{\infty} a_K\,e^{+jK\omega_0 t}$$

$$x^*(t) = \sum_{K=-\infty}^{\infty} a_K^*\,e^{-jK\omega_0 t}$$

Since, $x(t) = x^*(t)$; ∵ $x(t)$ is real.

$\Rightarrow \boxed{a_K = a_{-K}^*}$ $\Rightarrow$ If $a_1 = 1+2j$
$a_{-1} = 1-2j$

If $x(t)$ is real,

$$x(t) = a_0 + \sum_{K=1}^{\infty} 2A_K \cos(\theta_K + K\omega_0 t).$$

OR.

$$x(t) = a_0 + \sum_{K=1}^{\infty} B_K \cos K\omega_0 t - C_K \sin K\omega_0 t$$

> A_K, θ_K or B_K, C_K are derived from a_K.

Eg:- $x(t) = 5 + 2\cos(\omega_0 t)$.

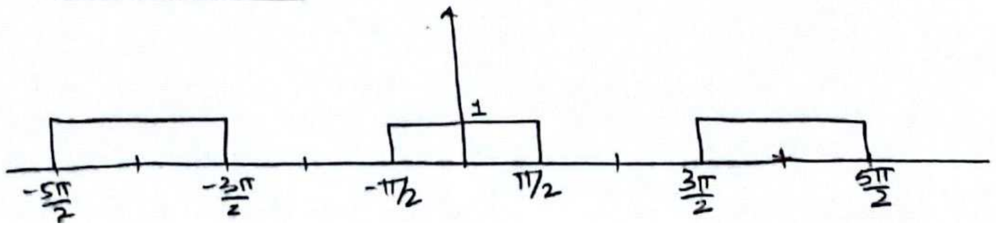
Sol:-

$$a_K = \frac{1}{T}\int_0^T x(t)\, e^{jK\omega_0 t}\, dt$$

$$x(t) = \underset{\overset{\downarrow}{a_0 e^{0j\omega_0 t}}}{5} + \underset{\overset{\downarrow}{a_1 e^{1j\omega_0 t}}}{e^{j\omega_0 t}} + \underset{\overset{\downarrow}{a_{-1} e^{-1j\omega_0 t}}}{e^{-j\omega_0 t}}$$

$\Rightarrow a_0 = 5,\ a_1 = 1,\ a_{-1} = 1$

Eg:- Pulse train



$T = 2\pi$

$\Rightarrow \omega_0 = \frac{2\pi}{2\pi} = 1$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, e^{-j0\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_0^T x(t)\, dt$$

we can choose any interval of width T since signal is periodic + area covered is the same.

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\,dt = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1\, dt = \frac{\pi}{2\pi} = \frac{1}{2}$$

$$a_K = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jK\omega_0 t}\, dt$$

This is not mathematically rigorous.

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\, e^{-jkt}\, dt = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-jkt}\, dt \quad \Big(= \frac{1}{2\pi}\cdot\frac{-1}{jk}\cdot e^{-jkt}\Big]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi K}\cdot\frac{1}{2j}\left[e^{jk\pi/2} - e^{-jk\pi/2}\right]$$

$$= \frac{\sin K\pi/2}{\pi K.} = \frac{1}{2}\operatorname{sinc}\frac{K\pi}{2}$$

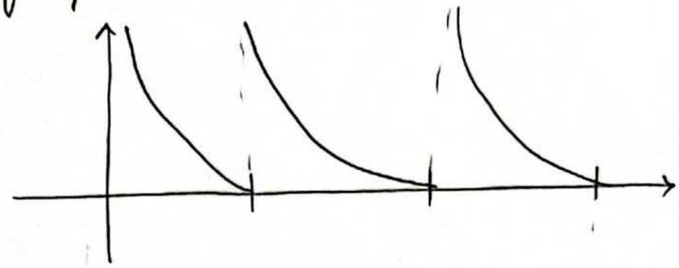
$$\Rightarrow a_1 = \sin\pi/2\,/\,\pi = \frac{1}{\pi},\quad a_{-1} = \frac{1}{\pi},\quad a_2 = 0,\ a_{-2} = 0;\ a_3 = a_{-3} = \frac{-1}{3\pi}$$

→ only odd terms.

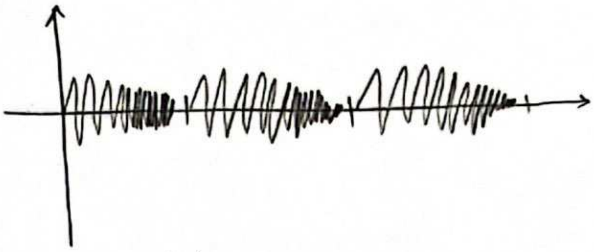
Properties & Notes

> When does it not work?

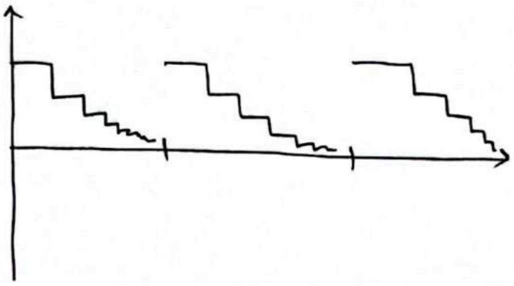
1) Asymptotic periodic signals. since area = ∞.
If area ≠ ∞ not sure?



2) ∞ wiggle.



3) ∞ Discontinuities.

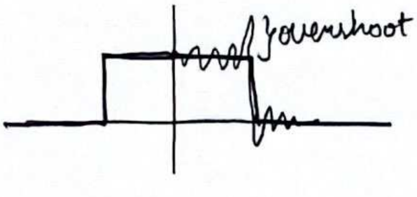


> What happens at a discontinuity?

a) FS converges at every continuous point.

b) Converges to the average value at every discontinuity.

Gibbs phenomenon



> Can never get the overshoot below ≈ 9% of the height of the discontinuity.

> **Note:** Complex exponentials are eigenfunctions of LTI systems. since if we input a complex fn only phase & amplitude change.

$$y = H\left[e^{j\omega_0 t}\right] = \underbrace{Ae^{j\phi}}_{\lambda \to \text{Eigenvalue.}} \cdot e^{j\omega_0 t}$$

Properties.

Suppose we have $x(t)$, period $= T$, F.S is given by $\{a_k\}$

1) Linearity:

$$x(t) \longleftrightarrow \{a_k\}$$
$$y(t) \longleftrightarrow \{b_k\}$$

$$\alpha x(t) + \beta y(t) \longleftrightarrow \{\alpha a_k + \beta b_k\}$$

2) Time shifting:

$$x(t) \longleftrightarrow \{a_k\}$$

let $y(t) = x(t - t_0)$

$$\Rightarrow \quad y(t) \longleftrightarrow \{a_k e^{-jk\omega_0 t_0}\}$$

$\Rightarrow$ F.S coeffs change in phase but mag. remains same.

$$|b_k| = |a_k|$$

3) Differentiation:

$$x(t) \longleftrightarrow \{a_k\}$$
$$x'(t) \longleftrightarrow \{jk\omega_0 a_k\}$$

4) Parseval's theorem

> Power of signal. $\frac{1}{T}\int_0^T |x(t)|^2 dt$ = Power of coefficien $\sum_{k=-\infty}^{\infty} |a_k|^2$

⇒ Same amount of power in time & frequency domain.

5) Convolution.

$$x(t) \longrightarrow \{a_k\}$$
$$y(t) \longrightarrow \{b_k\}$$

$$x(t)\, y(t) \longleftrightarrow \sum_{l=-\infty}^{\infty} a_l\, b_{k-l} = a * b$$

$$\int_0^T x(\tau)\, y(t-\tau)\, d\tau \longleftrightarrow T a_k b_k.$$

Lec 5 Fourier Transform.

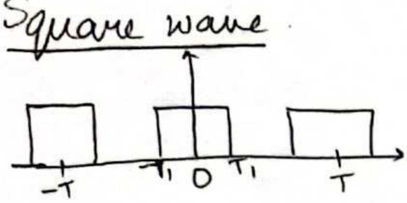
> Non periodic signals ⇒ periodic with ∞ period.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \rightarrow \text{Fourier series.} \quad \omega_0 = \frac{2\pi}{T}$$

> T ↑ ⇒ ω_0↑ ⇒ Frequencies become more closely spaced. At the limit T ⟶ ∞, Frequencies become a continuum.

$$\Rightarrow \quad x(t) = \int_{\omega=-\infty}^{\infty} \boxed{?}\, e^{j\omega t}\, dt$$

Square wave



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

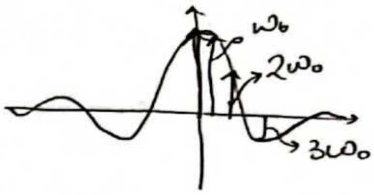
$$\Rightarrow Ta_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0} = \left.\frac{2\sin\omega T_1}{\omega}\right|_{\omega = k\omega_0}$$

↳ Sampling a continuous fn ω at $k\omega_0$ times.

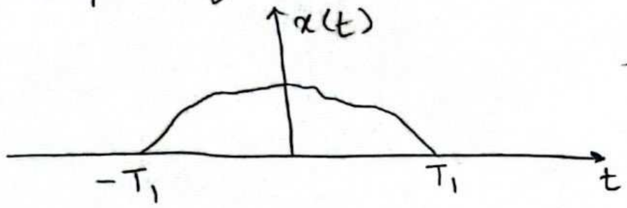
> Intuition

The $\{a_k\}$ are evenly spaced values of this continuous function of ω

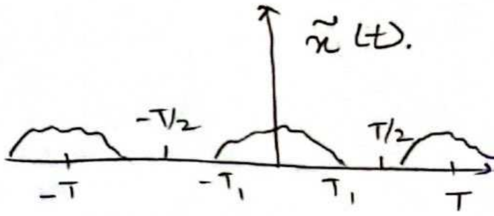
as $T \to \infty$, $\omega_0 \to 0$, samples get ∞ly close.



Math point of view



$\xrightarrow[\text{on xaxis}]{\text{make copies}}$



> We can take F.S for $\tilde{x}(t)$.

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

> Within $-T/2$ to $T/2$ $\tilde{x}(t) = x(t)$ $\Rightarrow$ $a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt$

> Changing limits to $-\infty$ to ∞ doesn't chang the $\int$ since $x(t)$ is 0 elsewhere $\}$ $\Rightarrow a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt$

Define $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$

$$\Rightarrow \quad a_k = \frac{1}{T}\, X(k\omega_0)$$

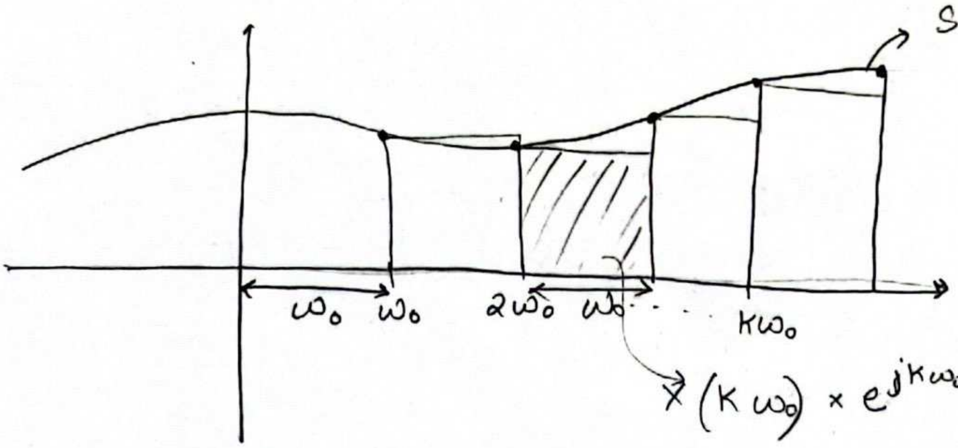
$$\Rightarrow \tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(k\omega_0) e^{jk\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T}$$

$$\Rightarrow \frac{1}{T} = \frac{\omega_0}{2\pi}$$

$$\Rightarrow \quad \tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left[X(k\omega_0)\, e^{jk\omega_0 t} \right] \omega_0$$

What is the sum?



Say this is $X(\omega)e^{j\omega t}$

$X(k\omega_0) \times e^{jk\omega_0 t} \cdot \omega_0$ } Area of each 'box' is each term in the summation

$\Rightarrow$ As $T \uparrow\uparrow$, $\omega_0 \downarrow\downarrow \Rightarrow$ the sum converges to the integral. This also pushes the copies to $\infty \Rightarrow$ $\tilde{x}(t) = x(t)$.

$\lim_{\omega \to 0}$ of both sides.

Derived

$$\boxed{\Rightarrow x(t) = \frac{1}{2\pi} \int_{-\infty} x(\omega)\, e^{j\omega t}\, d\omega}$$

Defined

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$

$x(t)$ continuous $\rightarrow$ Periodic $\Rightarrow$ F.S (Discrete a_K)

$x(t)$ continuous $\rightarrow$ Aperiodic $\Rightarrow$ F.T (continuous $X(\omega)$)

When is this "legal"?

1) $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$ Finite Energy.

2) Finite # of extrema.

3) Finite # of discontinuities.

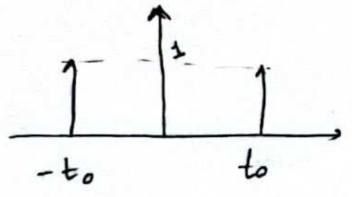
* We do allow the F.T of periodic signals. ⇒ contradicts the first rule since $\int$ of periodic signal is ∞.

Some signals.

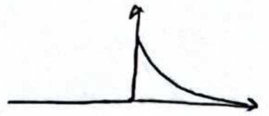
1) $x(t) = \delta(t)$.

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega(0)} = 1 \text{ for all } \omega.$$

2) $x(t) = \delta(t - t_0) = e^{j\omega t_0}$ → This also lies on the unit circle but phase is shifted.

3) 

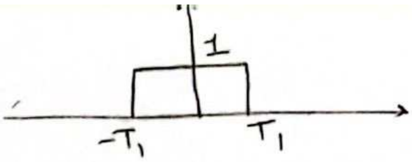
$\Rightarrow e^{j\omega t_0} + e^{-j\omega t_0} = 2\cos\omega t_0.$

4) $x(t) = e^{-at}u(t)$ 

$$\Rightarrow x(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt = \frac{-1}{a+j\omega}\left[e^{-(a+j\omega)t}\right]_{t=0}^{\infty}$$

$$= \underline{\frac{1}{a+j\omega}}$$

5)

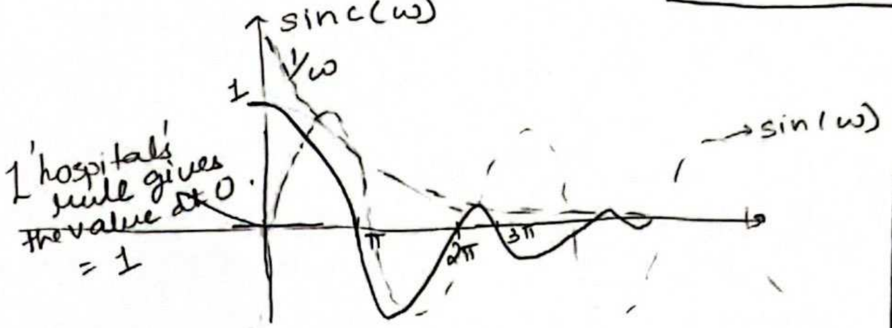


$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T_1}^{T_1} e^{-j\omega t}\, dt$$

$$= \frac{2 \sin \omega T_1}{\omega}$$

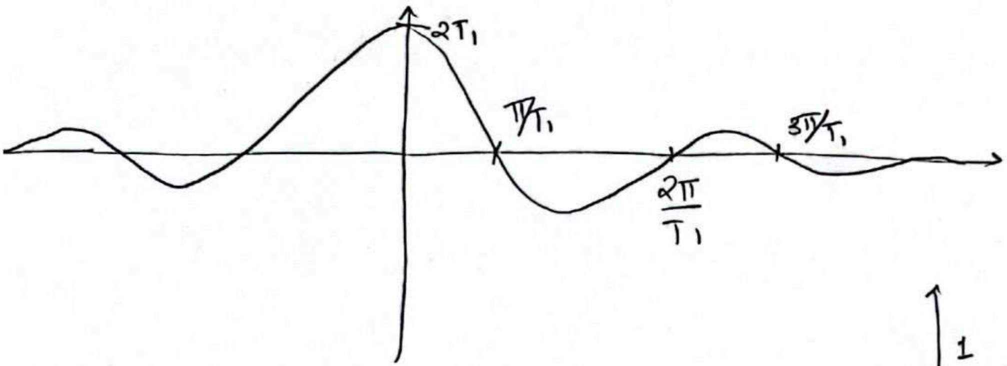
$$\boxed{X(\omega) = 2T_1 \operatorname{sinc} \omega T_1}$$



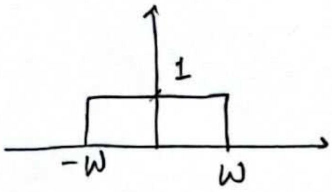
L'hospital's rule.

If $\lim_{x\to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\pm\infty$, $g'(x) \neq 0$ for all x in I with $x \neq c$. if $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists.

$$\Rightarrow \lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$



6) What if $X(\omega)$ is a pulse?

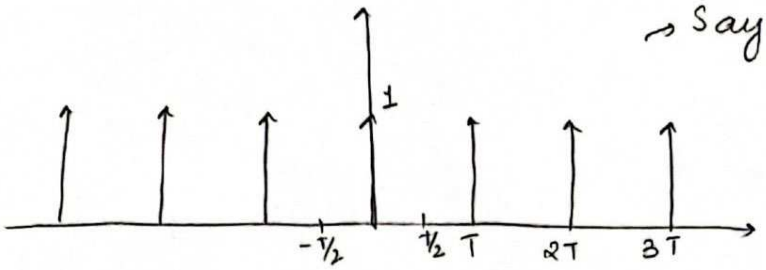


$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi\, jt} \cdot e^{jtw} \Big]_{-W}^{W} = \frac{1}{\pi t} \frac{\sin tw}{1} = \frac{W}{\pi} \operatorname{sinc} tw$$

$$\Rightarrow \boxed{x(t) = \frac{W}{\pi} \operatorname{sinc} tW}$$

Periodic signal F.T?

- We can take F.S or F.T.
-



Say we have an impulse train.

$$x(t) = \sum_{K=-\infty}^{\infty} \delta(t - KT)$$

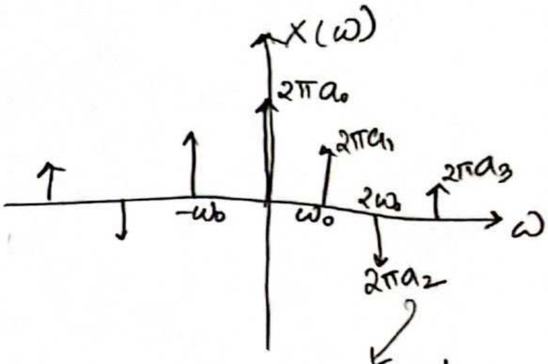
FS => $$a_K = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jK\frac{2\pi}{T}t} dt$$

$$a_K = \frac{1}{T} \quad \text{For all } K.$$

$$x(t) = \sum_{K=-\infty}^{\infty} a_K e^{jK\omega_0 t}$$

Say we have a signal whose F.T is :

$$X(\omega) = \sum_{K=-\infty}^{\infty} 2\pi \cdot a_K \cdot \delta(\omega - K\omega_0).$$



F.S coefficients show up here.

What is the corresponding $x(t)$?

Suppose $X(\omega) = \delta(\omega - K\omega_0)$

$$x(t) = \frac{1}{2\pi} \int \delta(\omega - K\omega_0) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} e^{jK\omega_0 t}$$

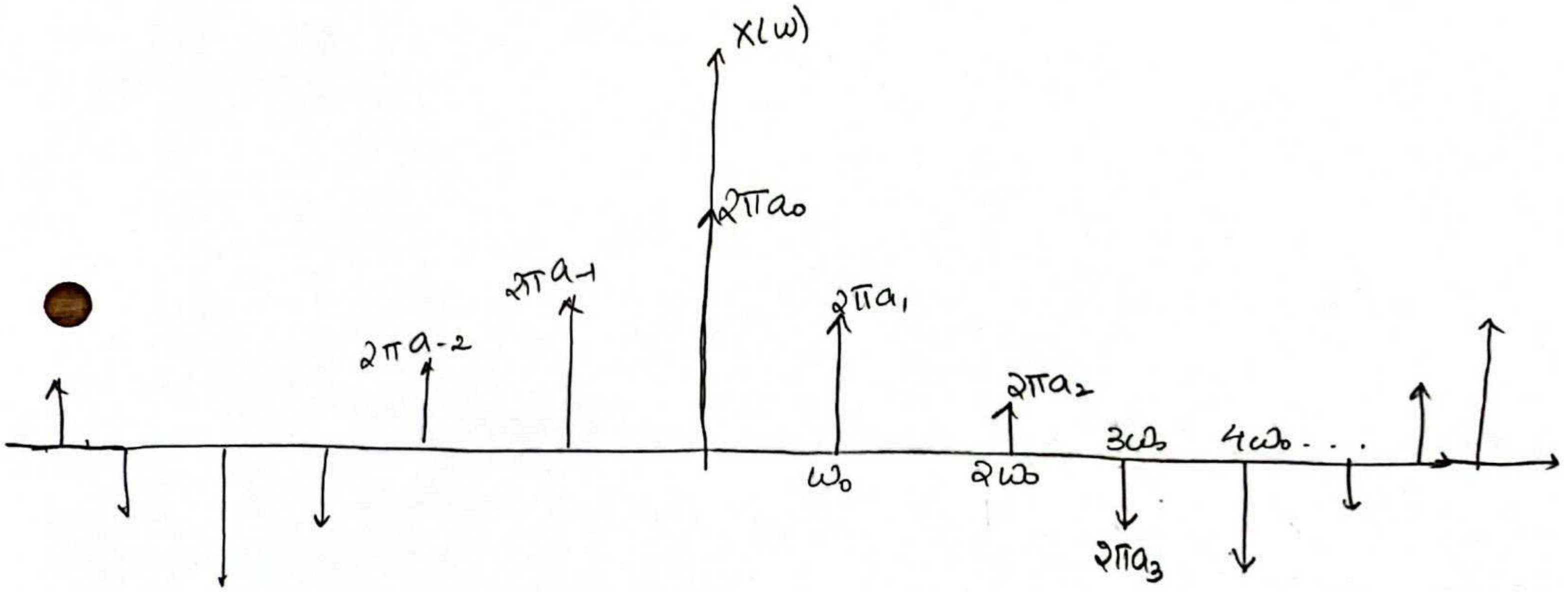
So for $X(\omega) = \sum 2\pi a_K \delta(\omega - K\omega_0)$

$$x(t) = \sum_{K=-\infty}^{\infty} a_K e^{jK\omega_0 t}$$

$= F \cdot S$ of $x(t)$ for a periodic signal with $T = \frac{2\pi}{\omega_0}$

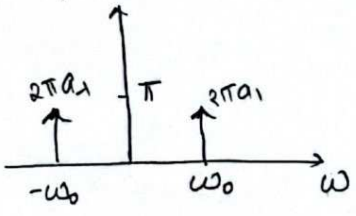
→ F.T of a periodic signal is a series of equally spaced impulses

* we can read off the coefficients from the heights of the impulses.



→ $x(t)$ is an impulse train ⇒ $a_K = 1/T$ for all K ⇒ $X(\omega)$ is also an impulse train of heights $\frac{2\pi}{T}$.

Eg:



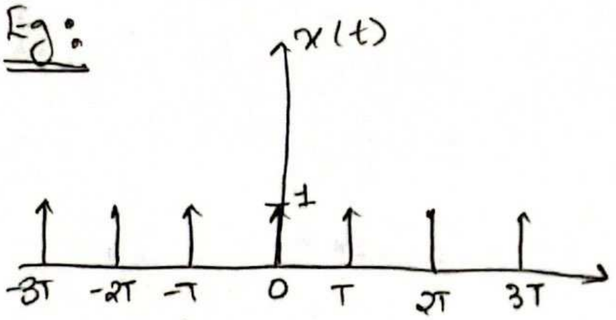
$x(t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

$= \cos\omega_0 t$.

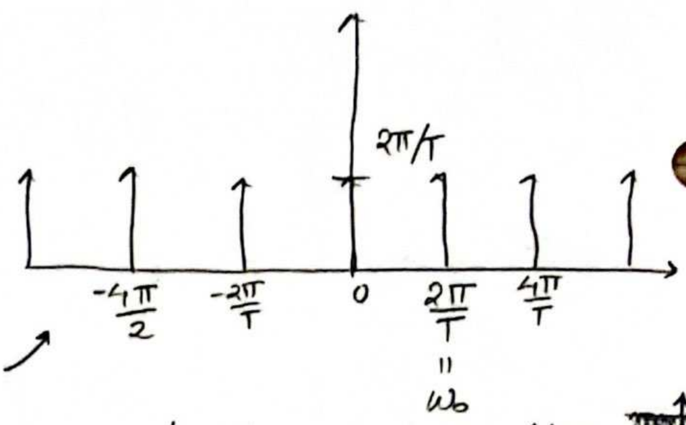
⇒

$$\left.\begin{array}{l} \omega = 0 \Rightarrow X(\omega) = 0 \\ \omega = \pm\omega_0 \Rightarrow X(\omega) = \text{delta height } \pi \\ \omega = \pm K\omega_0 \Rightarrow X(\omega) = 0 \quad (K>1) \end{array}\right\} \Rightarrow \begin{array}{l} a_0 = 0 \\ a_{\pm 1} = \frac{1}{2} \\ a_{\pm K} = 0 \ (K>1) \end{array}$$

Eg:



$\xrightarrow{F.T}$



↓

wider . T ⇒ narrower ω_0 ↗

At the limit if only one T ⇒ all deltas bunch up into a line

F.T properties

1) Linearity.

2) Time shift

$x(t-t_0) \longleftrightarrow X(\omega)e^{-j\omega t_0}.$

3) Symmetry properties.

$X(\omega) = R(\omega) + j\,I(\omega)$

If $x(t)$ is real, $Re(\omega) = Re(-\omega)$ Real part is even

$Im(\omega) = -Im(-\omega)$ Im part is odd.

$|X(\omega)|$ is even, $\angle X(\omega)$ is odd.

Can show $Even(x(t)) \longleftrightarrow Re(X(\omega))$

$Odd(x(t)) \longleftrightarrow j\,Im(X(\omega))$

$x(t)$ is real & even, $X(\omega)$ real., even

$x(t)$ is real & odd, $X(\omega)$ imaginary - odd.

4) Diff & Integration: $x'(t) \longleftrightarrow j\omega X(\omega)$; $\int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega$

5) Scaling: $x(at) \longleftrightarrow \frac{1}{|a|}X(\omega/a)$

5) Parsevals $$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Key Principle : Convolution :

If $y(t) = x(t) * h(t)$

$Y(\omega) = X(\omega)\ H(\omega)$

$\hookrightarrow$ Frequency Response.

Lec 6 Frequency Response

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\ h(t-\tau)\ d\tau$$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\ e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) d\tau \right) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t} dt \right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\ \underbrace{FT\left(h(t-\tau)\right)}\ d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\ \overbrace{H(\omega)\, e^{-j\omega\tau}}\ d\tau$$

$$= H(\omega) \cdot \underbrace{\int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau} d\tau}_{X(\omega)}$$

$$\boxed{Y(\omega) = H(\omega).X(\omega)}$$

→ Frequency response only works for $\underline{\text{linear systems}}$ (LTI infact) since we use convolution.

$$\delta(t) \xrightarrow{\text{system}} h(t).$$

$$F(h(t)) \xrightarrow{F.T} H(\omega).$$

→ $x(t) = e^{j\omega_0 t}$ → phase shift

$X(\omega) = 2\pi\, \delta(\omega - \omega_0)$ → freq. shift

$$Y(\omega) = H(\omega)\; X(\omega).$$

$$= H(\omega)\, 2\pi\, \delta(\omega - \omega_0)$$

$$= H(\omega_0) \cdot 2\pi\, \delta(\omega - \omega_0)$$

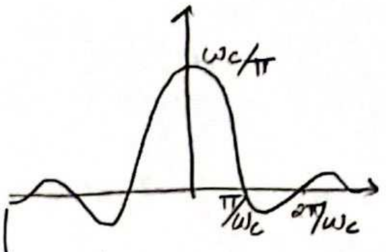
$$y(t) = \underset{\substack{\uparrow \\ \text{This is a} \\ \text{complex no.}}}{H(\omega_0)} \cdot e^{j\omega_0 t}$$

→ Therefore frequency content doesn't change. It is only scaled by a complex number ⇒ Amp & phase change.

If $x(t) = \cos\omega_0 t$

$$y(t) = |H(\omega_0)|\; \cos(\omega_0 t + \underline{\angle H(\omega_0)}).$$

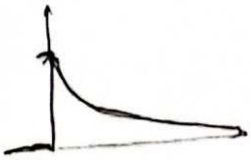
→ Impulse response of LPF = $\frac{\omega_c}{\pi}\,\text{sinc}(\omega_c t)$.



↳ Not causal

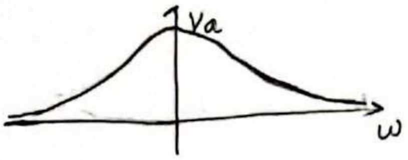
↳ wiggling goes on forever

→ $h(t) = e^{-at} u(t)$.



$$H(\omega) = \frac{1}{a+j\omega}$$

$$|H(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$



Lec 7 Discrete Time Fourier Transform.

