Complex Analysis - IITM - Balki

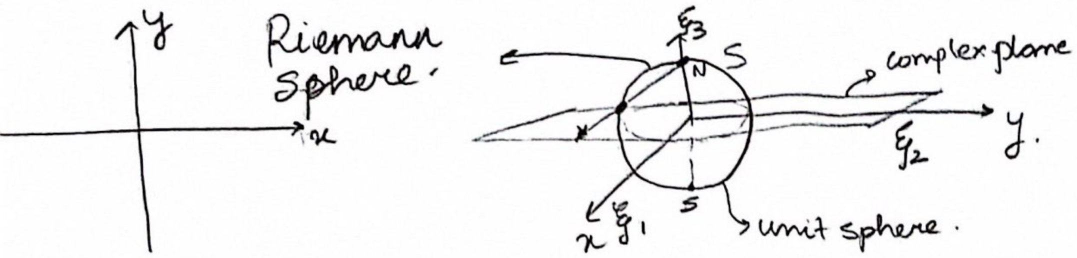
Lec 1

› $Z = x+iy \quad (x, y \in \mathbb{R})$

› $Z^* = x-iy \rightarrow$ linearly independent of $x+iy$

Analytic functions are fns of only $Z = x+iy$ & not of Z^*.

Stereographic projection. → projecting sphere onto a plane.



θ, φ are polar & azimuthal angle of the sphere.

Let ξ_1, ξ_2, ξ_3 be coordinates of the sphere. $\Rightarrow$ $\xi_1 = \sin\theta\cos\phi$

$\xi_2 = \sin\theta\sin\phi$

$\xi_3 = \cos\theta$

& $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$

› Projection from North pole onto complex plane is always associated to one point on the sphere

› map $x, y \rightarrow \xi_1, \xi_2$ & ξ_3 is not independant.

It is easy to show that $x = \dfrac{\xi_1}{1-\xi_3}$; $y = \dfrac{\xi_2}{1-\xi_3}$ $\Rightarrow$ $Z = \dfrac{\xi_1 + i\xi_2}{1-\xi_3}$ (not Z axis.)

$\Rightarrow x = \left(\cot\frac{\theta}{2}\right)\cos\phi$; $y = \left(\cot\frac{\theta}{2}\right)\sin\phi$; $Z = e^{i\phi}\cot\frac{\theta}{2}$

$\xi_1 = \dfrac{2x}{x^2+y^2+1}$ $\quad \xi_2 = \dfrac{2y}{x^2+y^2+1}$ $\quad \xi_3 = \dfrac{x^2+y^2-1}{x^2+y^2+1}$ $\Rightarrow \xi_1 = \dfrac{Z+Z^*}{|Z|^2+1}$; $\xi_2 = \dfrac{Z-Z^*}{i(|Z|^2+1)}$; $\xi_3 = \dfrac{|Z|^2-1}{|Z|^2+1}$

The origin is mapped to SP, all points at ∞ are mapped to NP

$\Rightarrow$ $\mathbb{C} : \{|Z| < \infty\}$ → complex plane.

$\hat{\mathbb{C}} : \{|Z| \leq \infty\}$ → extended complex plane

- If z_1 & z_2 are two points on the complex plane the chordal distance b/w the corresponding points on the sphere is $\dfrac{2|z_1 - z_2|}{\sqrt{(|z_1|^2+1)(|z_2|^2+1)}}$

> Now distance between any 2 pts. becomes finite.

> Distance (chordal) from z to ∞ = $\dfrac{2}{\sqrt{|z|^2+1}}$.

Analytic functions

$f(z) = u(x,y) + i\,v(x,y)$ is analytic if the Cauchy Riemann conditions are satisfied.

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ ; \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

$x = \dfrac{z+z^*}{2}$ $\quad y = \dfrac{z-z^*}{2i}$

we basically say $\dfrac{\partial f}{\partial z^*} = 0$ since f is not a fn of z^*

Wirtinger derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \ ; \ \frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\Rightarrow \frac{\partial f}{\partial z^*} = 0 \Rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \frac{i\,\partial u}{\partial y} - \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \ \& \ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

Eg: $f = x$ → not analytic

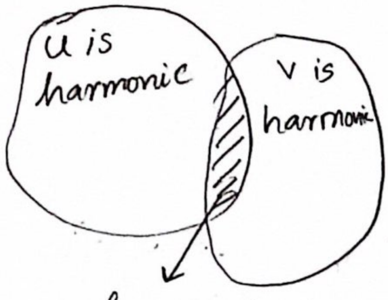
$f = y$ → No

$f = r = \sqrt{x^2+y^2}$ → No since $r = (zz^*)^{1/2}$

$f = \theta = \tan^{-1}\dfrac{y}{x} = \dfrac{1}{2i}\ln\left(\dfrac{z}{z^*}\right)$ → No

From CR equations we can see that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ & $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

$\Rightarrow u, v$ are harmonic functions in the region where f is analytic.

→ 

here $u+iv$ is analytic

⇒ Given u we can find v from CR eqns where $u+iv$ is analytic.

→ If f is analytic in $\mathbb{C}$ then f is called an entire function.

$f(z) = z^2$ → Yes! Entire!

$f(z) = z^n$ → yes, $f(z) = P_n(z)$ → yes. (↗ polynomial of degree n)

$f(z) = e^z$ → yes

$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ → all finite z, this is convergent ⇒ $|z| < \infty$

e^{-z} → yes.

$\frac{e^z + e^{-z}}{2} = \cosh(z)$ → yes; $\sinh(z)$ → yes; $\cos(z)$ → yes, $\sin(z)$ → yes.

$\tan z$ → not entire. (singularities ⇒ not differentiable)

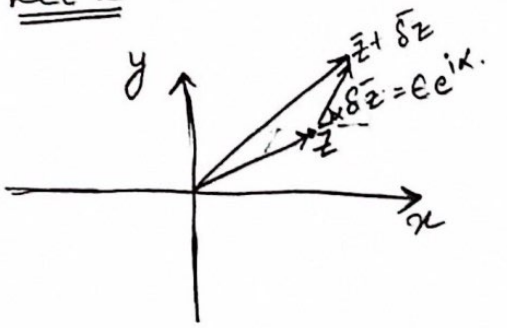
z^{-1} → not entire (singularity at $z=0$).

→ Any fn. that is entire in the extended complex plane, must be a constant.

<u>Liouville's Theorem</u>: A function that is bounded & entire must be constant.

⇒ If it is entire & not constant, it must be singular at ∞.

Lec 2



$$\frac{df}{dz} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

not sure how to go from here to there.

Basically since δz can be along arbitrary direction, this δz must must be coming in from any direction. Let $\delta z = \epsilon e^{i\alpha}$

$$\Rightarrow \frac{df}{dz} = e^{-i\alpha}\left\{\left(\frac{\partial u}{\partial x}\cos\alpha + i\frac{\partial v}{\partial y}\sin\alpha\right) - i\left(\frac{\partial u}{\partial y}\cos\alpha + i\frac{\partial v}{\partial x}\sin\alpha\right)\right\}$$

$$= e^{-i\alpha}\left\{\frac{\partial u}{\partial x}\cos\alpha + i\frac{\partial v}{\partial y}\sin\alpha - i\left(\frac{\partial u}{\partial y}\cos\alpha - i\frac{\partial v}{\partial x}\sin\alpha\right)\right\}$$

Apply CR

$$= e^{-i\alpha}\left\{\frac{\partial u}{\partial x}\left[\cos\alpha + i\sin\alpha\right] - i\frac{\partial u}{\partial y}\left[\cos\alpha + i\sin\alpha\right]\right\}$$

$$= \cancel{e^{-i\alpha}}\left\{\cancel{e^{i\alpha}}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)\right\}$$

$$\boxed{\frac{df}{dz} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}} \quad \text{or} \quad \frac{df}{dz} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$\Rightarrow$ Only when CR conditions are applied the derivative is independent of α.

$$\frac{df}{dz} = i\left(\frac{\partial v}{\partial x} - i\frac{\partial v}{\partial y}\right)$$

$$\Rightarrow 2\frac{df}{dz} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right).$$

$$\Rightarrow \frac{df}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f$$ $\rightarrow$ Wirtinger derivative from earlier.

> $f(x) = x|x|$ has first derivative at 0 but no 2^{nd} derivative.

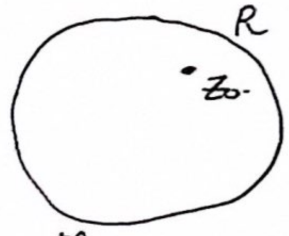
$$f(x) = \begin{cases} -x^2 & x > 0 \\ x^2 & x < 0 \end{cases}$$

> However if $f(z)$ is analytic $\Rightarrow$ first derivative exists $\Rightarrow$ all derivatives exist.

$\Rightarrow$ $f(z)$ can be represented as a power series in the region where it is analytic

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n} \rightarrow \text{Power series (Taylor series)}.$$

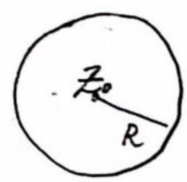
$$\boxed{a_n = \frac{1}{n!} \frac{d^n f(z)}{dz^n}\bigg|_{z = z_0}}$$



> Worst case convergence check is absolute convergence $\Rightarrow \sum_{n=0}^{\infty} |n^{th} \text{ term}| < \infty$.

> Every power series about z_0 has a circle of convergence with radius R. inside which the series converges absolutely.

$$\Rightarrow \sum_{n=0}^{\infty} |a_n (z - z_0)^n| < \infty$$



R = radius of convergence.

$$\boxed{R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|}$$

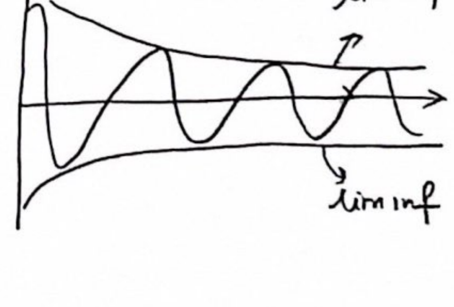
Ratio test: $\lim_{n \to \infty} \left| \frac{a_{n+1} (z - z_0)^{n+1}}{a_n (z - z_0)^n} \right| = L$

$L < 1 \Rightarrow$ absolute conv.

Better defn. ?

$$\boxed{R = \limsup_{n \to \infty} |a_n|^{1/n}}$$

least upper bound.



$R = \infty \Rightarrow$ entire functions.

Eg: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{z^{n+1}\, n!}{(n+1)!\, z^n}\right| = \lim_{n\to\infty}\left|\frac{z}{n+1}\right| \to 0$ for all $|z| < \infty$

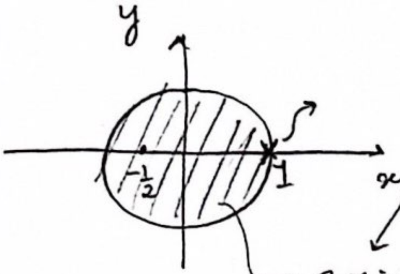
$\Rightarrow$ e^z is an entire function

Eg: $f(z) = 1 + z + z^2 + \dots = \sum_{n=0}^{\infty} z^n \rightarrow$ converges when $|z| < 1$

R = 1

$f(z) - 1 = z f(z)$ $\Rightarrow f(z)(1-z) = 1$

$\Rightarrow$ $\boxed{f(z) = \frac{1}{1-z}}$ when $|z| < 1$



series converges to $\frac{1}{1-z}$ value inside but diverges outside.

But, $\frac{1}{1-z}$ can have values at any point outside the region of convergence of the series. The Taylor series & the fn $f(z)$ match inside the ROC.

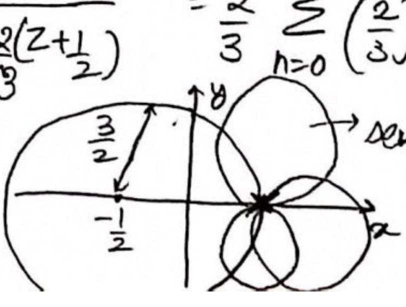
> $\frac{1}{1-z}$ blows up only at $z = 1$. In fact $\frac{1}{1-z}$ is analytic everywhere except at $z = 1$.

. $\frac{1}{1-z}$ is the analytic continuation of the series $\sum_{n=0}^{\infty} z^n$ since it has values outside the ROC & matches the series inside the ROC.

> Say we want a power series for $\frac{1}{1-z}$ about $-\frac{1}{2}$

$$\frac{1}{1-z} = \frac{1}{\frac{3}{2} - (z+\frac{1}{2})} = \frac{2}{3}\,\frac{1}{1-\frac{2}{3}(z+\frac{1}{2})} = \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n\left(z+\frac{1}{2}\right)^n$$

This converges when $|z+\frac{1}{2}| < \frac{3}{2}$



several series can be made

** > Therefore a given analytic function can have an infinite number of series representations each valid in a different region. Always specify ROC. Each of these series' are analytic continuations of each other.

> For every power series there is a circle of convergence (may be ∞) inside which it is absolutely convergent, outside which it is divergent & on which nothing can be said & investigation on a case by case basis is required.

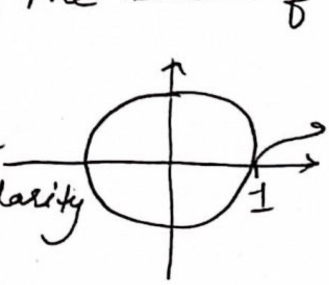
> Let $z=i \Rightarrow \sum z^n = 1+i-1-i+1+\dots$ Therefore we get 4 partial sums, 1, $1+i$, i, 0 depending on where you end the series. The arithmetic avg of these four sums is $\frac{1+i}{2}$ which is also given by substituting $z=i$ in $\frac{1}{1-z}$.

> This arithmetic mean of partial sums is guaranteed to match the value given by the analytic contn. fn. This arithmetic mean of partial sums is called Césaro sum.

> For every power series, the function that represents the power series must have atleast one singularity on its circle of convergence.

Eg: $\sum\limits_{n=1}^{\infty} \frac{z^n}{n^2}$ $\rightarrow$ converges even on the circle of convergence at all points.

If we find $f(z)$ for this series, we can show that there is indeed a 'subtle' singularity at $x=1$.



here value is max on COC & value is $\frac{\pi^2}{6} = \frac{1}{1}+\frac{1}{2^2}+\frac{1}{3^2}+\dots$

Eg:

$\lim_{z \to 1} (z-1) \ln(z-1) \longrightarrow 0$ but there is still a singularity.
Therefore singularity does not mean ∞.
Here there is a logarithmic branch pt.

Eg:

$f(z) = z + z^2 + z^4 + z^8 + \dots = \sum_{n=0}^{\infty} (z)^{2^n}$ → diverges everywhere on the COC. (shown below)

$f(z) = z + f(z^2) \Rightarrow f(z^2)$ is singular at both $+1, -1$. for sure

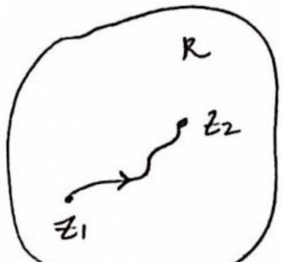
$f(z) = z + z^2 + f(z^4) \Rightarrow$ singular at $+1, -1, +i, -i$ for sure.

⇒ we can show there is a dense set of singularities on the unit circle & therefore, there is no way this series can be analytically continued outside the COC. The COC forms a "natural boundary". $f(z)$ has a "natural boundary".

> Such a series with no analytic continuation outside the COC is called a "lacunary series." Large gaps in powers give lacunary series.

Eg: $\sum_{n=0}^{\infty} z^{n!}$

Lec 3 Calculus of Residues.



$\int_{z_1}^{z_2} f(z)\, dz$ is independant of the actual path from z_1 to z_2.

> Looks like a conservative vector field
$\vec{u} = \nabla \phi(\bar{r})$

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{u} \cdot d\bar{r} = \int_{\bar{r}_1}^{\bar{r}_2} \nabla \phi \cdot d\bar{r} = \phi(\bar{r}_2) - \phi(\bar{r}_1)$$

> This is why potentials (pot energy) are used to describe a conservative force.

$\Rightarrow \oint \nabla \phi \cdot d\bar{r} = 0$. $\longrightarrow$ Cauchy's Theorem.

$\Rightarrow \boxed{\oint_C f(z)\, dz = 0}$ The integral of an analytic function over a closed contour is identically zero.

> This is also why potentials are used to describe E field.

$V = \int E \cdot dl$ & $\oint E \cdot dl = 0$ when $\frac{dB}{dt} = 0$ - Recall that

$E = -\frac{\partial A}{\partial t} - \nabla \phi \Rightarrow$ if $\frac{\partial A}{\partial t} = 0$ $\bar{E}$ becomes a conservative field

* can be written at $V_2 - V_1 \rightarrow$ voltage!

Singularities

> Removable singularities.

By defining the fn appropriately we can remove it.

Eg: $\frac{\sin z}{z} \rightarrow$ indeterminate at $z = 0$ but limit as $z \rightarrow 0$ is 1

$\Rightarrow$ define as $f(z) = \left\{ \begin{array}{ll} \frac{\sin z}{z} & z > 0 \\ 1 & z = 0 \end{array} \right\}$ $\Rightarrow$ fn becomes continuous.

Other singularities

> $f(z) = \underbrace{\frac{C_{-1}}{z-a}}_{\text{singular part}} + \underbrace{\sum_{n=0}^{\infty} C_n (z-a)^n}_{\text{regular part.}} \rightarrow$ Simple pole at $z = a$

Residue at $z-a$ = C_{-1} ! Super essential.

> $f(z) = \frac{\sin z}{z^2} = \frac{z - \frac{z^3}{3!} + \dots}{z^2} = \underbrace{\frac{1}{z}} - \frac{z}{3!} + \dots$

$\hookrightarrow$ simple pole with redire of 1.

> $f(z) = \frac{g(z)}{h(z)}$; $g(a) \neq 0$ & $h(a) = 0$

$\Rightarrow f(z) = \frac{g(a) + (z-a) g'(a) + \dots}{\cancel{h(a)} + (z-a) h'(a) + \dots} \Rightarrow \frac{g(a)}{(z-a) h'(a)} + \frac{g'(a)}{h'(a)} + \dots$

$\Rightarrow$ simple pole with residue of $\boxed{g(a)/h'(a)}$

$\rightarrow$ $f(z)$: simple pole at $z = a$

$\Rightarrow$ Residue : $\boxed{\lim_{z \to a} (z-a) f(z)}$ since this would extract C_{-1} & kill all other terms.

Eg : $\frac{1}{\sin \pi z} \rightarrow$ Singularities at all integers! All singularities are simple poles.

At $z \to 0$ $\frac{1}{\sin \pi z} \rightarrow \frac{1}{\pi z} \Rightarrow$ residue is $\frac{1}{\pi}$.

→ Residue of $\frac{1}{\sin \pi z}$ at $z = n$?

$$\lim_{z \to n} \left(\frac{z-n}{\sin \pi z} \right) = \frac{1}{\pi \cos \pi z} = \frac{(-1)^n}{\pi}$$

→ How about fns with multiple poles?

$$f(z) = \frac{C_{-m}}{(z-a)^m} + \dots + \frac{\overset{\text{Residue}}{C_{-1}}}{(z-a)} + \sum_{n=0}^{\infty} C_n (z-a)^n.$$

↳ pole of order m at $z = 0$. (more singular)

The residue is C_{-1} not $C_{-2}, C_{-3} \dots$ so we need to extract it.

$$C_{-1} = \mathrm{Res}\,(f(z))\Big|_{z=a}$$

$$\boxed{C_{-1} = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]}$$ → m is order of pole.

Essential singularity.

If $f(z)$ contains all negative powers of $(z-a)$ it has an essential singularity at $z=a$. if $f(z)$ can be expressed as,

$$f(z) = \sum_{n=1}^{\infty} \frac{C_{-n}}{(z-a)^n} + \sum_{n=0}^{\infty} C_n (z-a)^n$$. Again C_{-1} is the residue of $f(z)$.

↳ Such a series representation is called a Laurent series.

> Say the regular part converges inside a COC at a with radius r_1.

The singular part also converges outside some other COC around a.

If there is no overlap b/w these two ROCs, the Laurent series representation cannot be used.

> If $r_1 > r_2$ we can use the Laurent series ...

> $r_2 < |z-a| < r_1 \Rightarrow$ Laurent series converges.

Eg: $e^{\frac{1}{z}} = 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \cdots$

$$= \underbrace{\sum_{n=1}^{\infty} \frac{1}{n!z^n}}_{\text{singular}} + \underbrace{1}_{\text{regular.}} \rightarrow \text{converges in } |z|>0$$

$\Rightarrow r_2 = 0\ ;\ r_1 = \infty$

infact it also converges at ∞.

> Residue is 1

Eg: $e^{\frac{1}{z}} + e^{z} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n!z^n}}_{|z|>0} + 2 + \underbrace{\sum_{n=1}^{\infty} \frac{z^n}{n!}}_{|z|<\infty} \rightarrow$ converges $0<|z|<\infty$

Eg: $e^{\frac{1}{z^2}} \rightarrow$ essential singularity, residue $=0$.

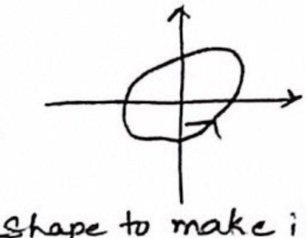
Eg: $e^{-\frac{1}{(z-2)^2}} \rightarrow$ " " - " "

Eg: $(z-2)\, e^{-\frac{1}{(z-2)^2}} \rightarrow$ essential singularity residue $=-1$

Eg: $(z-2)^4\, e^{-\frac{1}{(z-2)^2}} \rightarrow$ essential, residue $=0$.

Residues → Why is it so important?

→ Consider $\oint_C z^n \, dz$ → entire fn $\Rightarrow = 0$.
$n = 0, 1, 2 \dots$



$z = e^{i\theta}$ on the unit circle (we can distort the shape to make it unit circle)

$dz = e^{i\theta} d\theta \, i$

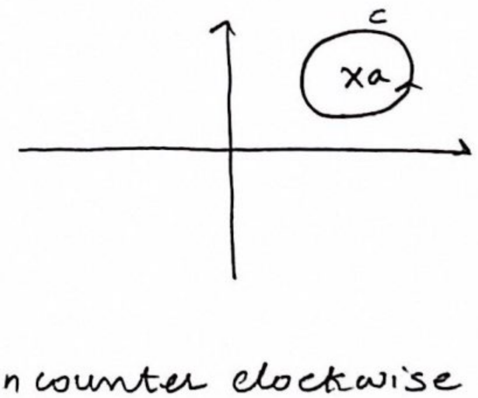
$$\Rightarrow \oint_0^{2\pi} e^{(n+1)i\theta} \, i \, d\theta = 0$$

→ Consider $\oint \frac{dz}{z^{n+1}}$ where $n = 0, 1, 2, 3 \dots$

$$\Rightarrow \oint \frac{dz}{z^{n+1}} = i\int_0^{2\pi} \frac{e^{i\theta} d\theta}{e^{(n+1)i\theta}} = i\int_0^{2\pi} d\theta \, e^{-ni\theta} = \begin{cases} 2\pi i & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

→ $$\boxed{\frac{1}{2\pi i} \oint_C \frac{dz}{z^{n+1}} = \delta_{n,0}}$$ → Knocker delta $\Rightarrow \delta_{n,0} \begin{cases} = 1 & \text{when } n = 0 \\ = 0 & \text{when } n \neq 0 \end{cases}$

Generalising



$$\boxed{\frac{1}{2\pi i} \oint_C \frac{dz}{(z-a)^{n+1}} = \delta_{n,0}}$$

Therefore when integrating a pole in counter clockwise direction, the only term that contributes is the $\frac{1}{z-a}$ term & hence that is the residue.

$$\boxed{\frac{1}{2\pi i} \oint_C f(z) \, dz = \operatorname{Res}_{z=a} [f(z)]}$$ → Cauchy's integral formula!

Eg: $\oint_C e^{1/z}\, dz = 2\pi i$ Despite having a horrible essential singularity at $z=0$, the value depends only on the residue $\Rightarrow$ only on $\frac{1}{z}$ term

$$\oint_C e^{1/z^2}\, dz = 0$$

Recursion relations Lec-4

$$C_{n+1} = \frac{C_n + C_{n+2}}{2} \quad ; \; n = 0, 1, 2, 3 \ldots$$

$\rightarrow$ C_{n+1} is the mean $\Rightarrow$ solution must be a linear function $\rightarrow$ equivalent of a harmonic function.

$$C_{n+2} - 2C_{n+1} + C_n = 0 \quad ; \quad C_0, C_1 \text{ are given.}$$

Define, $f(z) \longrightarrow$ a generating function.

$$f(z) \equiv \sum_{n=0}^{\infty} C_n z^n$$ $\rightarrow$ use coefficients to define this series.

$$\Rightarrow \underbrace{\sum_{n=0}^{\infty} z^n C_{n+2}}_{\frac{1}{z^2}(f(z) - C_0 - C_1 z)} - 2 \underbrace{\sum_{n=0}^{\infty} z^n C_{n+1}}_{\frac{2}{z}(f(z) - C_0)} + \underbrace{\sum_{n=0}^{\infty} z^n C_n}_{f(z)} = 0$$

$$\Rightarrow \frac{1}{z^2}(f(z) - C_0 - C_1 z) - \frac{2}{z}(f(z) - C_0) + f(z) = 0$$

$$\Rightarrow f(z)\left(\frac{1}{z^2} - \frac{2}{z} + 1\right) = \frac{C_0 + C_1 z}{z^2} - \frac{2C_0}{z}$$

$$\Rightarrow f(z) = \frac{C_0 + (C_1 - 2C_0) z}{(z-1)^2}$$

=> Substitute ICs c_0, c_1 & expand $f(z)$ as a power series, the coefficients now give c_n.

$$\longrightarrow \quad c_n = \left[\frac{1}{n!}\frac{d^n f(z)}{dz^n}\right]_{z=0}$$

→ directly gives c_n. But doesn't work if negative powers of z exist. ⇒ poles.

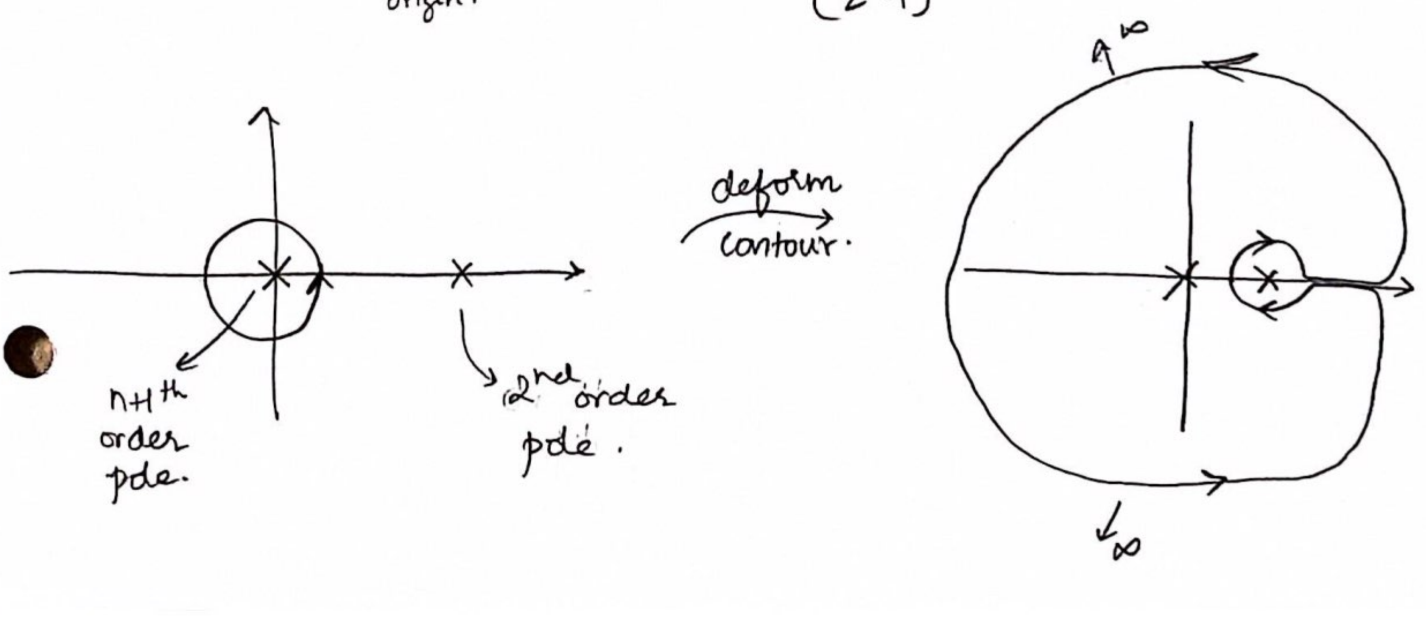
> To find c_{-1} say we can simply find the residue $\frac{1}{2\pi i}\oint_C f(z)\,dz$.

To find c_3 I can find the residue of $\frac{1}{2\pi i}\oint_C \frac{f(z)}{z^4}\,dz$ since c_3 goes from being the coefficient of z^3 to z^{-1} which is the residue.

$$\Rightarrow \quad \boxed{c_n = \frac{1}{2\pi i}\oint \frac{f(z)}{z^{n+1}}\,dz}$$

→ Always works even with singularities as long as series is Laurent.

$$\Rightarrow \quad c_n = \frac{1}{2\pi i}\oint_{\text{origin.}} \frac{dz}{z^{n+1}} \cdot \frac{c_0 + (c_1 - 2c_0)z}{(z-1)^2}$$

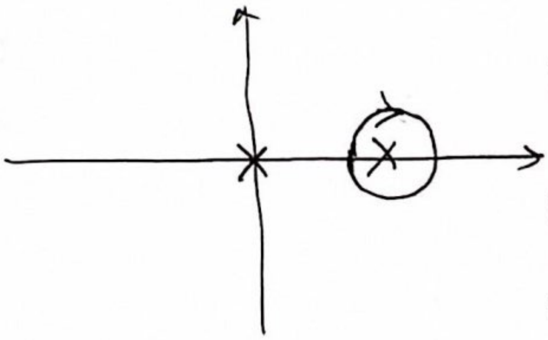


At infinity, $dz \to R$ $\quad z \to R$ $\quad \Rightarrow \dfrac{\cancel{R} \times \cancel{R}}{R^{n+1} \times \cancel{R^2}} \Rightarrow \dfrac{1}{R^{n+1}}$

$z^{n+1} \to R^{n+1}$ $\quad \dfrac{1}{(z-1)^2} \to R^2$

goes to 0 (even) for $n = 0, 1, \ldots$ $\Rightarrow$ contour at $\infty \to 0$.

$\Rightarrow$



Therefore we only need to find -ve of residue of $(z-1)^2$ pole.
Finding residue at origin $\Rightarrow$ we would need to differentiate n times since pole there is of order $n+1$.

$$\Rightarrow C_n = \frac{1}{2\pi i}(-2\pi i)\left[\text{Residue of } \frac{C_0 + (C_1 - 2C_0)z}{z^{n+1}(z-1)^2}\right] \text{ at } z = 1$$

$$= \frac{-2\pi i}{2\pi i}\left[\lim_{z\to 1} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}}\left[\frac{\cancel{(z-1)^2}\, C_0 + (C_1 - 2C_0)z}{z^{n+1}\, \cancel{(z-1)^2}}\right]\right]$$

$$= -\lim_{z\to 1} \frac{d}{dz}\left[\frac{C_0 + (C_1 - 2C_0)z}{z^{n+1}}\right]$$

$$= -\lim_{z\to 1} C_0(-n-1) z^{-n-2} + -n(C_1 - 2C_0) z^{-n-1}$$

$$= (n+1)C_0 + n(C_1 - 2C_0)$$

$$C_n = -nC_0 + C_0 + nC_1 = n(C_1 - C_0) + C_0.$$

$$\Rightarrow \boxed{C_n = n(C_1 - C_0) + C_0}$$

Eg: Try $C_{n+2} = C_n + C_{n+1}$. $\quad C_0 = 0$; $C_1 = 1$. $\Rightarrow$ Fibonacci!

Steps: 1> Define generating function.

2> Apply summation to given Diff. Eq.

3> Write in terms of $f(z)$.

4> Use $C_n = \frac{1}{2\pi i} \oint \frac{f(z)}{z^{n+1}} dz$

5> Distort contour to find residue over finite poles.

$$C_{-1} = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]$$ & residue $= 2\pi i (C_{-1})$

Soln: 1> $f(z) = \sum_{n=0}^{\infty} C_n z^n$

2> $$\underbrace{\sum_{n=0}^{\infty} z^n C_{n+2}}_{\frac{1}{z^2}[f(z) - C_0 - C_1 z]} - \underbrace{\sum_{n=0}^{\infty} z^n C_{n+1}}_{\frac{1}{z}[f(z) - C_0]} - \underbrace{\sum_{n=0}^{\infty} z^n C_n}_{f(z)}$$

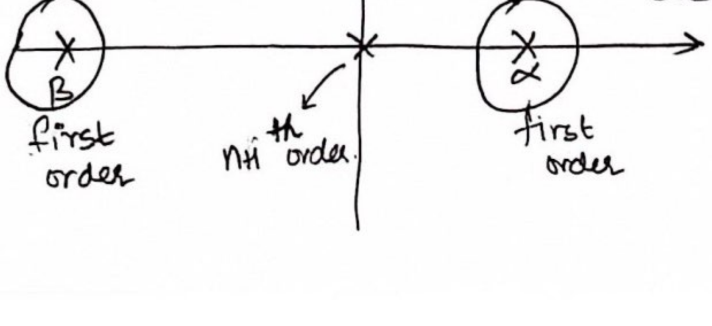
3> $\Rightarrow$ $$f(z) \left[\frac{1}{z^2} - \frac{1}{z} - 1 \right] = \frac{C_0 + C_1 z}{z^2} - \frac{C_0}{z}$$

$\Rightarrow$ $$f(z) \left[\frac{1 - z - z^2}{\cancel{z^2}} \right] = \frac{C_0 + C_1 z - C_0 z}{\cancel{z^2}}$$

$\Rightarrow$ $$f(z) = \frac{C_1 z + C_0 (1 - z)}{1 - z - z^2}$$ $\rightarrow$ Could use a binomial expansion & write it as $\sum_{n=0}^{\infty} C_n z^n$ to find C_n but let us use Complex Analysis.

4> $$C_n = \frac{-1}{2\pi i} \oint \frac{1}{z^{n+1}} \cdot \frac{C_1 z + C_0 (1 - z)}{\underbrace{z^2 + z - 1}} dz$$ $\rightarrow$ roots $= \alpha, \beta = \frac{-1 \pm \sqrt{5}}{2}$

5>



$\rightarrow$ distorting contour, $\int$ at ∞ goes to Zero even for $n=0$.

$$\Rightarrow \quad C_n = \frac{-1}{\cancel{2\pi i}} \times \overset{\text{clockwise.}}{-\cancel{2\pi i}} \times \left[\left\{ \lim_{z \to \alpha} (\cancel{z-\alpha}) \cdot \frac{C_1 z + C_0(1-z)}{z^{n+1}(\cancel{z-\alpha})(z-\beta)} \right\} + \left\{ \lim_{z \to \beta} (\cancel{z-\beta}) \cdot \frac{C_1 z + C_0(1-z)}{(\cancel{z-\beta}) z^{n+1}(z-\alpha)} \right\} \right]$$

$$\boxed{C_n = \left[\frac{C_1 \alpha + C_0(1-\alpha)}{\alpha^{n+1}(\alpha-\beta)} + \frac{C_1 \beta + C_0(1-\beta)}{\beta^{n+1}(\beta-\alpha)} \right]}$$

$$\alpha = \frac{\sqrt{5}-1}{2}$$

$$\beta = \frac{-\sqrt{5}-1}{2}$$

Fibonacci $\Rightarrow$ $C_0 = 0$; $C_1 = 1$

$$\Rightarrow C_n = \left[\frac{\cancel{\alpha}}{\alpha^{\cancel{n+1}}(\alpha-\beta)} - \frac{\cancel{\beta}}{\beta^{\cancel{n+1}}(\alpha-\beta)} \right] = \frac{1}{\alpha-\beta}\left[\frac{1}{\alpha^n} - \frac{1}{\beta^n} \right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{2^n}{(\sqrt{5}-1)^n} - \frac{2^n}{(-\sqrt{5}-1)^n} \right] = \frac{1}{\sqrt{5}}\left[\frac{2^n}{(\sqrt{5}-1)^n} - \frac{1}{(-1)^n} \cdot \frac{2^n}{(\sqrt{5}+1)^n} \right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{2^n(\sqrt{5}+1)^n}{4^n} - \frac{(-1)^n 2^n (\sqrt{5}-1)^n}{4^n} \right]$$

$$\boxed{C_n = \frac{1}{2^n\sqrt{5}}\left[(\sqrt{5}+1)^n - (-1)^n (\sqrt{5}-1)^n \right]}$$

$\frac{1+\sqrt{5}}{2} \to \phi$

Golden ratio.

1.618

Lec 5 Summation of Series

$$S(a) = \sum_{n=1}^{\infty} \frac{1}{n^2+a^2} \qquad (a>0, \text{ say})$$

$$= \frac{1}{2} \sum_{\substack{n\neq 0 \\ -\infty}}^{\infty} \frac{1}{n^2+a^2} \quad \text{since n is even.}$$

Note, the function $\pi \cot \pi z$ has simple poles at all integers & the residue is 1.

$$\pi \cot \pi z = \pi \frac{\cos \pi z}{\sin \pi z} \xrightarrow[z \to n]{} \frac{\pi \cancel{(-1)^n}}{\underbrace{(z-n)}\, \cancel{\pi (-1)^n}} + \text{regular part}$$

simple poles for all integers.

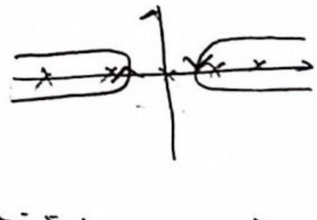
$\therefore$ Consider $f(z) = \dfrac{\pi \cot \pi z}{z^2+a^2}$ → this fn at $z=n$ would have a residue of $\dfrac{1}{n^2+a^2}$.

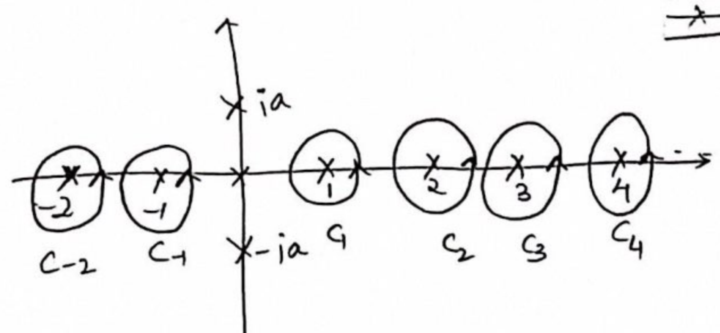
$$\Rightarrow S(a) = \frac{1}{2} \cdot \frac{1}{2\pi i} \sum_{\substack{n\neq 0 \\ -\infty}}^{\infty} \oint_{C} dz \frac{\pi \cot \pi z}{z^2+a^2}$$

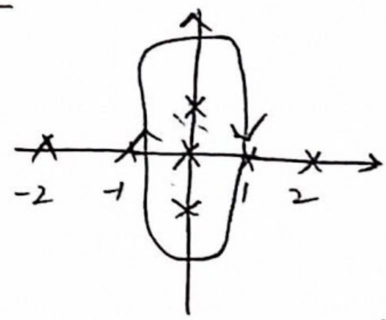
→ remains bounded, could be imag!

$\cot \pi z = \dfrac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}}$ NUM & DEN compensate

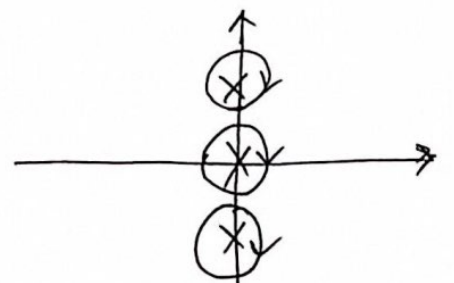
$\to R$, $\to R^2$







$\Rightarrow$



$$\Rightarrow S(a) = \frac{1}{4\cancel{\pi i}} (-2\cancel{\pi i}) \left[\frac{1}{a^2} + \frac{\pi \cot i\pi a}{2ia} + \frac{\pi \cot -i\pi a}{-2ia} \right]$$

Same

$$= -\frac{1}{2a^2} - \frac{\pi}{2\cancel{i}a} \cancel{i} \frac{e^{i(i\pi a)} + e^{-i(i\pi a)}}{e^{i(i\pi a)} - e^{-i(i\pi a)}}$$

$$\Rightarrow \boxed{S(a) = \frac{\pi}{2a}\left[\coth\pi a - \frac{1}{\pi a}\right]}$$

$\coth x - \frac{1}{x} \rightarrow$ Langevin func.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{2a}\left[\coth\pi a - \frac{1}{\pi a}\right]$$

$$\coth z \xrightarrow[z\to 0]{} \frac{\text{Residue}}{z} + \underbrace{\text{reg part}}_{\sum_{n=0}^{\infty} a_n z^n}$$

first term must be z^1 not z^0 since coth is odd

$$\frac{\cosh z}{\sinh z} = \frac{1+\frac{z^2}{2}+\cdots}{z\left(1+\frac{z^2}{6}+\cdots\right)} \cong \frac{\left(1+\frac{z^2}{2}+\cdots\right)\left(1-\frac{z^2}{6}\right)}{z} = \frac{1}{z}+\frac{z}{3}+\cdots$$

$$\Rightarrow S(a) \rightarrow \frac{\pi}{2a}\left[\cancel{\frac{1}{\pi a}} + \frac{\pi a}{3} + \cdots - \cancel{\frac{1}{\pi a}}\right]$$

$$\Rightarrow \boxed{S(a) = \frac{\pi^2}{6}} \rightarrow \zeta(2) \rightarrow \text{Zeta fn of } 2.$$

We already know this value.

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

Eg: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2+a^2)}$ $\Rightarrow$ we need a fn whose residue at each integer is $(-1)^n$

$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{z^2+a^2}$$

be careful at ∞

Eg: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$- 2\left[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right]$$

$$= \zeta(2) - \frac{2}{4}\zeta(2)$$

$$= \frac{\pi^2}{12}$$

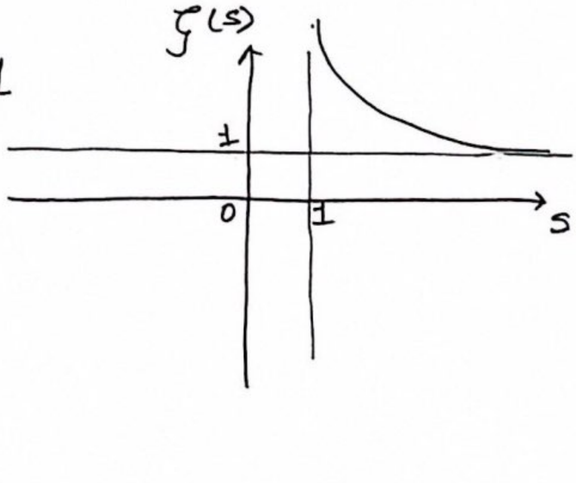
Eg: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} \Rightarrow \frac{\pi \cot \pi z}{z^4} \rightarrow$ bit messy $\rightarrow$ 5^{th} order pole at origin.

However $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4+a^4} \rightarrow \frac{\pi \cot \pi z}{z^4+a^4} \rightarrow$ 5 simple poles $\Rightarrow$ easier then set $a=0$.

> Also note that differentiating $S(a) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{(n^2+a^2)}$ gives $-\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n^2+a^2)^2}$ then setting $a=0$ (carefully) we can find $\zeta(4)$.

$\Rightarrow \dfrac{d}{da}\left[\dfrac{\pi}{2a}\left(\coth \pi a - \dfrac{1}{\pi a}\right)\right]$. Similarly further differentiation gives all even positive integer argument Zetas.

$\zeta(4) = \dfrac{\pi^4}{90}$ $\qquad \lim\limits_{k\to\infty} \zeta(k) \longrightarrow 1$



Lec 6 Dirichlet integral

$$\boxed{\int_0^\infty dx\, \frac{\sin x}{x} = \frac{\pi}{2}}$$

Note that $\left|\frac{\sin x}{x}\right|$ does not converge since the $\frac{dx}{x}$ grows logarithmically. But the + & − signs in $\sin x$ make it finite.

Also,
$$\int_0^\infty dx\, \frac{\sin bx}{x} \begin{cases} = \frac{\pi}{2} & \text{for all } b > 0 \\ = -\frac{\pi}{2} & \text{for all } b < 0 \end{cases}$$

$$\Rightarrow \int_0^\infty dx\, \frac{\sin bx}{x} = \frac{\pi}{2}\, \mathrm{sgn}(b)$$

Derivation

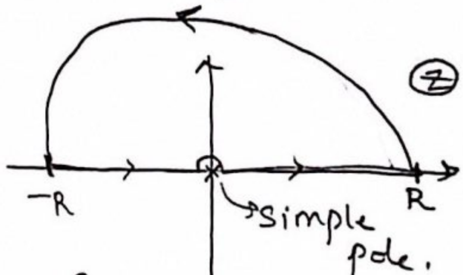
$$= \frac{1}{2}\int_{-\infty}^{\infty} dx\, \frac{\sin bx}{x} \longrightarrow \frac{e^{ibx} - e^{-ibx}}{2i}$$

, but one of these will always blow up if we close the contour in upper or lower Half plane. So we need to eliminate $\sin x$.

$$= \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty} dx\, \frac{e^{ibx}}{x}$$

But this integral blows up at the origin, unlike the case where we had $\frac{\sin x}{x}$

Consider, $\displaystyle\int_C dz\, \frac{e^{ibz}}{z}$ where C is



$$\Rightarrow 0 = \int_C dz\, \frac{e^{ibz}}{z} = \int_{-R}^{-\epsilon} dx\, \frac{e^{ibx}}{x} + \int_{\epsilon}^{R} dx\, \frac{e^{ibx}}{x} + \text{small contour} + \text{large contour}.$$

where on the small contour $z = \epsilon e^{i\theta} \Rightarrow dz = \epsilon e^{i\theta}\, i\, d\theta$.

$$\Rightarrow \text{Small contour} = \int_\pi^0 \cancel{\epsilon e^{i\theta}}\, i\, d\theta\, \frac{e^{ib\epsilon e^{i\theta}}}{\cancel{\epsilon e^{i\theta}}} = -i\pi \text{ as } \epsilon \to 0$$

$$\text{large contour} = \int_0^\pi \frac{\cancel{Re^{i\theta}}}{\cancel{Re^{i\theta}}}\, i\, d\theta\, e^{ibRe^{i\theta}}$$

, $z = Re^{i\theta}$ has +ve imaginary part

b should be positive.