Recall CTFT: $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

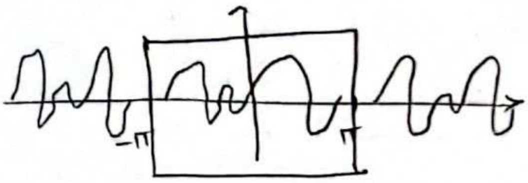
Define DTFT: $\boxed{X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}}$

> CTFTs have ∞ frequency range.

> DTFTs have a freq. range of 2π.

$$X(\omega+2\pi) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} e^{-2\pi n} = X(\omega).$$

⇒ DTFT is periodic with 2π range.



Recall CTIFT $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$

DTIFT $\boxed{x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega}$

→ to restrict limits

Proof :- Consider $\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$

$$= \int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m} e^{j\omega n} d\omega$$

Sufficient condition to switch the order of $\int$ + Σ :

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

$$\Rightarrow \int \Sigma = \sum_{m=-\infty}^{\infty} x[m] \underbrace{\int_{-\pi}^{\pi} e^{j\omega(n-m)} d\omega}_{\underbrace{\int_{-\pi}^{\pi} \cos \omega(n-m) + j \sin \omega(n-m) d\omega}}$$

$= 2\pi$ when $n = m$
$= 0$ otherwise.

$$\Rightarrow \int \Sigma = \sum_{m=-\infty}^{\infty} x[m] . \begin{Bmatrix} 2\pi, & n = m \\ 0, & n \neq m \end{Bmatrix}.$$

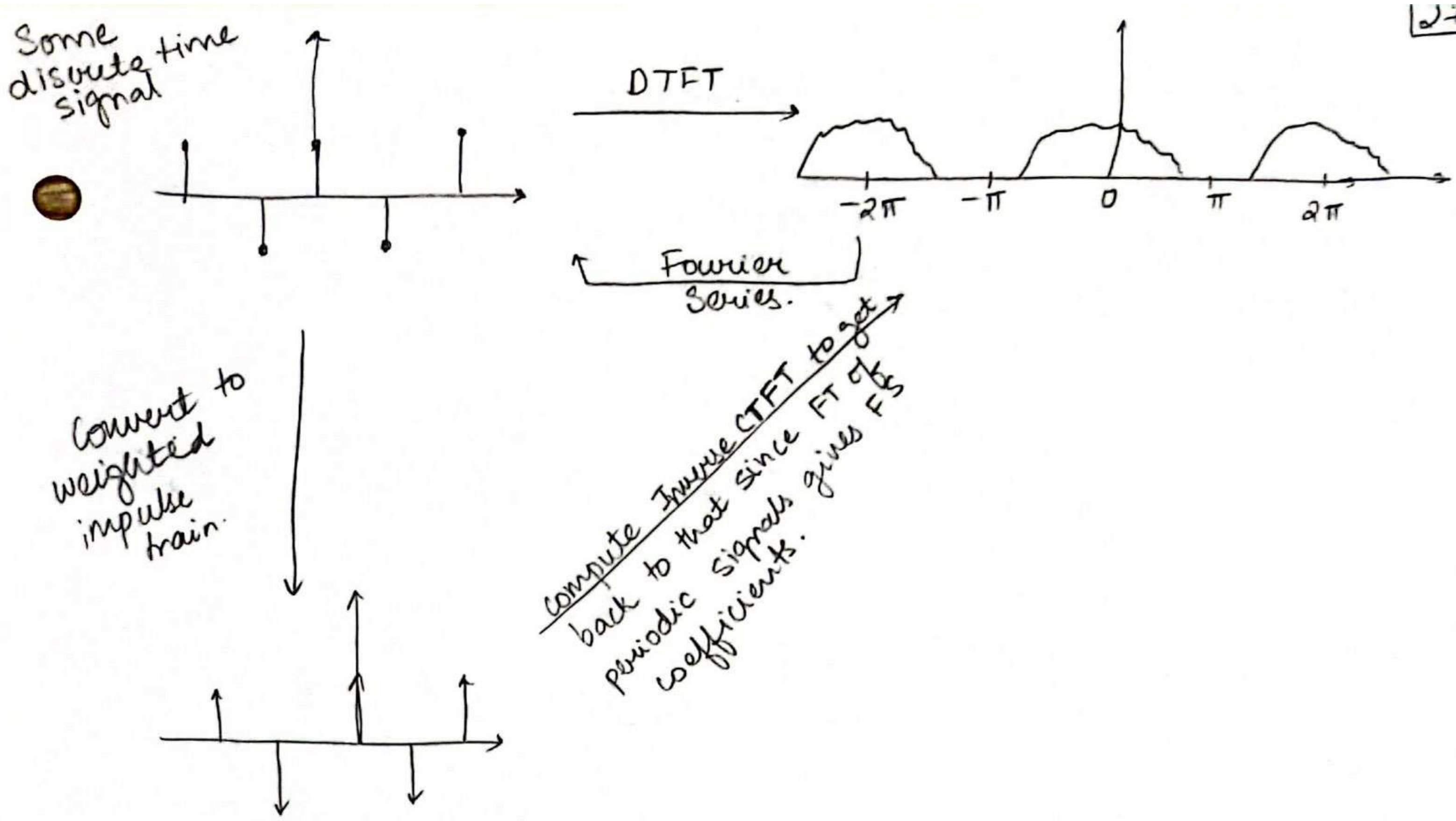
$$= x[n] . 2\pi.$$

$$\Rightarrow \boxed{x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega}$$

↳ This looks like a FS for a signal with period 2π.

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j\frac{2\pi}{T}kt} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t) e^{-jkt} dt.$$

Sign is based on how inverse & forward are defined.



→	continuous	discrete
periodic	FS (maybe FT)	DFT
nonperiodic.	FT	DTFT

Conditions for DTFT

→ If $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$

$$\lim_{N\to\infty} \overset{\to \text{maximum}}{\sup_{\omega}} |X(\omega) - X_N(\omega)| = 0$$

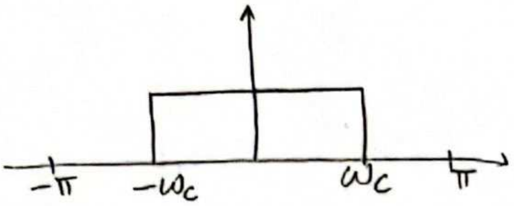
As N gets BIG, $X_N(\omega) = \sum_{n=-N}^{N} x[n] e^{-j\omega n}$ converges at each point to $X(\omega)$

> If $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$ (looser condition).

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} |X(\omega) - X_N(\omega)|^2 d\omega = 0 \longrightarrow \text{mean square convergence.}$$

Eg

$X(\omega) =$



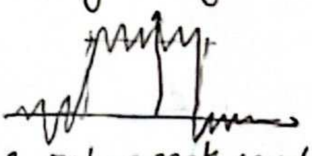
$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi j n}\left(e^{j\omega_c n} - e^{-j\omega_c n}\right) = \frac{\omega_c}{\pi}\text{sinc}(\omega_c n).$$

↳ satisfies $\sum |\,|^2 < \infty$
Not $\sum |\,| < \infty$

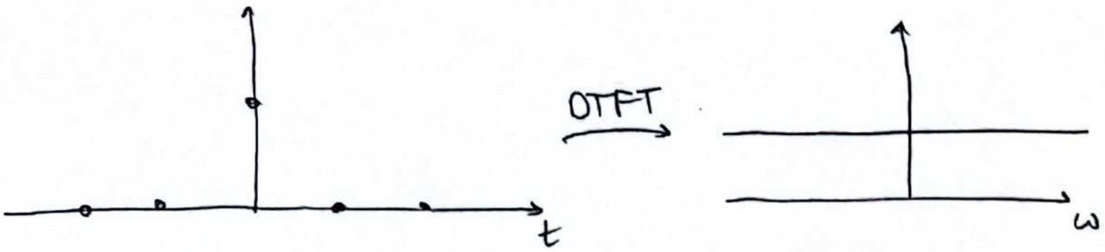
=> In reality we get



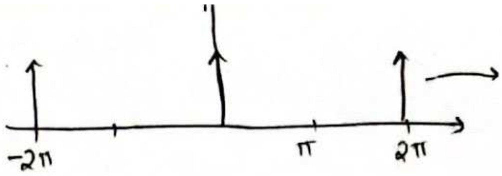
since only weak condition is satisfied.

Eg: $x[n] = \delta[n]$

$$X(\omega) = \sum_{n=-\infty}^{n=\infty} \delta[n] e^{-j\omega n} = 1$$



Eg $X(\omega) = \delta(\omega)$

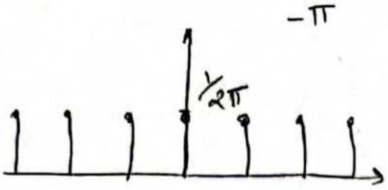


→ These copies show up!

(Same as doing CTFT of an impulse train!)

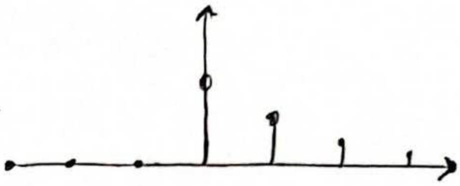
Anyway, $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n} = \frac{1}{2\pi}$

⇒



Eg $x[n] = a^n u[n]$ $|a| < 1$

↳ To satisfy those 2 conditions.



$$X(\omega) = \sum_{n=-\infty}^{\infty} a^n u[n]e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}. = \sum_{n=0}^{\infty}(ae^{-j\omega})^n$$

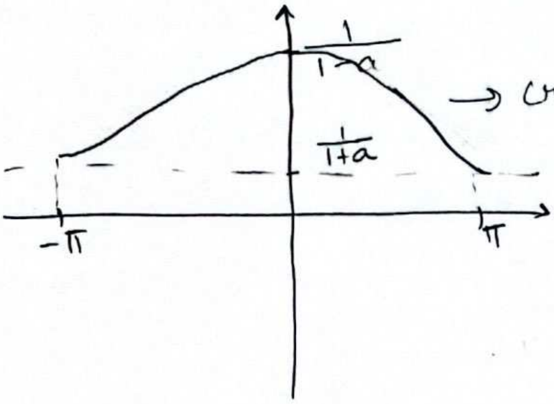
$$= \frac{1}{1-ae^{-j\omega}}$$

↳ A real signal can give complex F.T.

⇒ Magnitude response = $|X(\omega)|$

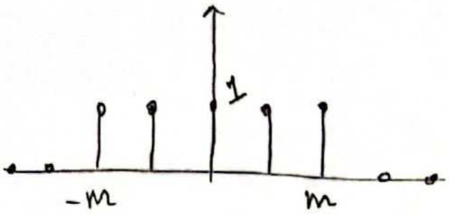
Phase response = $\angle X(\omega)$

$$|X(\omega)| = \left|\frac{1}{1-ae^{-j\omega}}\right| = \frac{1}{|(1-a\cos\omega)+ja\sin\omega|} = \frac{1}{\sqrt{(1-a\cos\omega)^2+a^2\sin^2}}$$



→ crappy LPF.

Pulse in time domain.



$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=-M}^{M} e^{-j\omega n} \quad = e^{j\omega M} \sum_{n=0}^{2M} e^{-j\omega n}.$$

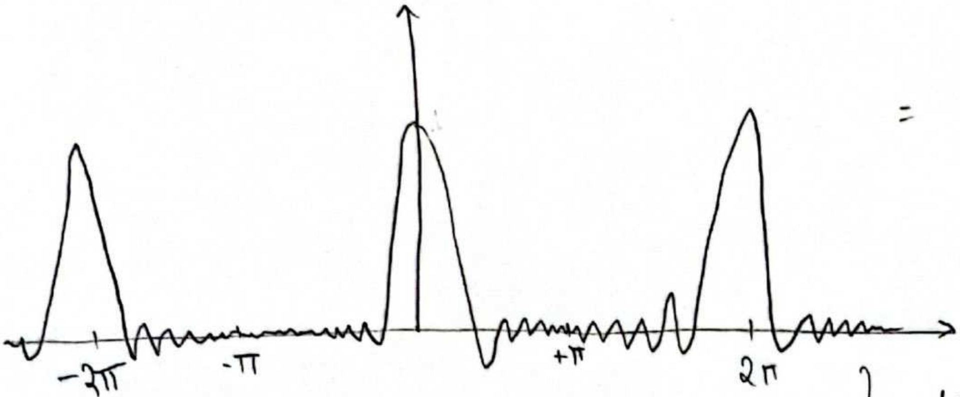
$$= e^{j\omega M}\left[\frac{1-e^{-j\omega(2M+1)}}{1-e^{-j\omega}}\right]$$

↳ From finite sums.

$$= \frac{\cancel{e^{j\omega M}}\cdot\cancel{e^{-j\omega\frac{2M+1}{2}}}}{\cancel{e^{-j\omega/2}}}\;\frac{\left(e^{j\omega\frac{2M+1}{2}} - e^{-j\omega\frac{2M+1}{2}}\right)}{\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

$$= \frac{\sin \omega\frac{2M+1}{2}}{\sin\frac{\omega}{2}}.$$

↳ Doesn't look like a Sinc. because Sinc is not periodic



↳ Kind of looks like a sinc.

- We want to use DTFT to study LTI systems.

Our convolution property still holds.

$$y[n] = x[n] * h[n].$$

$$Y(\omega) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} \quad = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[k]\,h[n-k]\,e^{-j\omega n}.$$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} h[n-k]\,e^{-j\omega n}$$

rename $m = n-k$.
$n = m+k$.

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{m=-\infty}^{\infty} h[m]\, e^{-j\omega(m+k)}$$

$$= \left(\sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k}\right)\left(\sum_{m=-\infty}^{\infty} h[m]\, e^{-j\omega m}\right)$$

$$\boxed{Y(\omega) = X(\omega)\, H(\omega).}$$

>

$$x[n] = A e^{j\omega_0 n}$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\; x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\; A e^{j\omega_0 (n-k)}$$

$$= A e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega_0 k}.$$

$$= H(\omega_0) \cdot A e^{j\omega_0 n}$$

$$A e^{j\omega_0 n} \longrightarrow \underset{\text{Amplitudescaling}}{A\,|H(\omega_0)|}\; e^{j(\underset{\text{phase shifting}}{\angle H(\omega_0)} + \omega_0 n)}.$$

> In the same way: $\cos(\omega_0 n + \phi) \longrightarrow |H| \cos(\omega_0 n + \phi + \angle H).$

if $h[n]$ is real.

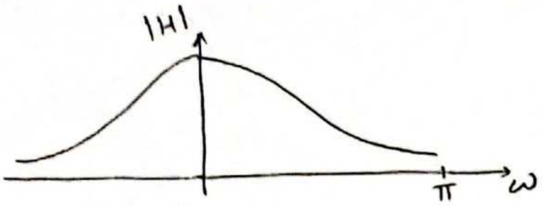
Eg

$$h[n] = \left(\tfrac{1}{3}\right)^n u[n] \longrightarrow H(\omega) = \frac{1}{1 - \frac{1}{3} e^{-j\omega}}$$

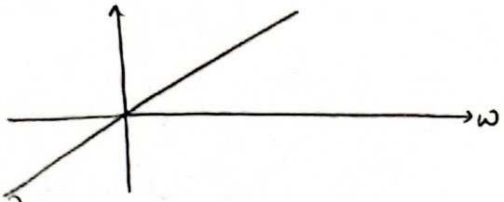
$$x[n] = 2 e^{j\pi/3\, n}. \rightarrow \frac{\pi}{3} \text{ valued sinusoid} \Rightarrow H\left(\frac{\pi}{3}\right) = \frac{1}{1 - \frac{1}{3} e^{-j\pi/3}} = \frac{1}{\frac{5}{6} + \frac{\sqrt{3}}{6} j}$$

$$\Rightarrow y[n] = 2\,|H(\pi/3)|\, e^{j(\frac{\pi}{3} n + \angle H(\frac{\pi}{3}))}.$$

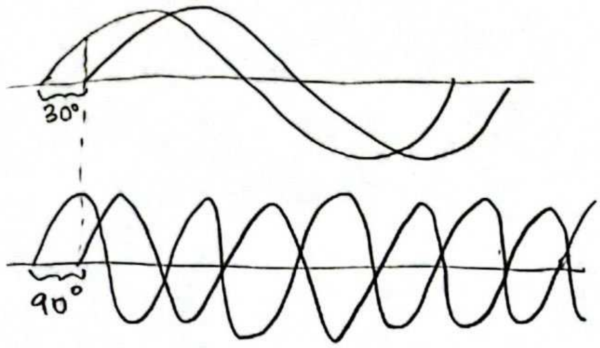
→ When $h[n]$ is real, $|H(\omega)|$ is even



When $h[n]$ is real $\angle H(\omega)$ is odd.



↳ We want this to be linear so that all cosines & sines are delayed by the same time in time domain. A ω↑ the phase delay also should ↑ to make sure the time delay is constant for all freq.



⇒ Same time delay requires different phase delay!

Mathematically: $Y(\omega) = X(\omega)\ H(\omega)$

$= X(\omega) \cdot |H|\ e^{j\angle H}$.

$= X(\omega) \cdot e^{-jc\omega}$

let $\angle H = -\overset{\text{constant}}{c}\omega$

& assume $|H| = 1$

⇒ $y[n] = \underbrace{x[n-c]}$ ↳ properties of IFT. ⇒ constant time delay!

⇒ $\boxed{\text{No distortion.}}$

→ Systems with no delay are non causal. The IFT would have non zero negative values. Also stability is an issue.

Z - Transform

Continuous	Discrete
$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ CTFT	$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}dt$ Laplace.	$X(Z) = \sum_{n=-\infty}^{\infty} x[n]Z^{-n}$. Z Transform.

$\uparrow$ $z = e^{+j\omega}$

Where z is a complex number.

$\Rightarrow$ $x[n] = z^n$.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k}.$$

$$= z^n \sum_{k=-\infty}^{\infty} h[k]\, z^{-k} = H(z)\, z^n.$$

$\Rightarrow$ z^n is the "special" signal for DT LTI systems. Like how e^{st} is special signal (with $\int$) in CT LTI systems.

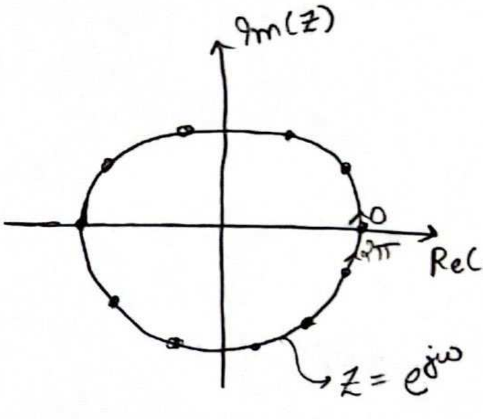
e^{st} $\downarrow$ Eigenfn of CT LTI Systems

$\rightarrow$ z^n is the Eigenfunction of DT LTI systems.

$\rightarrow$ $X(Z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= X(Z)\Big|_{z = e^{j\omega}}$$



$\rightarrow$ Why do we need z transform?

$\rightarrow$ The DTFT doesn't always converge or exist. Recall the sufficient conditions $\left[\sum_{n=-\infty}^{\infty} |x[n]| < \infty\right]$.

& the other "relaxed" condition.

z transform on this unit circle is the DTFT. Also shows why DTFT is periodic in 2π. This unit circle is key for DT systems like $j\omega$ axis for CT systems.

→ The z transform may converge when DTFT doesn't exist.

→ Notation is easier → polynomials in z or rational functions of z.

→ Very helpful while designing filters.

Region of Convergence (ROC)

Let $z = re^{j\omega}$ → polar form.

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]\, r^{-n}) . e^{-j\omega n}$$

⇒ DTFT of $(x[n] r^{-n})$

⇒ The z transform converges if $\sum_{n=-\infty}^{\infty} |x[n] r^{-n}| < \infty$

If $r = 1$ & z transform converges ⇒ DTFT also exists.

If $r = 1$ & z transform does not converge ⇒ DTFT does not exist.

⇒ we could ↑r to make z transform converge even though DTFT doesn't exist.

→ Range of r for which z transform converges defines the ROC.

Eg: input is $u[n]$ ⇒ $\sum_{n=-\infty}^{\infty} u[n] = \infty$ ⇒ DTFT doesn't exist.

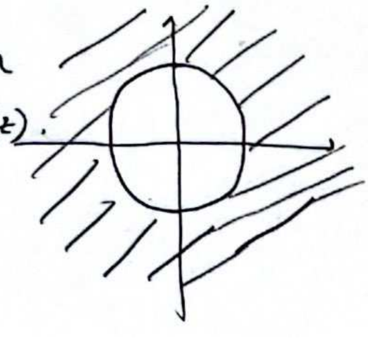
What about $u[n] r^{-n}$ for some real r?

$$\sum_{n=-\infty}^{\infty} u[n] r^{-n} = \sum_{n=0}^{\infty} r^{-n} = \frac{1}{1-r^{-1}} \text{ if } |r| > 1$$

⇒ ROC of the z transform of $u[n]$ is $|r| > 1$

⇒ Convergence of z transform depends only on the mag(z).
$|z| = r$

⇒ ROC is "circular"



> ROC usually comes in circles.

→ In general $r_1 < |z| < r_2$.

→ If ROC includes the unit circle ⇒ DTFT exists.

NOTE: For Laplace ROC.

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t) e^{-\sigma t} . e^{-j\omega t} dt = \mathcal{F}\{x(t)e^{-\sigma t}\}.$$

For $\mathcal{F}\{x'(t)\}$ to exist we need $\int_{-\infty}^{\infty} |x'(t)|\, dt < \infty$

⇒ ROC: $\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt < \infty$

→ The range of σ for which the above condition is satisfied is the ROC

→ ROC depends only on σ ⇒ $Re\{s\}$.

→ If ROC includes the $j\omega$ axis, F.T exists. and $X(s)|_{s=j\omega} = \mathcal{F}(x(t$

→ If the z transform of an impulse response $h[n]$ for an LTI system converges on the unit circle (DTFT exists) then the system is stable.

→ We can write $X(z) = \frac{N(z)}{D(z)}$ ⇄ Polynomials in z.

$N(z) = 0$ ⇒ $X(z) = 0$ ⇒ 'zeroes'

$D(z) = 0$ ⇒ $X(z) = \infty$ ⇒ 'poles'

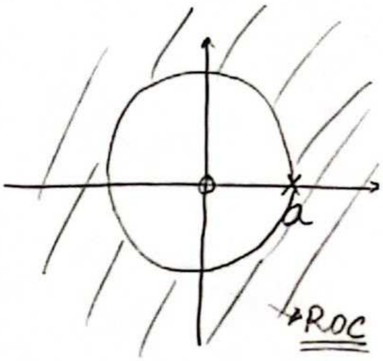
Eg Right sided exponential.

$x[n] = a^n u[n]$ (a<1 ?)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

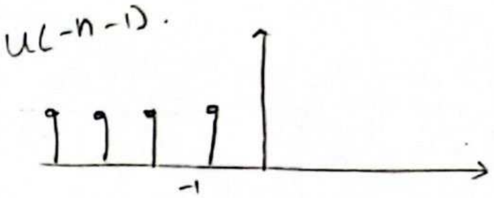
Converges if $\left|\frac{a}{z}\right| < 1 \Rightarrow |z| > |a|$

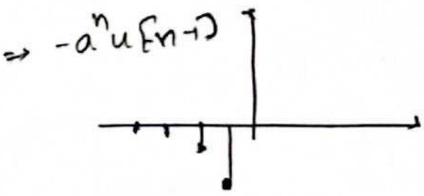
Then $X(z) = \frac{1}{1 - a/z} \boxed{= \frac{z}{z-a}}$ → zero ↳ pole.



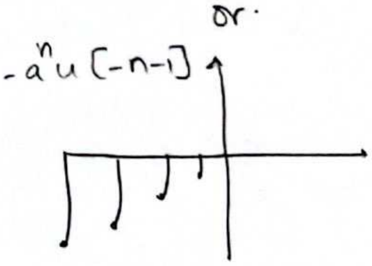
> $a<1 \Rightarrow$ F.T exists.

Eg Left sided exponential.





or.



$x[n] = -a^n u[-n-1]$

$$\Rightarrow X(z) = \sum_{n=-\infty}^{-1} -\left(\frac{a}{z}\right)^n$$

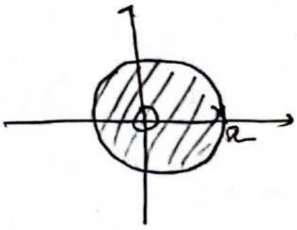
$$= \sum_{n=1}^{\infty} -\left(\frac{z}{a}\right)^n$$

Converges if $|z/a| < 1 \Rightarrow |z| < |a|$

$$X(z) = \frac{-1}{1-\left(\frac{z}{a}\right)} + 1$$

↳ since n=0.

$$= \frac{-a}{a-z} + \frac{a-z}{a-z} = \frac{-z}{a-z}$$



$$\boxed{X(z) = \frac{z}{z-a}}$$

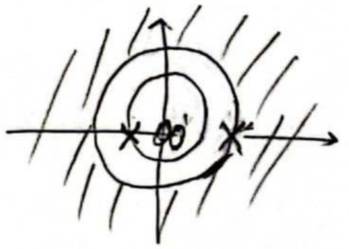
Same as earlier However, ROC is

→ Z transform consists of an algebraic form of $X(Z)$ and an ROC that says where this is valid.

→ **Eg:** $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{-1}{3}\right)^n u[n] \Rightarrow \frac{2z(z - 1/12)}{(z-\frac{1}{2})(z+\frac{1}{3})}$

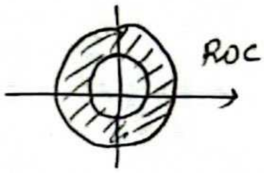
$\downarrow \quad \frac{z}{z-\frac{1}{2}}$ ROC: $|z| > \frac{1}{2}$

$\downarrow \quad \frac{z}{z+\frac{1}{3}}$ ROC: $|z| > \frac{1}{3}$



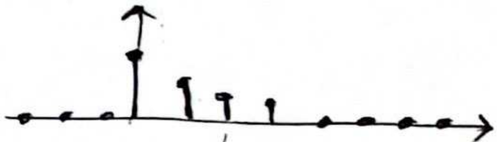
⇒ ROC $|z| > \frac{1}{2}$

$x[n] = \left(\frac{-1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1] \Rightarrow X(z) =$ same.



→ **Eg:** <u>Finite length exponential.</u>

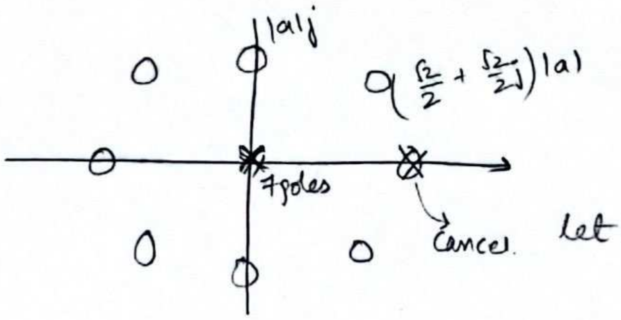
$$x[n] = \begin{cases} a^n & n \in [0, N-1] \\ 0 & \text{elsewhere.} \end{cases}$$



↳ ROC is not an issue since input is finite.

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \frac{1-(a/z)^N}{1-a/z} = \frac{z^N - a^N}{z^{N-1}(z-a)}$$

⇒ N-1 poles at $z = 0$
pole at $z = a$.
N zeroes at complex roots.



let $N = 8$

⇒ ROC is $|z| > 0$

7 poles & 7 zeroes.

Eg: $x[n] = 2^n \cos(3n)\, u[n]$

$$= 2^n \frac{e^{j3n} + e^{-j3n}}{2} u[n].$$

$$= \frac{1}{2}\left((2e^{3j})^n + (2e^{-3j})^n\right) u[n]$$

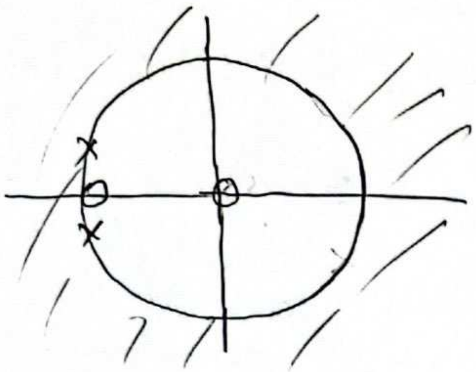
$$X(z) = \frac{1}{2}\left(\frac{1}{1-\frac{2e^{3j}}{z}}\right) + \frac{1}{2}\left(\frac{1}{1-2\frac{e^{-3j}}{z}}\right)$$

$$\vdots$$

$$= \frac{z^2 - (2\cos 3)z.}{z^2-(4\cos 3)z+4}$$

ROC: $x[n] = a^n u[n]$ was $|z| > |a|$

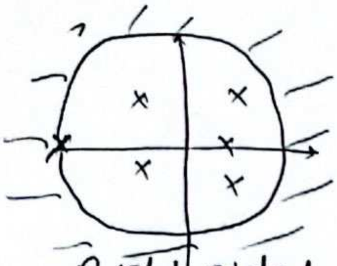
so $|a| = 2$ ⇒ $|z| > 2$ is the ROC.



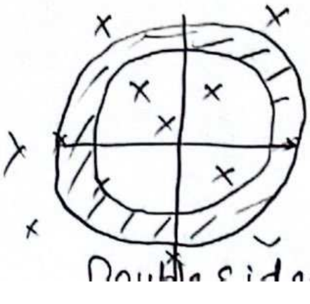
Rules about the ROC.