- On the line of integration if a singularity exists & a symmetric neighbourhood (infinitesimal) is excluded, the integral is called the <u>principle value integral</u>.

$$\Rightarrow P\int_{-R}^{R} dx\, \frac{e^{ibx}}{x} \quad \text{or} \quad ⨍_{-R}^{R} dx\, \frac{e^{ibx}}{x}$$

$$\Rightarrow \lim_{R\to\infty} P\int_{-R}^{R} dx\, \frac{e^{ibx}}{x} = i\pi$$

Equate imaginary parts $\Rightarrow$ $P\int_{-\infty}^{\infty} dx\, \frac{\sin bx}{x} = \pi$

But now we dont have a singularity since $\frac{\sin bx}{x}$ has a defined value at 0 so we can remove the P.

$$\Rightarrow \int_{-\infty}^{\infty} dx\, \frac{\sin bx}{x} = \pi \xrightarrow[\text{fn}]{\text{even}} \int_{0}^{\infty} dx\, \frac{\sin bx}{x} = \frac{\pi}{2}$$

- What is the physical significance of residues?

<u>Eg:</u> For a resonator with multiple natural frequencies, resonance frequencies correspond to the poles. At each pole the residue gives the "strength" of the resonance at that frequency.

Meromorphic function.

"A function whose only singularities in the finite part of the complex plane are poles".

> Recall that branch points are also singularities.

> Could be an entire function.

> You could express meromorphic functions as the sum of a singular part and an entire function.

Mittag-Leffler representation.

Recall $\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{2a}\left[\coth \pi a - \frac{1}{\pi a}\right]$, a is real.

> a can be complex but the equality is only valid where the fn has no poles.

> Let $a = iz$

$$\Rightarrow \frac{\pi}{2iz}\coth(i\pi z) + \frac{1}{2z^2} = \sum_{n=1}^{\infty}\frac{1}{n^2-z^2}$$

$$\frac{\cos \pi z}{i\sin \pi z} = -i\cot \pi z$$

$$\Rightarrow -\frac{\pi \cot(\pi z)}{2z} + \frac{1}{2z^2} = \sum_{n=1}\frac{1}{n^2-z^2}$$

$$\Rightarrow \boxed{\pi\cot(\pi z) = \frac{1}{z} + 2\sum_{n=1}^{\infty}\frac{z}{z^2-n^2}}$$

pole at zero

$(z+n)(z-n)$
$\Rightarrow$ at all non zero integers there are simple poles which is what $\cot \pi z$ has.

In the Mittag Leffler representation,

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left[\frac{1}{(z-n)} + \frac{1}{(z+n)} \right]$$

This is not the ML representation.

Here we cannot simply open the square brackets since $\sum \frac{1}{z-n}$ diverges as $\sum \frac{1}{n}$.

Eg: $\sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n}\right]$ cannot be seperated (my eg).

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \left[\underbrace{\left(\frac{1}{z-n} + \frac{1}{n}\right)} + \left(\frac{1}{z+n} - \frac{1}{n}\right) \right]$$

Now as $n \to \infty$ $\frac{1}{z-n} \to \frac{1}{-n}$ & it cancels with $\frac{1}{n}$ so the remainder grows as $\frac{1}{n^2}$ which converges. So we can now seperate them.

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n}\right) + \sum_{n=-\infty}^{-1} \frac{1}{z-n} + \frac{1}{n}$$

replace $n \to -n$.

$$\Rightarrow \boxed{\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=-\infty}^{\infty}{}' \left[\frac{1}{z-n} + \frac{1}{n}\right]}$$

$\Rightarrow n=0$ is excluded.

This is the ML representation in the correct form.

> The correct representation must include a term which regularizes the series (ie $\frac{1}{n}$) & must contain an entire fn (ie 0 in this case).

Differentiating both sides

$$\boxed{\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}}$$

just a sum of all the double poles!!

Lec 7 Linear response, dispersion relations.

Response $R(t)$, Stimulus/Force $F(t)$

$$R(t) = [\qquad] F(t).$$

$t \rightarrow$ causality $\Rightarrow$ force till t produces response at t.

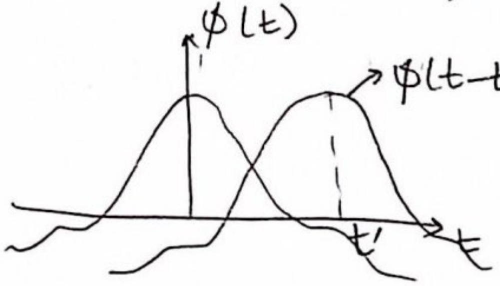
$$R(t) = \int_{-\infty}^{t} dt' \, \phi(t,t') \, F(t')$$

$\leftarrow$ linearity due to superposition integral.

t' is when the force is applied & t is when the response is measured. As long as system is invariant with time ϕ is only a function of $t-t'$ since it does not matter when t' is applied.

$$\Rightarrow LTI \Rightarrow R(t) = \int_{-\infty}^{t} dt' \, \underbrace{\phi(t-t')}_{\text{response fn}} F(t')$$

$\rightarrow$ retarded response.



Take input at t' as an impulse & multiply with the impulse response shifted to t', this shifting of the impulse response & weighting by the input & summing is the convolution.

$\rightarrow$ F, R could be vectors & ϕ would be a 2^{nd} order tensor.

$\rightarrow$ ϕ here is called the "response functions" in this framework.

Let $\tau = t - t'$ be time after force is applied.

$$\text{Let } R(t) = \int_{-\infty}^{\infty} d\omega \, e^{-i\omega t} \tilde{R}(\omega)$$

$$\Rightarrow \tilde{R}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, e^{i\omega t} \, R(t)$$

$$F(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \tilde{F}(\omega) \quad \& \quad \tilde{F}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, e^{i\omega t} F(t)$$

$$\Rightarrow \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \tilde{R}(\omega) = \int_{t'=-\infty}^{t} dt'\, \phi(t-t') \int_{\omega=-\infty}^{\infty} d\omega\, e^{-i\omega t'} \tilde{F}(\omega)$$

$$= \int_{-\infty}^{\infty} d\omega\, \tilde{F}(\omega) \int_{-\infty}^{t} dt'\, \phi(t-t')\, e^{-i\omega t'}$$

$$= \int_{-\infty}^{\infty} d\omega\, \tilde{F}(\omega) \int_{0}^{\infty} d\tau\, \phi(\tau)\, e^{-i\omega(t-\tau)}$$

$$= \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \tilde{F}(\omega) \int_{0}^{\infty} d\tau\, \phi(\tau)\, e^{+i\omega\tau}$$

$$\Rightarrow \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \left\{ \tilde{R}(\omega) - \left(\int_{0}^{\infty} d\tau\, e^{i\omega\tau} \phi(\tau) \right) \tilde{F}(\omega) \right\} = 0$$

$e^{-i\omega t}$ is a complete orthonormal set of functions of t.

$$\Rightarrow \quad \tilde{R}(\omega) = \left(\int_{0}^{\infty} d\tau\, e^{i\omega\tau} \phi(\tau) \right) \tilde{F}(\omega) \quad \text{for each real } \omega.$$

$$\boxed{\tilde{R}(\omega) = \chi(\omega)\, \tilde{F}(\omega)}$$

Called <u>Generalized Susceptibility</u>!

$$\boxed{\chi(\omega) = \int_{0}^{\infty} dt\, e^{i\omega t} \phi(t)}$$

only defined for $t > 0$ due to causality.

Neither a FT or LT. if it was e^{-st} it would be L.T

- Let's call it a Fourier Laplace transform.
- Assume the integral exists $\Rightarrow X(\omega)$ is a complex number.

$$X(\omega) = \int_0^\infty dt\, e^{i\omega t}\, \phi(t).$$

$$\Rightarrow \quad \left.\begin{aligned} \mathrm{Re}\{X(-\omega)\} &= \mathrm{Re}\{X(\omega)\} \quad \text{since } \cos(\omega t) \\ \mathrm{Im}\{X(-\omega)\} &= -\mathrm{Im}\{X(\omega)\} \quad \text{since } \sin(\omega t) \end{aligned}\right\} \text{ For real } \omega.$$

Later we see that $X(\omega)$ is analytic $\Rightarrow$ cannot be purely real or imag.
$\Rightarrow$ dissipation / energy storage always exists.

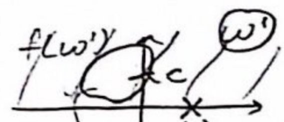
$X(\omega)$ also exists for complex ω as long as $\mathrm{Im}(\omega) \geq 0$ since that would be give decaying exponentials as $t \to \infty$

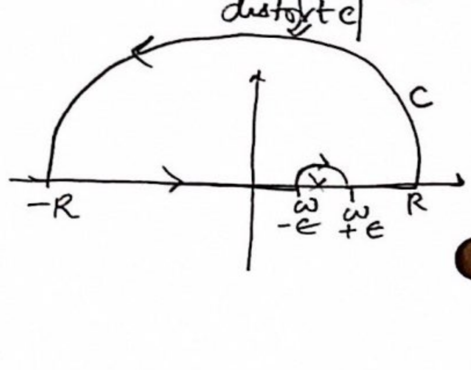
"$X(\omega)$ is an analytic function of ω in the UHP including real axis"!

$\Rightarrow \oint_C d\omega\, X(\omega) = 0$ if C is entirely in the UHP including real axis.

Define $f(\omega') = \dfrac{X(\omega')}{\omega' - \omega}$ where ω is some fixed real value.

$\Rightarrow$ In ω' plane $f(\omega')$ is analytic everywhere in UHP except at point ω





$$\oint_C f(\omega')\, d\omega' = 0 = \oint_C d\omega' \frac{X(\omega')}{\omega' - \omega}$$

$$\Rightarrow \int_{-R}^{R} d\omega' \frac{X(\omega')}{\omega' - \omega} + \text{Small semi circle} + \text{large semi circle} = 0$$

> On small semicircle $\omega' = \omega + \epsilon e^{i\theta}$ $\Rightarrow d\omega' = \epsilon e^{i\theta} i d\theta$.

$$\Rightarrow \int_{-R}^{R} \frac{d\omega' \, X(\omega')}{\omega' - \omega} + \int_{\pi}^{0} \frac{\cancel{\epsilon e^{i\theta}} i d\theta \, X(\omega + \epsilon e^{i\theta})}{\cancel{\epsilon e^{i\theta}}} + \int_{large} d\omega' \ldots = 0$$

Large integral $= \displaystyle\int_{0}^{\pi} \frac{R e^{i\theta} i d\theta . X(R e^{i\theta})}{R e^{i\theta} - \omega}$ as $R \to \infty$ if $X(Re^{i\theta})$ goes to 0, the integral goes to 0 $\Rightarrow$ susceptibility must vanish at ∞ frequency.

Small integral $= \lim_{\epsilon \to 0} \displaystyle\int_{\pi}^{0} i d\theta \; X(\omega + \epsilon e^{i\theta}) = i\pi X(\omega)$

$$\boxed{\Rightarrow \underbrace{X(\omega)}_{\text{real freq.}} = \frac{-i}{\pi} \int_{-\infty}^{\infty} \frac{d\omega' X(\omega')}{\omega' - \omega}}$$

→ real freq again.

$$\Rightarrow Re(X(\omega)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\omega' . \, Im \, X(\omega')}{\omega' - \omega}$$

$$Im(X(\omega)) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\omega' \, Re \, X(\omega')}{\omega' - \omega}$$

Note that these are the Kramers Kronig relations.

We see here that the Real and Imaginary parts of X are Hilbert transforms of each other.

<u>Hilbert transform</u>: $H(t) = \frac{1}{\pi} \displaystyle\int_{-\infty} \frac{u(\tau)}{t - \tau} d\tau$, it is the convolution of $u(t)$ with $\frac{1}{\pi t}$.

$\Rightarrow Re(X)$ & $Im(X)$ comprise a Hilbert transform pair!

* For stable systems, causality $\Rightarrow$ analyticity. Since causality only holds in UHP, the X is analytic in UHP. For functions that are analytic in UHP, the Kramers Kronig relations are true.

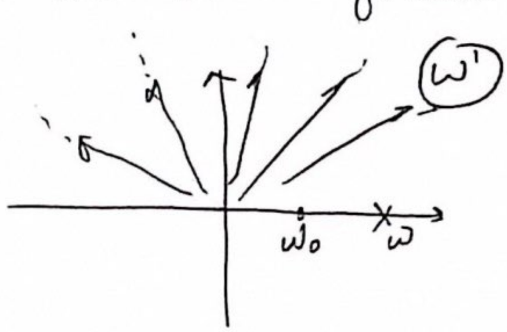
Lec 8 The Hilbert transform relations aka Dispersion relations.

If $\chi(\omega') \longrightarrow \chi_\infty$ $(\neq 0)$ as $|\omega'| \longrightarrow \infty$ in the UHP we can account for this since $\displaystyle\int_{\substack{\text{Large}\\\text{semi}\\\text{circle}}} \frac{\overset{\alpha R}{\cancel{d\omega'}}\, \chi(\omega')^{\to \chi_\infty}}{\underset{\alpha R}{\cancel{\omega' - \omega}}} \xrightarrow[R\to\infty]{\lim} i\pi\chi_\infty$

Subtracted dispersion relations.

If $\chi(\omega') \neq 0$ & goes to a different constant along different rays reaching ∞ or if it grows to ∞ but slower than linear growth we can define,

$$f(\omega') = \frac{\chi(\omega') - \chi(\omega_0)}{(\omega' - \omega)(\omega' - \omega_0)}$$



where $\chi(\omega_0)$ for some ω_0 must be known.

$d\omega'$ cancels with $\omega' - \omega$ & $\chi(\omega')$ must grow slower than $\omega' - \omega_0$.

> For faster growth we can add more terms if we know χ at other points.

Eg: LCR series circuit.

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_{-\infty}^{t} dt'\, I(t') = V(t)$$

FT $\Rightarrow$ $\left(-i\omega L + R + \frac{1}{-i\omega C}\right)\tilde{I}(\omega) = \tilde{V}(\omega)$

$\Rightarrow$ $\tilde{I}(\omega) = Y(\omega)\,\tilde{V}(\omega)$.

$\hookrightarrow$ Admittance.

This looks like the form $\tilde{R}(\omega) = \chi(\omega)\tilde{F}(\omega)$.
$\Rightarrow$ $Y(\omega)$ cannot have poles in UHP. It is analytic there.

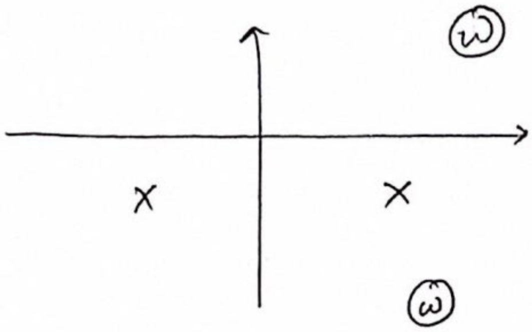
$$Y(\omega) = \frac{i\omega}{L\left(\omega^2 + \frac{R}{L}i\omega - \frac{1}{LC}\right)}$$

$$\Rightarrow Y(\omega) = \frac{i\omega}{L(\omega^2 + i\gamma\omega - \omega_0^2)}$$

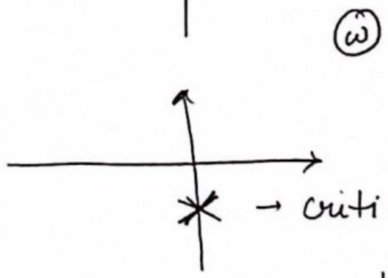
; $\omega_0 = \frac{1}{\sqrt{LC}}$ $\gamma = \frac{R}{L}$

$\hookrightarrow$ inverse of time const.

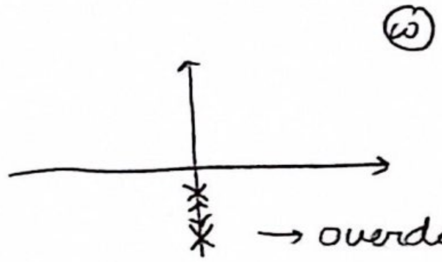
- > Poles of $Y(\omega)$ at $\omega = -\frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$ → underdamped case ($\Rightarrow \omega_0 > \frac{\gamma}{2}$)



underdamped.



→ critically damped
$\omega_0^2 = \frac{\gamma^2}{4}$



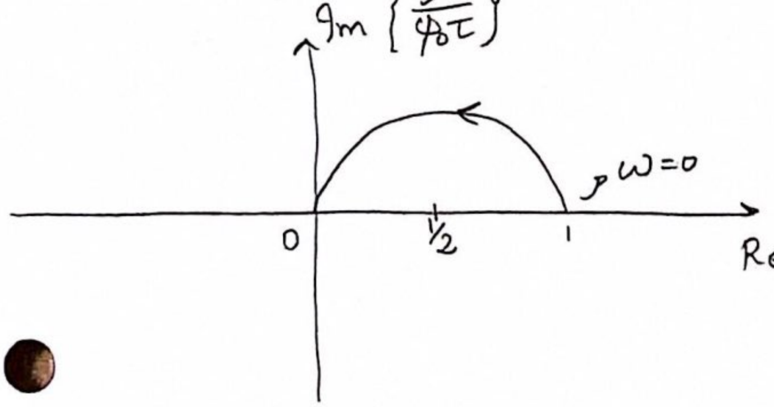
→ overdamped.

$Y(\omega)$ should satisfy K K relations

→ $\chi(\omega) = \int_0^\infty dt\, e^{i\omega t} \phi(t)$ Say, $\phi(t) = \phi_0 e^{-t/\tau}$ ↳ relaxation time
↳ Debye relaxation ⇒ single order relaxation

$$\chi(\omega) = \frac{\phi_0 \tau}{1 - i\omega\tau} = \phi_0\tau\left(\frac{1}{1+\omega^2\tau^2} + \frac{i\omega}{1+\omega^2\tau^2}\right)$$

Cole - Cole plot → Plot of $Im\left\{\frac{\chi}{\phi_0\tau}\right\}$ vs. $Re\left\{\frac{\chi}{\phi_0\tau}\right\}$



$Re\,\frac{\chi}{\phi_0\tau} = u$ & $Im\,\frac{\chi}{\phi_0\tau} = v$

$\Rightarrow \left(u - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}$

If measured u, v fall on this semicircle we can say that the relaxation process has a single relaxation time.

Lec 9 Analytic Continuation & Γ function.

Euler integral of second kind.

Euler noticed, $\int_0^\infty dt\, t^n e^{-t} = n!$ for $n = 1, 2, 3 \ldots$

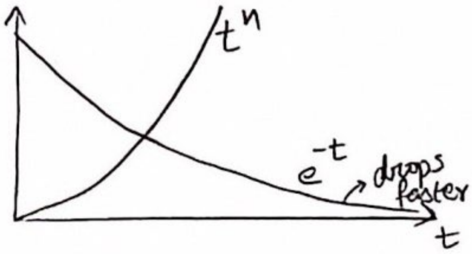
It is easy to show this using integration by parts. $\int uv\,dt = u\int v\,dt - \int u'(\int v\,dt)dt$

For $n=0$ $\Rightarrow$ $\int_0^\infty dt\, t^0 e^{-t} = 1$ → define this as $0!$.

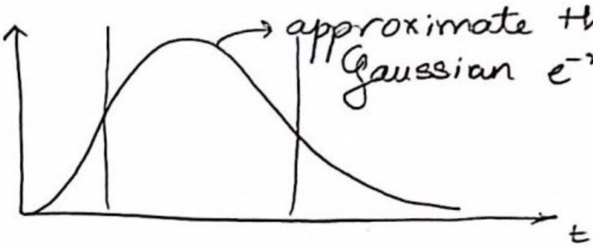
For large n use Sterling's formula.

$$n! = n^n e^{-n}\sqrt{2\pi n}\left\{1 + \frac{1}{12n} + O(n^2)\right\}$$

Gaussian integration approximation.



$\simeq\Rightarrow$



Note, $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$ & $\int_{-\infty}^{\infty} e^{-a(x+b)^2}dx = \sqrt{\frac{\pi}{a}}$.

$$\Rightarrow \int_0^\infty dt\, e^{-t} t^n = \int_0^\infty dt\, e^{-(t - n\log t)}$$

Where does $f(t) = t - n\log t$.

$f'(t) = 1 - \frac{n}{t} = 0 \Rightarrow t = n$

$$= \int_0^\infty dt\, e^{-\left[(n - n\log n) + \frac{(t-n)^2}{2n} + \cdots\right]}$$

→ expanding around maxima.

$f''(t) = \frac{n}{t^2} = \frac{1}{n}$ at $t = n$.

$$\Rightarrow n! = n^n e^{-n}\int_0^\infty dt\, e^{-\frac{(t-n)^2}{2n}}\{1 + \ldots\}$$

→ Gaussian

↳ could show that $\int_{-\infty}^{\infty}$ yield same result (not shown here though)

$$\Rightarrow \int_{-\infty}^{\infty} e^{-at^2} = \sqrt{\frac{\pi}{a}}$$

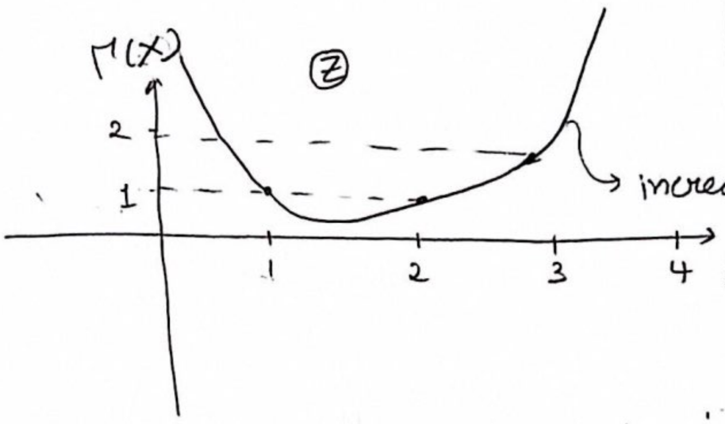
$$\Rightarrow n! \simeq n^n e^{-n} \sqrt{2\pi n} \rightarrow \text{sterling's approximation}$$

Going back;

Historical reason to make it $n-1$

$$\int_0^\infty dt\, t^n e^{-t} = n! \quad \& \quad \boxed{\int_0^\infty dt\, t^{n-1} e^{-t} = \Gamma(n)} \rightarrow \text{Gamma fn}$$

$\Gamma(n) = (n-1)!$ for $n = 1, 2, 3 \ldots$



$$\boxed{\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}}$$

If $x = 0$ $\int_0^\infty \frac{dt}{te^t}$ does not converge for $t = 0$. Same for $x < 0$

- $\Gamma(x)$ converges when $x - 1 > -1 \Rightarrow x > 0$
- $\Gamma(z)$ converges when $x > 0$ for $z = x + iy \Rightarrow$ RHP
- $\Gamma(z)$ is analytic in RHP + there are singularities on Im axis.

- $\Gamma(z) \equiv \int_0^\infty dt\, t^{z-1} e^{-t} \quad \text{Re}\, z > 0$

Integrate by parts $\Rightarrow$ $$\frac{t^z}{z} e^{-t} \Big|_{t=0}^{\infty} + \frac{1}{z} \int_0^\infty dt\, t^z e^{-t}, \quad \text{Re}\, z > 0$$

Vanishes at $t = 0$ iff $\text{Re}\, z > 0$ $\Rightarrow$ first term $= 0$

$$\Rightarrow \Gamma(z) = \frac{1}{z}\int_0^\infty dt\, t^z e^{-t}$$ $\rightarrow$ This representation is exact & equally valid.

> However now we have exposed the simple pole at $z=0$ that has a residue of 1. Also note that this integral is valid for all $\mathrm{Re}(z) > -1$

> Keep integrating by parts to move further into LHP.

$$\Gamma(z) = \frac{1}{z(z+1)}\int_0^\infty dt\, t^{z+1} e^{-t}$$ $\Rightarrow$ Another simple pole with residue $= -1$

> Therefore this function has simple poles at all non positive integers and residue is $(-1)^n$.

> Therefore, $\Gamma(z)$ is a meromorphic function with simple poles at $z=-n$ $(n=0,1,2,3\ldots)$ and residue $\frac{(-1)^n}{n!}$.

$$\Gamma(z) = \frac{1}{z(z+1)\cdots(z+n)}\int_0^\infty dt\, t^{z+n} e^{-t} \qquad \mathrm{Re}\, z > -n-1$$

$\boxed{\Gamma(z+1) = z\,\Gamma(z)}$ $\rightarrow$ Functional equation for $\Gamma(z)$. This enables the analytic continuation. Since $\Gamma(z+1)$ is valid for $\mathrm{Re}(z) > -1$. This can be continued.

Define $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$

$$= \frac{1}{\Gamma(z)} \frac{d\,\Gamma(z)}{dz}$$

$\psi(z+1) - \psi(z) = \frac{1}{z}$ → Functional equation.

Near $z = -n$ $\quad \Gamma(z) = \frac{(-1)^n}{n!(z+n)}$ + regular part

$$\Rightarrow \psi(z) = \frac{-1}{z+n} + \text{regular part}$$

(diff & divide by itself to give $\frac{-1}{z+n}$. & deriv of regular part is regular

⇒ $\psi(z)$ is meromorphic with simple poles at $-n$ but residue is always -1.

Can we write a Mittag Leffler representation since it is meromorphic and we have the poles?

Note that in $\Gamma(z) = \frac{1}{z(z+1)(z+2)\dots(z+n)} \int_0^\infty dt\, t^{z+n} e^{-t}$ we only have trouble in the $\int_0^1$ region since this is where t^{z+n} was blowing up.

Therefore, splitting the integral we can find the exact ML representation

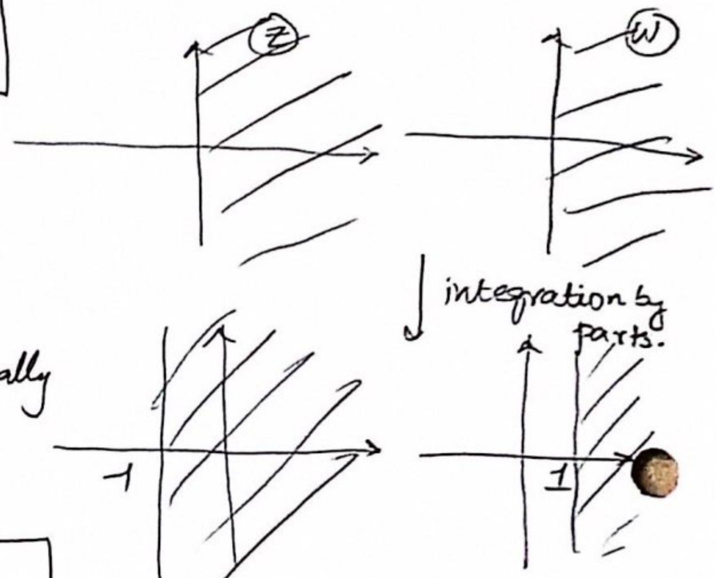
$$\Gamma(z) = \frac{1}{z(z+1)\dots(z+n)} \left[\int_0^1 dt\, t^{z+n} e^{-t} + \underbrace{\int_1^\infty dt\, t^{z+n} e^{-t}}_{\text{regular part.}} \right]$$

Beta function - Euler integral of first kind.

$$B(m,n) \equiv \int_0^1 dt \, t^{m-1} (1-t)^{n-1} \qquad m,n = 1,2,3 \ldots$$

diverges at 0 if $m=0$ — diverges at 1 if $n=0$

$$\boxed{B(z,w) = \int_0^1 dt \, t^{z-1} (1-t)^{w-1}} \Rightarrow \text{Re}\{z\} > 0 \text{ \& } \text{Re}\{w\} > 0.$$



- Integration by parts with $u = t^{z-1}$ & $v = (1-t)^{w-1}$ would improve for z but worsen for w. Therefore integration by parts cannot be used to analytically continued.

> But $$\boxed{B(z,w) = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}}$$ can be used.

Gaussian Integrals

$$\int_0^\infty du\, e^{-au^2} u^r \quad \text{where } \operatorname{Re} a > 0 \;;\; r > -1$$

Let $au^2 = t$; $u = \frac{\sqrt{t}}{\sqrt{a}}$ $\Rightarrow du = \frac{dt}{2\sqrt{at}}$

$$\Rightarrow \int_0^\infty \frac{dt}{2t^{1/2}a^{1/2}} e^{-t} \frac{t^{\frac{r}{2}}}{a^{r/2}} = \frac{1}{2a^{\frac{r+1}{2}}} \int_0^\infty dt\, t^{\frac{r-1}{2}} e^{-t}$$

$$\Rightarrow \boxed{\int_0^\infty du\, e^{-au^2} u^r = \frac{\Gamma\left(\frac{r+1}{2}\right)}{2a^{\frac{r+1}{2}}}} \quad \Rightarrow \int_0^\infty du\, e^{-au^2} = \frac{\Gamma\left(\frac{1}{2}\right)}{2a^{1/2}} = \frac{\sqrt{\pi}}{2\sqrt{a}}$$

↳ since integral from 0 to ∞.

Also, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$; $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$; $\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$; ...

Mittag Leffler representation of Γ

$$\Gamma(z) = \int_0^1 dt\, t^{z-1} \underbrace{e^{-t}}_{\text{expand}} + \int_1^\infty dt\, e^{-t} t^{z-1}$$

→ Only this breakup gives the right pole terms.

$$= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^1 dt\, t^{n+z-1} + \int_1^\infty dt\, e^{-t} t^{z-1}$$

$$\boxed{\Gamma(z) = \underbrace{\sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{1}{z+n}}_{\text{All poles with corresponding residues}} + \underbrace{\int_1^\infty dt\, e^{-t} t^{z-1}}_{\text{regular part}}}$$

Recall, $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$ & $\psi(z+1) = \frac{1}{z} + \psi(z)$

Near the poles $\Gamma(z) \cong \frac{(-1)^n}{n!(z+n)}$ + reg part.

$\Rightarrow \psi(z) \cong \frac{(-1)}{z+n}$ + reg part $\Rightarrow$ constant residue.

$$\Gamma(z) \underset{z \simeq 0}{\overset{\text{at}}{=}} \frac{1}{z} + \underbrace{\sum_{n=0}^{\infty} c_n z^n}_{\text{reg part}} \simeq \frac{1}{z} - \gamma + c_1 z + \dots$$

<u>Euler-Mascheroni constant</u>

$$\gamma = \lim_{N \to \infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \ln N \right)$$

$\sum$ ↳ diverges as $\ln N$

$-\ln N$ ↳ corrects this & makes this series converge

and $\gamma = 0.5772\ldots$ → conjectured to be irrational & also transcendental.

$$\psi(z) \simeq \underbrace{-\frac{1}{z}}_{\text{singular}} \underbrace{- \gamma + \dots}_{\text{regular}}$$

& $\psi(z+1) = \frac{1}{z} + \psi(z)$.

$\psi(1) = \cancel{\frac{1}{z}} - \cancel{\frac{1}{z}} - \gamma + \underbrace{\dots}_{\text{goes to 0 at } z=0}$

$$\Rightarrow \boxed{\psi(1) = -\gamma}$$

$\psi(2) = 1 - \gamma$

$\psi(n+1) = -\gamma + \sum_{j=1}^{n} \frac{1}{j}$

> $\Gamma(z)$ is meromorphic. & $\frac{1}{\Gamma(z)}$ is entire $\Rightarrow$ $\Gamma(z)$ has no zeroes

> Is there a representation such that all zeroes can be explicitly written down for $\frac{1}{\Gamma(z)}$? Yes.

Weierstass' Representation.

$$\frac{1}{\Gamma(z)} = \underbrace{z e^{\gamma z}}_{\text{entire fn.}} \underbrace{\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}}_{\text{zeroes.}}$$

Beta function

$$B(z,w) = \int_0^1 dt\, t^{z-1} (1-t)^{w-1} \qquad \begin{array}{l} \mathrm{Re}\, z > 0 \\ \mathrm{Re}\, w > 0 \end{array}$$

converges

Consider,

$$I(z,w) = \int_0^\infty du \int_0^\infty dv\; e^{-(u^2+v^2)}\, u^{2z-1}\, v^{2w-1}$$

$$\Rightarrow \int_0^\infty du\, e^{-u^2} u^{2z-1} \qquad \text{let } u^2 = t\, ;\ u = \sqrt{t}$$

$$= \frac{1}{2}\int_0^\infty dt\, e^{-t}\, t^{z-1} = \frac{1}{2}\Gamma(z)$$

$$\Rightarrow I(z,w) = \frac{1}{4}\Gamma(z)\Gamma(w)$$

Evaluating the integral in polar coordinates

$u = r\cos\theta \; ; \; v = r\sin\theta .$

$$\Rightarrow I(z,w) = \int_0^\infty dr\, r e^{-r^2} \int_0^{\pi/2} d\theta\, r^{2z+2w-2} (\cos\theta)^{2z-1} (\sin\theta)^{2w-1}$$

$$= \frac{1}{2}\int_0^\infty 2r\,dr\, e^{-r^2} r^{2(z+w-1)} \int_0^{\pi/2} d\theta (\cos\theta)^{2z-1} (\sin\theta)^{2w-1}$$

Let $r^2 = s$ & $\cos^2\theta = \xi$ $\Rightarrow -2\cos\theta\sin\theta = d\xi$

$$\Rightarrow \underbrace{\frac{1}{2}\int_0^\infty ds\, e^{-s} s^{z+w-1}}_{\Gamma(z+w)} \underbrace{\int_0^1 \frac{1}{2}\cdot d\xi\, \xi^{z-1}(1-\xi)^{w-1}}_{\frac{B(z,w)}{2}}$$

$$\Rightarrow I(z,w) = \frac{1}{4}\Gamma(z+w)\, B(z,w)$$

$$\Rightarrow \boxed{B(z,w) = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}}$$

$\rightarrow$ Now we can analytically continue the Beta function.

Put $w = 1-z$ $\Rightarrow B(z,1-z) = \dfrac{\Gamma(z)\Gamma(1-z)}{\Gamma(1)}$

$$\Rightarrow \boxed{B(z,1-z) = \Gamma(z)\,\Gamma(1-z)}$$

Could show $\boxed{\Gamma(z)\,\Gamma(1-z) = \dfrac{\pi}{\sin\pi z}}$ also, $\boxed{\psi(1-z) - \psi(z) = \pi\cot(\pi z)}$

negative z poles at integers

positive z poles at integers

all integer poles.

$\rightarrow$ Reflection formula for $\Gamma(z)$.

$$\underbrace{B(z,z)}_{\text{evaluate this}} = \frac{\Gamma(z)\,\Gamma(z)}{\Gamma(2z)}$$

$\Downarrow$

Doubling formula. $\boxed{\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\,\Gamma(z)\,\Gamma(z+\tfrac{1}{2})}$

$$\ln(\Gamma(2z)) = (2z-1)\ln 2 + \ln\Gamma(z) + \ln(\Gamma(z+\tfrac{1}{2})) - \tfrac{1}{2}\ln\pi$$

$$\Rightarrow \boxed{2\psi(2z) = 2\ln 2 + \psi(z) + \psi(z+\tfrac{1}{2})}$$ → Gives $\psi(\frac{1}{2})$, $\psi(\frac{3}{2})$ & so on.

→ Check the signs maybe wrong.

Mobius Transformation (Lec 11)

> If $w = f(z) = u + iv$. If $f(z)$ is analytic in some region, curves on this region in z plane preserve angles as they are mapped to w plane.

> What is the most general mapping from $z \to w$ such that the mapping is one to one $\Rightarrow$ it cannot involve higher powers $\Rightarrow$ must be a linear map. $\Rightarrow w = az + b$. This mapping preserves the 'point' at Infinity. It is basically a mapping which preserves the north pole of the Riemann sphere. However, if we are not restricted to this & we can map "all" points on one Riemann sphere to another Riemann sphere we get the Mobius Transformation.

$$\boxed{w = \frac{az+b}{cz+d}}$$

Where $a, b, c, d \in \mathbb{C}$ & the det must be nonzero to ensure map doesn't become degenerate $\Rightarrow ad - bc \neq 0$ | If $ad = bc$, $w = \frac{a}{c}$ $\Rightarrow$ Constant.

→ The Mobius transformation is analytic and one to one.

→ If $ad-bc = k$, divide a, b, c, d by $\sqrt{k}$ & we get $ad-bc=1$ this does not change the transform. Therefore we can always enforce $ad-bc=1$.

→ $z = -\frac{d}{c} \longrightarrow w = \infty$ & $\boxed{z = \frac{dw-b}{-cw+a}}$

→ $z = \infty \longrightarrow w = \frac{a}{c}$

Fixed Points

⇒ Solutions to $z = \frac{az+b}{cz+d}$ ⇒ $cz^2 + (d-a)z - b = 0$

$$\Rightarrow \xi_{1,2} = \frac{(a-d) \pm \sqrt{(a-d)^2 + 4bc}}{2c}$$

$$\boxed{\xi_{1,2} = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}}$$ since $ad-bc=1$

Case (1) ξ_1, ξ_2 are finite and distinct

⇒ $c \neq 0$ & $a+d \neq \pm 2$

Case (2) $\xi_1 = \xi_2 = \xi$

⇒ $c \neq 0$ & $a+d = \pm 2$ ⇒ $\xi = \frac{a-d}{2c}$

Case (3) ξ_1 finite, $\xi_2 = \infty$ $\Rightarrow$ linear translation

$\Rightarrow$ $c = 0$ & $ad = 1$ = $d = \frac{1}{a}$

$$\Rightarrow z = \frac{az+b}{d} = a^2 z + ab$$

$$\Rightarrow \xi_1 = \frac{ab}{1-a^2} \; ; \; \xi_2 = \infty$$

Case (4) $\xi_1 = \xi_2 = \infty$

$\Rightarrow$ $c = 0$ & $a = d$ & $ad = 1$ = $a = \pm 1$

$\Rightarrow$ $z \longrightarrow z + ab$

$\Rightarrow$ $z \longrightarrow z \pm b$ $\Rightarrow$ trivial translation

---

Cross ratio of 4 points.

Given z_1, z_2, z_3, z_4,

$$[z_1, z_2, z_3, z_4] \equiv \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

Let $z_i \longrightarrow \frac{az_i + b}{cz_i + d}$ & $z_j \longrightarrow \frac{az_j + b}{cz_j + d}$.

$$\Rightarrow z_i - z_j \longrightarrow \frac{ac\cancel{z_i}z_j + ad\,z_i + bc\,z_j + \cancel{bd} - ac\cancel{z_i}z_j - ad\,z_j - bc\,z_j - \cancel{bd}}{(cz_i+d)(cz_j+d)}$$

$$\Rightarrow z_i - z_j \longrightarrow \frac{z_i - z_j}{(cz_i+d)(cz_j+d)}$$

Now take $[z_i, z_j ; z_k, z_L]$ & substitute difference values

$\Rightarrow$ cross ratio doesn't change.

$\Rightarrow$ If $z_1 \to w_1$; $z_2 \to w_2$; $z_3 \to w_3$ & $z_4 \to w_4$

$$\boxed{[w_1, w_2 ; w_3 ; w_4] = [z_1, z_2 ; z_3, z_4]}$$

> Choose 4 arbitrary points & find the cross ratio, & find the cross ratio of the mapped points. They would be equal.

> From a set of 4 points we could write 24 cross ratios?

> Represent $[z_i, z_j ; z_k, z_l] \to [i,j ; k,l]$

Note, $[i\,j ; k\,l] = [k\,l ; i\,j]$

$= [j\,i ; l\,k]$

$= \dfrac{1}{[j\,i, k\,l]}$

$= \dfrac{1}{[i\,j, l\,k]}$

$= 1 - [i\,k, j\,l]$

> All 24 cross ratios are related like above so only one cross ratio is independent.

> $[\infty, z_2 ; z_3, z_4] = \dfrac{z_2 - z_4}{z_2 - z_3}$

> Given z_1, z_2, z_3 & their images w_1; w_2 & w_3 we can show that there exists a unique transformation that satisfies this mapping. Therefore, it maps circles to circles.

> $[z, z_1; z_2, z_3] = [w, w_1; w_2, w_3]$ & solve for w in terms of z & you will see that it is unique.

> ξ_1, ξ_2 distinct, finite.

$$\xi_{1,2} = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}$$

> Consider $[z, \infty; \xi_1, \xi_2] \Rightarrow [w, \frac{a}{c}; \xi_1, \xi_2]$

$$\Rightarrow \frac{(z-\xi_1)}{(z-\xi_2)} = \frac{(w-\xi_1)(\frac{a}{c}-\xi_2)}{(w-\xi_2)\cdot(\frac{a}{c}-\xi_1)}$$

$$\Rightarrow \boxed{\frac{w-\xi_1}{w-\xi_2} = \underbrace{\frac{(a-c\xi_1)}{(a-c\xi_2)}}_{k} \left(\frac{z-\xi_1}{z-\xi_2}\right)}$$

$\to k \to$ The multiplier.

$$\Rightarrow \boxed{\frac{w-\xi_1}{w-\xi_2} = k\,\frac{z-\xi_1}{z-\xi_2}}$$

$\longrightarrow$ The normal form of the Mobius Transform.

'A more natural representation'

→ Making n successive Mobius Transformation

$$\frac{z^{(n)}-\xi_1}{z^{(n)}-\xi_2} = \underbrace{k^n \frac{(z^{(0)}-\xi_1)}{(z^{(0)}-\xi_2)}}_{\text{Let} = A}$$

$$\Rightarrow z^{(n)}-\xi_1 = A\,(z^{(n)}-\xi_2)$$

$$\Rightarrow z^{(n)} = \frac{\xi_1 - A\,\xi_2}{1-A}$$

$$\boxed{z^{(n)} = \frac{\xi_1 - \xi_2 k^n \left(\frac{z^{(0)}-\xi_1}{z^{(0)}-\xi_2}\right)}{1 - k^n\left(\frac{z^{(0)}-\xi_1}{z^{(0)}-\xi_2}\right)}}$$

→ After n iterations point $z^{(0)}$ goes to $z^{(n)}$.

→ This also works for negative n values.

$$k = \frac{a-c\,\xi_1}{a-c\,\xi_2} = \frac{a-\cancel{c}\left[\frac{a-d+\sqrt{(a+d)^2-4}}{2\cancel{c}}\right]}{a-c\left[\frac{a-d-\sqrt{(a+d)^2-4}}{2c}\right]}$$

$$\boxed{k = \frac{T-\sqrt{T^2-4}}{T+\sqrt{T^2-4}}}$$

Where $T = a+d$ is the trace of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

If $|k|>1$ as $n\to\infty$, $A\to\infty$ $\Rightarrow$ $\boxed{z^{(n)} \xrightarrow[n\to\infty]{\lim} \xi_2}$

⇒ Then every point except ξ_1 falls on ξ_2 as $n\to\infty$

> If $|k|<1$, ξ_1 becomes attractor & ξ_2 becomes repeller.

> If $|k|=1$, we will see.

> If $\xi_1 = \xi_2 = \xi = \frac{a-d}{2c}$ $\quad (a+d = \pm 2)$

We can show that $\boxed{\frac{1}{w-\xi} = \frac{1}{z-\xi} \pm c}$ → Normal form.

depends on

> If $c=0 \Rightarrow d = \frac{1}{a}$; $\xi_1 = \frac{ab}{1-a^2}$; $\xi_2 = \infty \Rightarrow z \longrightarrow a^2 z + ab$.

$$\boxed{w - \xi_1 = \underbrace{k}_{a/d} (z - \xi_1).}$$ → Normal form.

> If $c=0 \Rightarrow a=d \Rightarrow a = \pm 1 \Rightarrow z \rightarrow z \pm b$.

$$\Rightarrow \boxed{z^{(n)} = z \pm nb}$$

Note; $w = \frac{az+b}{cz+d} \Rightarrow \left|\frac{dw}{dz}\right| = \left|\frac{a(cz+d) - c(az+d)}{(cz+d)^2}\right| = \frac{1}{|cz+d|^2}$ → Stretch factor.

If $|cz+d| = 1 \Rightarrow$ This curve is not not stretched.
Note that this is a circle with radius $=1$ & centre at $-\frac{d}{c}$. This is called the <u>isometric circle</u>.

Lec 13 Recall, $K = \dfrac{T-\sqrt{T^2-4}}{T+\sqrt{T^2-4}}$ where, $T = a+d$

i) Note, if $-2 < T < 2 \Rightarrow |T-\sqrt{T^2-4}| = |T+\sqrt{T^2-4}| \Rightarrow \underline{\underline{|K| = 1}}, K \neq 1$

Then we get an <u>elliptic transformation</u>.

ii) Note if $T = \pm 2 \Rightarrow K = 1 \Rightarrow$ <u>parabolic transformation</u>.

iii) T is real & $|T| > 2 \Rightarrow K$ is real & not 1 $\Rightarrow$ <u>hyperbolic transformation</u>.

iv) T is complex, $T \notin [-2, 2] \Rightarrow$ <u>Loxodromic transformation</u>.
Note iii) is a special case of iv).

→ The isometric circle $|cz+d| = 1$ is mapped onto $|-cw+a| = 1$

→ The centre of $|cz+d| = 1$ is mapped onto ∞ since if $z = -\frac{d}{c}$, $w = \infty$.
Same goes for centre of $|-cw+a| = 1$ in the Inverse transform.

→ Points inside $|cz+d| = 1$ are mapped outside circle $|-cw+a| = 1$ & vice versa.