- > ROC is a ring / disc centred at the origin.
- > ROC contains no poles.
- > If $x[n]$ is finite length, ROC is entire z plane except possibly at $z=0$, $z=\infty$.
- >

Left sided



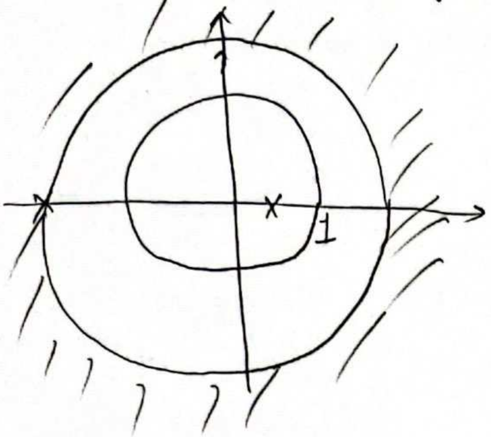
Right sided



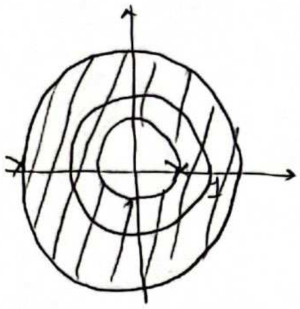
Double sided

› ROC for a T·F H(Z), can tell us about causality & stability.

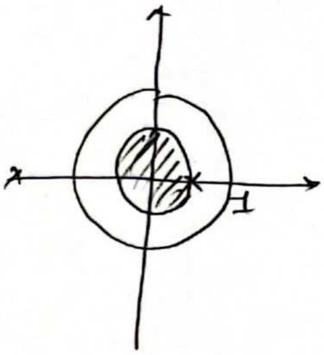
- DT FT exists!
- Stable ⇒ ROC includes unit circle.
- Causal ⇒ Impulse response is right sided. ⇒ ROC goes to ∞.



Causal
Not Stable.

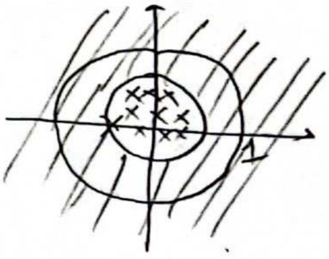


Stable
Not causal



Not stable
Not causal.

- Best case ⇒ Causal & stable ⇒ ROC goes to ∞ & contains the unit circle. & all poles lie inside the unit circle



Lec 9 Inverse Z Transform, Poles and Zeroes.

$$\alpha^n u[n] \xleftrightarrow{z} \frac{1}{1-\alpha z^{-1}} \qquad |z|>|\alpha|$$

$$\cos\omega_0 n\, u[n] \xleftrightarrow{z} \frac{1-(\cos\omega_0)z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}} \qquad |z|>1$$

$$r^n \sin\omega_0 n\, u[n] \xleftrightarrow{z} \frac{r\sin\omega_0\, z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}} \qquad |z|>r$$

Inverse Z transform.

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

↳ Complex contour integral for some $|z| = r$ in the RoC. ↳ Difficult to compute so we use patterns.

Eg 1

$$X(z) = \frac{7 - 13z^{-1}}{1 - 2z^{-1} - 3z^{-2}} \qquad |z| > 1$$

$$= \frac{7 - 13z^{-1}}{(1-3z^{-1})(1-z^{-1})}$$

$$= \frac{2}{1-3z^{-1}} + \frac{5}{1+z^{-1}}$$

We know it is right sided since ROC looks like a starburst

$$= 2\left(\frac{1}{3}\right)^n u[n] + 5(-1)^n u[n]$$

Eg 2 $X(z) = \dfrac{3z}{z^2+2z+4}$ Right sided signal. $\Longrightarrow$ roots are $-1 \pm \sqrt{3}j = \underset{\underset{r}{\downarrow}}{2} e^{\pm \underset{\underset{\omega_0}{\downarrow}}{\frac{2\pi}{3}} j}$

$$= \frac{3z^{-1}}{1+2z^{-1}+4z^{-2}} \longleftrightarrow \text{match with } r^n \sin \omega_0 n \, u[n]$$

$$x[n] = \sqrt{3}\left(2^n \sin \frac{2\pi}{3} n \, u[n]\right)$$

Eg:- $X(z) = 3z^{-2} + 5z^{-1} - \frac{1}{2} + 3z^3$ Roc: $0 < |z| < \infty$

Recall.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots + x[-1]z + x[-2]z^2 + \ldots$$

$$x[n] = -\frac{1}{2}\delta[n] + 5\delta[n-1] + 3\delta[n-2] + 3\delta[n+3].$$

Eg:- $X(z) = e^{z}$

Use power series.

$$X(z) = \underset{\substack{\downarrow \\ x(0)}}{1} + \underset{\substack{\downarrow \\ x(-1)}}{z} + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \dots$$

$x(0)$ $x(-1)$ .... $\Rightarrow$ left sided signal.

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) z^n.$$

$$\Rightarrow x[n] = \begin{cases} 0 & n \geq 1 \\ \frac{1}{(-n)!} & n \leq 0 \end{cases}$$

> Long division is another way to go.

$$X(z) = \frac{1+2z^{-1}}{1+z^{-1}}$$

$$= 1+z^{-1} \overline{\smash{)}\,1+2z^{-1}} \quad 1 + z^{-1} - z^{-2} \dots$$

$$\frac{1+z^{-1}}{}$$

$$z^{-1}$$

$$\frac{z^{-1}+z^{-2}}{}$$

$$-z^{-2}$$

Could use time shift

$$X(z) = \frac{1}{1+z^{-1}} + 2z^{-1}\frac{1}{1+z^{-1}}$$

$$x[n] = (-1)^n u[n] + 2(-1)^{n-1}u[n-1]$$

Properties of Z transform.

1) Linearity.

2) Time Shift: $x[n-n_0] \longleftrightarrow z^{-n_0}X(z)$

3) Scaling: $a^n x[n] \longleftrightarrow X\left(\frac{z}{a}\right)$

4) Time reversal: $x[-n] \longleftrightarrow X\left(\frac{1}{z}\right)$

*5) Convolution: $x[n] * h[n] \longleftrightarrow X(z)H(z)$

6) Differentiation: $n\,x[n] \longleftrightarrow -z\,\dfrac{dX(z)}{dz}$

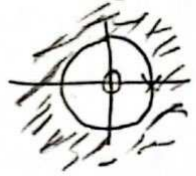
7) Initial value theorem: $x(0) = \lim_{z \to \infty} X(z)$.

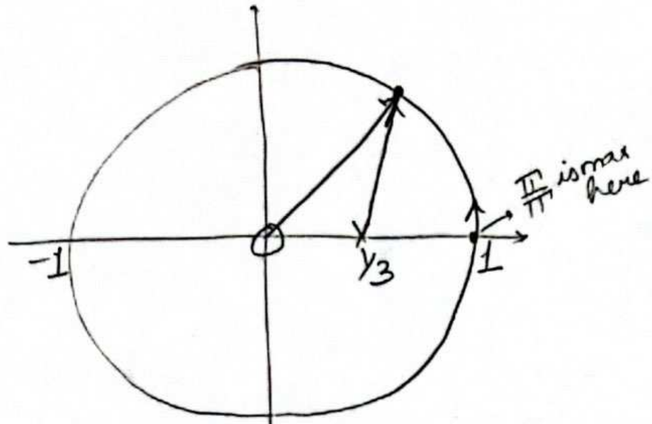
$$\underset{\text{Transfer Function}}{H(z)} = \frac{N(z)}{D(z)} \leftarrow \text{polynomials}$$

$N(z) = 0 \Rightarrow H(z) = 0 \Rightarrow$ zeroes.

$D(z) = 0 \Rightarrow H(z) = \infty \Rightarrow$ poles.

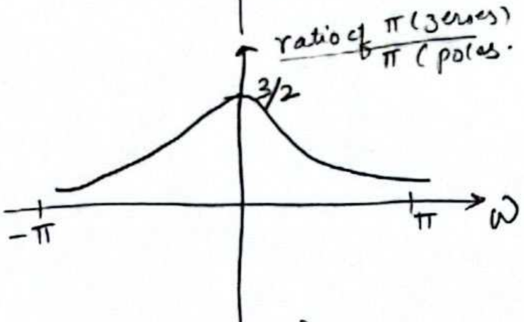
$h[n] = \left(\frac{1}{3}\right)^n u[n]$ $\qquad H(z) = \dfrac{1}{1 - \frac{1}{3}z^{-1}} = \dfrac{z}{z - \frac{1}{3}}$ $\qquad |z| > \frac{1}{3}$





$$\left|H(e^{j\omega_0})\right| = \left|\frac{N(e^{j\omega_0})}{D(e^{j\omega_0})}\right|$$

$$= \frac{\prod \left(\text{length of vector from each zero to } e^{j\omega_0}\right)}{\prod \left(\text{length of vector from each pole to } e^{j\omega_0}\right)}$$

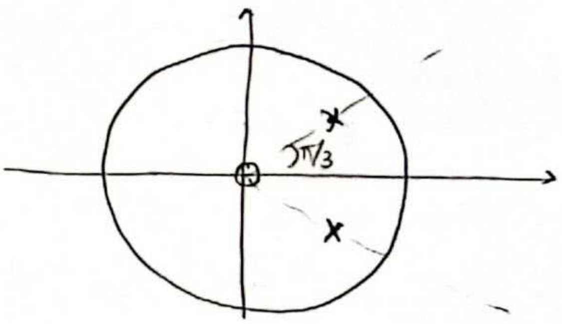


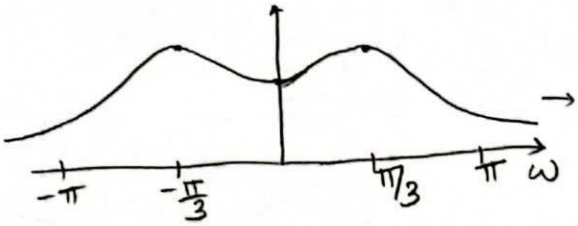
Imagine the zero is a "nail" on the fabric & the pole is a "tentpole". The height of the fabric on the unit circle follow $\frac{\pi}{\pi} \Rightarrow |H|$.

↳ Frequency response magnitude

→ $h[n] = \left(\frac{5}{6}\right)^n \sin\left(\frac{\pi}{3}n\right) u[n]$

$$H(z) = \frac{\frac{5}{6}\sin\frac{\pi}{3} z^{-1}}{1 - \frac{10}{6}\cos\frac{\pi}{3} z^{-1} + \left(\frac{5}{6}\right)^2}$$





→ Crude BPF

→ Looking at pole zero plot can give intuition about freq. response.

→ z transform is also used for solving difference equations.

$$y[n] + \frac{1}{4} y[n-1] = x[n] + \frac{1}{5} x[n-1]$$

$$\Rightarrow Y(z) + \frac{1}{4} z^{-1} Y(z) = X(z) + \frac{1}{5} z^{-1} X(z).$$

$$\Rightarrow H(z) = \frac{1 + \frac{1}{5} z^{-1}}{1 + \frac{1}{4} z^{-1}} \qquad |z| > \frac{1}{4}$$

If input is unit step?

$X(z) = \frac{1}{1-z^{-1}}$ ⇒ $Y(z) = H(z) \cdot X(z)$. & use partial fractions & inverse z transform.

Lec 10 Discrete Fourier Transform.

	Continuous	Discrete
Periodic	FS	1) DFT or DTFS 2) FFT
Nonperiodic	FT	DTFT

Intuition :- Recall Fourier series.

$$x(t) = \sum_{K=-\infty}^{\infty} \underset{a_K}{X[K]}\, e^{jk\frac{2\pi}{T}t} \qquad t \in [0,T]$$

What about periodic discrete signals?

$$x[n] = \sum_{K=-\infty}^{\infty} X[k]\, e^{jk\frac{2\pi}{N}n} \qquad n = 0, 1, \dots, N-1$$

$$e^{jk\frac{2\pi}{N}(n+N)} = e^{jk\frac{2\pi}{N}n} \cdot e^{jk2\pi}$$

$$= e^{jk\frac{2\pi}{N}n}.$$

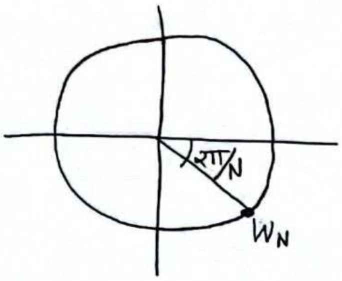
⇒ There are only N unique complex exponentials of period N.

Define. the discrete fourier transform (DFT) as :

$$X[K] = \sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n} \qquad K = 0, 1, \dots, N-1$$

$$x[n] = \frac{1}{N}\sum_{K=0}^{N-1} X[k]\, e^{jk\frac{2\pi}{N}n} \qquad n = 0, 1, \dots, N-1$$

Let $W_N = e^{-j\frac{2\pi}{N}}$



$(W_N)^N = 1$ ⇒ W_N is the Nth root of 1.

Therefore DFT:

$$X[K] = \sum_{n=0}^{N-1} x[n]\, W_n^{kn}$$

$$x[n] = \frac{1}{N}\sum_{K=0}^{N-1} X[K]\, W_N^{-kn}$$

$W_4 = -i$ $\quad W_4^3 = i$ $\quad$ The 4 roots of 1 are W_4, W_4^2, W_4^3, W_4^4.

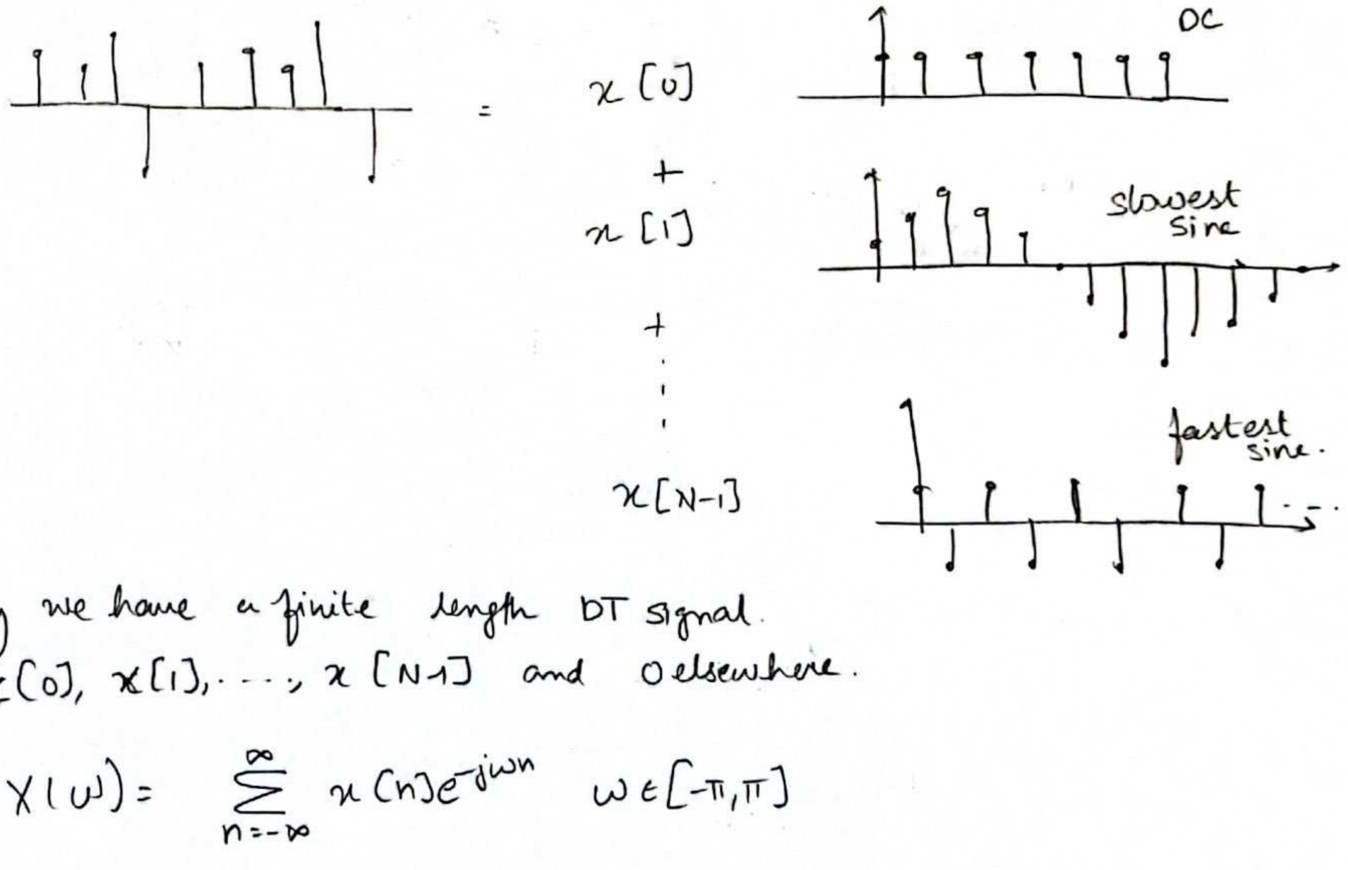
$W_4^2 = -1$ $\quad W_4 = 1$

→ We are going from N numbers in n to N numbers in K, which is simply a change of basis in N^th dimensional vectors in complex plane:

$$\underbrace{\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ \vdots \\ X[N-1] \end{bmatrix}}_{\substack{N \times 1 \\ \text{complex} \\ \text{vector}}} = \underbrace{\begin{bmatrix} W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & \cdots & W_N^{2(N-1)} \\ \vdots & & & \\ W_N^0 & \cdots & \cdots & W_N^{(N-1)^2} \end{bmatrix}}_{\substack{N \times N \\ \text{complex} \\ \text{matrix}}} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ \vdots \\ x[N-1] \end{bmatrix}}_{\substack{N \times 1 \\ \text{Real} \\ \text{vector}}} \rightarrow \text{Could be complex.}$$

$$\underset{N\times 1}{X} = \underset{N\times N}{F} \underset{N\times 1}{x}$$

F is orthogonal ⇒ columns are all perpendicular.



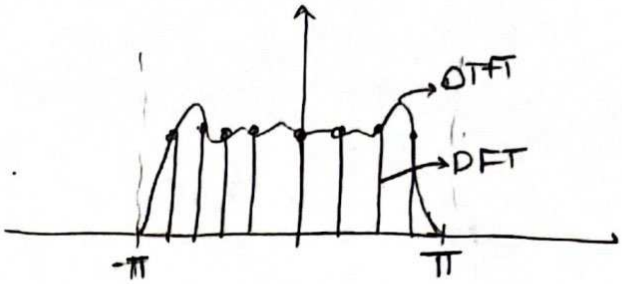
→ Say we have a finite length DT signal. x[0], x[1], ..., x[N-1] and 0 elsewhere.

DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \qquad \omega \in [-\pi, \pi]$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \rightarrow \text{Same as DFT if } \omega = \frac{2\pi}{N} k$$

$$\therefore \underset{\text{DFT}}{X[K]} = X\left(\frac{2\pi k}{N}\right)$$



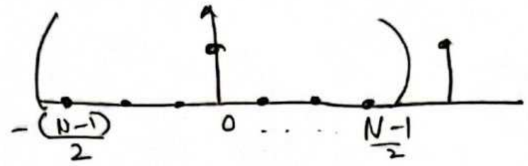
⇒ Sampling the DTFT gives DFT.

⇒ DFT is the DTFT of a discrete time finite length signal evaluated at N equally spaced points $\omega = \frac{2\pi k}{N}$, $K = 0, 1, \ldots N-1$.

Example DFTs.

> In DFTs both $x[n]$ and $X[K]$ are always period. $x[n]$ has period N and $X[K]$ has period 2π

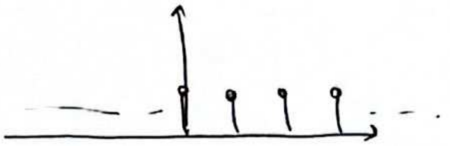
> Eg:- $\delta[n] \rightarrow$



$$X[K] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

$$= W_N^{0k} = 1 \quad \forall k.$$

⇒



> Eg:- Constant signal $x[n] = 1$

$$X[K] = \sum_{n=0}^{N-1} W_N^{nk}$$

$$= W_N^{0} + W_N^{k} + W_N^{2k} \cdots + W_N^{(N-1)k}$$

$$\text{if } K \neq 0 \Rightarrow \frac{1-(W_N^k)^N}{1-W_N^k} = \frac{1-W_N}{1-W_N^k} = \frac{0}{1-W_N^k} = 0$$

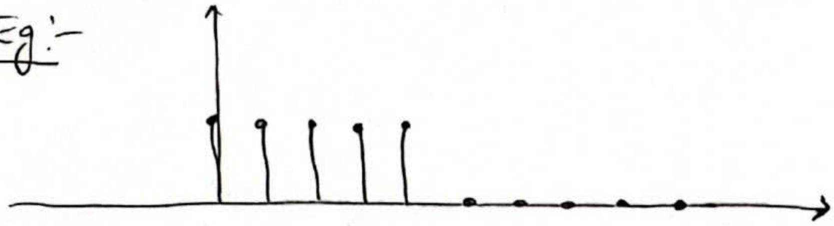
$$\text{if } K = 0 \Rightarrow X[K] = N.$$

$$\Rightarrow X[K] = \begin{cases} N & \text{when } k = 0 \\ 0 & \text{when } k \neq 0 \end{cases}$$

→ Orthogonality property

$$\sum_{n=0}^{N-1} W_N^{Mn} = \begin{cases} N & \text{if } M \text{ is an integer multiple of } N \\ 0 & \text{otherwise} \end{cases}$$

Eg:-



DTFT

$$X(\omega) = \sum_{n=0}^{4} e^{-j\omega n} = \frac{1-e^{-5j\omega}}{1-e^{-j\omega}} = \frac{e^{-5j\omega/2}\left(e^{5j\omega/2}-e^{-5j\omega/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2}-e^{-j\omega/2}\right)}$$

$$= e^{-2j\omega} \cdot \frac{\sin\left(\frac{5\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

Assume it was length 10 discrete time period signal.

$$\Rightarrow X[k] = \sum_{n=0}^{9} x[n] \underbrace{e^{-j\frac{2\pi}{10}kn}}_{W_{10}}$$

$$= \sum_{n=0}^{4} e^{-j\frac{2\pi}{10}kn} = \sum_{n=0}^{4} W_{10}^{kn} = \frac{1-W_{10}^{5k}}{1-W_{10}}$$

$$= \frac{1-e^{-jk\frac{2\pi 5}{10}}}{1-e^{-jk\frac{2\pi}{10}}}$$

$$= \frac{\left(e^{jk\frac{\pi}{10}}-e^{-jk\frac{9\pi}{10}}\right)\left(e^{-jk\frac{\pi}{10}}\right)}{e^{-jk\frac{\pi}{10}}\left(e^{jk\frac{\pi}{10}}-e^{-jk\pi/10}\right)} = e^{-j\frac{4k\pi}{10}} \cdot \frac{\sin\frac{\pi k}{2}}{\sin\frac{\pi k}{10}}$$

(Same as DTFT evaluated at $\omega = \frac{2\pi k}{10}$

Properties.

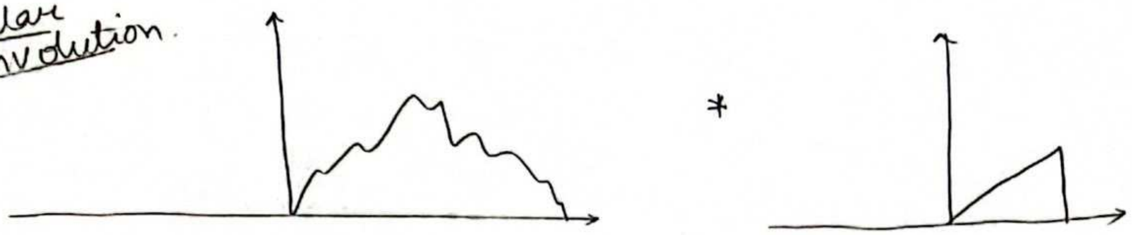
- Linearity, Symmetry
- Shifting.

$$x[n-m] \xleftrightarrow{DFT} W_N^{km} \cdot X[k].$$

↳ This is a cyclic shift since signal is periodic.

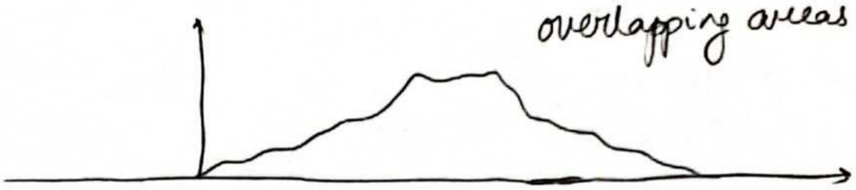
Cyclic Convolution. ⊛

Regular convolution.

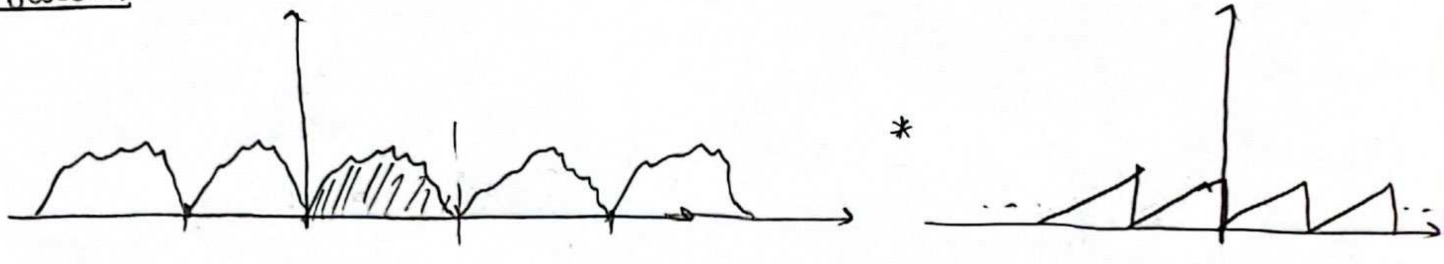


↳ Flip, Drag & add the overlapping areas.

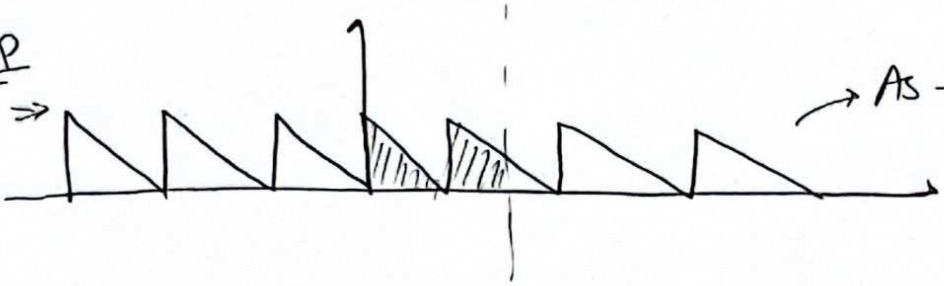
Result might look →



Cyclic convolution



Flip



→ As this moves from -∞ to ∞, the convolution also becomes periodic. since signals wrap around.

∴ For DFT convolution ⟷ Multiplication property holds but they are cyclic.

$$x[n] \circledast h[n] \longleftrightarrow X[k]H[k].$$

(all are length N signals)

→ How to get regular convolution we need for LTI systems.

$$\left.\begin{array}{cccccc} x[0] & x[1] & \cdots & & x[N-1] & \\ h[0] & h[N-1] & \cdots & h[2] & h[1] & \end{array}\right\} \rightarrow \text{gives } y[0].$$

$$\begin{array}{ccccc} h[1] & h[0] & \cdots & h[3] & h[2] \end{array} \rightarrow \text{gives } y[1]$$

$$\therefore y[0] = h[0]x[0] + h[N-1]x[1] + \cdots + h[1]x[N-1]$$

$$y[1] = h[1]x[0] + h[0]x[1] + \cdots + h[2]x[N-1]$$

$$\vdots$$

$$y[N-1] = h[N-1]x[0] + \cdots + h[0]x[N-1].$$

matrix

$$\begin{bmatrix} y[0] \\ \vdots \\ \vdots \\ y[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & h[N-1] & \cdots & h[1] \\ h[1] & h[0] & \cdots & h[2] \\ \vdots & \ddots & \ddots & \ddots \\ h[N-1] & h[N-2] & \cdots & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ \vdots \\ \vdots \\ x[N-1] \end{bmatrix}$$

↳ Circulant matrix N values are repeated along diagonals.

→ This matrix vector product corresponds to circular convolution.

→ Regular convolution.

$$y[0] = x[0]h[0]$$

$$y[1] = x[0]h[1] + x[1]h[0]$$

$$y[2] = x[0]h[2] + x[1]h[1] + x[2]h[0]$$

$$\vdots$$

$$\underset{7\times1}{\begin{bmatrix} y[0] \\ \vdots \\ \vdots \\ \vdots \\ y[6] \end{bmatrix}} = \underset{7\times4}{\begin{bmatrix} h[0] & 0 & 0 & 0 \\ h[1] & h[0] & 0 & 0 \\ & \vdots & \ddots & \\ 0 & 0 & 0 & h[3] \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}}$$

Assuming $x[0] \cdots x[3]$
$h[0] \cdots h[3]$.

make it 7×7 ⇒ zero padding.