→ To map $|z| = 1$ to $|w| = 1$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ must be of the form

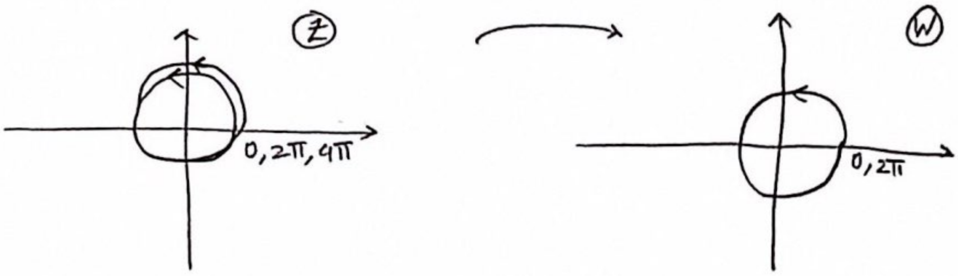
$\begin{pmatrix} a & b \\ a^* & b^* \end{pmatrix}$ with $|a|^2 - |b|^2 = 1$ or $\begin{pmatrix} a & b \\ -b^* & -a^* \end{pmatrix}$ with $-|a|^2 + |b|^2 = 1$.

<u>Unitary matrix</u> $\Rightarrow U^\dagger U = UU^\dagger = I \Rightarrow U^\dagger = U^{-1}$ where $U^\dagger$ is the conjugate transpose (or) Hermitian adjoint.

Also, $|\det U| = 1 \Rightarrow \det U = e^{i\theta}$ where $\theta \in \mathbb{R}$.

Lec 14 Multivalued Functions; Integral representations.

Eg: $f(z) = z^{1/2} = W \Rightarrow W = \sqrt{r}\, e^{i\frac{\theta}{2}}$



> θ must go from 0 to 4π to map onto the full W plane.

> The 2 copies of the z plane are called Riemann sheets.

> For $z^{1/3}$ we need 3 Riemann sheets.

> Topology of Riemann Sheets

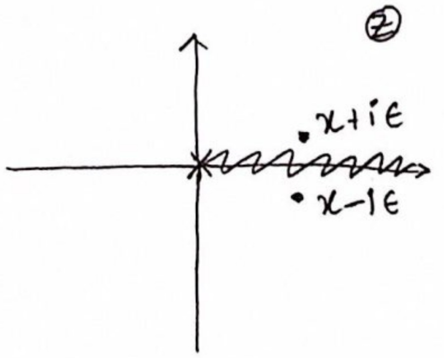
$W = z^{1/2} \longrightarrow$ on top sheet,

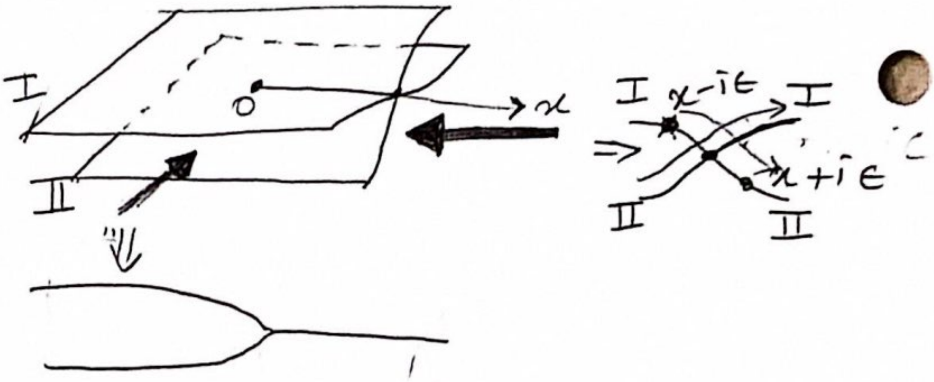
$W = e^{i\pi} z^{1/2} \rightarrow$ on the bottom sheet since $\theta \longrightarrow \theta + 2\pi$ & $\frac{\theta}{2} \rightarrow \frac{\theta}{2} + \pi$

$\Rightarrow W = -z^{1/2}$

> At $z = 0$ the 2 branches coincide & this point is called a "branch point" for this function. Note that $z = \infty$ is also a branch point.

> The two sheets must be glued together at the branch points since outputs coincide. Draw a line joining the branch points, this line can go to ∞ in any direction, but let's choose along positive x. This line joining the 2 branch points is called a "branch cut". The trajectory descends/ascends onto the next sheet at the branch cut.





> By continuity $\lim_{\epsilon\to 0} f_{I}(x-i\epsilon) = \lim_{\epsilon \to 0} f_{II}(x+i\epsilon)$

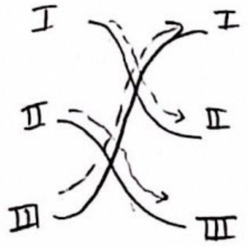
→ Now we don't need to specify $\pm\sqrt{z}$. We have a unique value for $\sqrt{z}$ from $0 \to 2\pi$ in W plane.

→ $\text{dis } f(z)\Big|_{z=x>0} = f_I(x+i\epsilon) - f_I(x-i\epsilon) = f_I(x+i\epsilon) - f_{II}(x+i\epsilon)$

Therefore there is a discontinuity as we jump b/w the sheets.
The disc vanishes on the branch cut & increases as $\epsilon\uparrow$.

Eg: $f(z) = z^{1/n}$ where $n = 2, 3 \ldots$ all have branch points at 0 & ∞.
These branch points are called <u>algebraic branch point</u>

For $n=3$,



Eg: $z^{p/q} = f(z) \Rightarrow q$ sheets

Eg: $f(z) = z^{\alpha}$; α is real. If it is irrational we have an infinite sheeted structure & $z=0$ is a "<u>winding branch point</u>"

Eg: $f(z) = \ln z = \ln r + i\theta$ $\Rightarrow$ we get an additive constant of $2\pi n i$ for each sheet. $n=0$ sheet is called the principle sheet.

$\Rightarrow$ $z = 0, \infty$ are "<u>logarithmic branch points</u>"

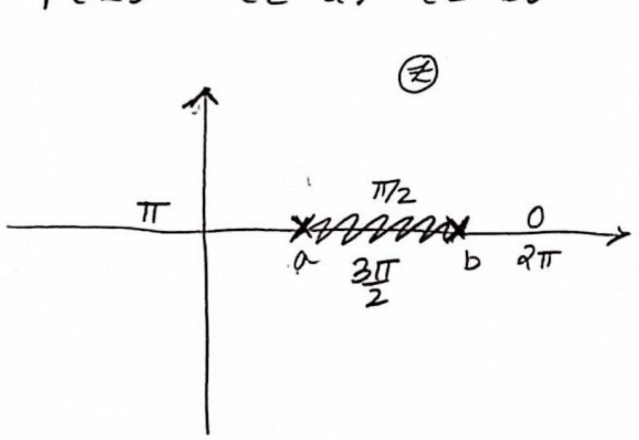
Eg: $f(z) = \ln(z-1)$ $\Rightarrow$ branch point (log) at $z = 1, \infty$.

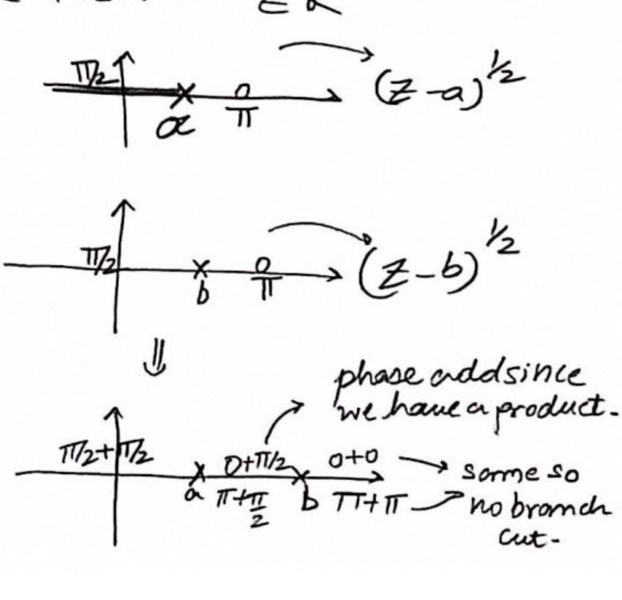
Eg: $f(z) = \frac{1}{z}\ln(1-z)$ at $z=0$ $\quad \ln(1-z) \cong -z - \frac{z^2}{2} - \cdots$

$\Rightarrow$ it looks like the singularity at $z=0$ is removable. But, remember that this is only true on the principle sheet. For all other sheets there is a $2\pi n i$ factor. $\ln 1 = 0$ only on the principle sheet.

$\Rightarrow$ $f(z) = \frac{1}{z}\ln(1-z)$ has a simple pole on every sheet except the principle sheet.

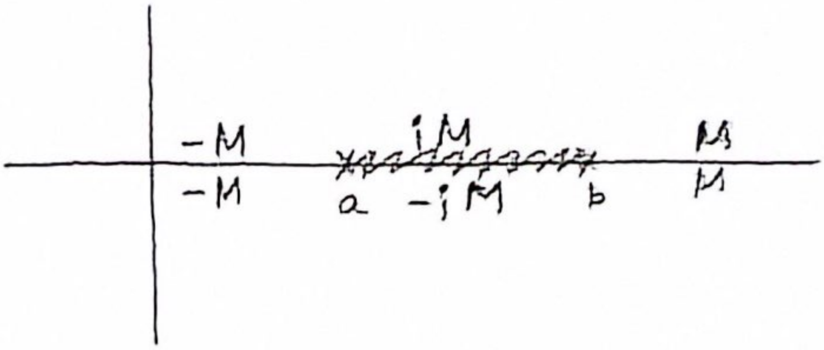
Eg: $f(z) = (z-a)^{1/2}(z-b)^{1/2}$ $\qquad 0 < a < b < \infty \in \mathbb{R}$



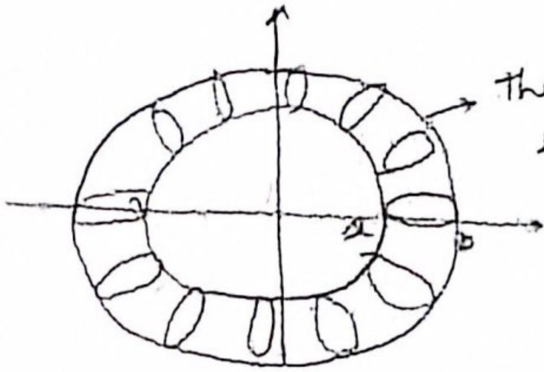


Let $M = |(z-a)(z-b)|^{1/2}$.

$\Rightarrow$ No branch point at ∞.

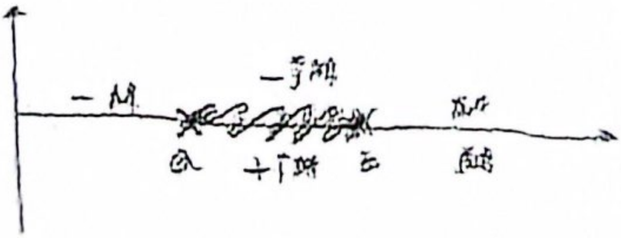


$\Rightarrow$



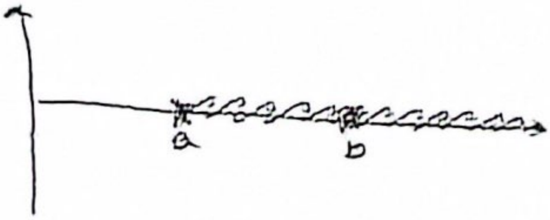
The 2 surfaces form a 'tube' between a & b & are attached elsewhere.

Eg: $f(z) = \dfrac{(z-a)^{1/2}}{(z-b)^{1/2}}$



Eg: $f(z) = (z-a)^{1/3}(z-b)^{1/3}$ $\rightarrow$ Here branch cut goes from $a \rightarrow \infty$ & $b \rightarrow \infty$

Discontinuities are diff b/w a & b & b/w b & ∞



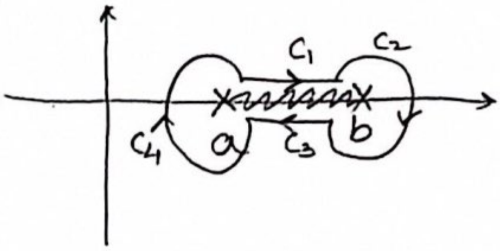
Eg:- $f(z) = (z-a)^{1/3}(z-b)^{-1/3}$ $\Rightarrow$ Cut only b/w a & b.

$\rightarrow$ Drawing closed contours must take into account the sheeted structure.

Lec15

Eg: $\displaystyle\int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}}$ using contour integration.

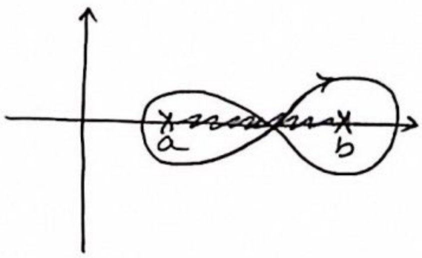
Consider $\displaystyle\frac{1}{\sqrt{(z-a)(z-b)}}$



Answer is 2π → Solve it.

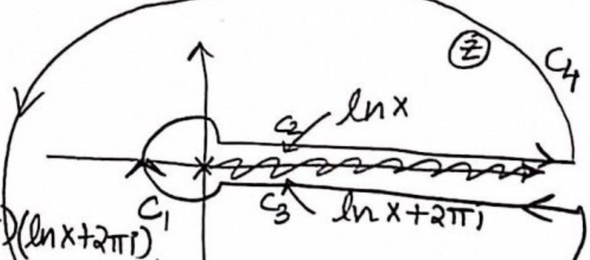
Note that C_2 & C_4 should evaluate to zero. Then blow up the contour to ∞ & evaluate the integral.

> Could choose another contour.
Try this



Eg: $\displaystyle I=\int_0^\infty dx\ \frac{P(x)}{q(x)}$ where $q(x)$ & $p(x)$ are polynomials & $\deg(q)-\deg(P)$ is ≥ 2 & no real positive zeroes of q.
↳ on the line of integration.

Consider, $\displaystyle\int_C dz\ \frac{P(z)\ln z}{q(z)}$



$$\Rightarrow = \int_0^\infty dx\ \frac{P(x)\ln x}{q(x)} + \int_\infty^0 \frac{dx\ P(x)(\ln x+2\pi i)}{q(x)}.$$

$$= \int_0^\infty dx\left[\frac{P(x)\ln x}{q(x)} - \frac{P(x)(\ln x+2\pi i)}{q(x)}\right] = -2\pi i\, I$$

Around origin $z=\epsilon e^{i\theta}$

$\Rightarrow dz \to \epsilon$; $\ln z \to \ln\epsilon$

$\lim_{\epsilon\to 0} \epsilon\ln\epsilon \longrightarrow 0$ since $\epsilon\to 0$ is faster than $\ln\epsilon \longrightarrow -\infty$.

$\Rightarrow C_1$ contribution is 0.

C_4 also has zero contribution since den has $R^{\deg(q)}$ & num has $R^{\deg(p)+1}$ $\Rightarrow$ zero as $R\to\infty$

$$\Rightarrow I = -\frac{1}{2\pi i}\oint_C \frac{dz\ P(z)\ln z}{q(z)} = -\frac{1}{2\pi i}\times 2\pi i \sum_{\text{poles at } q(z)} \text{Residues of } \frac{P(z)}{q(z)}\ln(z).$$

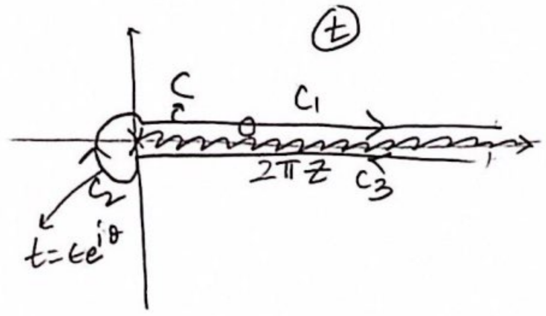
→ Note that $q(z)$ has zeroes outside of the real positive axis.

& $\boxed{I = -\sum \text{residues of } \frac{p(z)}{q(z)} \ln z.}$

Eg: $\int_0^\infty \frac{dx}{x^n+1}$ where $(n \geq 3)$ ⇒ poles at zeroes of $z = \sqrt[n]{-1}$.

Going back to $\Gamma(z)$

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t} \quad (\mathrm{Re}\, z > 0)$$



$t^{z-1} = |t|^{z-1}$ above.

$|t|^{z-1} e^{2\pi i (z-1)}$ below

↳ $e^{2\pi i z}$

$$\int_C dt\, t^{z-1} e^{-t} = \underset{C_1}{\int_0^\infty} dt\, t^{z-1} e^{-t} + \int_\infty^0 dt\, t^{z-1} e^{2\pi i z} e^{-t}$$

$$+ \int_{2\pi}^{0} \cancel{\underbrace{\epsilon e^{i\theta}\, i\, d\theta}_{dt}\, (\epsilon e^{i\theta})^{z-1} e^{-\epsilon e^{i\theta}}} \to 0$$

↳ $\epsilon^z \to 0$ as $\epsilon \to 0$ since ($\mathrm{Re}\, z > 0$) for $\Gamma(z)$.

$$\Rightarrow \int_C dt\, t^{z-1} e^{-t} = \left(1 - e^{2\pi i z}\right) \Gamma(z), \quad \mathrm{Re}(z) > 0$$

$$\Rightarrow \boxed{\Gamma(z) = \frac{1}{1 - e^{2\pi i z}} \oint_C dt\, t^{z-1} e^{-t}}$$

FOR ALL z → Since this integral is valid for all z it is the analytic continuation.

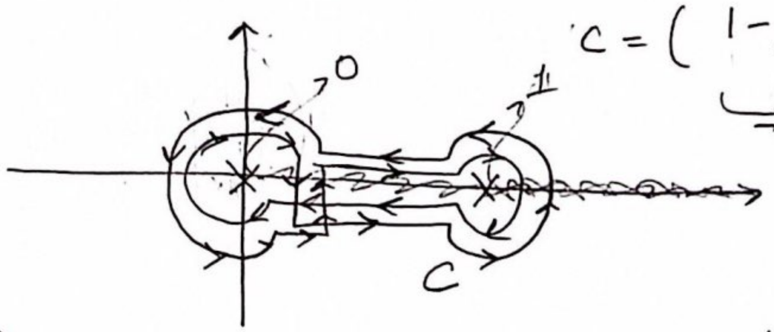
"Master representation of $\Gamma(z)$" → This contour integral converges for all z since the problem with $\mathrm{Re}\, z < 0$ only arose at $t = 0$ (ie C_2). However, this contour does not touch that point. Therefore this representation is valid for all values of z. When z is an integer the branch cut disappears > $1 - e^{2\pi i z}$ ⇒ we get poles at -ve integers.

Lec 16 **Beta function.**

$$B(z,w) = \int_0^1 dt\, t^{z-1}(1-t)^{w-1} \quad (\mathrm{Re}\, z>0,\ \mathrm{Re}\, w>0)$$

> Recall integration by parts doesn't improve overall ROC.

> There are branch points at $t=0,1$ since z, w can take on arbitrary values.. In fact this branch pt. is a winding point since z, w are arbitrary



Pochhammer contour.

$C = (1-, 0-, 1+, 0+)$

encircle 1 clockwise the 0 clockwise then 1 anticlockwise & then 0 anticlockwise.

1− picks up $-2\pi i w$ from $(1-t)^{w-1}$ term
since clockwise

0 − picks up $-2\pi i z$ from t^{z-1} term.

1+ picks up $+2\pi i w$ & 0+ picks up $+2\pi i z$

$\Rightarrow M \to M e^{-2\pi i w} \to M e^{-2\pi i (w+z)}$
$\to M e^{-2\pi i z} \to M$.
So we are sure that despite crossing branch cuts we are back to the original plane.

> $dt \to \epsilon$; $t^{z-1} \to \epsilon^{z-1}$; $(1-t)^{w-1} \to 1$ $\Rightarrow$ integrand $\to \epsilon^{z}$ which goes to 0 as $\epsilon \to 0$ for $\mathrm{Re}\, z>0$. Similarly contribution from circles around 1 also give 0.

$$\Rightarrow \boxed{B(z,w) = \frac{1}{(1-e^{2\pi i z})(1-e^{2\pi i w})} \oint_C dt\, t^{z-1}(1-t)^{w-1}}$$

This is valid for all z, w since this representation converges everywhere. Note that C never hits the poles it is always arbitrarily close to them so we are fine.

Riemann Zeta Function $\zeta(z)$

$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$, $Re(z) > 1$ $\longrightarrow$ Analytic Continuation?

Start with,

$$\Gamma(z)\,\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \int_0^{\infty} du\, u^{z-1} e^{-u} \quad , \quad Re(z) > 1$$

Put $u = nt$

$$\Rightarrow \Gamma(z)\,\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{\cancel{n^z}} \int_0^{\infty} \cancel{n} dt\, \cancel{n^{z-1}}\, t^{z-1} e^{-nt}$$

$$= \int_0^{\infty} dt\, t^{z-1} \sum_{n=1}^{\infty} e^{-nt} \quad ;$$

$\sum_{n=1}^{\infty} (e^{-t})^n$ is a geometric series.

$$= \frac{1}{1-e^{-t}} - 1$$

$$= \frac{e^{-t}}{1-e^{-t}} \quad \text{Since } e^{-t} < 1 \Rightarrow t > 0 \text{ in integral.}$$

$$= \frac{1}{e^t - 1}$$

$$= \int_0^{\infty} \frac{dt\, t^{z-1}}{e^t - 1}$$

$Re(z) > 0$ still since $t^{-1} \to \infty$ at $t = 0$

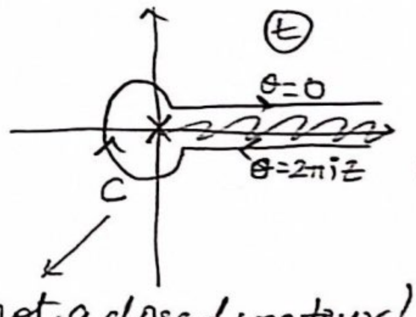
But $Re(z) > 1$ still since $e^t - 1 \simeq t\left(1 + \frac{t}{2} + \cdots\right)$

$\frac{t^{z-1}}{e^t - 1} \simeq \frac{t^z}{t^2(\cdots)} \Rightarrow z = 1 \Rightarrow \frac{1}{t}$ blows up since $\ln t$ is $-\infty$ at 0.

$\therefore z > 1 \Rightarrow \frac{t^z}{t^2(1\cdots)}$ does not blow up. at $t = 0$

$$\Rightarrow \Gamma(z)\,\zeta(z) = \int_0^{\infty} \frac{dt\, t^{z-1}}{e^t - 1} \quad , \quad Re(z) > 1$$

Now use contour integral in t plane.



not a closed contour!

$$= \frac{1}{1-e^{2\pi i z}} \int_C dt \, \frac{t^{z-1}}{e^t - 1}$$

same as earlier.

$$\Rightarrow \boxed{\zeta(z) = \frac{1}{\Gamma(z)\,(1-e^{2\pi i z})} \int_C dt \, \frac{t^{z-1}}{e^t-1}}$$

C does not touch Z so this rep is valid for all Z.

Recall $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$

Rewriting,

$$\zeta(z) = \frac{i\,e^{-i\pi z}}{2\Gamma(z)\sin\pi z} \int_C dt \, \frac{t^{z-1}}{e^t-1}$$

$$\boxed{\zeta(z) = \frac{i e^{-i\pi z}\,\Gamma(1-z)}{2\pi} \int_C \frac{dt\, t^{z-1}}{e^t-1}}$$

has singularity at $z=1,2,3\ldots$ but Zeta has pole only at $z=1$. So why the discrepancy.

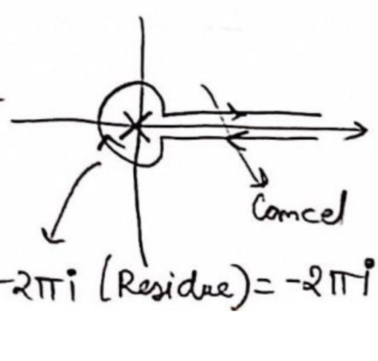
$$\zeta(z) \xrightarrow[z\to 1]{} \frac{i\,e^{-i\pi}}{2\pi} \cdot \underbrace{\frac{1}{1-z}}_{\text{Gamma fn.}} \cdot \int_C \frac{dt\, t^0}{e^t-1}$$

$t^0 \to 1 \Rightarrow$ branch point disappears. but $e^t - 1 \to t(\cdots)$ which has a singularity at $t=0$.

$\Gamma(w) \xrightarrow[w\to 0]{} \frac{1}{w}$

$$= \frac{i e^{-i\pi}}{2\pi} \cdot \frac{-1}{z-1} \cdot \int_C \frac{dt}{t}\left(1+\frac{t}{2}+\cdots\right)^{-1}$$

simple pole. residue $=1$



$$\Rightarrow \zeta(z) = \frac{\cancel{i}}{\cancel{2\pi}} \cdot \frac{1}{z-1} \cdot (\cancel{-2\pi i}) + \text{Reg part.}$$

$$\zeta(z) = \frac{1}{z-1} + \text{Reg part}$$

$\Rightarrow$ Simple pole at $z=1$ with residue of 1

For $z = n$; $n = 2, 3, 4 \ldots$

$\Gamma(1-z)$ has poles,

$$\Rightarrow \zeta(z) = \frac{i e^{-i\pi z}\, \Gamma(1-z)}{2\pi} \underbrace{\int_C \frac{dt\, t^{z-1}}{e^t - 1}}$$

let $t^{z-1} = e^{(z-1)\ln t}$

& diff it for HOT

$$= \frac{i}{2\pi} \frac{\cancel{(-1)^{n-1}}}{(n-1)!} \cdot \frac{\cancel{(-1)^{n-1}}}{\cancel{z-n}} \left[\underbrace{\cancel{\int_C \frac{dt\, t^{n-1}}{e^t - 1}}}_{0} + \cancel{(z-n)} \int_C \frac{dt\, t^{n-1} \ln t}{e^t - 1} \ldots \right]$$

go to zero as $z \to n$

$$= \frac{i}{2\pi (n-1)!} \left[\underbrace{\int_C \frac{dt\, t^{n-1} \ln t}{e^t - 1}} \right]$$

recall that this integral the $\ln t$ $-\ln t - 2\pi i \to -2\pi i$

$$\Rightarrow \frac{\cancel{i}}{\cancel{2\pi}(n-1)!} \int_C \frac{dt\, t^{n-1} \cancel{(2\pi i)}}{e^t - 1} = \frac{1}{(n-1)!} \int_C \frac{dt\, t^{n-1}}{e^t - 1}$$

$$\Rightarrow \boxed{\zeta(n) = \frac{1}{(n-1)!} \int_C \frac{dt\, t^{n-1}}{e^t - 1}} \quad (n = 2, 3, \ldots)$$

$\Rightarrow$ the poles from Γ fn disappear

What happens at $n = 0, -1, -2 \ldots$? Let's look at Bernoulli numbers.

We know, $\zeta(2) = \frac{\pi^2}{6}$; $\zeta(4) = \frac{\pi^4}{90}$; ...

Bernoulli numbers

Consider, $\frac{t}{e^t - 1} = \frac{t}{\left(1 + t + \frac{t^2}{2!} + \cdots\right) - 1} = \frac{t}{t\left(1 + \frac{t}{2!} + \cdots\right)} = \frac{1}{1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots}$

$$\Rightarrow \boxed{\frac{t}{e^t - 1} \equiv \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}}$$

B_n are the Bernoulli numbers.

$B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$

$B_6 = \frac{1}{42}$ ...

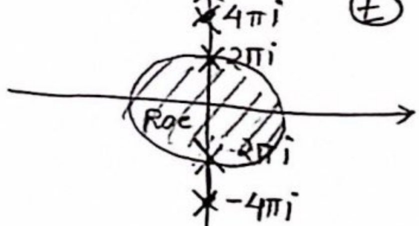
> All odd $B_n = 0$; $n \geq 3$.

> Looks like RHS converges for all t since B_n seems to decrease & $n!$ blows up.

> But on LHS pole at $t = 0$ is removable. But $e^t = 1$ for $t = 2\pi n i$

> Therefore ROC is only 2π for the series expansion.

(t)



& in fact the Bernoulli numbers blow up after few values! (faster than $n!$).

Going back to $\zeta(z)$ $\quad \zeta(z) = \frac{i e^{-i\pi z}}{2\pi} \Gamma(1-z) \int_C \frac{dt\, t^{z-1}}{e^t - 1}$

$$\zeta(0) = \underbrace{\frac{i}{2\pi}}\ \underbrace{e^0}_{1}\ \underbrace{\Gamma(1)}_{1} \int_C \frac{dt}{t^2} \underbrace{\frac{t}{e^t - 1}}_{\text{Bernoulli}} = \frac{i}{2\pi} \oint_C \frac{dt}{t^2} \cdot \left[B_0 + B_1 t + \frac{B_2 t^2}{2!} + \cdots\right]$$

contour around origin.

$\hookrightarrow$ residue $= B_1$

$$= \frac{i}{\cancel{2\pi}} (\cancel{-2\pi i}) B_1 = B_1 \Rightarrow \boxed{\zeta(0) = -\frac{1}{2}}$$

Similarly,

$$\zeta(-1) = \frac{i}{2\pi} \oint_C \frac{dt}{t^3} \left[B_0 + B_1 t^2 + \frac{B_2 t^2}{2!} + \cdots \right]$$

$$= \frac{\cancel{i}}{\cancel{2\pi}} (\cancel{-2\pi} i) \frac{B_2}{2!}$$

$$\boxed{\zeta(-1) = \frac{-1}{12}}$$

$$\zeta(-2n) = \frac{i}{2\pi} \Gamma(2n+1) \int_C \frac{dt}{t^{2n+2}} \left[B_0 + B_1 t + \frac{B_2 t^2}{2!} + \frac{B_4 t^4}{4!} + \cdots \right]$$

Since there are no odd powers in the B_n terms there is no residue $\Rightarrow$ $\zeta(-2n)$ for $n = 1, 2, 3 \ldots = 0$. These are the trivial zeroes of the zeta function.

Legendre Functions

$$\left[(1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} + \nu(\nu+1)\right]\phi(z) = 0 \longrightarrow \text{Legendre's Equation.}$$

where, ν & $z \in \mathbb{C}$

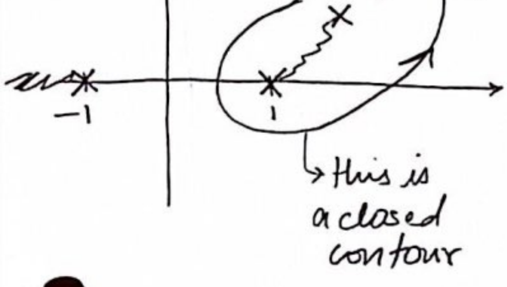
> The 2 solutions are $P_\nu(z)$, $Q_\nu(z)$. They are linearly independant.
(ν → order, z → argument.)

> For $-1 \leq z \leq 1$ for $\nu = n$ where $n \in \mathbb{Z}$, $P_n(z)$ are the Legendre Polynomials.

> Legendre Eq. can be recast as an Eigenvalue problem.

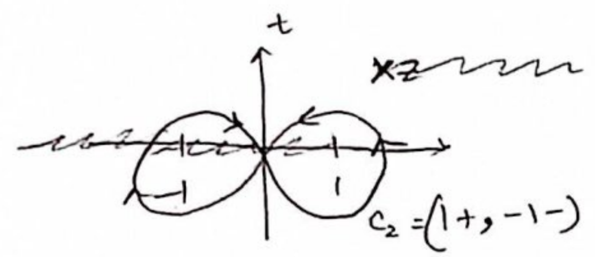
$$\frac{d}{dz}\left[(1-z^2)\frac{d}{dz}\phi\right] = -\nu(\nu+1)\phi$$

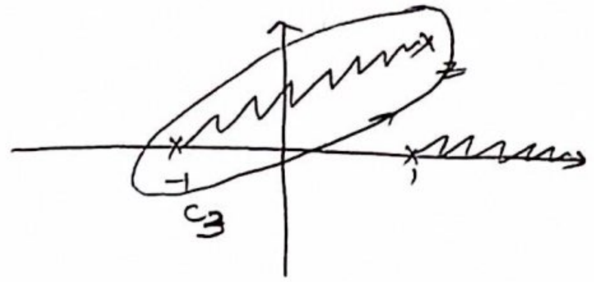
Consider, $\left[\frac{t^2-1}{2(t-z)}\right]^\nu$ as a fn of ν for a given z. This function has winding points at $t = \pm 1$ & $t = z$ since ν is arbitrary.



Encircling $t=1$ or $t=-1$ counterclockwise gives $e^{2\pi i\nu}$ whereas encircling $t=z$ counterclockwise gives $e^{-2\pi i\nu}$. So choosing the contour wisely can form a closed contour. This allows us to choose cuts appropriately.

$$C_1 = (1+, z+)$$





> C_1, C_2, C_3 are the only 3 independant cuts. C_1 & C_3 are almost same

> $$\boxed{P_\nu(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{dt}{(t-z)} \cdot \left[\frac{t^2-1}{2(t-z)}\right]^\nu} \longrightarrow \text{Contour integral representation}$$

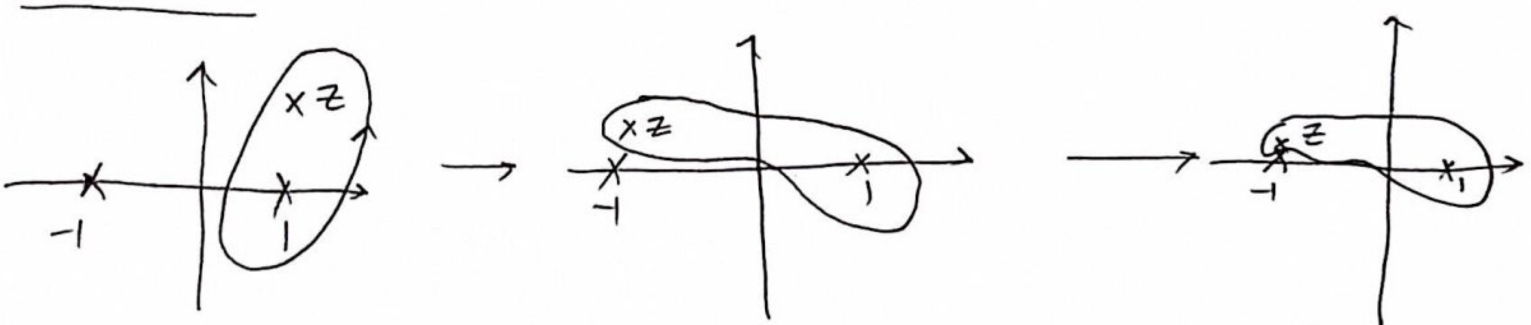
$$\boxed{Q_\nu(z) = \frac{1}{\pi(e^{2\pi i\nu}-1)} \oint_{C_2} \frac{dt}{(t-z)} \left[\frac{t^2-1}{2(t-z)}\right]^\nu}$$

Let $\nu \neq \mathbb{Z}$,

When $z = -1$ the contour is "pinched b/w" the 2 poles (in t plane) & we get a pole in the z plane.

$\Rightarrow P_\nu(z)$ is singular at $z = -1$

Pictorially,

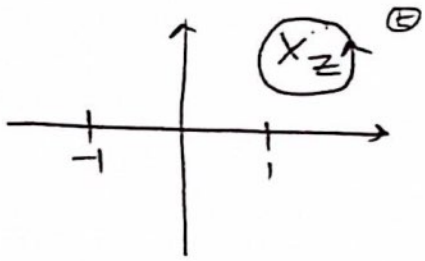


When $z \to -1$, the contour can no longer be distorted & gets pinched b/w the 2 poles in t plane. We can show that this produces a singularity in the z plane at $z = -1$.

> Similarly a singularity occurs at $z = +1, -1$ for $Q_\nu(z)$ since the contour C_2 is pinched when $z \to \pm 1$.

i) > Let $\nu = l = 0, 1, 2 \ldots$ the branch cut b/w 1 & z disappears, since it was only there for fractional ν.

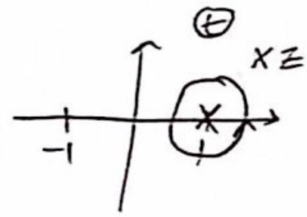
$$P_l(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{dt\,(t^2-1)^l}{2^l \cdot (t-z)^{l+1}}$$ $\Rightarrow$ pole of order $l+1$ at $t = z$



$P_l(z)$ is a polynomial

$$\boxed{P_l(z) = \frac{1}{2^l\, l!} \frac{d^n}{dt^n} (t^2-1)^n \Big|_{t=z}}$$

$$\Rightarrow \boxed{P_l(z) = \frac{1}{2^l\, l!} \frac{d^n}{dz^n} (z^2-1)^n}$$ $\longrightarrow$ Rodrigues formula for $P_l(z)$ true for all z.



ii) > $P_{-l-1}(z)$ for $l = 0, 1, 2 \ldots$ $\Rightarrow$ pole at $t = 1$

$$\Rightarrow \boxed{P_{-l-1}(z) = P_l(z)}$$

> Recall $l(l+1) = \left(l+\frac{1}{2}\right)^2 - \frac{1}{4}$ $\Rightarrow$ Symmetry around $-\frac{1}{2}$ since $\left(l+\frac{1}{2}\right)^2 - \frac{1}{4}$
from Diff Eq. is even around $-\frac{1}{2}$.

$\Rightarrow$ Symmetry about $-\frac{1}{2}$ is true for any ν $\Rightarrow$ $\boxed{P_{-\nu-1}(z) = P_\nu(z)}$

iii) <u>Let's look at $Q_\nu(z)$</u>

> $\nu = l = 0, 1, 2 \dots$

Since the poles at $t = \pm 1$ vanish, the contour integral $\to 0$ but $(e^{2\pi i \nu} - 1)$ also vanishes. So we need to look at the limiting behaviour to find the value.

$$Q_l(z) = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) P_l(z) + R_{l-1}(z)$$

$\hookrightarrow$ polynomial of degree $l-1$ in z.

$\to$ Q_l has a logarithmic singularity at $z = \pm 1$.. Remember pinching.

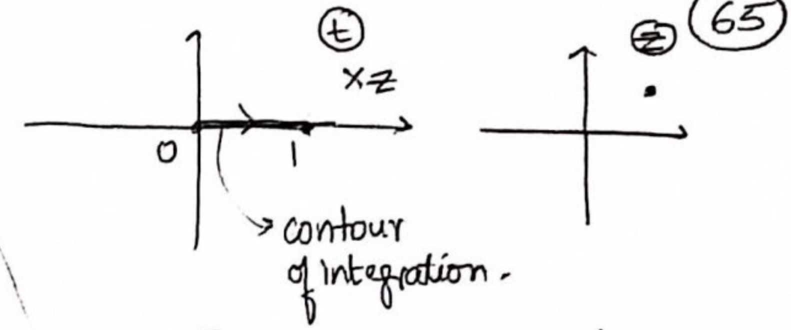
> $Q_l(z)$ is singular for negative l.

> Also, $Q_\nu(z) = \dfrac{P_l(z)}{\nu + l + 1} + \text{reg part}$

$\hookrightarrow$ there are singularities in ν plane.

> $$Q_l(z) = \frac{1}{2} \int_{-1}^{1} dt \, \frac{P_l(t)}{z-t} \quad (l = 0, 1, 2 \dots)$$

Eg: $f(t) = \int_0^1 \frac{dt}{t-z}$



> As pt. in z plane moves, the pole in t plane moves. As long as the pole does not hit 0, 1, the contour can be distorted to maintain the integral. If $z \to \infty$ the contour goes to ∞ & this may cause the integral to diverge.

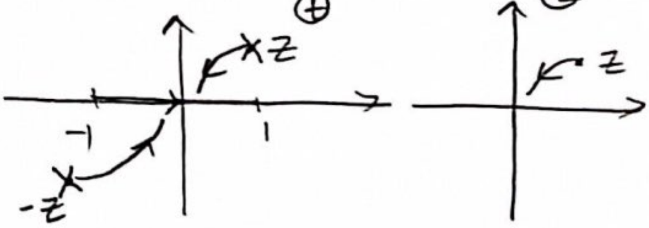
End point singularities at $z = 0, 1$

> Such singularities are called "end point singularities". When a singularity of the integrand attacks an end point of integration

Solving, $\ln|t-z| \Big|_{t=0}^{t=1} = \ln \frac{z-1}{z} \rightarrow$ This explicitly shows the 2 singularities.

Therefore, we can recognize end point singularities without explicitly solving the integral.

Eg: $f(t) = \int_{-1}^{1} \frac{dt}{t^2 - z^2}$



As $z \to \pm 1$ we have end point singularities. However even when $z \to 0$ the 2 poles "pinch off" the contour. This issue only occurs at $z \to 0$. This singularity is called a "pinched singularity".

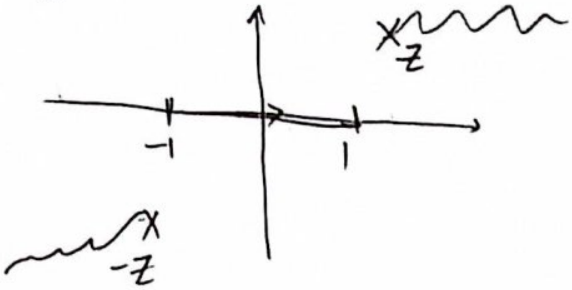
Solving the integral,

$$f(t) = \int_{-1}^{1} dt \left(\frac{1}{t-z} - \frac{1}{t+z} \right)$$

$$f(t) = \frac{1}{z} \ln \frac{1-z}{1+z}$$ $\Rightarrow$ singularities at $z = \pm 1$ & 0. But the simple pole occurs for all sheets except the principle sheet.

Eg: $$f(t) = \int_{-1}^{1} \frac{dt}{\sqrt{z^2 - t^2}}$$ singularities at $\pm z$ are branch points.

Try solving it.



$\rightarrow$ $$Q_\ell(z) = \frac{1}{2} \int_{-1}^{1} \frac{dt\, P_\ell(t)}{z-t}$$ $\ell = 0, 1, 2 \ldots$

$\rightarrow$ polynomial $\Rightarrow$ no singularities.

$\Rightarrow$ we have end point singularities that end up as log singularities.

Partial Differential Equations - V. Balakrishnan

Lec 23 Poisson's Equation

$\nabla^2 f(\bar{r}) = g(\bar{r}) \rightarrow$ eg: electrostatics $\nabla^2 \phi(\bar{r}) = \frac{-\rho(\bar{r}')}{\epsilon_0}$

∇^2 is a scalar operator $\Rightarrow$ invariant under rotations.
$\hookrightarrow$ aka elliptic partial differential operator.

Green's function.

Let D_x be a differential operator acting on $f(x)$.

$$D_x f(x) = g(x) \quad \text{where } g(x) \text{ is given.}$$

In abstract notation, $D_x \rightarrow D$

$f(x) \rightarrow |f\rangle$ $\rightarrow$ Ket vector notation from Bra-ket notation.

$g(x) \rightarrow |g\rangle$ Kets $\rightarrow |\ \rangle$ are vectors.

Bras $\rightarrow \langle\ |$ are functionals

$\Rightarrow |f\rangle = D^{-1}|g\rangle + \sum_i c_i |h_i\rangle$

where $D|h_i\rangle = 0$ $\hookrightarrow$ linear comb. of all vectors $|h_i\rangle$ such that when D acts on it, it is annihilated.

Therefore $D^{-1}|g\rangle$ is the particular integral.

& $\sum_i c_i |h_i\rangle$ is the complementary fn (since $D|h_i\rangle = 0$ is homogeneous).

Going back to position space $f(x)$ is $\langle x|f\rangle \Rightarrow$ representative of f vector in x basis.

$$\Rightarrow f(x) = \int dx'\, G(x,x')\, g(x') + \sum_i c_i h_i(x)$$

Here $G(x,x')$ represents the D_x^{-1} and is called the Green's function of D_x. Explicitly,

$$G(x,x') = \langle x|D^{-1}|x'\rangle$$

We have, $|f\rangle = D^{-1}|g\rangle + \sum_i c_i |h_i\rangle$

$$\Rightarrow \langle x|f\rangle = f(x) = \langle x|D^{-1}|g\rangle + \sum_i c_i \underbrace{\langle x|h_i\rangle}_{h_i(x).}$$

insert a complete set of states $\int dx' \, |x'\rangle\langle x'| \to$ Identity operator.

$$\Rightarrow \int dx' \underbrace{\langle x|D^{-1}|x'\rangle}\underbrace{\langle x'|g\rangle}$$
$$= \int dx' \, G(x,x') \quad g(x')$$

Finding $G(x,x')$

$DD^{-1} = \mathbb{1} \to$ identity operator.

$$\Rightarrow \mathcal{D}_x \, G(x,x') = \langle x|x'\rangle = \delta(x-x').$$

$\Rightarrow \mathcal{D}_x f(x) = g(x)$ can be fully solved by finding

$$\boxed{\mathcal{D}_x \, G(x,x') = \delta(x-x')}$$ $\to$ Basically an impulse response.

The Boundary conditions would determine how much of the CF must be used in the full soln.

Eg: $\dfrac{d^2 f(x)}{dx^2} = g(x)$ $\quad x \in [0,1]$; $f(0) = a$, $f(1) = b$

$$\frac{d^2 G(x,x')}{dx^2} = \delta(x-x')$$

$\Rightarrow$ in $0 \le x \le x'$

$\Rightarrow G(x,x') = A_1 x + A_2$ since $\dfrac{d^2 G}{dx^2} = 0$

in $x' < x \le 1$

$\Rightarrow G(x,x') = A_3 x + A_4$

2 BC & 4 constants.
But we know more about G.

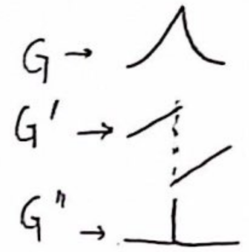
→ We can also show that $G(x, x')$ is continuous. The second derivative has an ∞ jump & first derivative has a finite jump.

$$\frac{d^2 G(x,x')}{dx^2} = \delta(x-x')$$

$$\Rightarrow \int_{x'-\epsilon}^{x'+\epsilon} dx \, \frac{d^2 G(x,x')}{dx^2} = \left.\frac{dG}{dx}\right|_{x'+\epsilon} - \left.\frac{dG}{dx}\right|_{x'-\epsilon} = 1 \quad \text{— (1)}$$

⇒ Apply (1) to A_1, A_2, A_3, A_4 eqns. to get another B.C.

⇒ G must have a cusp (it is still continuous) this gives the fourth B.C ie continuity of G at x'



$A_1 x' + A_2 = A_3 x' + A_4$ BC 4

$A_1 - A_3 = 1$ BC 3

Going back to Poisson's Equation.