Can add anything here since it multiplies to zeroes.

$$\begin{bmatrix} y[0] \\ \vdots \\ \vdots \\ y[6] \end{bmatrix} = \left[\begin{array}{cccc|ccc} h[0] & 0 & 0 & 0 & \underbrace{h[3] & h[2] & h[1]} \\ h[1] & h[0] & 0 & 0 & & & \\ & \vdots & \ddots & & & & \\ 0 & 0 & 0 & h[3] & h[2] & h[1] & h[0] \end{array}\right] \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now this looks like a circulant matrix.
↳ But still it is linear convolution.

⇒ To do linear convolution with DFT.

$h[n]$ has length N. ⟶ zero pad with $M-1$ zeros ⟶ $N+M-1$ DFT

$x[n]$ has length M. ⟶ zero pad with $N-1$ zeros ⟶ $N+M-1$ DFT

$\otimes$ ⟶ $N+M-1$ DFT

Inverse ↓

$h * x$
linear convolution

Lec 11 Radix-2 Fast Fourier Transform.

Revisiting the DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

$n = 0, 1, \ldots\ldots, N-1$
$K = 0, 1, \ldots\ldots, N-1$

$X[k]$ ↓ N terms. ↓ ②

$$= \sum_{n=0}^{N-1} \underbrace{x[n]\, W_N^{kn}}_{N \text{ multiplies} — ①}$$

where $W_N = e^{-j\frac{2\pi}{N}}$
↳ N^{th} root of 1

> $W_N, W_N^2, W_N^3, \ldots\ldots, W_N^N$ are N^{th} roots of 1.

From ① and ② we see we have N^2 complex multiplies. and $N(N-1)$ complex additions. Order of operations is around N^2.

> **FFT** :- Reduce the number of operations to $O(N \log_2 N)$.

Main ideas:- > Decompositions into smaller DFTs.
> Simplifications $W_N^{(kN)} = 1$

$$W_N^{N/2\,(\text{odd } k)} = -1$$

> $$W_N^{n(k+N)} = W_N^{k(n+N)} = W_N^{kn} \rightarrow \text{periodicities.}$$

> $$W_N^{k(N-n)} = W_N^{-nk} = (W_N^{kn})^*$$

Decimation in Time (N is even) ↳ even number

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$$

$$= \sum_{n\,\text{even}} x[n]\, W_N^{nk} + \sum_{n\,\text{odd}} x[n]\, W_N^{nk}$$

⇒ $n = 2r$

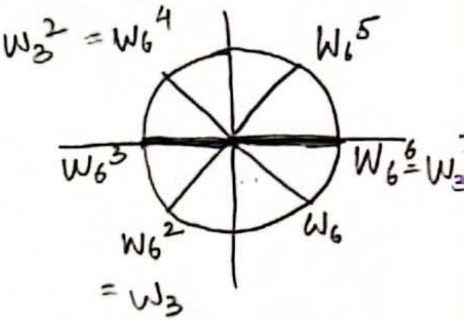
$$= \sum_{r=0}^{N/2-1} x[2r]\, W_N^{2rk} + \sum_{r=0}^{N/2-1} x[2r+1]\, W_N^{(2r+1)k}$$

$$= \sum_{r=0}^{N/2-1} x[2r](W_N^2)^{rk} + W_N^K \sum_{r=0}^{N/2-1} x[2r+1](W_N^2)^{rk}$$

$$= \underbrace{\sum_{r=0}^{N/2-1} x[2r](W_{N/2})^{rk}}_{\text{Length } N/2 \text{ DFT of the even entries.}} + W_N^K \underbrace{\sum_{r=0}^{N/2-1} x[2r+1] W_{N/2}^{rk}}_{N/2 \text{ DFT of the odd entries.}}$$

$$G[K] + W_N^K H[K]$$

Say N = 6

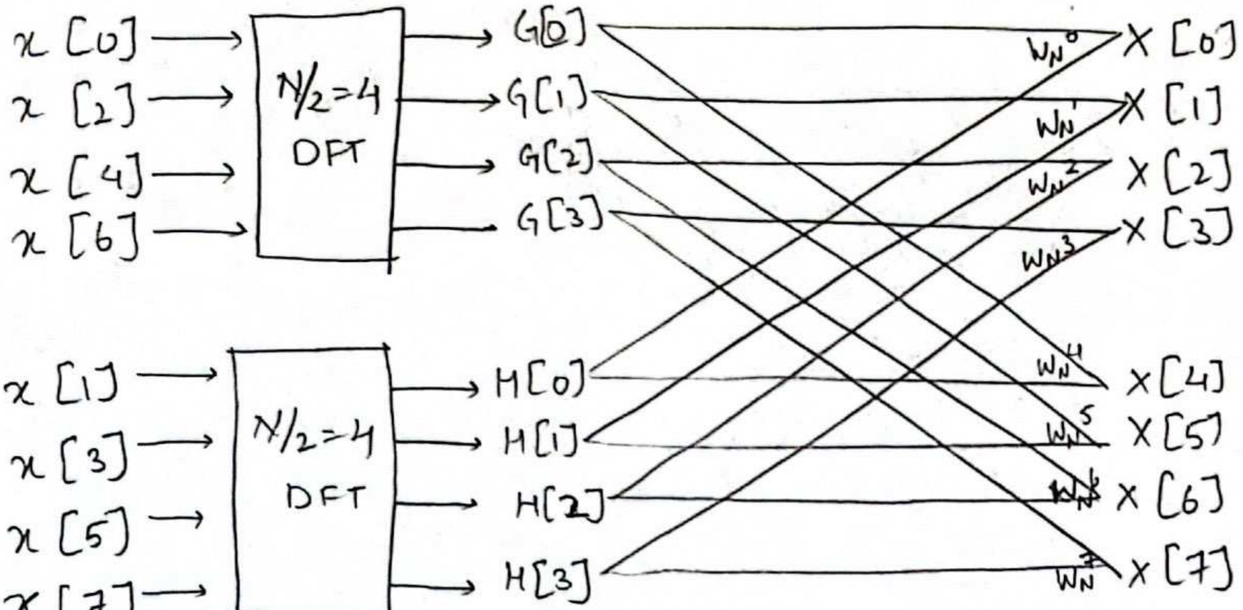


$$\Rightarrow W_N^2 = W_{N/2}$$

$$X[K] = G[K] + W_N^K H[K] \quad , K = 0, 1, \ldots N-1$$

N = 8 pt DFT

$(\frac{N}{2})^2$ multiplications.

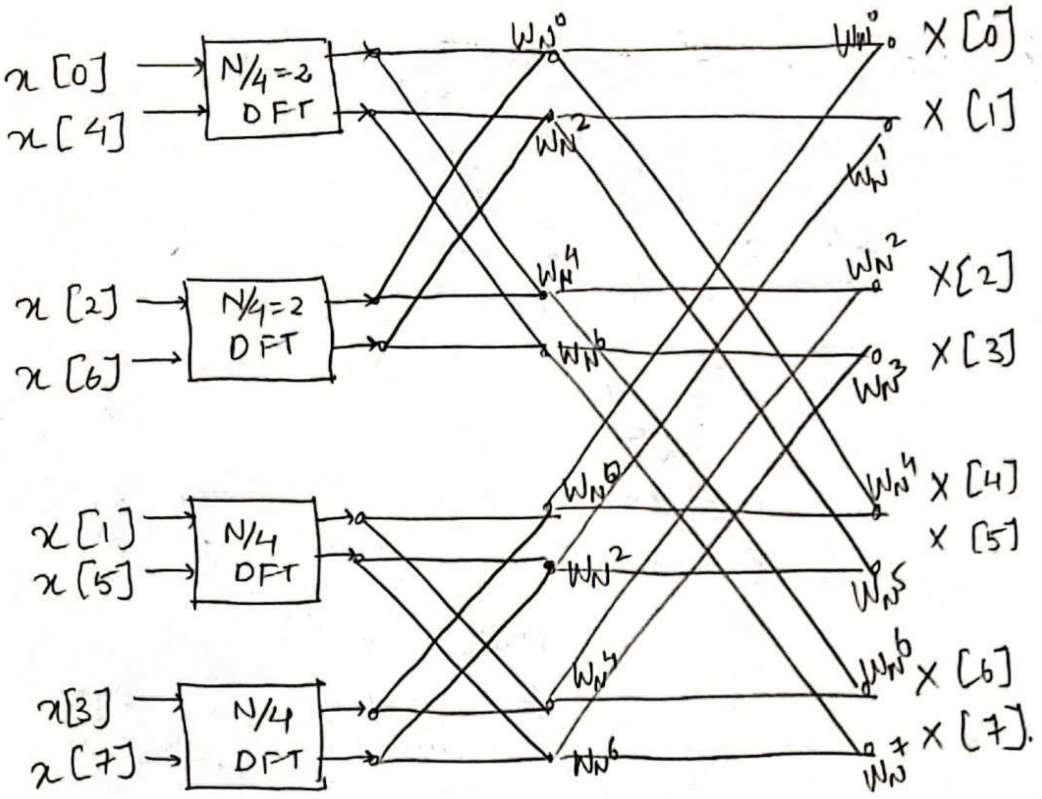


$(\frac{N}{2})^2$ multiplications

N multiplications.

$$\Rightarrow \frac{N^2}{4} + \frac{N^2}{4} + N \simeq \frac{N^2}{2} \text{ multiplications.}$$

→ Could redo this again as an FFT.



Length 2 DFT :

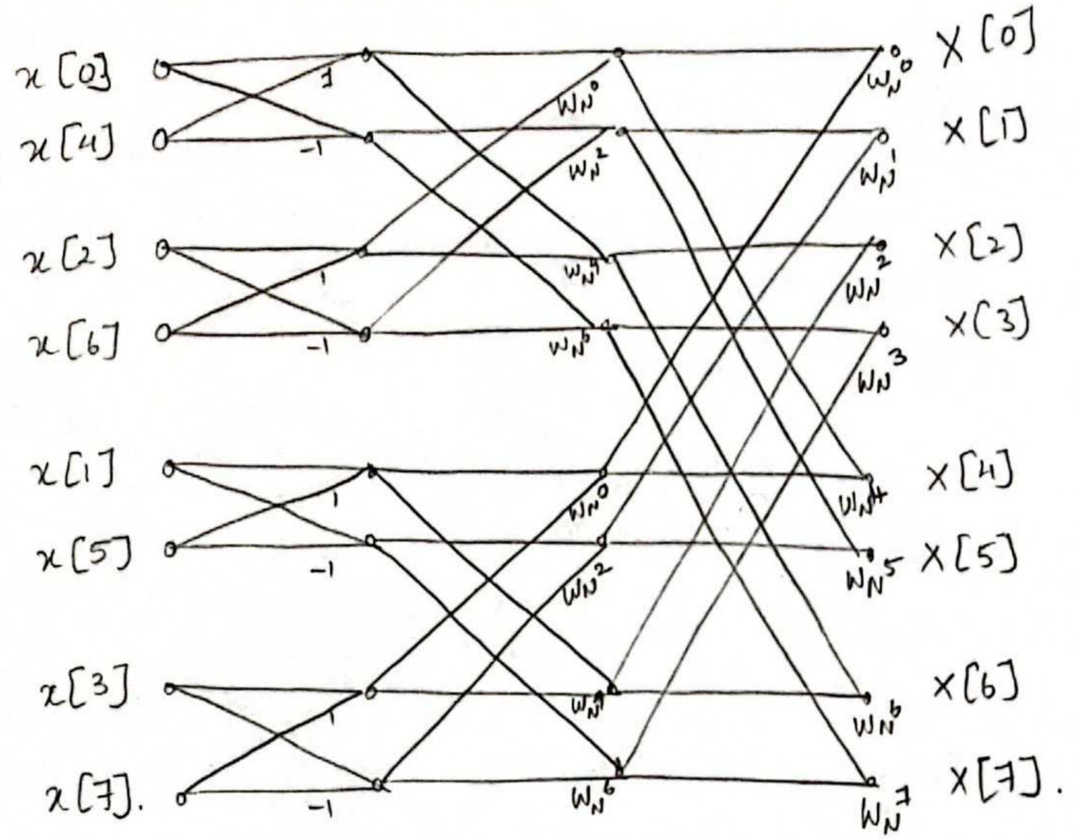
$x[0], x[1] \longrightarrow X[0], X[1].$

$$X[k] = \sum_{n=0}^{1} x[n]\, W_2^{nk}.$$

$$= \sum_{n=0}^{1} x[n]\, (-1)^{nk}.$$

$$\left.\begin{aligned} X[0] &= x[0] + x[1]. \\ X[1] &= x[0] - x[1] \end{aligned}\right\} \text{No multiplication.}$$

PTO.

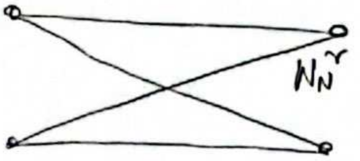


Butterfly diagram.

→ If $N = 2^k \Rightarrow$ we need k stages. $\Rightarrow \log_2 N$ stages.

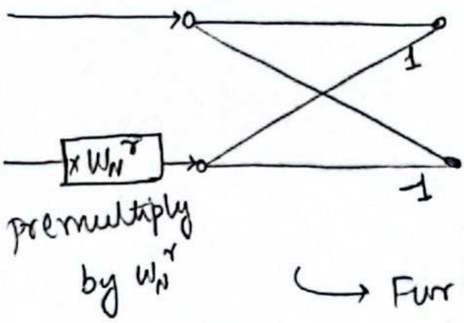
→ Each stage has N computations (multiplies) $\Rightarrow \boxed{N \log_2 N}$

→ Notice, All computations are of the form.



$$W_N^{r+(N/2)} = W_N^r \cdot W_N^{N/2} = -W_N^r$$

Butterfly



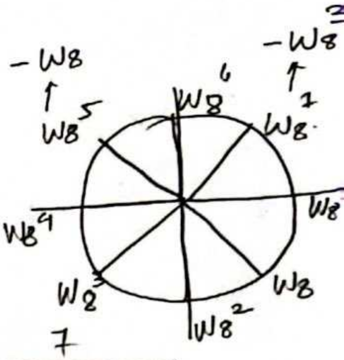
→ Also computations can simply update the elements of the length N vector after each stage, saving memory space.

↪ Further reduces multiplications by a factor of 2.

→ Also notice the order of inputs in binary.

$x[0]$	000		000
$x[4]$	100		001
$x[2]$	010		010
$x[6]$	110	$\xrightarrow[\text{order}]{\text{reverse}} \Rightarrow$	011
$x[1]$	001		100
$x[5]$	101		101
$x[3]$	011		110
$x[7]$	111		111

⇒ "Bit reversed order" is very easy to achieve.



→
$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix} = \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \hline \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_8 & -j & W_8^3 & -1 & -W_8^3 & j & -W_8 \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & W_8^3 & j & W_8 & -1 & -W_8^3 & -j & -W_8 \\ \hline 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -W_8 & W_8^2 & -W_8^3 & -1 & W_8 & -W_8^2 & W_8^3 \\ 1 & -W_8^2 & -1 & W_8^2 & 1 & -W_8^2 & -1 & W_8^2 \\ 1 & -W_8^3 & -W_8^2 & -W_8 & -1 & W_8^3 & W_8^2 & W_8 \end{array}\right] \end{array} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}$$

Even columns ⇒ top half = bottom half.

Odd columns ⇒ top half = -ve bottom half.

→ Could rewrite.

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix} = \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} 1 & & & \\ & W^8 & & \\ & & W_8^2 & \\ & & & W_8^3 \end{bmatrix} \end{bmatrix} \begin{bmatrix} F_4 & 0 \\ 0 & F_4 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \\ x[4] \\ x[6] \\ x[1] \\ x[3] \\ x[5] \\ x[7]. \end{bmatrix}$$

$\downarrow$ **Proof**

This process is called decimation in time.

Even columns.

$$\begin{bmatrix} I_{4\times4} \\ I_{4\times4} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_8^2 & -1 & -W_8^2 \\ 1 & -1 & 1 & -1 \\ 1 & -W_8^2 & -1 & W_8^2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} I \\ I \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ 1 & -1 & -1 & -1 \\ 1 & -W_4 & -1 & W_4 \end{bmatrix} \rightarrow \text{same as DFT matrix for } N=4.$$

same.

Odd columns.

$$\begin{bmatrix} I_{4\times4} \\ -I_{4\times4} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ W_8 & W_8^3 & -W_8 & -W_8^3 \\ W_8^2 & -W_8^2 & W_8^2 & -W_8^2 \\ W_8^3 & W_8 & -W_8^3 & -W_8 \end{bmatrix} = \begin{bmatrix} I \\ -I \end{bmatrix} \begin{bmatrix} 1 & & & \\ & W_8 & & \\ & & W_8^2 & \\ & & & W_8^3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ 1 & -1 & -1 & -1 \\ 1 & -W_4 & -1 & W_4 \end{bmatrix}$$

$$\therefore F_8 = \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} I & \\ & \begin{bmatrix} 1 & & & \\ & W^8 & & \\ & & W_8^2 & \\ & & & W_8^3 \end{bmatrix} \end{bmatrix} \begin{bmatrix} F_4 & \\ & F_4 \end{bmatrix} \begin{bmatrix} X_{even} \\ X_{odd} \end{bmatrix}$$

length 4 DFTs.

Butterfly

Decimation in frequency FFT

→ Shuffle output order instead

$$X[K] = \sum_{n=0}^{N-1} x[n] \, W_N^{nk}.$$

Even samples of $X[K]$: $X[2r]$ $r = 0, 1, \ldots \frac{N-1}{2}$

$$X[2r] = \sum_{n=0}^{N-1} x[n] \, W_N^{2nr}$$

$$= \sum_{n=0}^{N/2-1} x[n] \, W_N^{2nr} + \sum_{n=\frac{N}{2}}^{N-1} x[n] \, W_N^{2nr}$$

$$= \sum_{n=0}^{N/2-1} x[n] \, W_{N/2}^{nr} + \sum_{n=0}^{N/2-1} x[n+N/2] \, W_N^{2(n+\frac{N}{2})r}.$$

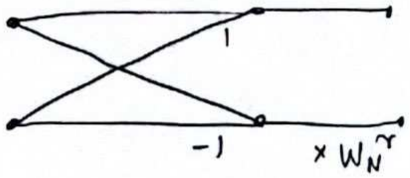
$$= \sum_{n=0}^{N/2-1} \left(x[n] + X[n+N/2]\right) W_{N/2}^{nr}$$

→ Like N/2 DFT of summed input (top half + bottom half).

→ Similarly for odd entries of $X[K]$, can show,

$$X[2r+1] = \sum_{n=0}^{N/2-1} \left(x[n] - x[n+\tfrac{N}{2}]\right) \underbrace{W_N^{n}}_{\substack{\text{twiddle} \\ \text{factor.}}} W_{N/2}^{nr}$$

Butterfly



Decimation in Frequency.

Lec 12 Cooley Tukey and Good Thomas FFT

> If n is small, we can zero pad to get to 2^{n}.
> What if n is large?

Cooley Tukey.

DFT of length N

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

$n = 0, 1, \ldots, N-1$
$K = 0, 1, \ldots N-1$
$W_N = e^{\frac{-2\pi}{N} j}$

Assume $N = n_1 n_2$

Write $n = n_1 i + j$ $\quad i = 0, 1, \ldots n_2 - 1$
$j = 0, 1 \ldots n_1 - 1$

$N = 12 \Rightarrow n_1 = 3;\ n_2 = 4$

$n = 3i + j$ $\quad i = 0, 1 \ldots 3$
$j = 0, 1 \ldots 2$

	$j=0$	$j=1$	$j=2$
$i=0$	0	1	2
⋮	3	4	5
	6	7	8
$i=3$	9	10	11

> Write $K = n_2 a + b$ $\quad a \in 0, 1, \ldots, n_1 - 1$
$b \in 0, 1, \ldots, n_2 - 1$

$N = 12,\ n_1 = 3,\ n_2 = 4$
$K = 4a + b$

	$b=0$ . . .			$b=3$
$a=0$	0	1	2	3
$a=1$	4	5	6	7
$a=2$	8	9	10	11

> Substituting for n and k.

$$X[k] = \sum_{i=0}^{n_2-1} \sum_{j=0}^{n_1-1} x[n_1 i + j]\, W_N^{K(n_1 i + j)}$$

$$X[n_2a+b] = \sum_{i=0}^{n_2-1}\sum_{j=0}^{n_1-1} x[n_1 i + j]\, W_N^{(n_2a+b)(n_1i+j)}$$

Taking $W_N^{(n_2a+b)(n_1i+j)} = \underbrace{\underbrace{W_N^{\overbrace{n_1n_2}^{N}ai}}_{1 \text{ since } W_N^N = 1}}_{1} \; W_N^{n_1ib} \; W_N^{n_2aj} \; W_N^{bj}$

$$W_N = e^{-\frac{2\pi}{N}\overset{\sqrt{-1}}{j}} \Rightarrow W_N^{n_1} = W_{n_1n_2}^{n_1}$$

$$e^{-\frac{2\pi}{\cancel{n_1}n_2}j\cancel{n_1}} = e^{-\frac{2\pi}{n_2}j} = W_{n_2}$$

$$\therefore \; W_N^{(n_2a+b)(n_1i+j)} = W_{n_2}^{ib}\, W_{n_1}^{aj}\, W_N^{bj}$$

$$X[n_2a+b] = \boxed{\sum_{j=0}^{n_1-1} \boxed{\sum_{i=0}^{n_2-1} x[n_1i+j]\, W_{n_2}^{ib}}\; W_N^{bj}\, W_{n_1}^{aj}}$$

→ Twiddle Factor

→ An n_2 length DFT with fixed j

→ An n_1 length DFT.

→ We need n_1 length n_2 DFTs
n_2 length n_1 DFTs
N multiplications by Twiddle factors.

→ Assuming naive smaller length DFTs.

$$n_1(n_2^2) + n_2(n_1^2) + N$$

$$= N(n_1 + n_2 + 1).$$

In general, if $N = \prod_{i=1}^{l} n_i$

mults = $N(\sum_{i=1}^{l} n_i)$

→ If $N = 2^r$, Radix 2 FFT had $O(N \log_2 N)$ mults.

Cooley Tukey FF is like a generalization with some kind of efficiency but N need not be 2^r.

Algorithm

→ $N = 15$, $n_1 = 5$, $n_2 = 3$.

$$x = \begin{bmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ 14 \end{bmatrix} \rightarrow$$

	j=0	1	2	3	4
i=0	0	1	2	3	4
1	5	6	7	8	9
2	10	11	12	13	14

(i, j)

Step ①

→ Do 5 length 3 DFTs down the columns.

Step ②

→ Multiply the 3 × 5 array by a 3 × 5 element wise array W_N^{bj} $b = 0, 1, 2$. $j = 0 \dots 4$.

Step ③

→ Do 3 length 5 DFTs.

Step ④

→ Put 2D elements back into a 1D vecto in a different order.

0	3	6	9	12
1	4	7	10	13
2	5	8	11	14

→ Could further reduce/decompose each row/column DFTs

→ Break down DFT to prime factors.

→ How to get rid of the Twiddle Factor term? L6

Good Thomas FFT

Math stuff

1) If a and b are integers, not both 0, then the greatest common divisor $(a,b) = \gcd(a,b)$ exists. There are integers m_0 and n_0 so that $(a,b) = m_0 a + n_0 b$

If $(a,b) = 1$, a and b are called relatively prime/coprime.

There exists $m_0, n_0 \in \mathbb{Z}$ S.T $1 = m_0 a + n_0 b$

Eg:- 3,4 ⇒ gcd = 1 ; $(-5)3 + (4)4 = 1$

MATLAB

```
>> [G,C,D] = gcd(3,4)
>> G = 1 ; C = -1 ; D = 1
```

$$C = -1 \;(m_0) \; ; \; D = 1 \;(n_0)$$

> Chinese remainder theorems.

Thm 1) Given a set of integers, $m_0, m_1, \ldots m_K$ that are pair-wise relatively prime, and a set of integers $C_0, C_1, \ldots, C$ with $0 \le C_i < m_i$; then the system $C_i = C \bmod \overset{\to\text{remainder.}}{m_i}$ $i = 0 \ldots k$ has at most one solution for C in the interval $0 \le C \le \prod_{i=0}^{K} m_i$.

Eg:- $m_0 = 3$; $m_1 = 4$; $m_2 = 5$

$3.4.5 = 60$.

Find x so that

$x \bmod 3 = \overset{\text{remainder.}}{1}$ → 3 choices.

$x \bmod 4 = 2$ → 4 choices

$x \bmod 5 = 0$ → 5 choices.

} 60 choices.

$\underbrace{x \bmod 5 = }$ → In this case $x = 10$.

0 → ⇒ x has to be a multiple of 5.

$x \in [0, 1, \dots, 59]$

Thm 2) Let $M = \prod_{i=0}^{k} m_i$ be a product of relatively prime integers.

Let $M_i = \frac{M}{m_i}$

M= 60

m_i	M_i
3	20
4	15
5	12

Let N_i satisfy

$\underbrace{N_i M_i + n_i m_i}_{\text{Relatively prime}} = 1$

, Then $C_i = C \bmod m_i$ is uniquely solved

$i = 0, 1, \dots k$

by :

$$C = \left(\sum_{i=0}^{k} C_i N_i M_i \right) \bmod M.$$

$M = 60$,

m_i	M_i	N_i	n_i	$M_i N_i$
3	20	-1	7	-20
4	15	-1	4	-15
5	12	-2	5	-24

↑ to go from remainder to big

Eg:- $C \bmod 3 = 2$

$C \bmod 4 = 1$

$C \bmod 5 = 2$.

$C = 2(-20) + 1(-15) + 2(-24). = -103 \bmod 60$

$= \underline{17}$

Going back to FFT

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$$

Write $n = i(N_2 n_2) + j(N_1 n_1)$,

Let $N = n_1 n_2$ where n_1, n_2 are coprime.

$i = n \bmod n_1$ $\quad i = 0, 1, \ldots, n_1 - 1$

$j = n \bmod n_2$ $\quad j = 0, 1, \ldots, n_2 - 1$

$k_1 = k N_2 \bmod n_1$

$k_2 = k N_1 \bmod n_2$

$k = [n_2 k_1 + n_1 k_2] \bmod N$.

$$X[n_2 k_1 + n_1 k_2] = \sum_{j=0}^{n_2-1} \sum_{i=0}^{n_1-1} x[i N_2 n_2 + j N_1 n_1]\, W_N^{(n_2 k_1 + n_1 k_2)(i N_2 n_2 + j N_1 n_1)}$$

$$W_N^{i k_1 n_2^2 N_2}, \quad W_N^{j k_2 n_1^2 N_1}$$

$$\Downarrow \qquad \Downarrow$$

$$W_{n_1}^{i k_1} \qquad W_{n_2}^{i k_2}$$

Note, $W_N^{n_1^2 N_1} = (W_N^{n_1})^{n_1 N_1} = W_{n_2}^{n_1 N_1} = W_{n}$

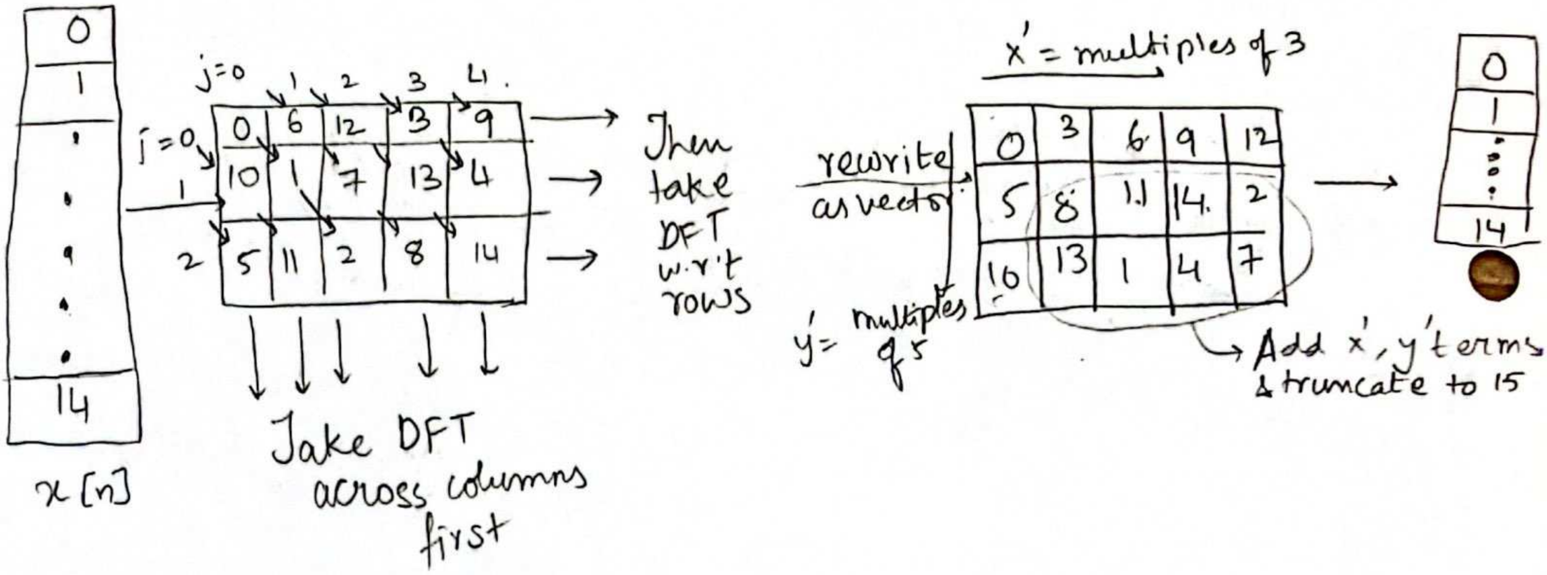
because $n_1 N_1 + n_2 N_2 = 1$

$$X[k_1][k_2] = \underbrace{\sum_{j=0}^{n_2-1} \left(\underbrace{\sum_{i=0}^{n_1-1} x[i][j]\, W_{n_1}^{ik_1}}_{\text{length } n_1 \text{ DFT}} \right) W_{n_2}^{jk_2}}_{\text{length } n_2 \text{ DFT}}$$

=> no "twiddle factors!"

Again with $N = 15$, $n_1 = 3$; $n_2 = 5$

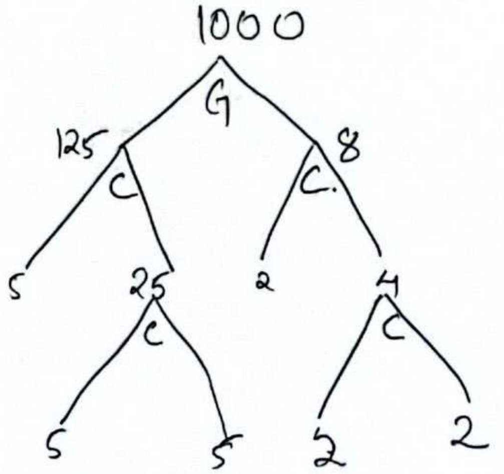
$N_1 = -3$ $N_2 = 2$

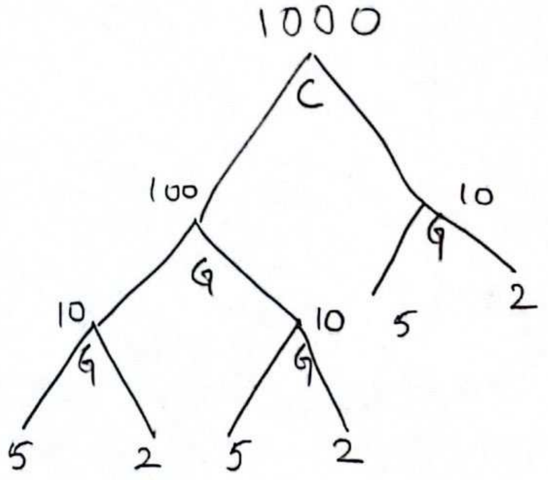


0	6	12	3	9
10	1	7	13	4
5	11	2	8	14

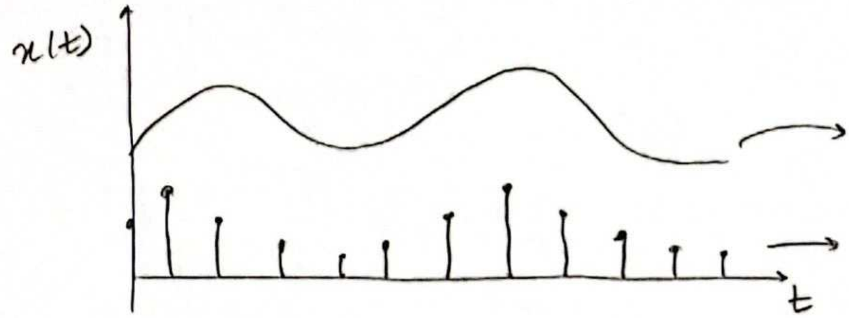
0	3	6	9	12
5	8	11	14	2
10	13	1	4	7

→ Length 1000 DFT:





Lec 13 The Sampling Theorem



Periodic sampling.

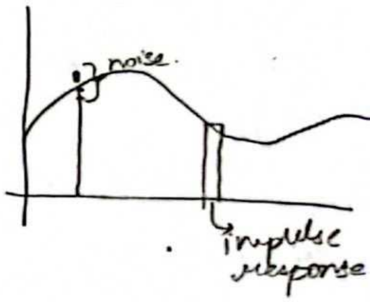
$x[n] = X_c(nT)$

n is an integer.
T = sampling period.
ω_s = sampling frequency (radians)

Non ideal effects

1> Practically not possible to sample exactly one point. We get the impulse response of the sampler.

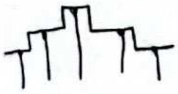
2> Noise while sampling ⇒ $x[n] = y(nT) + \underbrace{z[n]}_{\text{noise}}$.



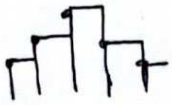
• How to recreate the C.T signal given $x[n]$?

> There are many possibilities... how to choose the 'best' fit.

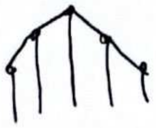
Simple estimates.



Nearest neighbour. > Need to look into future ⇒ not causal.

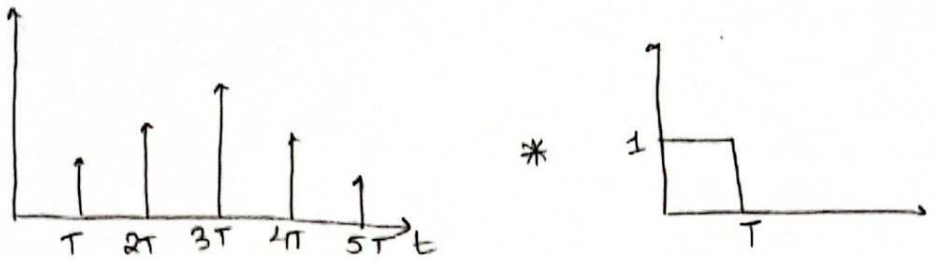


Zero order hold.



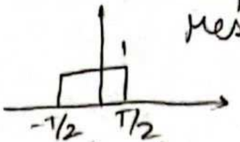
First order hold or Linear interpolation.

Zero order hold



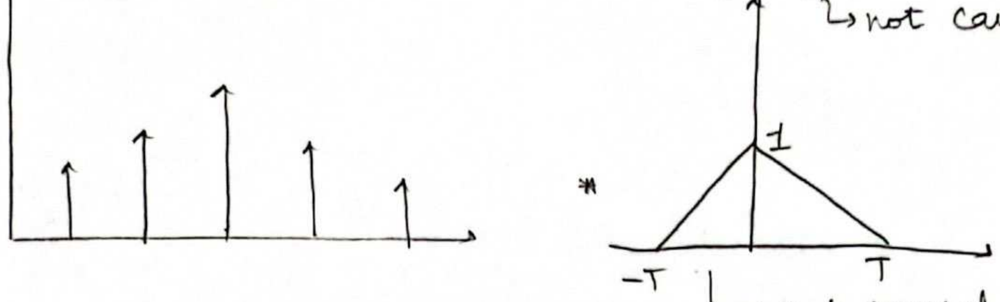
= gives zero order hold.

↳ causal. impulse response.



= nearest neighbour.

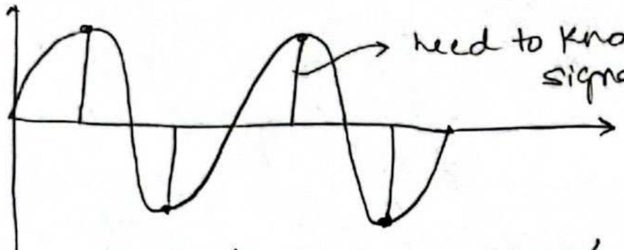
↳ not causal.



= First order hold.

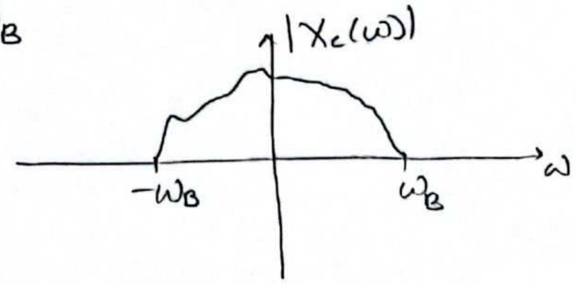
↳ not causal.

Eg:



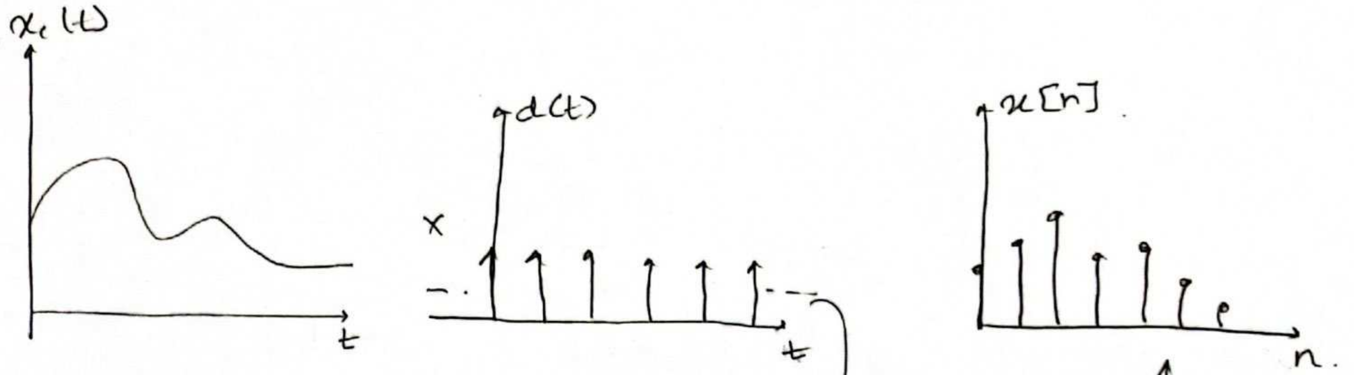
need to know something about the original signal's frequency to decide the sampling rate.

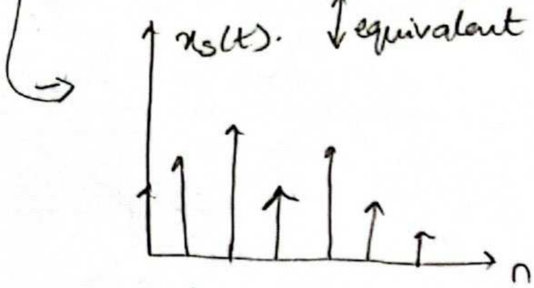
> A signal is band limited if there's some frequency ω_B such that $X_c(\omega)=0$ for $|\omega| > \omega_B$



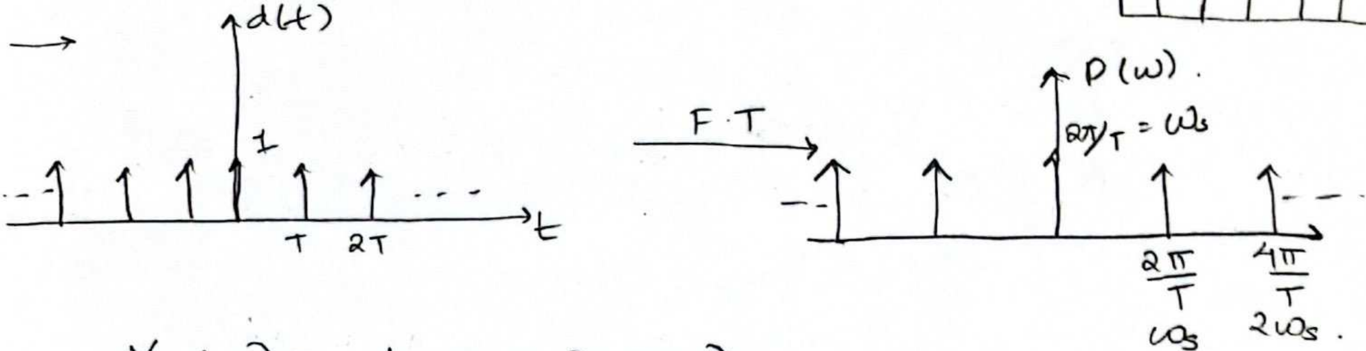
Sampling Theorem. → Shannon 1949 and Nyquist 1928.

A bandlimited signal with max freq ω_B can be perfectly reconstructed from evenly spaced samples if the sampling frequency ω_s satisfies $\boxed{\omega_s > 2\omega_B}$. $2\omega_B$ is called the Nyquist rate.





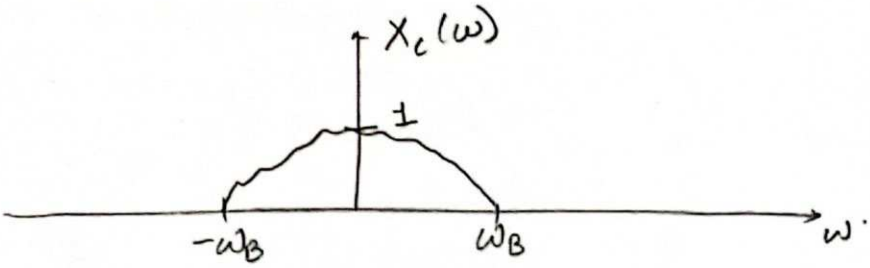
$$x_s(t) = x_c(t) \times d(t).$$



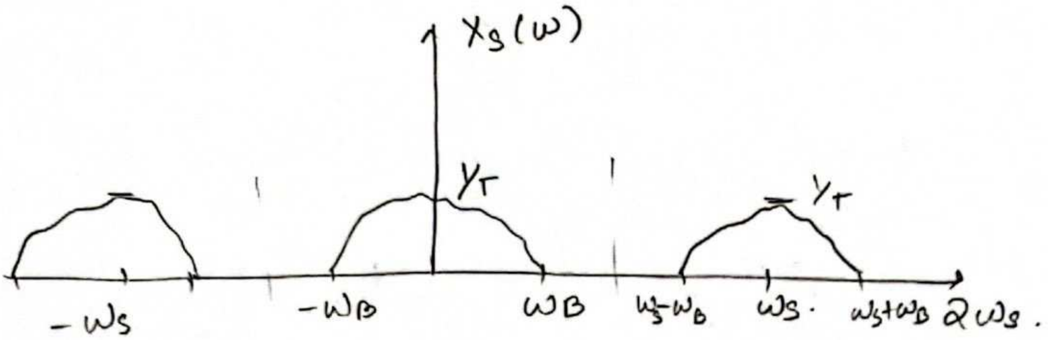
$$X_s(\omega) = \frac{1}{2\pi} X_c(\omega) * D(\omega).$$

$$= \frac{1}{2\pi} X_c(\omega) * \left[\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \right]$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(\omega - k\omega_s)$$ → copies of the original signal at the deltas!



↓ after Sampling.

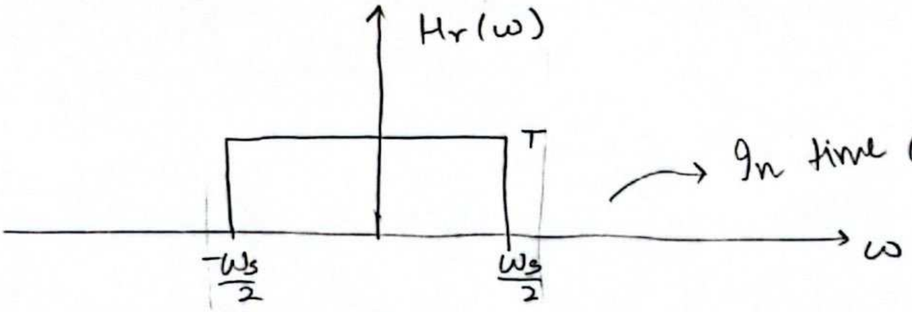


aliasing.

High freq. components appear/combine/fold into lower frequency components.

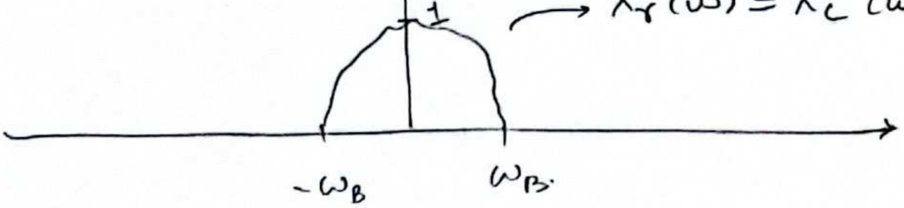
if $\omega_s > 2\omega_B$, there is no overlap.

↓ Multiply with a low pass filter. (notch).



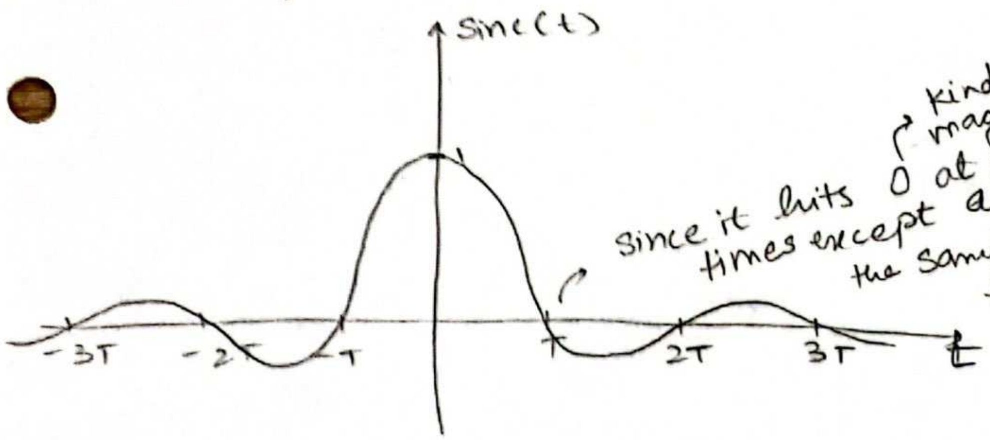
→ In time domain we need a perfect sinc function

↓ $X_r(\omega) = X_s(\omega) \cdot H_r(\omega)$.



→ $X_r(\omega) = X_c(\omega)$ → reconstructed.

Sinc function.



Since it hits 0 at all the discrete times except at $t=0$, convolving it with the sampling waveform will produce the original signal.

Kind of works like magic in time domain.
But obvious in frequency domain

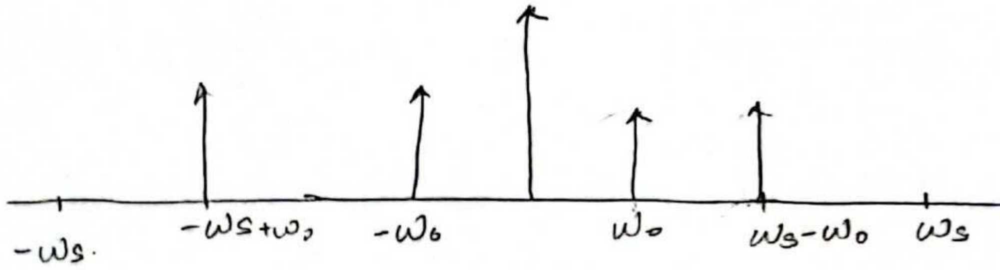
$$h_r(t) = \text{sinc}\left(\frac{\pi}{T} \cdot t\right)$$

$$x(t) = x_s(t) * h_r(t).$$

$$= \sum_{n=-\infty}^{\infty} x[n] \text{ sinc}\left(\frac{\pi}{T}(t-nT)\right)$$

Easy example.

$x(t) = \cos \omega_0 t$



Phase reversal.

> Can show if $\omega_s < 2\omega_B$.

> $x(t) = \cos(\omega_0 t + \phi)$.

$$\xrightarrow[\text{reconstructed}]{} x_r(t) = \cos((\omega_s - \omega_0)t \underbrace{-\phi}_{}).$$

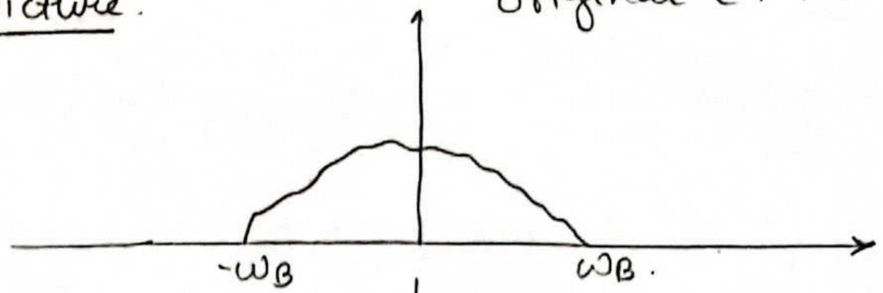
going backward.
(aliasing/
Stroboscopic effect)

> gives really good practical examples
* gives a nice demo of aliasing.

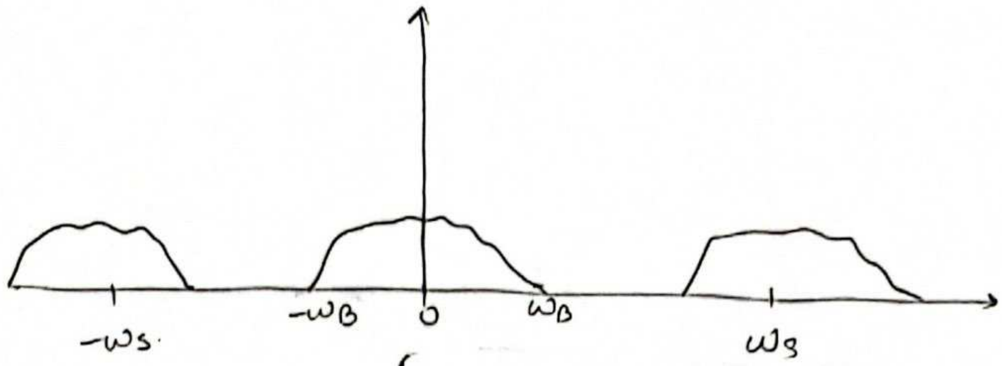
> Aliasing changes the high frequency to low freq and also reverses the phase. → recall stroboscopic effect.

The full picture.

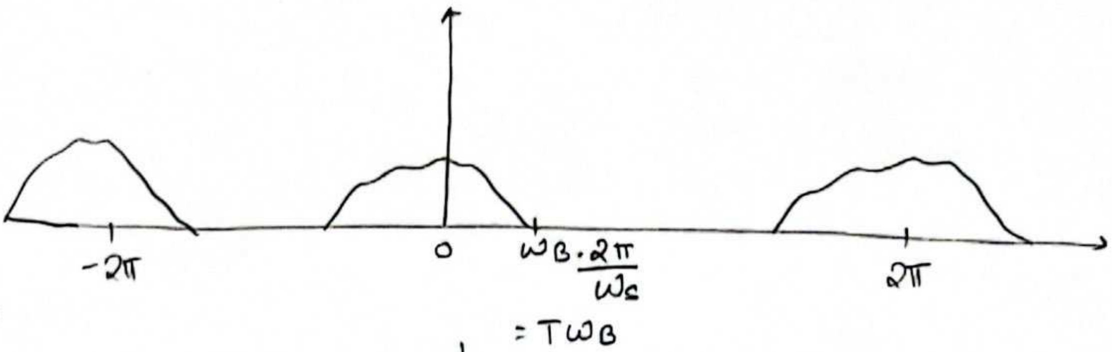
original CT FT



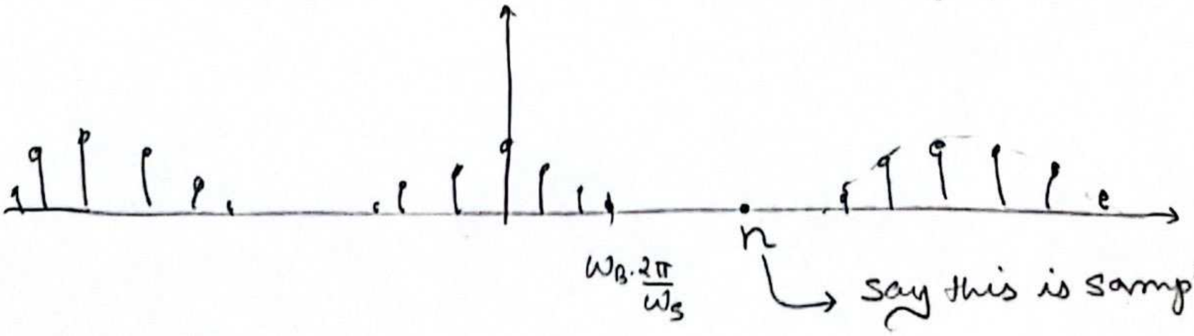
sampling at ω_s.



DTFT in MATLAB.



DFT → takes samples of the DTFT.
at every $\frac{2\pi}{N}$

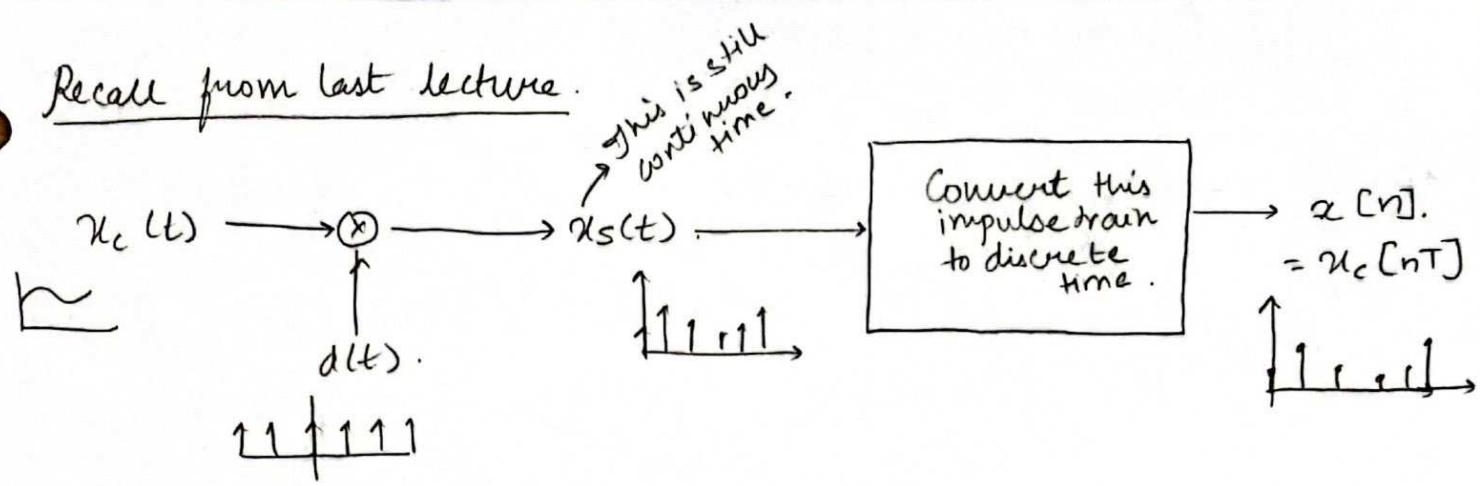


say this is sample n.

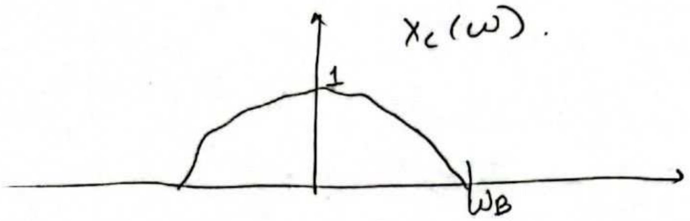
Sample n corresponds to $\frac{2\pi n}{N} \cdot \frac{\omega_s}{2\pi} = \frac{n\omega_s}{N}$ on the CTFT axis

Lec 14 : Continuous Time Filtering with digital systems.

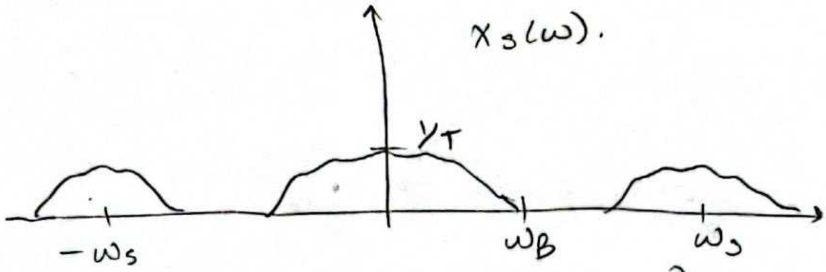
Recall from last lecture.



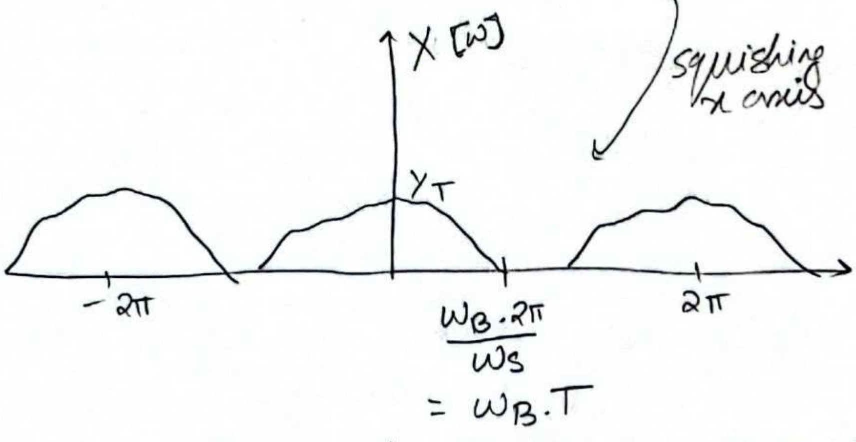
$$X_s(\omega) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X_c(\omega - K\omega_s) \qquad \omega_s = \frac{2\pi}{T}$$



CTFT of $X_c(t)$.