$\nabla^2 \phi(\bar{r}) = -\frac{\rho(\bar{r})}{\epsilon_0}$; we know the soln. is Coulomb's law,

$$\Rightarrow \phi(\bar{r}) = \frac{1}{4\pi\epsilon_0} \int_{r'} d\bar{r}' \, \frac{\rho(\bar{r}')}{|\bar{r}-\bar{r}'|}$$

($d\bar{r}' \to d^3 r'$)

Therefore in general

$$f(\bar{r}) = \frac{-1}{4\pi} \int_{r'} d\bar{r} \, \frac{g(\bar{r}')}{|\bar{r}-\bar{r}'|}$$

→ Need to derive this more formally.

B.C used is $\bar{r} \to \infty \Rightarrow G \to 0$

Formal solution to Poisson's Equation.

$$\nabla^2 f(\bar{r}) = g(\bar{r})$$

$$\Rightarrow f(\bar{r}) = \int d^3_{r'}\, G(\bar{r},\bar{r}')\, g(\bar{r}') . + C.F \rightarrow \text{ignoring this}$$

$$\nabla_r^2\, G(\bar{r},\bar{r}') = \delta^{(3)}(\bar{r}-\bar{r}')$$

> Since ∇_r^2 is a scalar function if $\bar{R} = \bar{r}-\bar{r}'$ we can write,

$\nabla_R^2 = \nabla_r^2$. Invariant to translation & rotation.

> RHS is a fn of only $\bar{r}-\bar{r}'$ so is G also only a fn of $\bar{r}-\bar{r}'$?

We need to ensure that the B.C is also translation invariant. Therefore we need $\vec{r} \rightarrow \infty \Rightarrow \vec{G} \rightarrow 0$ ie natural B.C.

$$\Rightarrow \nabla_r^2\, G(\bar{r}-\bar{r}') = \delta^{(3)}(\bar{r}-\bar{r}')$$

$$\Rightarrow \boxed{\nabla_R^2\, G(\bar{R}) = \delta^{(3)}(\bar{R})} \quad -(1)$$

$$G(\bar{R}) = \frac{1}{(2\pi)^3}\int d^3k\, e^{i\bar{K}\cdot\bar{R}}\, \tilde{G}(\bar{K}) \quad -(2) \rightarrow$$ defining a fourier transform in 3D.

$$\Rightarrow \tilde{G}(\bar{K}) = \int d^3R\, e^{-i\bar{K}\cdot\bar{R}}\, G(\bar{R})$$

$$\nabla_R\, e^{i\bar{K}\cdot\bar{R}} = i\bar{K}\, e^{i\bar{K}\cdot\bar{R}} \Rightarrow \nabla^2 = -|K|^2 e^{i\bar{K}\cdot\bar{R}}$$

Remember,

$$\nabla\cdot\bar{a}\, e^{i\bar{K}\cdot\bar{R}} = i\bar{K}\cdot\bar{a}\, e^{i\bar{K}\cdot\bar{R}}$$

$$\nabla\times\bar{a}\, e^{i\bar{K}\cdot\bar{R}} = i\bar{K}\times\bar{a}\, e^{i\bar{K}\cdot\bar{R}}$$

Replace LHS in ① by ② & replace RHS by its analog of Eq ②.

$$\Rightarrow \frac{1}{(2\pi)^3}\int_K d^3k\, e^{i\bar{k}\cdot\bar{R}}\left[-k^2\tilde{G}(k)\right] = \frac{1}{(2\pi)^3}\int d^3k\, e^{i\bar{k}\cdot\bar{R}} \quad \text{③}$$

Since $\bar{k}$ forms a complete set in $\bar{R}$ space they must be equal by components.

$$\Rightarrow -k^2\tilde{G}(k) = 1$$

$$\Rightarrow \boxed{\tilde{G}(k) = \frac{-1}{k^2}}$$

③ can be rewritten as

$$\frac{1}{(2\pi)^3}\int_k d^3k\, e^{i\bar{k}\cdot\bar{R}}\left[\tilde{G}(k)\right] = \frac{-1}{(2\pi)^3}\int_k \frac{d^3k}{k^2}\, e^{i\bar{k}\cdot\bar{R}}$$

$$\Rightarrow \boxed{G(\vec{R}) = \frac{-1}{(2\pi)^3}\int_k \frac{d^3k}{k^2}\, e^{i\bar{k}\cdot\bar{R}}}$$

Since $G(R)$ is a scalar, we can use rotational invariance property & choose polar axis of $\bar{k}$ space (ie $\hat{k}_z$) along $\bar{R}$ $\Rightarrow \bar{k}\cdot\bar{R} = kR\cos\theta$

$$\Rightarrow G(\bar{R}) = \frac{-1}{(2\pi)^{\cancel{3}\,2}}\int_0^\infty \frac{\cancel{k^2}dk}{\cancel{k^2}}\int_0^{2\pi} d\phi \int_0^{\pi} e^{ikR\cos\theta}\cdot\sin\theta\, d\theta.$$

$$\int_0^{2\pi} d\phi \to \cancel{2\pi}$$

$$\int_0^{\pi} e^{ikR\cos\theta}\sin\theta\, d\theta \to -\int_1^{-1} e^{ikR\cos\theta}\, d(\cos\theta)$$

$\cos\theta$ is even

$$\frac{e^{ikR} - e^{-ikR}}{ikR} \leftarrow \int_{-1}^{1} e^{ikRx}\, dx \leftarrow \int_{-1}^{1} e^{ikR\cos\theta}\, d(\cos\theta)$$

$2\sin kR$

$$\Rightarrow \quad G(\bar{R}) = -\frac{1}{2\pi^2 R} \underbrace{\int_0^\infty dk \frac{\sin kR}{kR}}_{\text{Dirichlet's integral}}$$

$$= -\frac{1}{4\pi R}$$

$$\boxed{G(\bar{R}) = \frac{-1}{4\pi|r-r'|}} \rightarrow \text{Coulomb potential!}$$

Free space green's function for Poisson Equation.

$$\Rightarrow \boxed{f(\bar{r}) = -\frac{1}{4\pi}\int d^3r' \frac{g(\bar{r}')}{|\bar{r}-\bar{r}'|}}$$

CF $\rightarrow 0$ due to BC that $G \rightarrow 0$ as $\bar{r} \rightarrow \infty$

$\frac{1}{|\bar{r}-\bar{r}'|}$ is called the Coulomb kernel.

$$\frac{1}{|\bar{r}-\bar{r}'|} = \frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l \underbrace{P_l(\cos\gamma)}_{\text{Legendre polynomial.}}$$

where γ is angle b/w r, r' & $r_>$ is r if $r > r'$ or $r_>$ is r' if $r' > r$.

$$= \sum_{l=0}^{\infty}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l \sum_{m=-l}^{l} \underbrace{Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta',\phi')}_{\text{spherical harmonics.}}$$

Greens function of Poisson's Eqn. in d spatial dimensions.

- Harmonic fns in 1D are lines $ax+b$ since $\frac{d}{dx^2}(ax+b)=0$.
- These fns have the property that their value at any pt is the mean of all points around them that lie on a sphere centered at that point. In higher dimensions the shapes are not trivial like in 1D. Since we can have $\frac{d^2f}{dx^2} = -\frac{d^2f}{dy^2}$

$$f(x,y) = \frac{f(x+\Delta x,y) - f(x-\Delta x,y) + f(x,y+\Delta y) - f(x,y-\Delta y)}{4}$$

↳ from this we can derive $\frac{d^2f}{dx^2} + \frac{d^2f}{dy^2} = 0$ in the limit. (Proof)

- $\nabla^2 G(\underset{\sim}{R}) = \delta^{(d)}(\underset{\sim}{R})$ $\quad$ $\underset{\sim}{R}$ is a vector in d dimensions.

$$G(\underset{\sim}{R}) = \frac{1}{(2\pi)^d} \int d^d k \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{R}}}{k^2}$$

$d=2 \Rightarrow d^2k = k\,dk\,d\phi \Rightarrow \frac{1}{(2\pi)^d}\int_0^\infty \frac{k\,dk\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{R}}}{k^2}$

When $k=0$, $\int$ is unbounded. ⇒ Blows up. aka infrared divergence since it happens at low frequencies ⇒ $k\downarrow$
Similarly we could have a UV divergence.

Polar coordinates in d dimensions. (Hyperspherical polar coordinates)

K, $0 \leq K < \infty$

$\theta_1\ \theta_2\ \theta_3 \dots \theta_{d-2} \in [0, \pi]$

$\phi \quad 0 \leq \phi \leq 2\pi$

$$d^d K = K^{d-1} (\sin\theta_1)^{d-2} (\sin\theta_2)^{d-3} \dots (\sin\theta_{d-2})^1 \, dK \, d\theta_1 \, d\theta_2 \dots d\theta_{d-2} \, d\phi$$

Choose $\underset{\sim}{R}$ along polar axis of $\underset{\sim}{K}$

$\Rightarrow \underset{\sim}{K} \cdot \underset{\sim}{R} = |\underset{\sim}{K}| \, |\underset{\sim}{R}| \cos\theta_1$

$\Rightarrow \int_0^\pi d\theta (\sin\theta)^r$ is left for all other angles & it is solvable.

Only θ_1 is problematic since $e^{iKR\cos\theta_1}$ is also present & it can be written in terms of Bessel fns. Then put that in $\int dK$.

In the end we get,

$$G(\underset{\sim}{R}) = (\text{const}) \int_0^\infty dK \, K^{\frac{d-4}{2}} J_{\frac{d}{2}-1}(KR)$$

$\rightarrow$ Bessel fn of first kind.

$\int^\infty dK \, K^r < \infty$ if $r < -1$; if $r = -1 \Rightarrow$ log divergence.

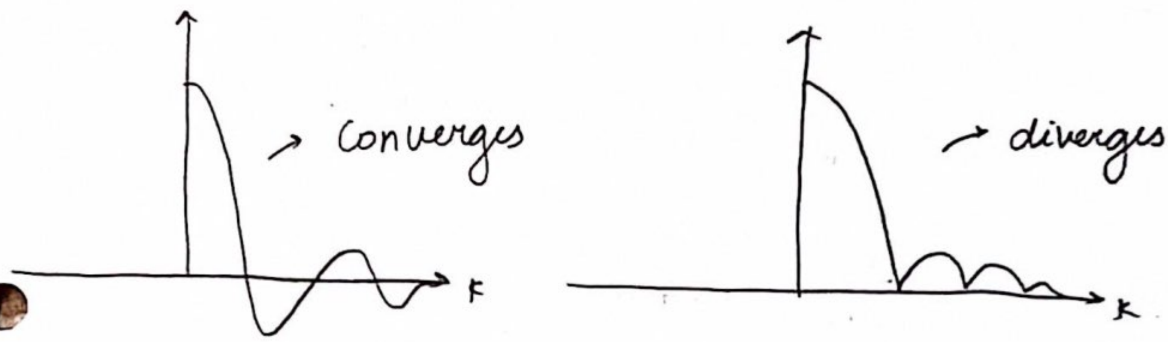
When $K \to \infty$ $J \propto \frac{1}{\sqrt{K}}$ $\Rightarrow \frac{d-4}{2} - \frac{1}{2} < -1$ for convergence.

$\Rightarrow d < 3$ but we already saw that for $d=3$ we have a convergence.

Therefore, the problem is that the power counting approach ignores the changes in sign of of the Bessel fn.

In 3D, $\int_0^\infty \frac{\sin kR}{k}\, dk$ is a sinc-fn & the integral is finite.

However if we ignore the sign of sinc, say $\int_0^\infty \frac{|\sin kR|}{k}\, dk$, num is bounded by 1 & the integral diverges logarithmically due to $\frac{1}{k}$.



→ So if $d=3$ we barely made it, when $d>3$ it diverges & it is called an ultraviolet divergence since it diverges for large k.

How about small k?

$\int_0 dk\, k^{\gamma}$ converges when $\gamma > -1$ since $\gamma = -1 \Rightarrow \log k \to -\infty$

$\Rightarrow$ looks like $\int_0^\infty dk\, k^{\gamma}$ never exists.

But here the Bessel fn saves us.

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{\nu+2n} (-1)^n}{\Gamma(\nu+n+1)\, n!}$$ which for small k is $\propto k^{\frac{d}{2}-1}$

$\Rightarrow \gamma = \frac{d-4}{2} + \frac{d}{2} - 1 > -1 \Rightarrow d > 2$

∴ Roughly speaking or naive power counting implies the integral for $G(\underset{\sim}{R})$ is convergent when $2 < Re(d) < 3$. Here d is replaced with $Re(d)$ since its imag. part corresponds to some oscillatory term. Let d be a complex variable although we only care about $d = 2, 3, 4 \cdot 2 \ldots$

Therefore we solve for $G(\underset{\sim}{R})$ b/w 2 & 3 & use analytic continuation for higher orders if possible. This is called dimensional regularization.

<u>Dimensional Regularization</u>.

Find $G(\underset{\sim}{R})$ as a fn of d in $d \in (2,3)$ & use analytic continuation for all other values.

Going back to $\nabla^2 G^{(d)}(R) = \delta^{(d)}(R)$ → translation invariant since sd. is spherically symmetric.

Integrating over a hypersphere

$$\int dV \, \nabla \cdot \nabla G^{(d)} = 1$$

Use Div Thm. " $\int_S \nabla G^{(d)} \cdot ds$ & ∇G we know is radially outward due to spherical symmetry. Therefore $\nabla G \cdot ds$ has no cosine term.

$$\Rightarrow \int_S \nabla G^{(d)} \cdot ds = \frac{dG^{(d)}}{dR} \int_S ds$$

↳ integral of hypersphere of radius R.

Since radial component of gradient is the only one that survives due to spherical symmetry of G

$$\frac{dG^{(d)}}{dR}\int_S dS = \frac{dG^{(d)}}{dR}\cdot\frac{2\pi^{\frac{d}{2}}R^{d-1}}{\Gamma(\frac{d}{2})}$$

⟶ surface area of sphere of d dimensions

$\Gamma(z+1) = z\,\Gamma(z)$

$\Gamma(n) = (n-1)!$

$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$ if $Re(z)>0$

$\Gamma(1) = \int_0^\infty e^{-x}dx = -e^{-x}\Big|_0^\infty = 1$ Therefore $0! = 1$

$\Gamma(\frac{1}{2}) = \sqrt{\pi} \Rightarrow \Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$

$d=3 \Rightarrow \dfrac{2\pi^{\frac{3}{2}}\cdot R^2}{\frac{1}{2}\sqrt{\pi}} = 4\pi R^2$

$d=2 \Rightarrow 2\pi R$

$$\Rightarrow \frac{dG^{(d)}}{dR} = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}R^{d-1}}$$

What is the volume of hypersphere

$$S_d(R) = \frac{2\pi^{\frac{d}{2}}R^{d-1}}{\Gamma(\frac{d}{2})}$$

$$V_d(R) = \frac{\pi^{\frac{d}{2}}R^d}{\Gamma(\frac{d}{2}+1)}$$

$$\frac{dV_d(R)}{dR} = \frac{d\,\pi^{\frac{d}{2}}R^{d-1}}{\Gamma(\frac{d}{2})\cdot\frac{d}{2}} = \frac{2\pi^{\frac{d}{2}}R^{d-1}}{\Gamma(\frac{d}{2})} = S_d(R)$$

$$\Rightarrow \int_R^\infty dR'\,\frac{dG^{(d)}(R')}{dR'} = -G^{(d)}(R)$$ since G is 0 at infinity.

$$\Rightarrow G^{(d)}(R) = -\int_R^\infty dR'\,R'^{1-d}\cdot\frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}}$$

$$\Rightarrow G^{(d)}(R) = \frac{-\Gamma(\frac{d}{2})}{2\pi^{d/2}}\,\frac{(R')^{2-d}}{2-d}\Bigg|_{R'=R}^{R'=\infty}$$

$d > 2 \Rightarrow \int$ vanishes at ∞.

$$\Rightarrow \boxed{G^{(d)}(R) = \frac{-\Gamma\left(\frac{d}{2}\right)}{2\pi^{\frac{d}{2}}(d-2)\,R^{d-2}}} \longrightarrow$$ Free space Green's function for Laplace's Equation in d dimensions.

It is an analytic function in d with $d=2$ pole $\rightarrow$ IR divergence. This is only valid when $2<d$. UV divergence disappeared? Yes.

$$d = 3 \Rightarrow G^{(3)}(R) = -\frac{1}{4\pi R}$$

We know $\dfrac{dG^{(2)}}{dR} = \dfrac{1}{2\pi R}$

$$\Rightarrow \boxed{G^{(2)} = \frac{\log_e R}{2\pi}}$$ potential of an ∞ line of charge.

$$G(R) = \int \frac{1}{S(R)}\, dR.$$

Applying the prescription of Regularization for $d=2$.
Expand $G^{(d)}(R)$ in terms of the singular part & a regular part.
Which is essentially a Taylor expansion of $\frac{1}{R^{d-2}}$ around $d=2$.

Let $R^{-(d-2)} = e^{-(d-2)\log R} \rightarrow$ expand as Taylor series.

$$\Rightarrow G^{(d)}(R) = \frac{-1}{2\pi(d-2)}\left[1 - (d-2)\log R + \underbrace{O\left((d-2)^2\right)}_{\rightarrow 0 \text{ as } d\rightarrow 2}\right]$$

$$= -\frac{1}{2\pi(d-2)} + \frac{\log R}{2\pi}.$$

Regularization says ignore singular part.

$$\Rightarrow \boxed{G^{(d)}(R) = \frac{\log R}{2\pi}}$$

The Diffusion Equation - Lec 25

- $\rho(\bar{r},t)$ is the conc. of solute in solvent. Follows certain laws.

① $\frac{\partial \rho}{\partial t} + \nabla \cdot \bar{J} = 0 \rightarrow$ Fick's I Law (Eqn. of continuity).

② $\bar{J}(\bar{r},t) = -D\nabla\rho(\bar{r},t) \rightarrow$

↳ Flow of solute current is from high conc. to low concentration.
$D \rightarrow$ diffusion constant. (l^2/t dimension)

$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} = D\nabla^2 \rho} \longrightarrow$ Diffusion Equation. (macroscopic perspective)

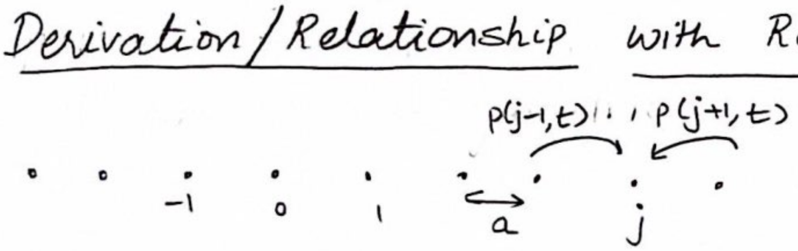
> Since it is first order in time, it is not time reversible. Note that M.E are also f.o in time but also f.o in space $\Rightarrow$ time reversible.

1D in time & 2D in space $\Rightarrow$ parabolic DE $\Rightarrow$ not time reversible.

- 2D in time & 2D in space $\Rightarrow$ elliptic DE $\Rightarrow$ time reversible

ρ may be temperature, D the diffusivity $\Rightarrow \frac{\partial u}{\partial t} = \alpha \nabla^2 u$ (temp; $\alpha \rightarrow$ diffusivity.)

Derivation / Relationship with Random Walks



$$\frac{dP(j,t)}{dt} = \frac{\lambda}{2}\left[P(j+1,t) + P(j-1,t) - 2P(j,t)\right]$$

jumping out to left or right.

50% prob from left or right.

$\lambda \rightarrow$ rate at which jumps occur

When $a \rightarrow 0$ & $\lambda \rightarrow \infty$ $P(j,t)$ goes from probability to probability density.

-

$$\Rightarrow \frac{dP(ja,t)}{dt} = \frac{\lambda}{2}\left[P(ja+a,t) + P(ja-a,t) - 2P(ja,t)\right]$$

$$= \frac{a^2\lambda}{2}\left[\frac{P(ja+a,t) - P(ja,t)}{a} - \left(\frac{P(ja,t) - P(ja-a,t)}{a}\right)\right]$$

$$= \frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2} \quad \text{where, } D \equiv \lim_{\substack{\lambda\to\infty \\ a\to 0}}\left(\frac{1}{2}\lambda a^2\right)$$

In 3D $\frac{\partial P(\bar{r},t)}{\partial t} = D\nabla^2 P(\bar{r},t)$ → microscopic perspective.

B.C ⇒ $p(\bar{r},t) \to 0$ as $\vec{r} \to \infty$.

I C ⇒ $p(\bar{r},0) = \delta^{(d)}(\bar{r})$ in d dimensions → gives the Green's function.

> Since t goes from 0 to ∞ we use Laplace transform in t.

$$\mathcal{L}[p(\bar{r},t)] = \tilde{p}(\vec{r},s)$$

$$\Rightarrow s\tilde{p}(\bar{r},s) - p(\bar{r},0) = D\nabla^2\tilde{p}(\bar{r},s)$$

$$\Rightarrow (s - D\nabla^2)\tilde{p}(\bar{r},s) = \delta^{(d)}(\bar{r}).$$

in d dimensionsions.

> Since $\bar{r}$ goes from $-\infty$ to ∞ we use Fourier transform in r.

$$\Rightarrow \tilde{p}(\bar{r},s) = \frac{1}{(2\pi)^d}\int d^dk\, \tilde{f}(\bar{k},s)\, e^{i\bar{k}\cdot\bar{r}}$$

$$\tilde{f}(\bar{k},s) = \int d^dr\, \tilde{p}(\bar{r},s)\, e^{-i\bar{k}\cdot\bar{r}}$$

$$\Rightarrow (s + Dk^2)\tilde{f}(\bar{k},s) = 1$$

$$\Rightarrow \tilde{f}(\bar{k},s) = \frac{1}{s + Dk^2}$$

> Inverting F.T would be painful.

> I$\mathcal{L}$.T is much easier first

$$\Rightarrow f(\bar{k},t) = \mathcal{L}^{-1}\frac{1}{s + Dk^2} = e^{-Dk^2t}$$

$$p(r,t) = \frac{1}{(2\pi)^d}\int d^dk \, e^{i\vec{k}\cdot\vec{r}} e^{-Dk^2t}$$

$$k^2 = k_1^2 + k_2^2 + \dots + k_d^2$$

It is much easier in Cartesian.

$$\Rightarrow p(r,t) = \frac{1}{(2\pi)^d}\int_{-\infty}^{\infty} dk_1 \, e^{ik_1x_1 - Dk_1^2t} \cdot \int_{-\infty}^{\infty} dk_2 \, e^{ik_2x_2 - Dk_2^2t} \cdots \int_{-\infty}^{\infty} dk_d \, e^{ik_dx_d - Dk_d^2t}$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_1 \, e^{-Dt\left(k_1^2 - i\frac{k_1}{Dt}x_1 + \frac{i^2x_1^2}{4(Dt)^2} - \frac{i^2x_1^2}{4(Dt)^2}\right)}$$

$$\Rightarrow \frac{e^{-\frac{x_1^2}{4Dt}}}{2\pi}\int_{-\infty}^{\infty} dk_1 \, e^{-Dt\left(k_1 - \frac{ix_1}{2Dt}\right)^2}$$ & change variables

$$= \frac{e^{-\frac{x_1^2}{4Dt}}}{2\pi} \cdot \frac{\sqrt{\pi}}{\sqrt{Dt}}$$

$$= \frac{e^{-\frac{x_1^2}{4Dt}}}{\sqrt{4\pi Dt}}$$

$$\Rightarrow \boxed{p(r,t) = \frac{e^{-\frac{r^2}{4Dt}}}{(4\pi Dt)^{d/2}}}$$

This is in fact the Green's fn of the diffusion equation.