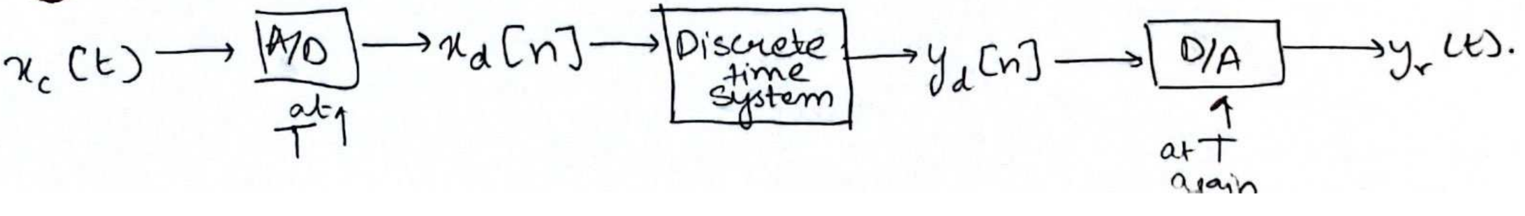


CTFT of $X_s(t)$.



DTFT of $x[n]$

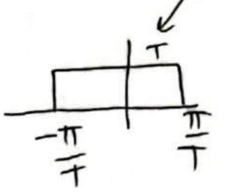
> Discrete time processing of continuous time signals.



① $x_d(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega - 2\pi k}{T}\right)$

③ $\sum_{n=-\infty}^{\infty} y_d[n] \, \text{sinc}\left(\frac{t-nT}{T}\right)$

$Y_r(\omega) = H_r(\omega) \, Y_d(\omega T)$



② $Y_d(\omega) = H(\omega) X_d(\omega)$

↖ Freq. response of discrete time system.

$Y_r(\omega) = \underset{\uparrow}{H_r}(\omega) \, Y_d(\omega T)$

← Reconstruction filter (pulse).

$= H_r(\omega) \, H(\omega T) \, X_d(\omega T)$

$= H_r(\omega) \, H(\omega T) \, \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(\omega - \frac{2\pi k}{T}\right)$

If $X_c(\omega) = 0$ for $|\omega| > \pi/T$, then $Y_r(\omega) = \begin{cases} H(\omega X_c(\omega) & \text{for } |\omega| < \frac{\pi}{T} \\ 0 & \text{elsewhere} \end{cases}$

> So the effective freq. response is:

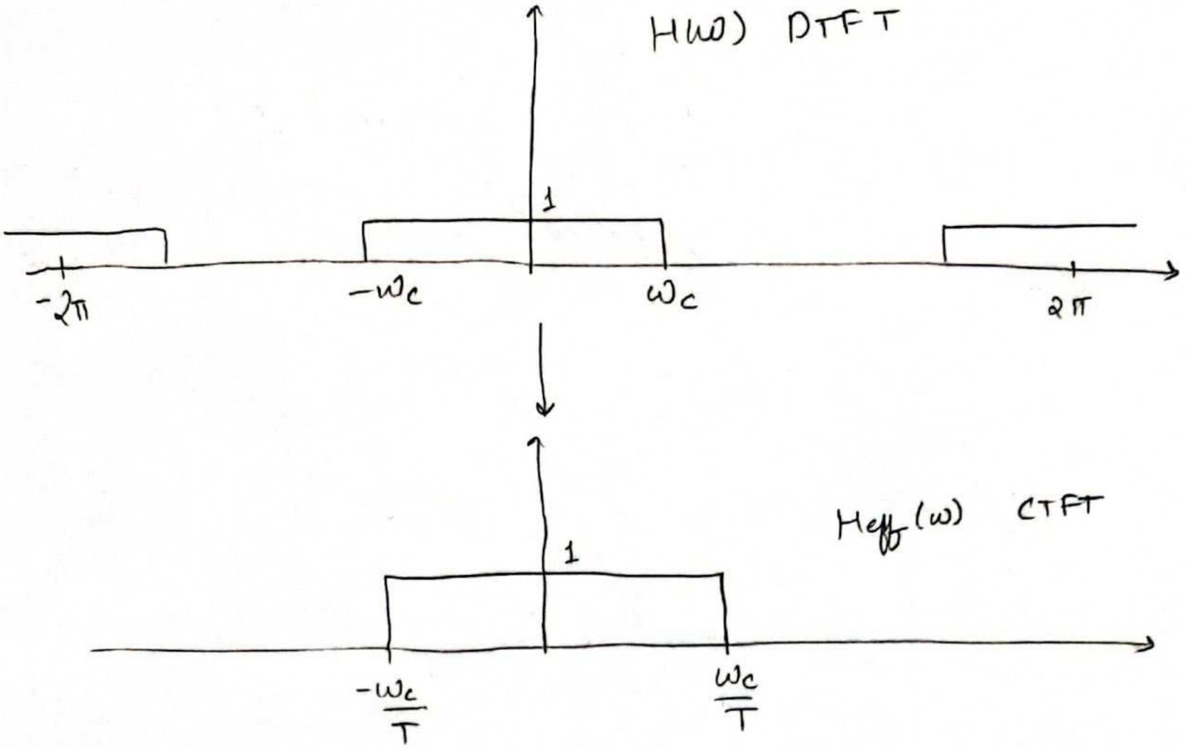
$Y_r(\omega) = H_{eff}(\omega) \, X(\omega) \longrightarrow$ CTFTs

$H_{eff}(\omega) = \begin{cases} H(\omega T) & |\omega| < \pi/T \\ 0 & \text{else.} \end{cases}$

effective CT filter.

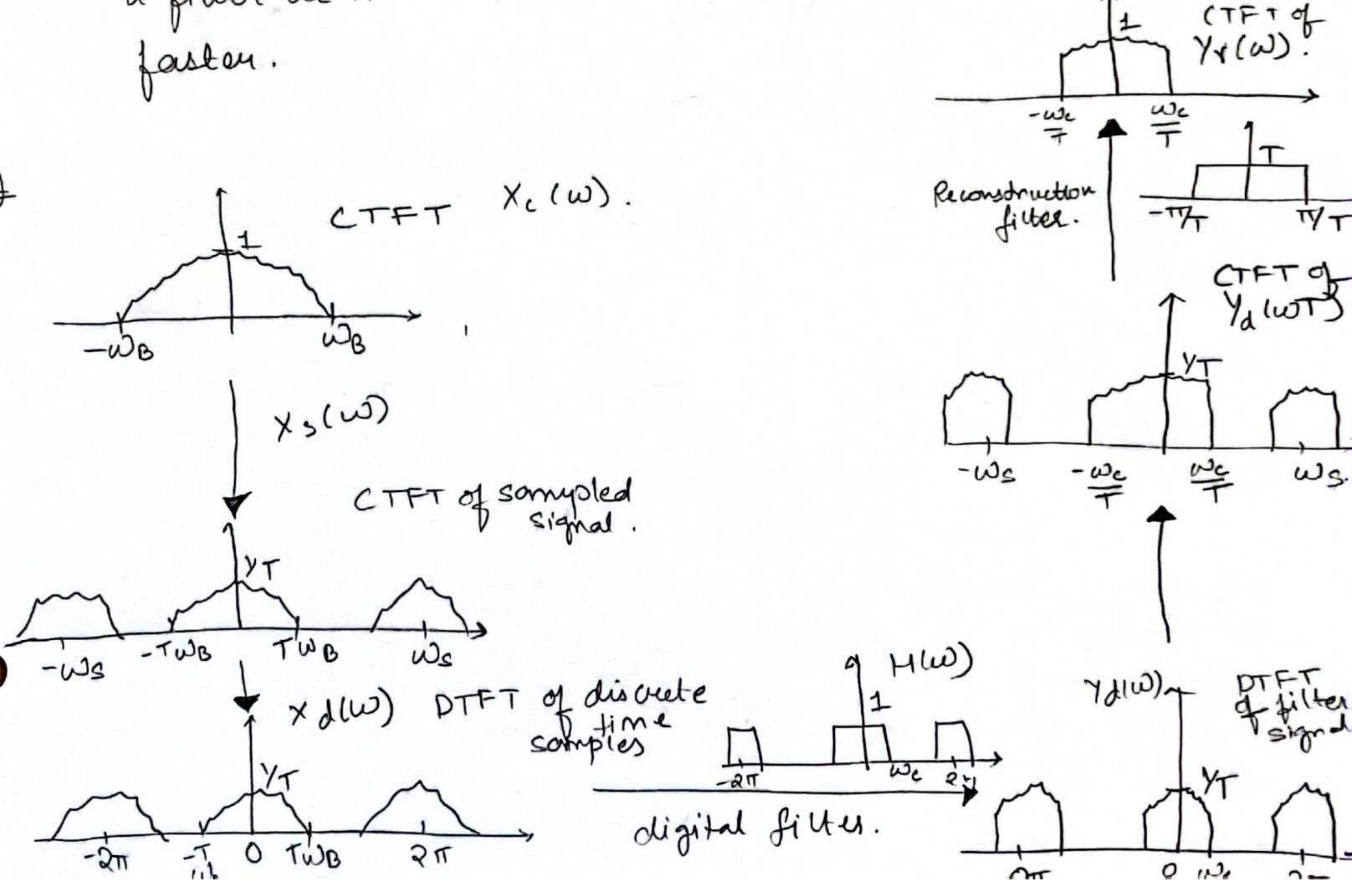
discrete time filter. } Scaled by a factor of T.

> Consider H(ω) as a digital LPF.



→ If we want a cut off of $\frac{\omega_c}{T}$ in CT world we need to design a filter ωc in DT. ⇒ We can extend CT cutoff by sampling faster.

Eg



→ One digital filter of high quality can do many C.T filterin

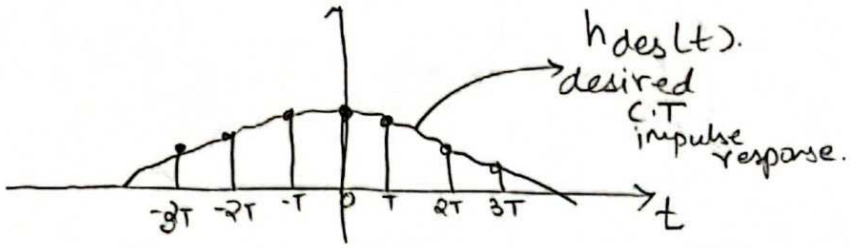
by modifying sampling rate.

Precautions.

1) Ensure D.T system is LTI.

2) Sample above Nyquist rate.

How are the impulse responses related?



$h[n] = T \; h_{des}(nT)$. → "Impulse invariance".

Upsampling/Down sampling.

Common problem: $x[n] = x_c(nT)$. → we got this

$x'[n] = x_c(nT')$ → But we want this.

$T' \neq T$.

→ Changing b/w samples purely in digital.

Downsampling at an integer rate.

$x_{down}[n] = x[nM] = X_c(n(MT))$. →Take every M^{th} sample. → Careful of Nyquist

$$\underset{T}{x[n]} \longrightarrow \underset{\text{"Compressor"}}{\boxed{\downarrow M}} \longrightarrow \underset{MT}{X_{down}[n]}$$

→ What happens in freq. domain?

$$\text{DTFT} \quad X(\omega) = \frac{1}{T} \sum_{K=-\infty}^{\infty} X_c\left(\frac{\omega}{T} - \frac{2\pi K}{T}\right)$$

$$X_{down}(\omega) = \frac{1}{T'} \sum_{K=-\infty}^{\infty} X_c\left(\frac{\omega}{T'} - \frac{2\pi K}{T'}\right)$$

$$X_{down}(\omega) = \frac{1}{MT}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{MT} - \frac{2\pi k}{MT}\right)$$

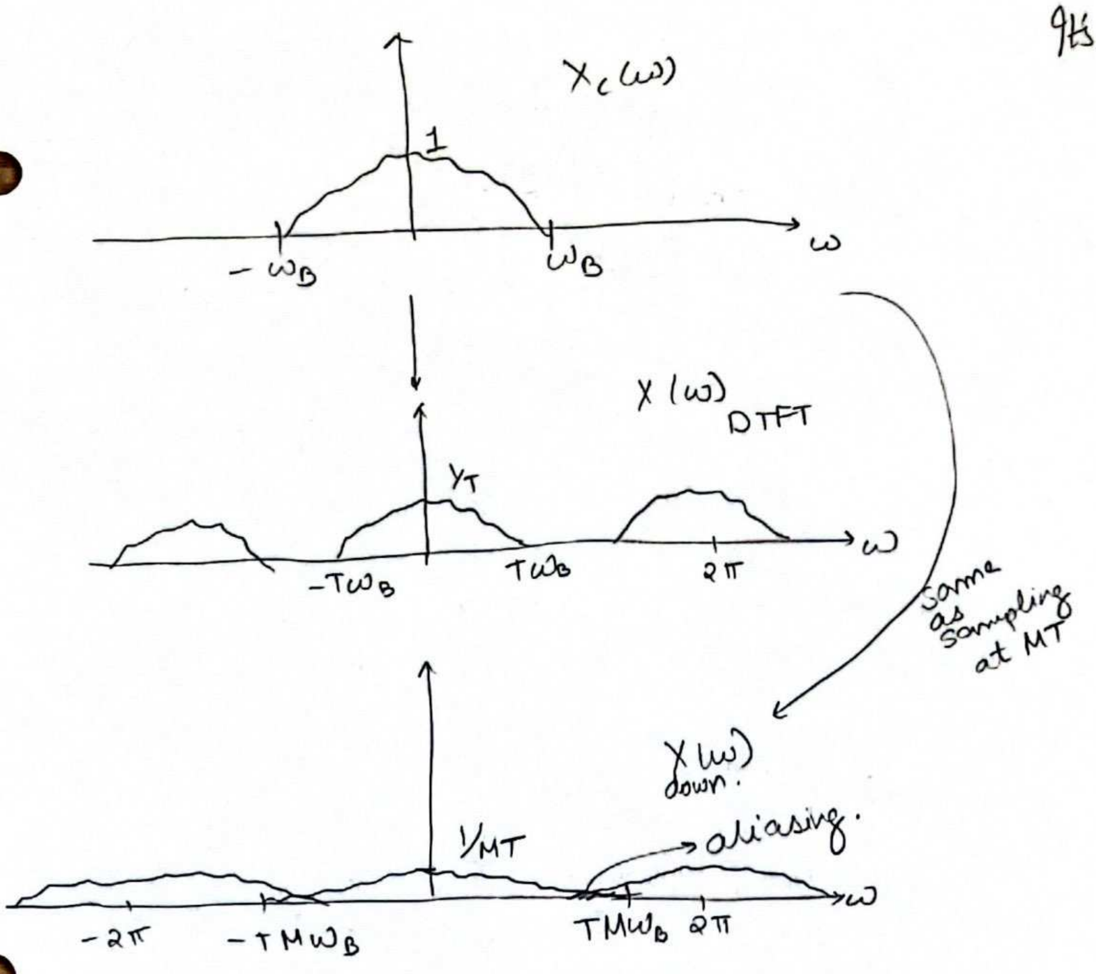
$$= \frac{1}{M}\sum_{i=0}^{M-1} \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{MT} - \frac{2\pi(i+kM)}{MT}\right)$$

$$= \frac{1}{M}\sum_{i=0}^{M-1}\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega-2\pi i}{MT} - \frac{2\pi k}{T}\right)\right]$$

$$= \frac{1}{M}\sum_{i=0}^{M-1} X\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)$$

↳ Scaled & shifted copies.

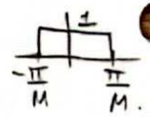
It's like sampling a sampled signal.



Aliasing can occur!

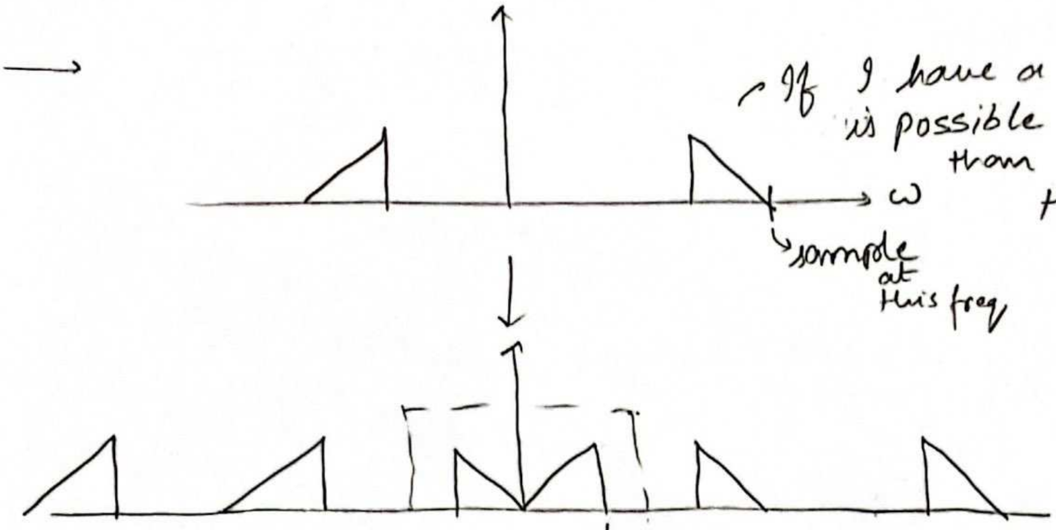
> We would have needed to sample the original signal at atleast M × Nyquist.

> Could prefilter to prevent aliasing, B.w of filter = $\frac{\pi}{M}$.



But information is lost

→



If I have a signal like this it is possible to sample much lower than Nyquist & reconstruct the signal.

could reconstruct from here.

→ So Nyquist rate depends $\omega_{B_{max}}$ AND $\omega_{B_{min}}$.

→ Briefly talks about nonuniform sampling. around 52:00 min

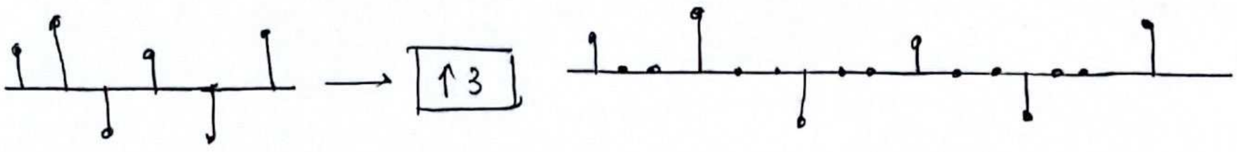
Upsampling.

We have $x[n] = x_c(nT)$.

We want $x_{up}[n] = x_c(n\frac{T}{L})$ L integer.

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_e[n].$$

expander.



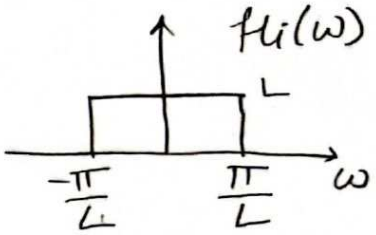
→ To generate the intermediate values, we need the full info.

→ as long as we have sampled above Nyquist we can.

$$\underset{\text{expanded.}}{x_e[n]} = \begin{cases} x[n/L] & n = 0, \pm L, \pm 2L, \dots \\ 0 & \text{elsewhere.} \end{cases}$$

How to create $x_{up}[n]$ from $x_e[n]$?

→ Pass it through a low pass filter!



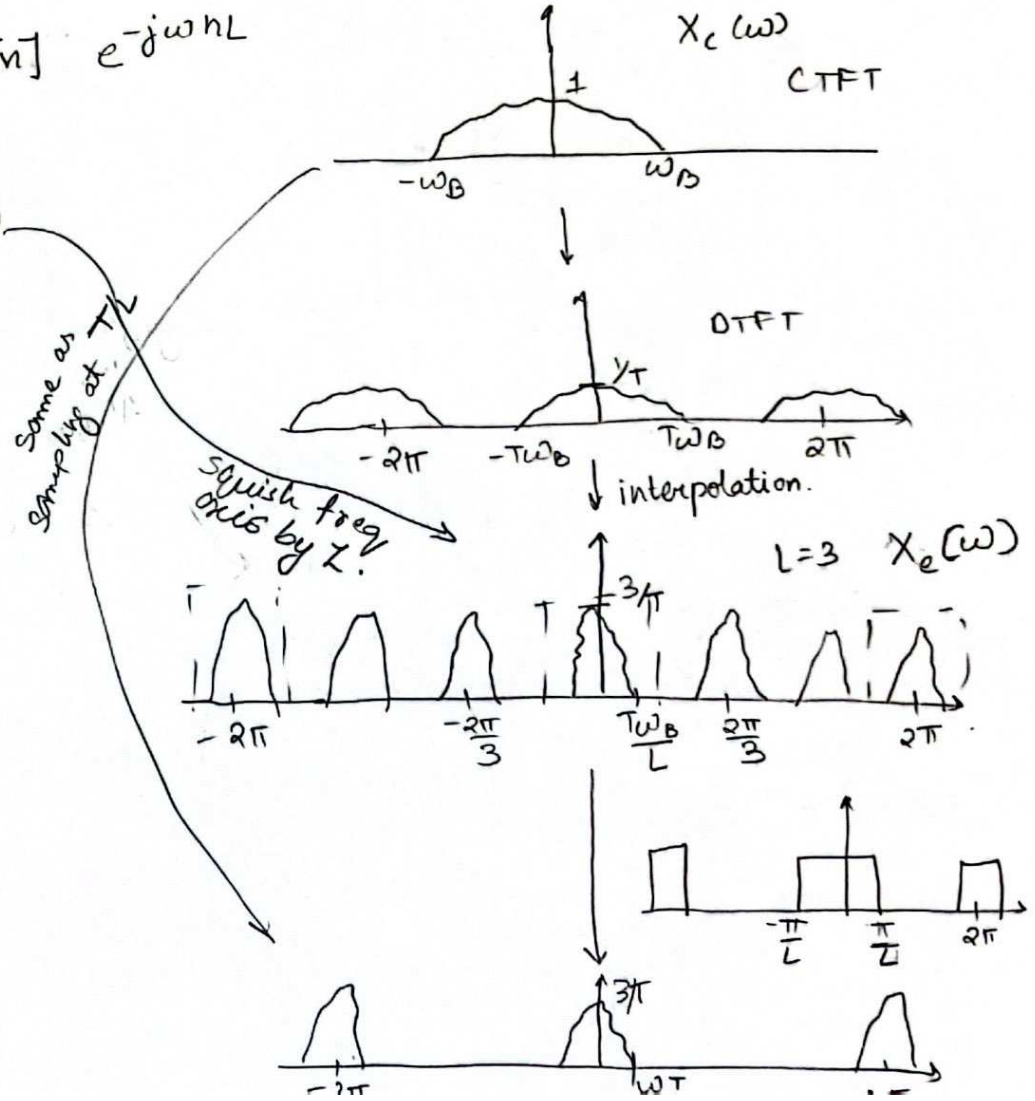
↳ Interpolation filter.

$$X_e(\omega) = \sum_{n=-\infty}^{\infty} x_e[n]\, e^{-j\omega n}$$

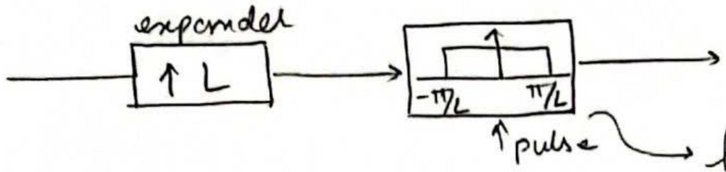
$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega nL}$$

$$= X(\omega L)$$

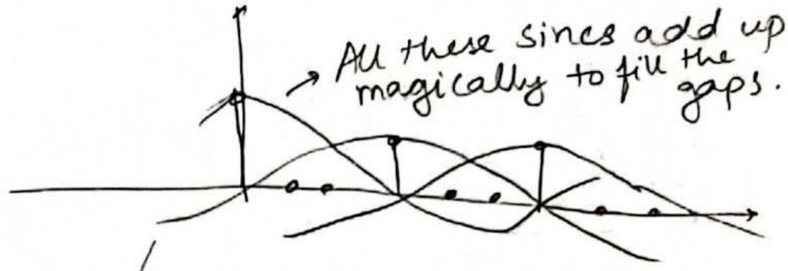
→ The LPF magically fills in the values for the zeroes!



Interpolator.



⤳ $h_i[n]$ = sinc.



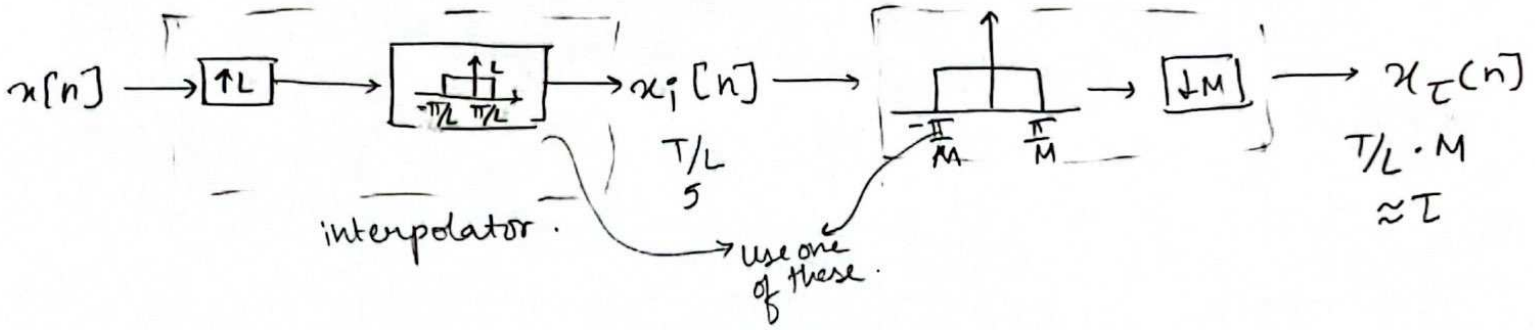
→ All these sincs add up magically to fill the gaps.

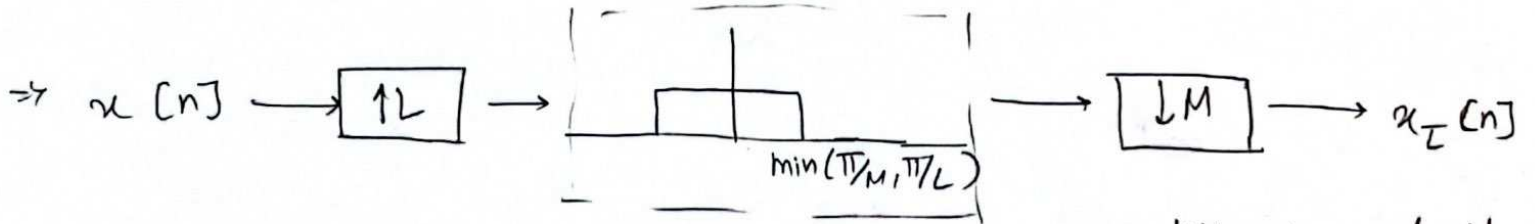
→ Same as reconstructing back to C·T but not going all the way.

> What about non integer ↑ & ↓ sampling?

Lec15 Multirate signal processing.

> Non integer factor $T = \frac{M}{L}$ → downsample by M, upsample by L } M, L are integers.





M > L ⇒ net reduction in sampling rate ⇒ prefilter is needed to prevent aliasing.

M < L ⇒ net increase in sampling rate ⇒ no aliasing.

> Even if $I \approx 1$, $(I = 1.01)$, $M = 101$, $L = 100$

⇒ Big changes to get a small change.

> <u>Solution:</u> Multirate signal processing. → Avoids unnecessary computations.

<u>Useful identities.</u>

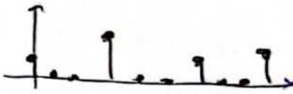
$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_a[n] \longrightarrow \boxed{H(z)} \longrightarrow y_a[n]$$

$$x[n] \longrightarrow \boxed{H(z^m)} \longrightarrow x_b[n] \longrightarrow \boxed{\downarrow M} \longrightarrow y_b(n)$$

are equivalent statements.

→ What is $H(z^m)$, $H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$

$$H(z^m) = \sum_{n=-\infty}^{\infty} h[n] z^{-Mn}$$ ↳ Taking Mth value same as taking z-Transform of expanded signal



$$= \mathcal{Z}(h_e[n])$$

<u>Proof</u>

$$X_b(\omega) = X(\omega) H(\omega M)$$

$$\Rightarrow Y(\omega) = \frac{1}{M} \sum_{i=0}^{M-1} X_b\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)$$ spread out & shrunk copies.

$$= \frac{1}{M} \sum_{i=0}^{M-1} X\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right) H(\omega - 2\pi i)$$ ↳ H(ω) periodic in 2π

$$= H(\omega) \frac{1}{M} \sum_{i=0}^{M-1} X\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)$$

$$= H(\omega) . X_a(\omega)$$

Also,

$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow x_a[n] \longrightarrow \boxed{\uparrow L} \longrightarrow y_a[n]$$

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_b[n] \longrightarrow \boxed{H(z^L)} \longrightarrow y_b[n].$$

↳ computationally inefficient since computations on the zeroes are not very useful.

$$Y_a(\omega) = X_a(\omega L) = X(\omega L)\, H(\omega L)$$

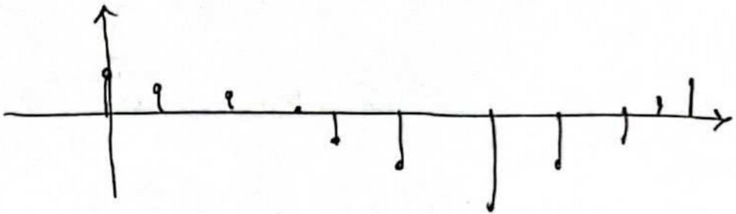
$$X_b(\omega) = X(\omega L)$$

$$Y_b(\omega) = X(\omega L)\, H(\omega L)$$

→ Can interchange the process of introducing zeroes and filtering by using a slightly different filter.

<u>Key idea</u> :- <u>Polyphase decomposition.</u>

Consider $h[n] =$



Decompose this as a sum of M subsequences.

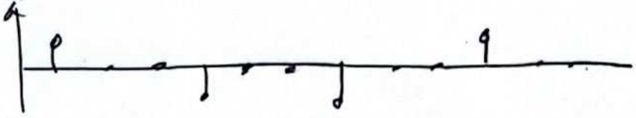
$$h_k[n] = \begin{cases} h[n+k] & n = (\text{integer})M. \\ 0 & \text{else.} \end{cases}$$

Eg: $M = 3$

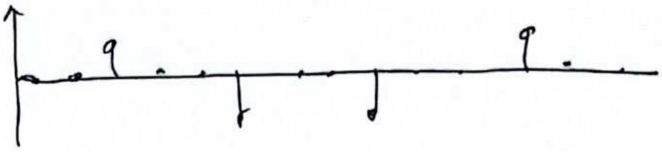
$h_0[n]$

Every 3rd element..

$h_1[n-1]$



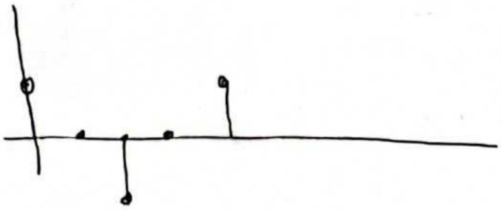
$h_2[n-2]$



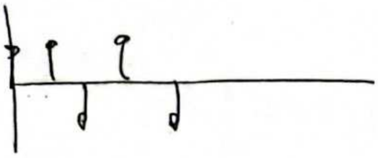
∴ $$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$

- $e_k[n] = h_k[nM]$ → removing zeroes from h_k.

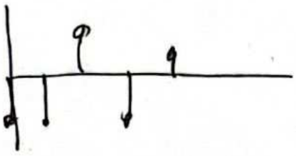
$e_0[n] =$

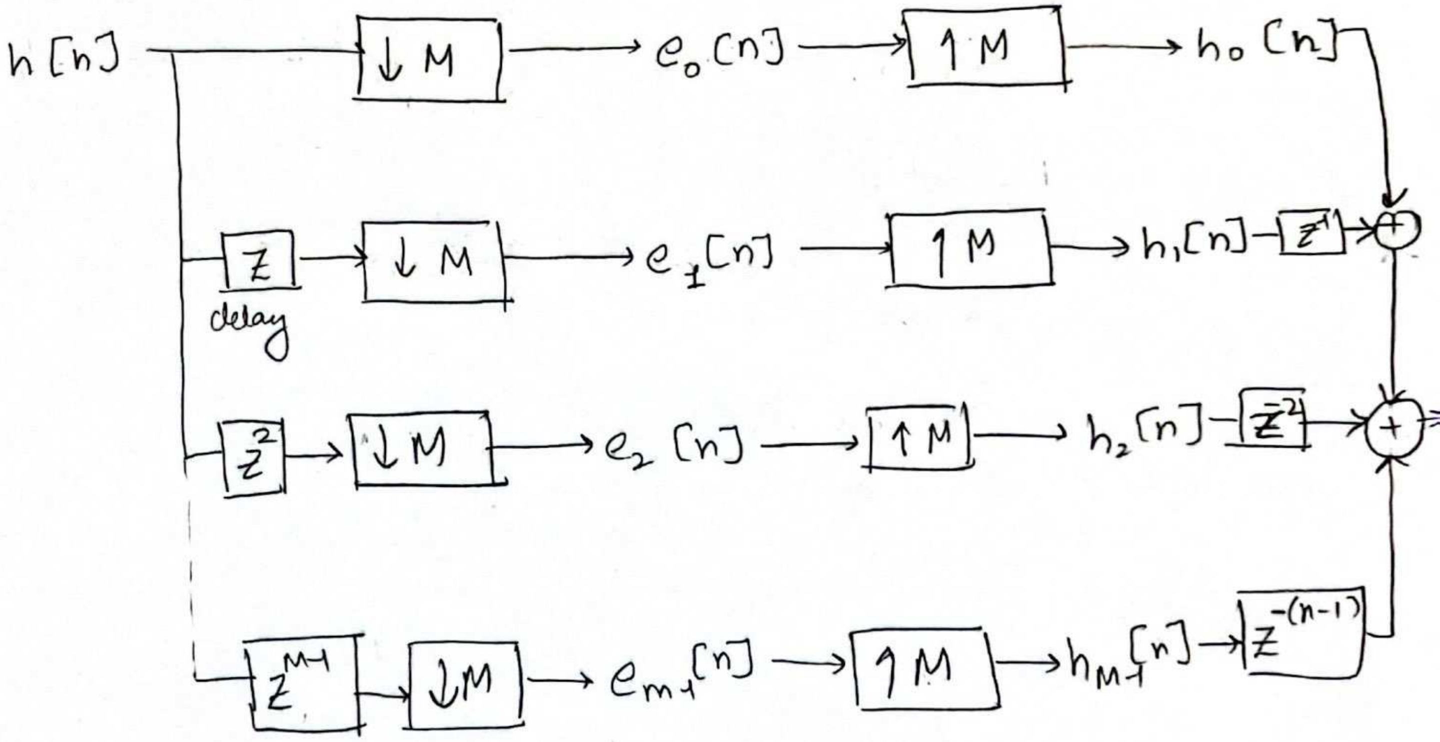


$e_1[n] =$

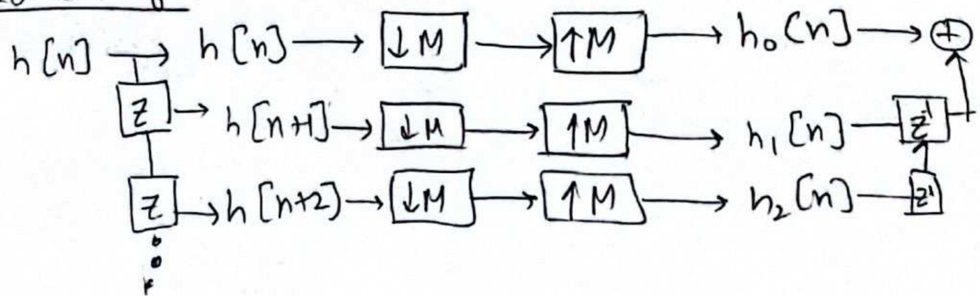


$e_2[n] =$





Redrawing.



Polyphase components of $h[n]$

$$e_k[n] = h[nM+k] = h_k[nM]$$

$$H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

$$= \sum_{l=-\infty}^{\infty} h[lM] z^{-lM} + \sum_{l=-\infty}^{\infty} h[lM+1] z^{-(lM+1)}$$

$$+ \ldots$$

$$= \sum_{k=0}^{M-1} \sum_{l=-\infty}^{\infty} h[lM+k] z^{-(lM+k)}$$

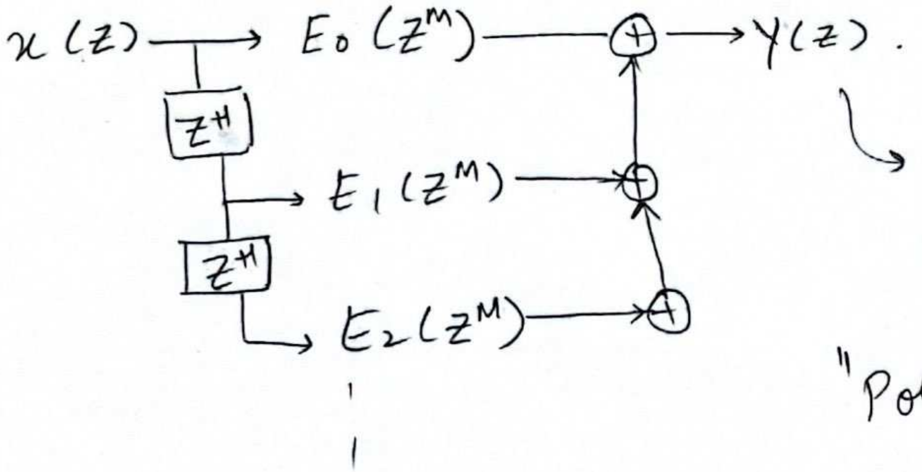
$$= \sum_{k=0}^{M-1} z^{-k} \sum_{l=-\infty}^{\infty} h[lM+k] z^{-lM}$$

$$= \sum_{k=0}^{M-1} z^{k} E_k(z^M)$$

In the transfer function world.

$$H(z) = \sum_{k=0}^{M-1} z^{-k} E_k(z^M)$$

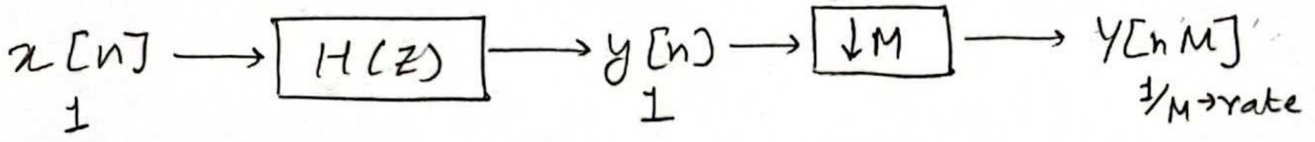
→



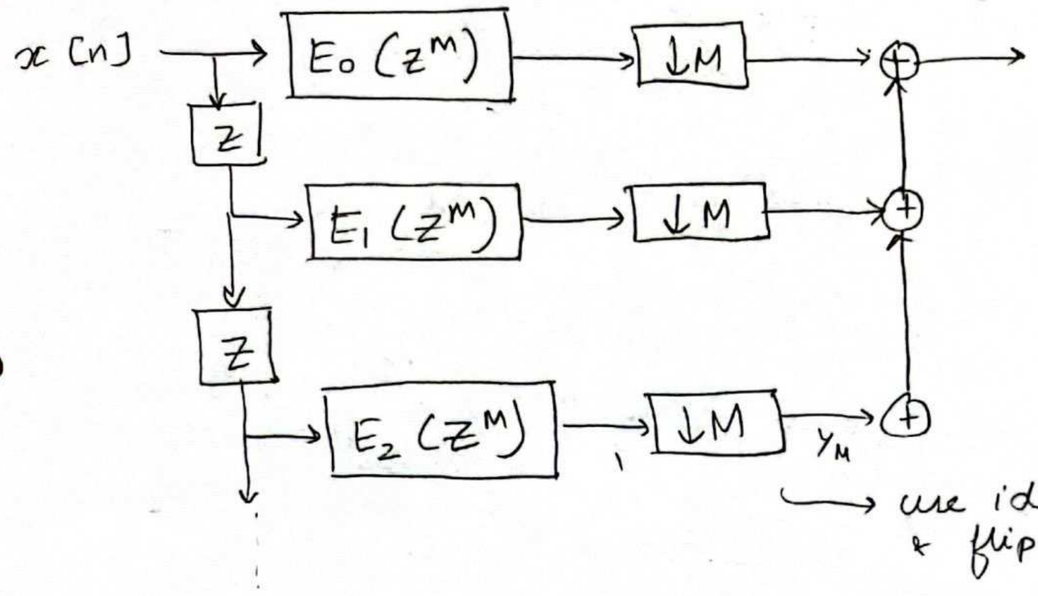
→ Decompozing a z transform into M easier z transforms.
"Polyphase realizations of $H(z)$"

Why are we doing all this?

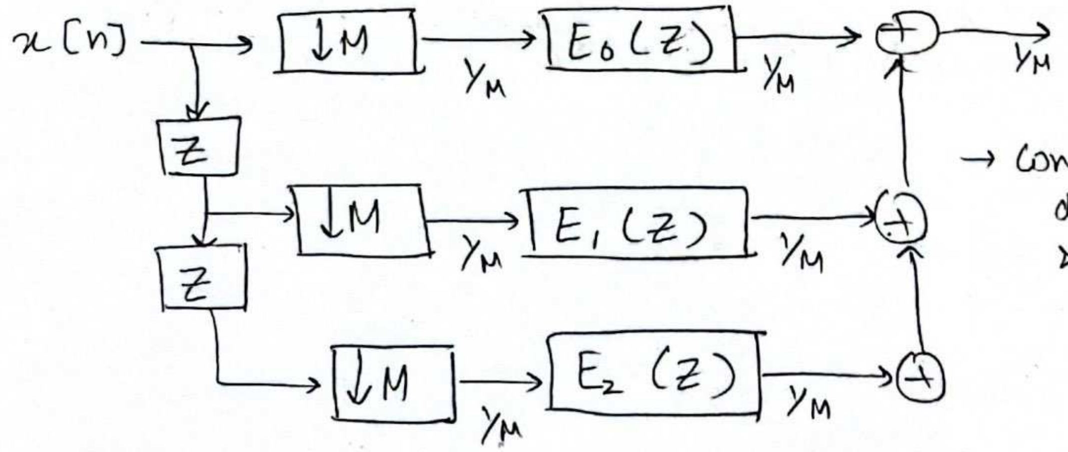
- Polyphase implementation of decimation / interpolation filters.



In this form we are generating & discarding $\frac{M-1}{M}$ values.



→ use identity & flip the 2 blocks.



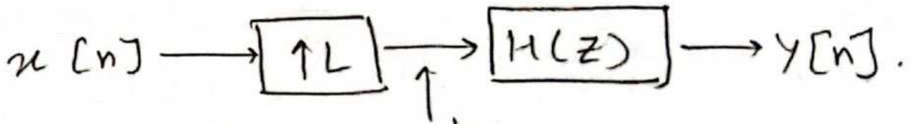
→ Computations are done at rate $1/M$ & not at 1

- Eg:- If original $h[n]$ was an N-TAP FIR FILTER, each of $e_q[n]$ is an N/M tap filter.

Original realisation:- N multiplications / unit time.

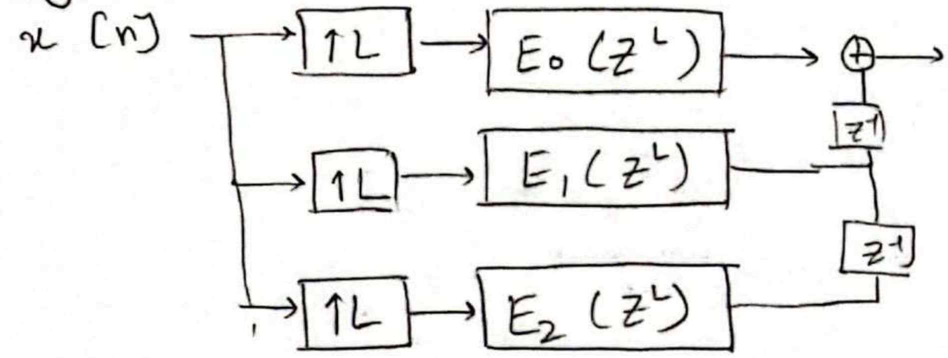
polyphase realization:- N/M mults / time.

Upsample case.

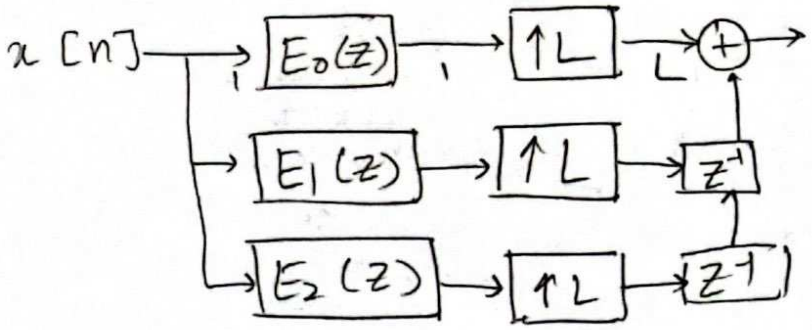


$\frac{L-1}{L}$ values = 0 ⇒ we would be filtering mostly zeroes!

Polyphase.

⇒ 

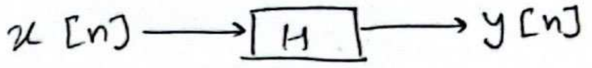
↳ use identity & flip.



– Computations are done at 1 instead of L rate

⇒ No zeroes are explicitly filtered.

Lec 16 FIR filter design using least squares.



$y[n] = x[n] * h[n]$

$Y(\omega) = X(\omega) H(\omega)$

$|Y(\omega)| = |X(\omega)| \cdot |H(\omega)|$

$\angle Y(\omega) = \angle X(\omega) + \angle H(\omega)$

Ideal filter.

$$H_{lp}(\omega) = \begin{cases} 1 & \omega < |\omega_c| \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

Time domain $h_{lp}[n] = \frac{\sin \omega_c n}{\pi n}$ $-\infty < n < \infty$

↳ goes on till infinity

→ Problems with sinc function.

- Infinite length impulse response.
- Not causal (-ve values)

→ Input signal phases should not be altered. → ideally.

→ Ideal delay $h_d[n] = \delta[n-n_d]$

$$H_d(\omega) = e^{-j\omega n_d}.$$

$$\Rightarrow |H_d(\omega)| = 1 \qquad \angle H_d(\omega) = -\omega n_d.$$

↳ phase shift is a linear fn of ω. ⇒ no distortion.

⟶ Output sequence = input shifted in time.

$$H_{lp}(\omega) = \begin{cases} e^{-j\omega n_d} & |\omega| < \omega_c \\ 0 & \omega_c \leq |\omega| \leq \pi \end{cases}$$

$$\Rightarrow |H_{lp}(\omega)| = \begin{cases} 1 \\ 0 \end{cases} \qquad \measuredangle H_{lp}(\omega) = -\omega n_d.$$

$$h_{lp}[n] = \frac{\sin \omega_c (n-n_d)}{\pi (n-n_d)}$$ → not centred at zero.

↳ still not finite.

Group delay.

$$\text{group delay} = \frac{-d}{d\omega}(\arg H(\omega))$$

group delay is constant ⇒ no distortion.

> Instead of $|H(\omega)|$ & $\measuredangle H(\omega)$ we use $\overset{\text{amplitude.}}{A(\omega)}$ & arg $H(\omega)$

↳ phase unwrapping.

Filter Design Process.

1) Choose desired frequency response.
2) Choose allowable class of filters. (eg. Length N FIR filter)
3) Choose a measure of Quality "How close".
4) Apply method/algorithm to find the optimal values.

> We want real, causal digital filters, of the form.

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_n z^{N}}.$$

$\longrightarrow$ if $\{a_i\} = 0$, FIR filter.

$\longrightarrow$ otherwise IIR filter.

In general,

$$\min_{a,b} \| E(z) \| = \left\| H_{des}(z) - \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}} \right\|$$

→ Consider $h[n]$, a length n FIR filter, assume it has a linear phase $\theta(\omega) = K_1 + K_2 \omega$

$$H(\omega) = \sum_{n=0}^{N-1} h[n] e^{-j\omega n}$$

$$= e^{-j\omega M} \sum_{n=0}^{N-1} h[n] e^{-j\omega n} e^{j\omega M}$$

$$= e^{-j\omega M} \sum_{n=0}^{N-1} h[n] e^{j\omega (M-n)}$$

$$= e^{-j\omega M} \Big((h[0] + h[N-1]) \cos \omega M + j(h[0] - h[N-1]) \sin \omega M$$

$$+ (h[1] + h[N-2] \cos \omega (M-1) + j(h[1] - h[N-2] \sin \omega (M-1)$$

$$+ \ldots$$

- When can this be put in the form,

$$H(\omega) = \underset{\text{real amplitude.}}{A(\omega)}\, e^{j(k_1 + k_2\omega)} \;?$$

↳ linear phase.

- If $h[n] = h[N-n-1]$, then all the sin terms drop away

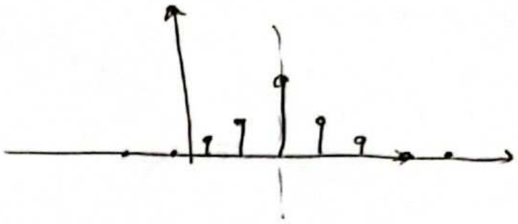
$$H(\omega) = e^{-j\omega M}\big((h[0] + h[N-1]\cos\omega M + (h[1] + h[N-2])\cos\omega(M-1) + \dots$$

≫ $A(\omega) = \sum_{n=0}^{M-1} 2h[n]\cos\omega(M-n) + h[M]$

if N is odd.

⇒ $h[n]$ is symmetric around it's middle element.

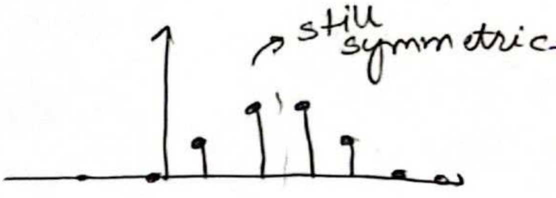
If $N=5$, $M = \frac{N-1}{2} = 2$



⇒ Linear phase filters must be symmetric.

- If N is even,

$$A(\omega) = \sum_{n=0}^{N/2-1} 2h[n]\cos(\omega(M-n))$$

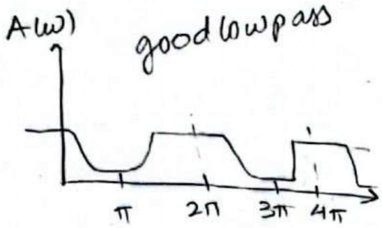


Linear phase filters.

Focus Here

Type I

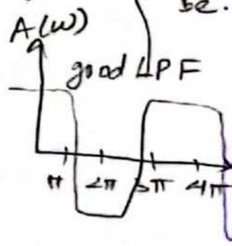
- N. odd
- symmetric



|Amplitude| is periodic just like DFT: supposed to be.

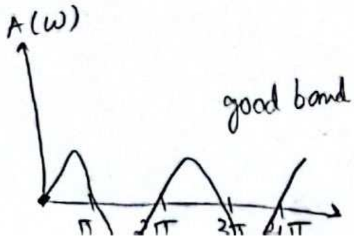
Type II

- N even
- Symmetric



Type III

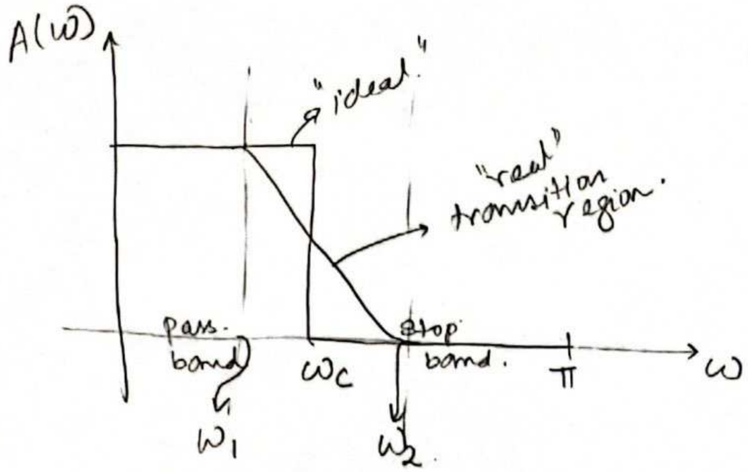
- N odd
- Antisymmetric

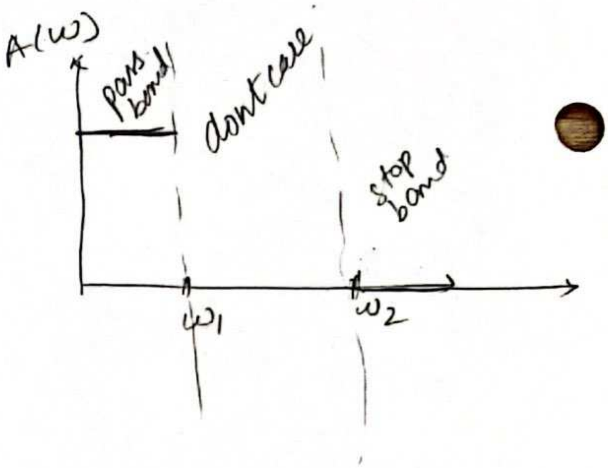


Type IV

- N even.
- Antisymmetric.

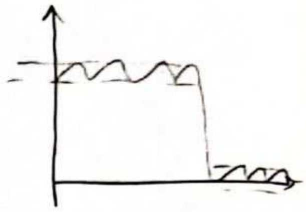
→ We have a desired amplitude response.





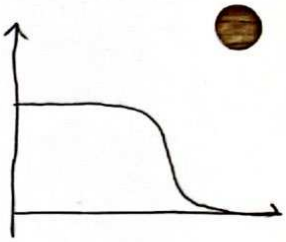
Approximation criteria.

1) Average or squared error in frequency domain.
⇒ least squares approximation.



2) Maximum error over the passband or stopband
⇒ Chebyshev approximation.

3) Taylor series approximation. to desired response.
⇒ maximally flat.
⇒ Butterworth filter.



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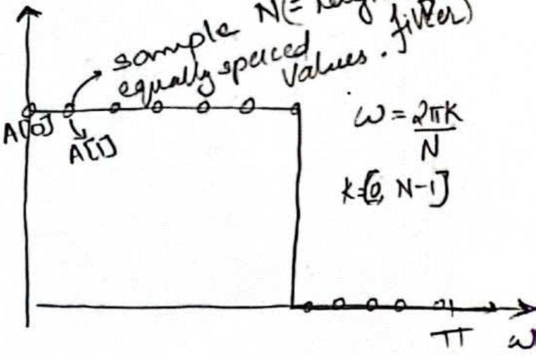
1) Frequency Sampling Design.

→ Sampling the DTFT = DFT.

⇒ we can get an FIR filter that exactly interpolates these samples with the Inverse DFT.



$$h[n] = \frac{1}{N} \sum_{k=0}^{N-1} H[k] e^{\frac{2\pi j k n}{N}}$$

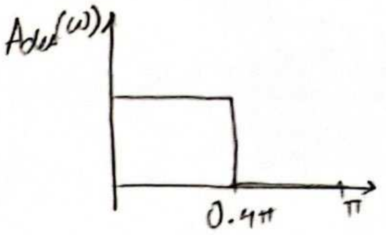
For Type 1 filter

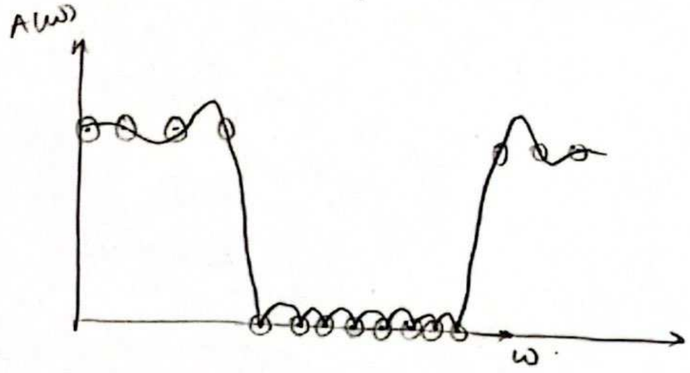
$$= \frac{1}{N} \left[A[0] + \sum_{k=1}^{M} 2A[k] \cos \frac{2\pi (n-M) k}{N} \right]$$

Can go directly from $A(n] \rightarrow h[n]$

Eg:- $N = 15 \Rightarrow M = 7$ & $\omega_c = 0.4\pi$

↳ Watch video for MATLAB example.





> Instead, what about minimizing the error at $L > N$ samples?
More unknowns than equations.

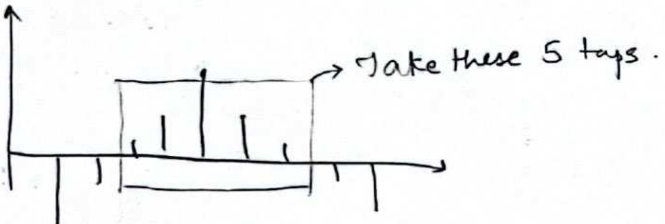
$$E = \sum_{k=0}^{L-1} | A(\omega_k) - A_{des}(\omega_k) |^2 \qquad \omega_k = \frac{2\pi k}{L}$$

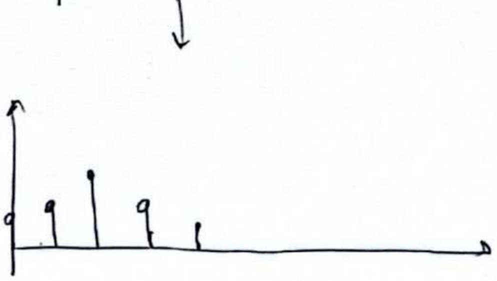
Recall Parseval's Theorem & use it

$$\Rightarrow E = \sum_{n=0}^{L-1} | h[n] - h_d[n] |^2$$

where $h_d[n]$ is the length L FIR filter that goes through the L desired samples.

> To minimize, chose $h[n]$ to agree with $h_{des}[n]$ on the middle N filter taps (like truncating $h_{des}[n]$).





→ Watch video for MATLAB example.

Matrix Formulation

$$A(\omega) = \sum_{n=0}^{M-1} 2\, h[n] \cos(\omega(M-n)) + h[M].$$

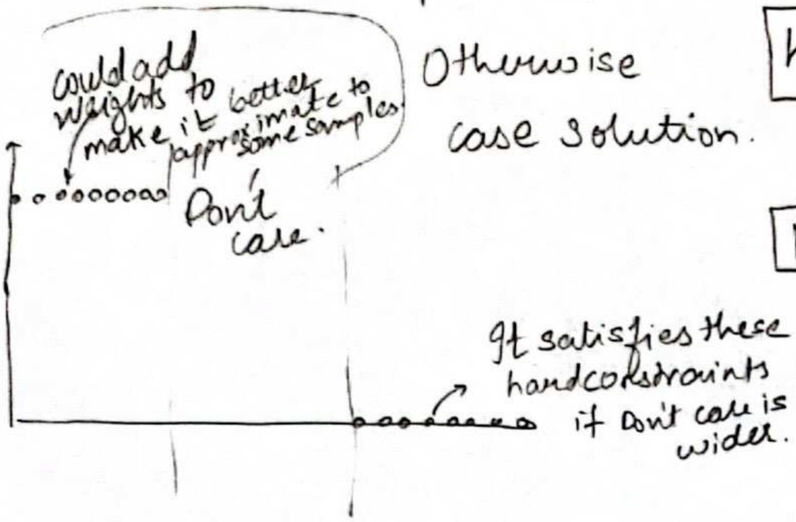
$$\underset{\substack{a_d \\ L\times 1}}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} = \underset{\substack{L \times (M+1) \\ \text{matrix of} \\ \text{Cosines}}}{\begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}} \underset{M+1}{\begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[M] \end{bmatrix}}$$

$$Fh = a_d. \Rightarrow \boxed{h = F^{-1} a_d.}$$ if F is square.

Otherwise $\boxed{h = (F^TF)^{-1} F^T a_d}$ is the least case solution.

↳ pseudo inverse.

$\boxed{h = F \setminus a_d}$ → general command in MATLAB for both cases.

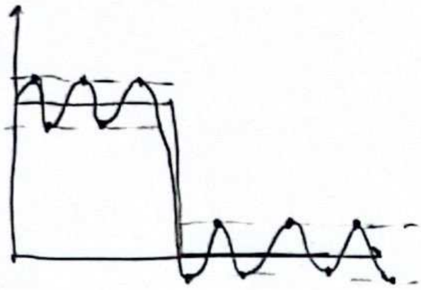


Lec 17 FIR filter design. (Chebyshev)

- How to control what happens between interpolation points.

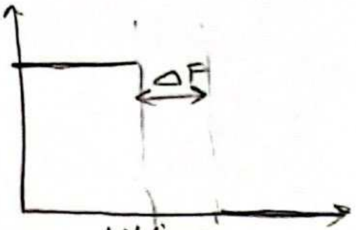
→ $\min_{\text{error}} E = \max_{\omega \in [0,\pi)} |A(\omega) - A_d(\omega)|$ → Forces the max deviation to be minimum.

→ Filters that are optimal w.r.t this criteria are called <u>Equiripple</u>.



\> An acceptable frequency response will have:

1) Linear phase, FIR.
2) Width ΔF of the transition band.
3) Deviation from 1 in pass band of $\pm \delta_1$
4) Deviation from 0 in stop band of $\pm \delta_2$



\> The amplitude response of a Type I linear phase filter can be written as

$$A(\omega) = \sum_{k=0}^{r-1} C_k \cos \omega_k . \qquad r = \frac{N+1}{2}$$

\> The approximation problem.

Given :- • A set of frequency bands in $[0, \pi]$

- A desired real valued $A_{des}(\omega)$ in these bands
- A positive weight function. $W(\omega)$
- The form of $A(\omega)$

$$A(\omega) = \sum_{K=0}^{r-1} C_K \cos \omega_K \qquad r = \frac{N+1}{2}$$

Find $\{C_K\}$

Solve using the Alternation Theorem.

\> If $A(\omega)$ is a sum of r cosines, then a necessary and sufficient condition for $A(\omega)$ to be the unique, best weighted Chebyshev approximation to $A_{des}(\omega)$ on the given bands is that:

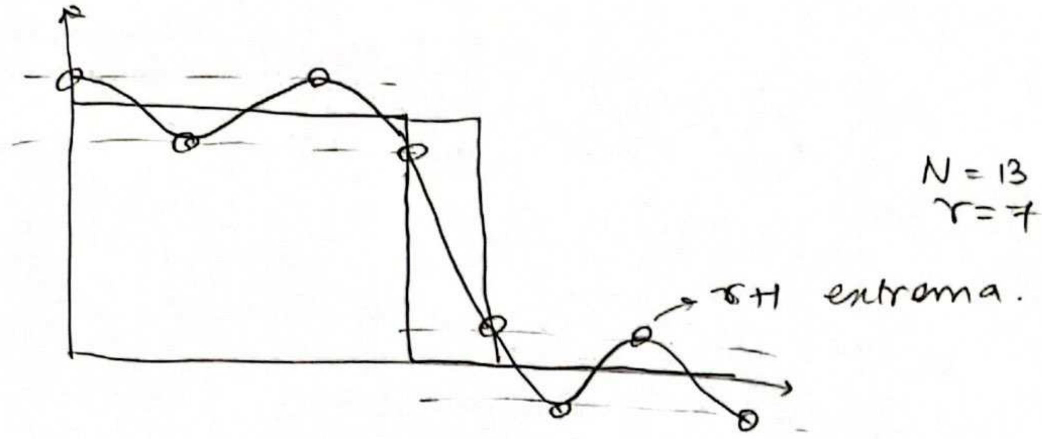
$$E(\omega) = W(\omega) \left| A(\omega) - A_d(\omega) \right|$$

↳ Exhibits atleast $r+1$ extremal frequencies in the given bands.

Extremal frequencies are a set of places such that the error at each of these places = max error.

$$\{\omega_1, \omega_2 \dots \omega_{r+1}\} \text{ such that}$$
$$E(\omega_i) = \max E(\omega)$$
$$E(\omega_i) = -E(\omega_{i+1}) \quad i = 1 \dots r.$$

Eg



→ Remez Exchange Algorithm.

Lemma: $E(\omega) = A_d(\omega) - \sum_{k=0}^{r-1} c_k \cos \omega k$. can be made to take on the values $\pm\delta$ for any given set $\{\omega_1 \dots \omega_{r+1}\}$

$$A_d(\omega_i) = \sum_{k=0}^{r-1} c_k \cos \omega_i k + (-1)^i \delta \quad \text{———} \quad (*)$$
$$i = 1 \dots r+1.$$

This has a unique solution for $\{c_k,\ k=0, \dots r-1\}$ and δ.

Algorithm:

$T_0 = \{\omega_1 \dots \omega_{r+1}\}$ → set of initial guesses for extremal frequencies.

1) Solve the linear equations in (*). Solution has an error that oscillates with amplitude δ_K on T_K.
2) Interpolate to find frequency response on all of $[0,\pi]$.
3) search $[0,\pi]$ to see if/where the $|\text{Error}| > \delta_K$.
4) If max error $= \delta_K$, done. Else, take $r+1$ max. error points as T_{K+1} and go back to step 1)

Sidenote: Sum of cosines $\Longleftrightarrow$ Sum of polynomials using a change of variables.

$$S(x) = A(\cos^{-1} x)$$

$$= \sum_{K=0}^{r-1} c_K \underbrace{\cos(K \cos^{-1} x)}$$

⟶ Turns out to be a polynomia
in x
"Chebyshev Polynomials".

$$= \sum_{K=0}^{r-1} d_K X^K$$

Eg: Try to approximate x^2 "Ades (ω)" by $d_0 + d_1 x$ ove
$[0,1]$ using chebishev error.

ie. $\min_{d_0, d_1} \max_{[0,1]} |x^2 - (d_0 + d_1 x)|$

2 functions ⇒ 3 extremal points

Guess: $T_0 = \{\frac{1}{4}, \frac{1}{2}, 1\}$

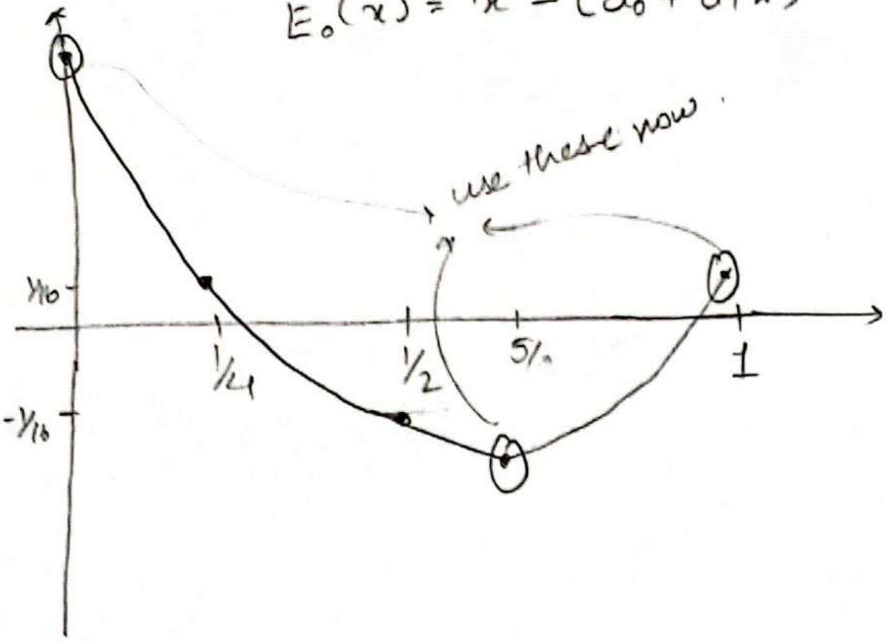
Make error oscillate:

$$x_i^2 = d_0 + d_1 x_i + (-1)^i \delta$$

$$\begin{bmatrix} 1 & 1/4 & 1 \\ 1 & 1/2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ \delta \end{bmatrix} = \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

$$d_0 = \frac{-5}{16}, \quad d_1 = \frac{5}{4}, \quad \delta = \frac{1}{16}$$

$E_0(x) = x^2 - (d_0 + d_1 x)$



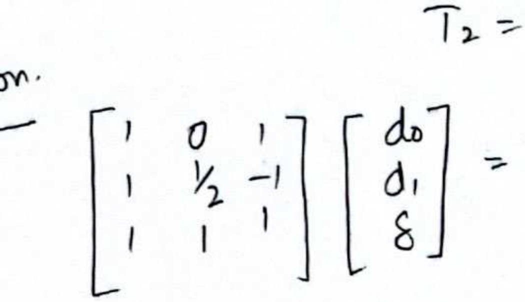
$\max E_0 = \frac{5}{16} \neq \frac{1}{16} = \delta$

Redo

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 5/8 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{25}{64} \\ 1 \end{bmatrix}$$

$\Rightarrow d_0 = \frac{-15}{128}$; $d_1 = 1$; $\delta = \frac{15}{128}$

$E_1(x) = x^2 - \left(\frac{-15}{128} + x\right)$



$\max E = \frac{17}{128} > \frac{15}{128}$

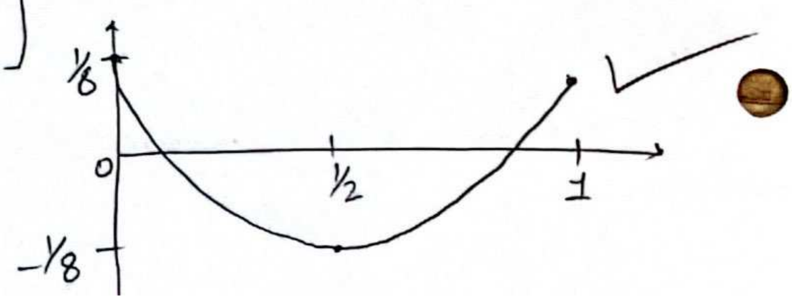
$T_2 = \{0, 1/2, 1\}$.

iteration #3

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1/2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ 1/4 \\ 1 \end{bmatrix}$$

$d_0 = -\frac{1}{8}$, $d_1 = 1$, $\delta = \frac{1}{8}$

$E_2(x) = x^2 - (-1/8 + x)$



→ Remez always converges to an equiripple approx.
but it may not have the passband/stopband characteristi
needed for a given N.

Heuristic:

Given: N: filter length.
F_p, F_s: passband, stopband edges.
δ_1: Deviation in pass band from 1.
δ_2: Deviation in stopband from 0.

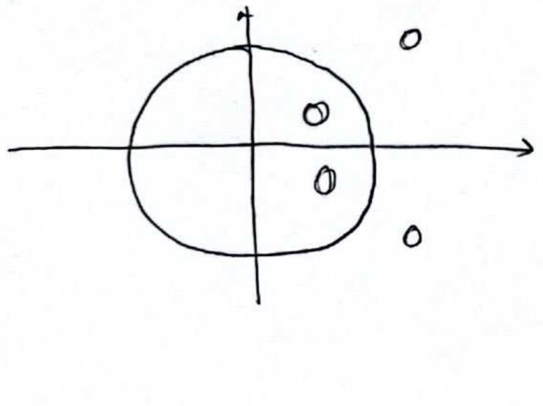
$$N \approx \frac{-20 \log_{10} \sqrt{\delta_1 \delta_2} - 13}{14.6 (F_s - F_p)} + 1$$

→ Small deviation => N ↑

↳ steep transition => N ↑

- Use function <u>firpm</u> to design equiripple filters.
AKA Parks McClellan algorithm.
- Demonstrates a MATLAB example.
- Tradeoff b/w no. of taps, ripple, width of transition region.

<u>Zero locations</u> - for linear phase FIR filters.
If z is a zero, then $\bar{z}$ is a zero and so is $\frac{1}{z}$, $\frac{1}{\bar{z}}$.



<u>FIR advantages</u>
- Can achieve exactly linear phase.
  ↳ symmetric
- Easy to design & implement using linear methods.
- Always stable since No poles.

<u>FIR disadvantages.</u>
- Need large N to achieve good respons
- May need many operations per outpu
  storage of many coefficients.

- In MATLAB use command : >> filterbuilder.

Order mode : min → least no. of taps for given specs.

→ or else use command : >> fdatool → GUI version.

Lec 18 IIR filter design

$$y[n] = \sum_{m=0}^{M} b[m]x[n-m] - \sum_{k=1}^{N} a[k]y[n-k]$$

→ feedback of previou output values.

$$H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$$

can be shown.

$$= \frac{\sum_{n=0}^{M} b[n]z^{-n}}{\sum_{n=0}^{N} a[n]z^{-n}}$$

$$= \frac{B(z)}{A(z)}$$

Differences (Assuming causality)

- Can't do linear phase. → since infinite impulse response canno be symmetric around some point.
- Low orders of IIR filters are sufficient to implement "tight" specs.

Design Process.

- Choose design response $H(\omega)$
- Choose class of filters (IIR, order N in NUM/DEN)
- Choose a distance measure b/w $H_{des}(\omega)$ and $H_{actual}(\omega)$
- Find the optimal filter in the class that minimizes the error.
- For IIR filters we commonly start by designing an analog filter & then convert to digital.

Direct digital IIR. design.

Prony's method.

Given a desired infinite impulse response:

$h_{des}[n]$ (or) $h_d[n]$ $\quad 0 \le n \le \infty$

$$y[n] = -\sum_{k=1}^{N} a[k]\, y[n-k] + \sum_{m=0}^{M} b[m]\, x[n-m]$$

$$H_{des}(z) = \sum_{n=0}^{\infty} h_d[n] z^{-n}.$$

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{\underset{a_0=1}{1} + a_1 z^{-1} + \dots + a_N z^{-N}}.$$

$$(1 + a_1 z^{-1} + a_2 z^{-2} + \dots)(h[0] + h[1] z^{-1} + \dots) = b_0 + b_1 z^{-1} + \dots$$

$$\begin{array}{r} M+1 \leftarrow \\ \\ K-M \leftarrow \\ (K+1)\times 1 \end{array}
\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_M \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}
=
\begin{bmatrix} h_0 & 0 & \dots & & 0 \\ h_1 & h_0 & 0 & \dots & 0 \\ h_2 & h_1 & h_0 & 0 \dots & 0 \\ \vdots & & & & \vdots \\ h_N & - & - & - & h_0 \\ h_K & & & & h_{K-N} \end{bmatrix}
\begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}_{(N+1)\times 1}$$

Rewriting.,

$$\begin{matrix} M+1 \\ K-M \end{matrix}\left\{\begin{bmatrix} b \\ 0 \end{bmatrix}\right. = -\left[\begin{array}{c|c} \multicolumn{2}{c}{\overset{(M+1)\times(N+1)}{H_1}} \\ \hline \underset{(K-M\times1)}{h_1} & \underset{(K-M)\times N}{H_2} \end{array}\right]\begin{bmatrix} 1 \\ a^* \end{bmatrix}\begin{matrix} 1 \\ N \end{matrix} \qquad a* = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}$$

$$b = H_1 a^*$$

$$0 = h + \underset{(K-M)\times N}{H_2} \underset{(N\times1)}{a^*}$$

If $K-M = N$
$\Rightarrow K = M+N$
(IIR has $M+N+1$ values)

$$H_2 a^* = -h_1$$

$$\boxed{\begin{aligned} a^* &= -H_2^{-1} h_1 \\ b &= -H_1 H_2^{-1} h_1 \end{aligned}}$$

→ discovered in 1790 by Prony while studying properties of gases.

→ What do we do with the rest of the impulse response?

$$\begin{bmatrix} b \\ 0 \end{bmatrix} + \underbrace{\overset{\text{error vector}}{e}}_{\text{minimize } \|e\|_2} = \left[\begin{array}{c|c} \multicolumn{2}{c}{H_1} \\ \hline h_1 & H_2 \end{array}\right]\begin{bmatrix} 1 \\ a^* \end{bmatrix}$$

→ least squares solution.

$$a^* = -\underbrace{(H_2^T H_2)^{-1} H_2^T}_{\text{pseudoinverse}} h_1$$

$$b = H_1 a^*$$

What we may really want is to minimize $\hat{e} = h[n] - h_d[n] \cdot (a$

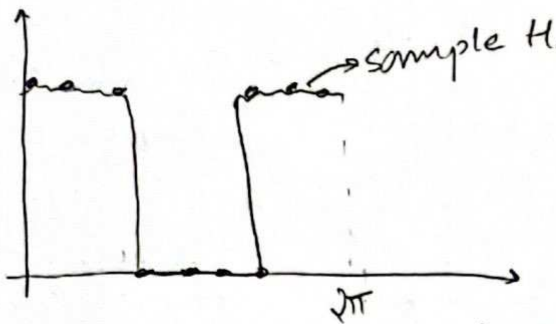
(Related to e by $e = \begin{bmatrix} a_0 & & \\ a_1 & \ddots & \\ a_2 & \ddots & a_0 \\ & \ddots & a_1 \\ & & a_2 \end{bmatrix} \hat{e}$)

→ Shows a MATLAB example.
>> help prony

Frequency sampling design of IIR filters.

> We have a $H_d(\omega)$. Find the best a, b to satisfy $H(z) = \frac{B(}{A(}$

> $L = M+N+1$ unknowns



→ sample $H_d(\omega)$ at $\frac{2\pi K}{L+1}$

→ No of samples = No. of unknowns
⇒ good.

> Doing this is like taking DFT of $h[n]$.

> $H[K] = \frac{\text{DFT}(b[n])}{\text{DFT}(a[n])} = \frac{B[K]}{A[K]}$

Element by element division.

← length L+1 DFT.

> Hence, take length L+1 IDFT of H[K] samples.

> $B[K] = H[K]A[K]$

> In time domain, circular convolution $b = h \circledast a$

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_M \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} g_0 & g_L & g_{L-1} & \cdots & g_1 \\ g_1 & g_0 & \ddots & & \vdots \\ g_2 & g_1 & \ddots & & g_L \\ \vdots & & \ddots & & g_0 \\ g_L & & & & g_1 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_N \\ 0 \\ 0 \\ \vdots \end{bmatrix} \qquad g_i = \text{IDFT}\left\{ H_d\left(\frac{2\pi K}{L+1} \right) \right\}$$

$$\begin{bmatrix} b \\ 0 \end{bmatrix} = \left[\begin{array}{c|c} \multicolumn{2}{c}{G_1} \\ \hline g_1 & G_2 \end{array}\right] \begin{bmatrix} 1 \\ a^* \end{bmatrix} \rightarrow$$

> Similar to Prony
> But approach is different.

$$a^* = -G_2^{-1} b$$

$$b = G_1 \; a$$

> Extend to having more samples of $H_d(\omega)$ than $M+N+1$

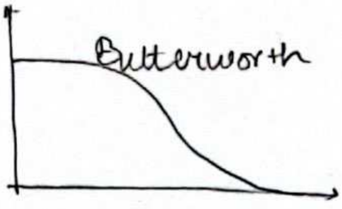
→ Least squares design.

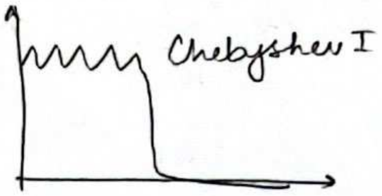
(Like FIR, extension of Prony solution).

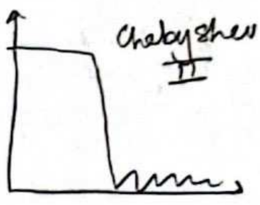
> Note: $H_d(\omega)$ should be consistent with real $h_d[n]$ so that $a[n]$, $b[n]$ are real.

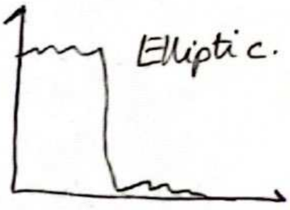
> No guarantees that designed filters are stable.

> Designing from Analog filters. → Common technique.





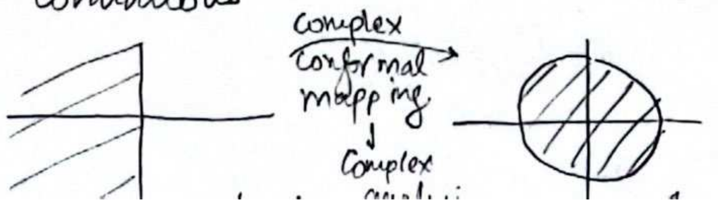




> Closed form solutions exist for such types of filters in continuous time.

> How to convert these into discrete time?

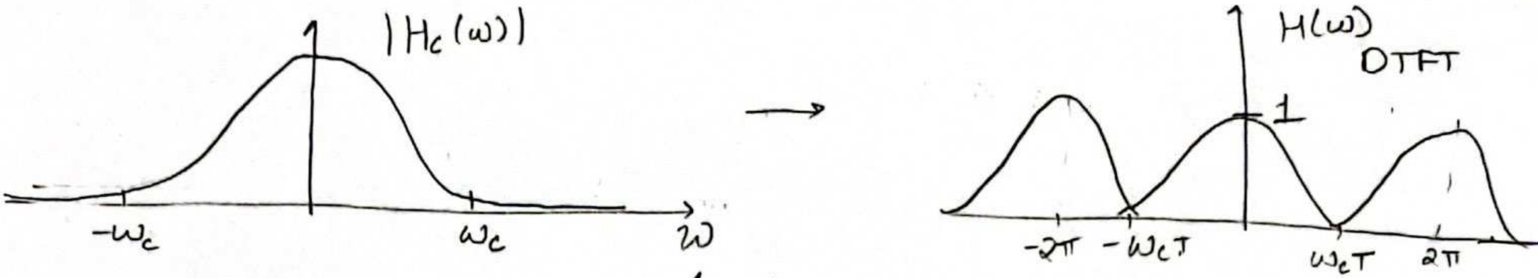
$$\begin{array}{ccc} H_c(s) & \xrightarrow{\quad T \quad} & H(z) \\ h_c(t) & & h[n] \\ \text{continuous} & & \text{discrete.} \end{array}$$



2 main approaches

1) Impulse invariance.

Given $h_c(t)$ and create $h[n] = T\, h_c(nT)$ ↱ since after sampling it is scaled by $1/T$.



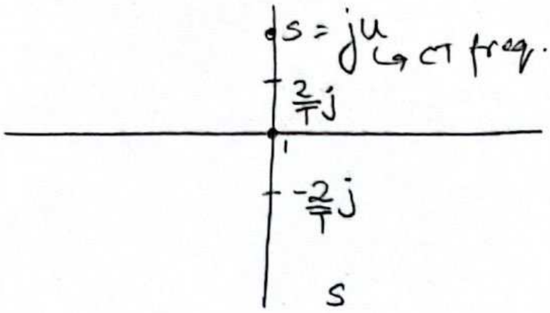
> sampling rate needs to be high enough.

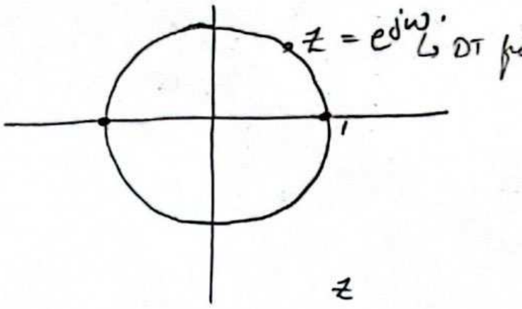
2) Bilinear Transformation.

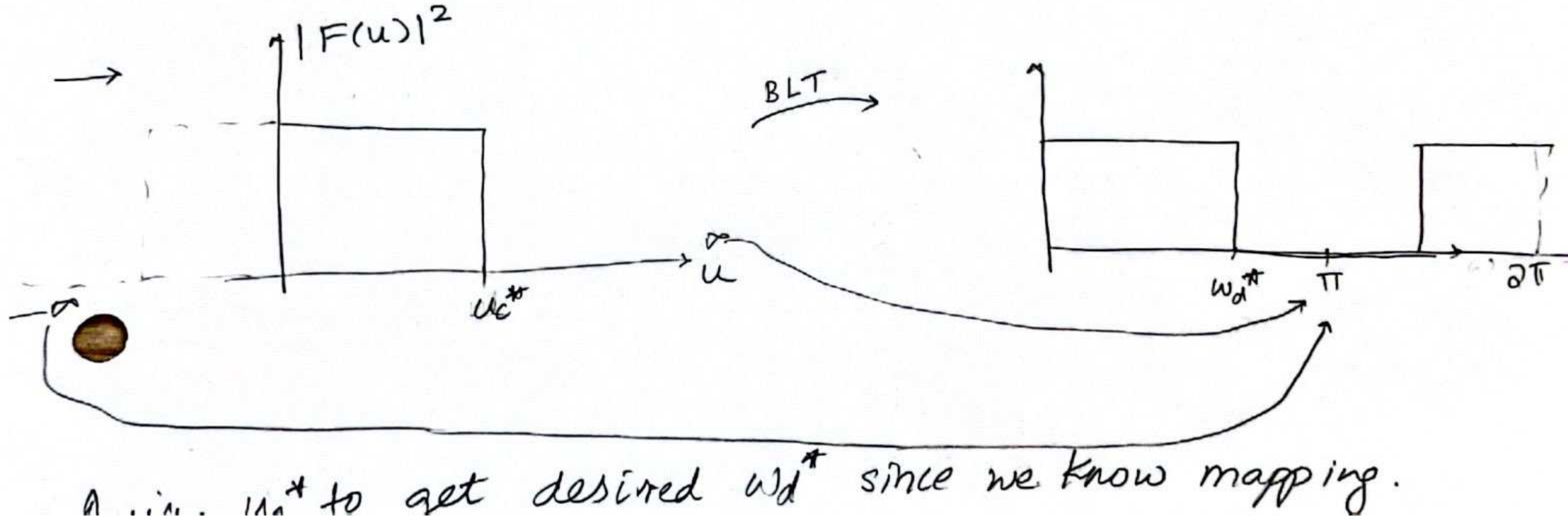
Consider a transformation of the s plane into the z plane.

$$S = \frac{2}{T}\,\frac{z-1}{z+1} \qquad z = \frac{\frac{2}{T}+s}{\frac{2}{T}-s}$$

s	z	$\omega \in [0,2\pi]$
0	1	0
$\pm\infty$	-1	π
$\frac{2}{T}j$	j	$\frac{\pi}{2}$
$\frac{-2}{T}$	0	N/A

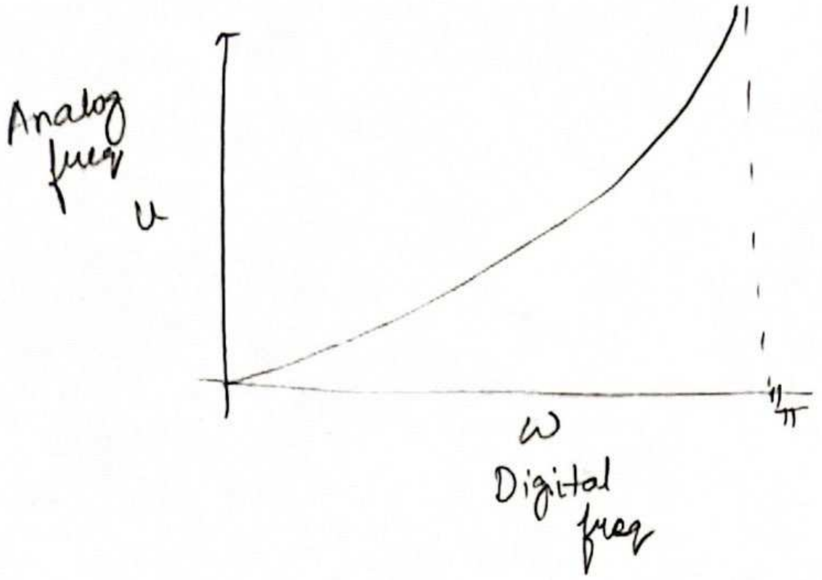






u_c^* to get desired ω_d^* since we know mapping.

> y axis doesnt change after BLT , so ripple specs remain unchanged while designing in analog.



> Gives matlab example : >> fda tool.