Gaussian in r & the variance increases linearly in time. Wow

Recall Gaussian $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$$\boxed{\langle r(t)^2\rangle = 2dDt}$$

→ mean square displacement $\propto t$.

Zero mean Gaussian $= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x^2}{2\sigma^2}}$

$\Rightarrow \sigma^2 = 2Dt$ in 2 dimensions.

In general $\sigma^2 = 2dDt$

Green's function for diffusion equation.

$$\left(\frac{\partial}{\partial t} - D\nabla^2\right) G(\bar{r},t) = \delta^{(d)}(\vec{r})\,\delta(t)$$ → But this is what we just solved.

Suppose $P_{init}(\vec{r},0)$ given. Integrate $G(\bar{r},t)$ to get $P(\bar{r},t)$

$$\Rightarrow \boxed{p(\vec{r},t) = \frac{1}{(4\pi Dt)^{d/2}} \int_{r'} e^{-\frac{(\bar{r}-\bar{r}')^2}{4Dt}} P_{init}(\vec{r},0)\, d^d r}$$

In general given $P_{init}(\bar{r},t_0)$ if $t > t_0$.

$$\Rightarrow \boxed{p(\bar{r},t) = \frac{1}{(4\pi D(t-t_0))^{d/2}} \int_{r'} e^{-\frac{(\bar{r}-\bar{r}')^2}{4Dt}} P_{init}(\vec{r},t_0)\, d^d r'}$$

> Say we are in 1D & want a finite B.C. ⇒ particles reflect off the boundary ⇒ current at boundary $=0$ like open circuit. ⇒ $\frac{\partial P}{\partial x} = 0$ or in 3D $\nabla P = 0$ at boundary.

> If particle disappears at the ends ⇒ $P=0$ at the boundary like S-C.

If P at $x_{boundary}$ is 0 & $\frac{\partial P}{\partial x}$ is 0 at boundary is P identically zero at the boundary? I think yes although just inside the boundary it is not 0. It could be a discontinuous jump maybe?

Lec 26 – Diffusion Equation contd.

$$\frac{\partial p(\bar{r},t)}{\partial t} = D\nabla^2 p(\bar{r},t)$$

$$p(\bar{r},t) = \frac{1}{(4\pi Dt)^{d/2}} e^{\frac{-r^2}{4Dt}}$$

Given $P_{init}(\bar{r},t')$

$$P(\bar{r},t) = \frac{1}{[4\pi D(t-t')]^{d/2}} \int d^d r' e^{\frac{-(\bar{r}-\bar{r}')^2}{4D(t-t')}} P_{init}(\bar{r}',t')$$

For a free particle the Schrodinger Eqn is the same.

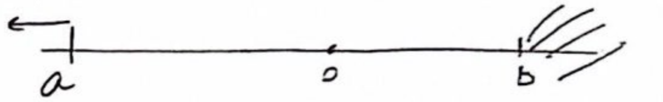
$$\frac{\partial \psi(\bar{r},t)}{\partial t} = \frac{i\hbar}{2m}\nabla^2\psi(\bar{r},t) \qquad \text{"}D\text{"} = \frac{i\hbar}{2m}$$

$$\Rightarrow \psi(\bar{r},t) = \int d^3r' \underbrace{\frac{e^{\frac{im(\bar{r}-\bar{r}')^2}{2\hbar(t-t')}}}{\left[2\pi\frac{i\hbar}{m}(t-t')\right]^{3/2}}}_{\text{Propagator } K(\bar{r},t;\bar{r}',t')} \psi(\bar{r}',t')$$

Applying finite Boundaries

1-D

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2}$$



absorbing BC $P|_{x=b} = 0$

No flux { Reflecting B.C $\Rightarrow \frac{\partial p}{\partial x}\Big|_{x=b} = 0$

Semi infinite region $(-\infty \le x \le b)$

Use method of images & uniqueness.

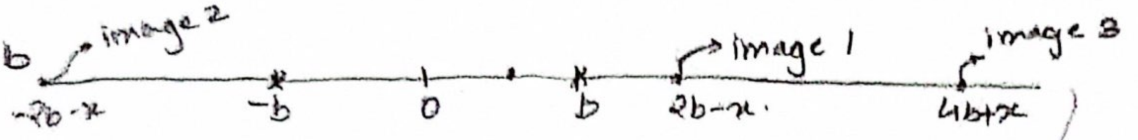
RBC at $x=b \Rightarrow p(\bar{x},t) = \frac{1}{\sqrt{4\pi Dt}}\left[e^{\frac{-x^2}{4Dt}} + e^{-(2b-x)^2/4Dt}\right]$

↳ satisfies the B-

- Absorbing BC at b

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}}\left[e^{-\frac{x^2}{4Dt}} - e^{-(2b-x)^2/4Dt}\right]$$ $x=b \Rightarrow$ annihilation.

- RBC at $x = \pm b$



We have ∞ no. of images. $\Rightarrow \int_{-b}^{b} dx\, p(x,t) = 1$ for all $t \geq 0$

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}} \sum_{n=-\infty}^{\infty} e^{-(x+2nb)^2/4Dt}$$ for RBC

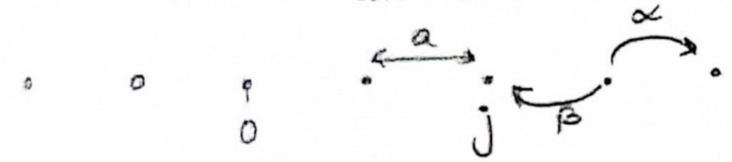
∞ no. of images

$$p(x,t) = \frac{1}{4\pi Dt} \sum_{n=-\infty}^{\infty} (-1)^n e^{-(x+2nb)^2/4Dt}$$ for ABC $\rightarrow \int_{-b}^{b} dx\, p(x,t) = S(t)$

Survival probability
$t=0 \Rightarrow S(t)=1$
$t=\infty \Rightarrow S(t)=0$

Diffusion with drift $\Rightarrow$ (diffusion under a constant force)

Random walker with bias



$\tau \rightarrow$ timestep.

$\alpha \rightarrow$ prob of right jump
$\beta \rightarrow$ prob of left jump
$\alpha+\beta=1$

$$P(ja, n\tau) = \alpha\, P(ja-a, (n-1)\tau) + \beta\, P(ja+a, n\tau-\tau)$$

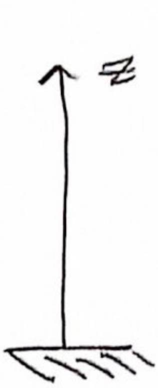
$$P(ja,n\tau) - P(ja, n\tau-\tau) = \alpha P(ja-a,(n-1)\tau) + \beta P(ja+a, n\tau-\tau) - P(ja, n\tau-\tau)$$

divide by τ & apply $a\to 0$; $\tau \to 0$ & $\alpha-\beta \to 0$ (bias $\to 0$).

$$\Rightarrow \frac{a^2\alpha}{\tau} \to D \quad \& \quad \frac{a(\alpha-\beta)}{\tau} \to c \qquad ja \to x,\ n\tau \to t$$

$$\boxed{\Rightarrow \frac{\partial p(x,t)}{\partial t} = -\underbrace{c}_{\text{avg. drift velocity}} \frac{\partial P(x,t)}{\partial x} + \underbrace{D}_{\text{avg diffusion constant}} \frac{\partial^2 P(x,t)}{\partial x^2}}$$

Smoluchowski equation



Sedimentation (Diffusion with drift)

- $$\frac{\partial p(z,t)}{\partial t} = -c\frac{\partial p}{\partial z} + D\frac{\partial^2 p}{\partial z^2}$$

Corresponding continuity equation.

$$\frac{\partial p}{\partial t} + \frac{\partial j}{\partial z} = 0 \;;\text{ where } j(z,t) = c\,p(z,t) - D\frac{\partial p(z,t)}{\partial z}$$

Drift velocity $c = -v$

B.C at $z=0$ is $j(0,t) = 0 \Rightarrow v\,p(z,t) - D\left.\frac{\partial p(z,t)}{\partial z}\right|_{z=0} = 0$

In thermal equilibrium $p(z,t) \longrightarrow p(z) \Rightarrow \frac{\partial}{\partial t}p(z,t) = 0.$

$\Rightarrow \frac{\partial j}{\partial z} = 0 \Rightarrow$ j is independant of $z \Rightarrow$ current is same everywhere. but at $z=0$ it is 0 so it must be z everywhere.

- $$D\frac{dp^{eq}}{dz} + v\,p^{eq} = 0 \Rightarrow p^{eq} \propto e^{-\frac{v}{D}z}$$

$\Rightarrow p^{eq}(z) \propto e^{-\frac{mgz}{K_BT}} \longrightarrow$ Barometric distribution of air.

$\Rightarrow \frac{v}{D} = \frac{mg}{K_BT} \Rightarrow D = \frac{v K_B T}{mg}$

Using Stoke's relation $6\pi R\eta v = mg$. where $\eta \rightarrow$ viscocity
$R \rightarrow$ radius
$v \rightarrow$ terminal vel.
for spherical particles. with stick bound. cond.

$\Rightarrow D = \frac{K_BT}{6\pi R\eta} \longrightarrow$ derived by Einstein & earlier by Sutherland. & Nernst.

Helm holtz Equation - Lec 29

$$(\nabla^2 + k^2) f = g$$

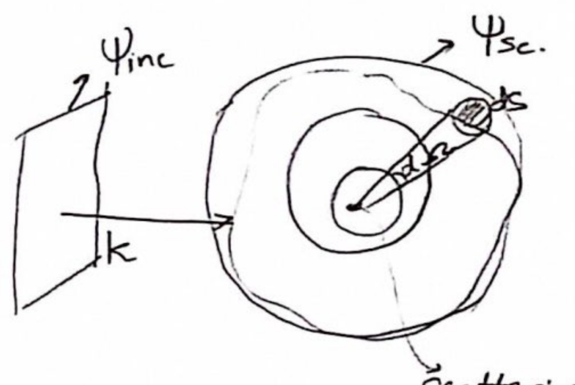
- If $\bar{r} \in \mathbb{R}$ then the homogeneous HH equation $(\nabla^2 + k^2) u = 0$ gives the normal modes of vibration for given B.C (say $u = 0$ on the boundary of $R \longrightarrow$ Dirichlet boundary condition).
- Say u is the z displacement of points on a drum, & $u=0 \Rightarrow$ clamping the boundary of the drum. Normal modes are the possible values of $k^2 \longrightarrow$ eigen value problem. $\nabla^2 u = -k^2 u$.

Green's Function (free space) for HH Eqn

The physical problem is of non relativistic (low speed) scattering of a quantum mechanical particle from a fixed static potential.

not a fn of ϕ θ.

Assume there is a scattering face center with some central potential around it. The incoming particle must be described by a wave function with some incident wave vector. Let incident ψ be a plane wave state.



$$\Rightarrow \psi_{inc}(\bar{r}) = e^{i\bar{k}\cdot\bar{r}}$$

Outgoing $\psi_{sc}(\bar{r})$ are spherical waves.

$$\psi(\bar{r}) = \psi_{inc}(\bar{r}) + \psi_{sc}(\bar{r})$$

We want to know what is the flux at some solid angle $d\Omega$.

$$\psi_{sc}(\bar{r}) \xrightarrow[\bar{r}\to\infty]{} \frac{e^{ikr}}{r} f(k,\theta,\phi)$$

$$E = \frac{\hbar^2 k^2}{2m}$$

since $\bar{k} \parallel \bar{r}$ as $\bar{r} \to \infty$ for coherent waves.

If $\bar{k}$ is $= +\hat{z} \Rightarrow \psi_{sc}$ has no dependance on ϕ. (Symmetry).

$r \to \infty \Rightarrow kr \gg 1$

$$\Rightarrow \psi(\bar{r}) = e^{i\bar{k}\cdot\bar{r}} + \underset{\substack{\uparrow \\ \text{scattering amplitude.}}}{f(k,\theta)} \frac{e^{ikr}}{r} \longrightarrow \text{Soln. of Sch.E we are looking for}$$

$$\frac{\text{Flux through } ds}{\text{incident flux}} = d\sigma \rightarrow \text{differential cross section.}$$

$$\bar{j}_{inc} = \frac{\hbar}{2mi}\left(\psi^*_{inc}\nabla\psi_{inc} - \psi_{inc}\nabla\psi^*_{inc}\right) \rightarrow \text{std. formula in QM.}$$

S.E (time dep version)

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{(i\hbar)^2}{2m}\nabla^2\psi + V(r)\psi \quad \times\psi^*$$

$$\Rightarrow i\hbar\frac{\partial\psi^*}{\partial t} = \frac{(-i\hbar)^2}{2m}\nabla^2\psi^* + V(r)\psi \quad \times\psi$$

& Subtract

$$\Rightarrow |\psi|^2 = \frac{(i\hbar)}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) = -\nabla\cdot j \Rightarrow j \text{ is as above.}$$

prob den.

$$\Rightarrow \bar{j}_{inc} = \frac{\hbar\bar{k}}{m} \quad \text{since } \psi_{inc} = e^{i\bar{k}\cdot\bar{r}} \quad \& \quad \nabla\psi_{inc} = i\bar{k}e^{i\bar{k}\cdot\bar{r}}$$

$$\nabla\psi^*_{inc} = -ik\,e^{i\bar{k}\cdot\bar{r}}$$

incident flux

momentum $\bar{p}$

Scattered flux (only looking at radial component).

$$\bar{j}_{sc} = \frac{\hbar}{2mi}\left(\psi^*_{sc}\frac{\partial\psi_{sc}}{\partial r} - \psi_{sc}\frac{\partial\psi^*_{sc}}{\partial r}\right) \qquad \psi_{sc} = f(k,\theta)\frac{e^{ikr}}{r}$$

$$\Rightarrow \bar{j}_{sc} = \frac{\hbar}{2mi}|f(k,\theta)|^2\cdot\left[\frac{e^{-ikr}}{r}\left\{ik\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^2}\right\} - \frac{e^{ikr}}{r}\left\{-ik\frac{e^{-ikr}}{r} - \frac{e^{-ikr}}{r^2}\right\}\right]$$

$$\bar{j}_{sc} = \frac{\hbar^2}{2mi} |f|^2 \left[\frac{2ik}{r^2}\right]$$

$$\bar{j}_{sc} = \frac{\hbar k}{m} \frac{|f|^2}{r^2}$$

$$\bar{j}_{sc} \cdot dS = \frac{\hbar k}{m} |f|^2 d\Omega$$

$$\Rightarrow d\sigma = \frac{\frac{\hbar k}{m} |f|^2 d\Omega}{\frac{\hbar k}{m}} = |f|^2 d\Omega$$

Therefore it does not depend on amplitude of incident wave & ∴ we could just assume that it is 1 which is what we did here.

∴ Differential cross section $\boxed{\frac{d\sigma}{d\Omega} = |f|^2}$

Total cross section (notice it has dimension of area) $\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = 2\pi \int_{-1}^{1} |f|^2 \, d(\cos\theta)$ → find this.

↳ Same as $\sin\theta \, d\theta$ from $0 \to \pi$

SE

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(\bar{r}) + \lambda V(\bar{r}) \psi(\bar{r}) = E \psi(\bar{r})$$

$E = \frac{\hbar^2 k^2}{2m}$

↳ λ coupling constant

$$\nabla^2 \psi(\bar{r}) + \frac{2mE}{\hbar^2} \psi(\bar{r}) = \lambda \left(\frac{2m}{\hbar^2}\right) V(r) \psi(\bar{r})$$

$$(\nabla^2 + k^2) \psi(\bar{r}) = \lambda U(r) \psi(\bar{r})$$ → Looks like Helmholtz. but RHS is unknown.

Say we know RHS.

$$\Rightarrow \psi(\bar{r}) = \underbrace{e^{i\bar{k}\cdot\bar{r}}}_{C.F} + \underbrace{\lambda \int d^3r' \, G(\bar{r}, \bar{r}') \, U(\bar{r}') \, \psi(\bar{r}')}_{P.I}$$

$\rightarrow$ We are able to write down the C.F because we know that we are looking for solns. of the form $e^{i\bar{k}\cdot\bar{r}} + f(k,\theta)\frac{e^{ikr}}{r}$.

$\rightarrow$ So now we find $G(\bar{r},\bar{r}')$. This does not solve for $\psi(\bar{r})$ but gives us an integral equation for $\psi(\bar{r})$.

Finding the Green's Function.

$$\underbrace{(\nabla^2+k^2)}_{\text{Translation inv}} G(\bar{r},\bar{r}') = \delta^{(3)}(\bar{r}-\bar{r}') \quad \text{with } G(\bar{r},\bar{r}') \to 0 \text{ as } r \to \infty$$

$\Rightarrow G(\bar{r},\bar{r}') \rightarrow G(\bar{r}-\bar{r}')$.

$\Rightarrow (\nabla^2+k^2)\, G(\bar{R}) = \delta^{(3)}(\bar{R})$.

Use Fourier Transform $\rightarrow$ it converts ∇^2 to $-q^2$ & since $e^{i\bar{q}\cdot\bar{R}}$ forms a complete set we can equate the coefficients.

$$\Rightarrow G(\bar{R}) = \frac{1}{(2\pi)^3}\int d^3q\, e^{i\bar{q}\cdot\bar{R}}\, \tilde{G}(\bar{q}).$$

$$\delta^{(3)}(R) = \frac{1}{(2\pi)^3}\int d^3q\, e^{i\bar{q}\cdot\bar{R}}$$

$$\Rightarrow (-q^2+k^2)\,\tilde{G}(\bar{q}) = 1$$

$$\Rightarrow \tilde{G}(\bar{q}) = \frac{-1}{q^2-k^2}$$

$$\Rightarrow G(\bar{R}) = \frac{-1}{(2\pi)^3}\int d^3q\, \frac{e^{i\bar{q}\cdot\bar{R}}}{q^2-k^2}$$

Choose polar axis in $\bar{q}$ space along $\bar{R}$

$$= -\frac{1}{4\pi^2}\int_0^\infty \frac{q^2\,dq}{q^2-k^2}\, 2i\,\frac{\sin qR}{iqR} \quad = \frac{-1}{2\pi^2}\int_0^\infty dq\, \frac{q\sin qR}{q^2-k^2}.$$

We need to shift the poles off the real axis by maintaining the boundary conditions.

$\hookrightarrow$ poles at $+k$ & $-k$

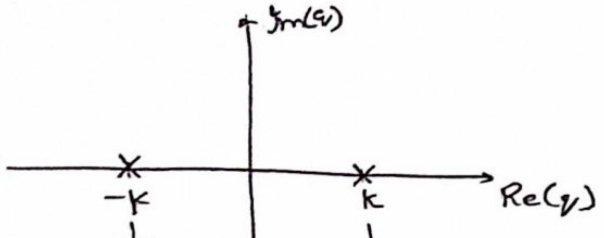
$$\Rightarrow G(R) = \frac{-1}{2\pi^2 R}\int_0^{\infty} dq\, \frac{q \sin qR}{q^2 - K^2}$$

no longer a fn of $\bar{R}$

$\rightarrow$ even fn in q $\Rightarrow$ Change limits to $-\infty$ to ∞

The solution from here on is complex analysis magic...

We are looking for solns. of the form e^{ikR}.



Rewriting $\sin q \rightarrow \frac{e^{iqR} - e^{-iqR}}{2i}$.

$$G(R) = \frac{-1}{8\pi^2 R i}\int_{-\infty}^{\infty} dq\, \frac{q\,(e^{iqR} - e^{-iqR})}{q^2 - K^2}$$

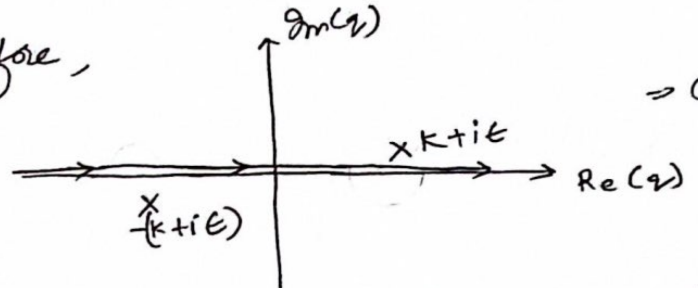
Contributes to e^{-iqR} to give e^{ikR}

Contributes to e^{iqR} to give e^{ikR}.

need to close contour in UHP due to $+i$

need to close in LHP

Therefore,



$$\Rightarrow G(R) = \frac{-1}{8\pi^2 R i}\lim_{\epsilon \to 0}\int_{-\infty}^{\infty} dq\, \frac{q\,(e^{iqR} - e^{-iqR})}{q^2 - (k+i\epsilon)^2}$$

There are 4 choices on how to move these poles but only the one chosen here gives e^{ikR} $\Rightarrow$ outgoing waves. Using Residue Thm after closing contours over UHP & LHP for the 2 terms.

$$\Rightarrow G(R) = \frac{-1}{8\pi^2 R i}\left[\frac{2\pi i\, k e^{ikR}}{2k} - \frac{(-2\pi i)(-k)e^{ikR}}{-2k}\right]$$

$$\boxed{G(R) = \frac{-e^{ikR}}{4\pi R}}$$

$$\Rightarrow G(R) = \frac{-1}{4\pi}\frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}$$

$\rightarrow$ in 530 we got $\frac{1}{4\pi R}e^{ikR}$ since RHS of HH was chosen as $-\delta(\bar{r}-\bar{r}')$

$$\therefore\ \psi(\bar r) = \underbrace{e^{i\bar k\cdot\bar r}}_{\text{incident wave}} - \frac{\lambda}{4\pi}\int d^3r'\, \frac{e^{ik|\bar r-\bar r'|}}{|\bar r-\bar r'|}\, U(r')\psi(\bar r')$$

$U(r') \longrightarrow \frac{2m}{\hbar^2}V(r')$ (Central potential)

Integral equation for scattering. (inhomogeneous)

Lec 31

The above equation can be written as,

$$f(x) = \underbrace{g(x)}_{\text{given fn}} + \int_a^b \underbrace{K(x,y)}_{\text{Kernal fn.}} f(y)\, dy \rightarrow \text{Fredholm equation of 2}^{\text{nd}}\text{ kind (Inhomogeneous)}$$

> If $f(x)=0$ ⇒ we get Fredholm eqn. of first kind.

> If $g(x)=0$ ⇒ we get an eigenvalue equation (Homogeneous Fredholm eqn)

⇒ In abstract notation $|f\rangle = \lambda K|f\rangle$ where K is the integral operator.

$|\bar r-\bar r'| \xrightarrow[r\to\infty]{} r$ in amplitude factor.

$$|\bar r-\bar r'| = (r^2+r'^2-2\bar r\cdot\bar r')^{1/2} = r\left(1+\cancel{\frac{r'^2}{r^2}}^{\,0}-\frac{2\bar r\cdot\bar r'}{r^2}\right)^{1/2}$$ assuming $r \gg r'$

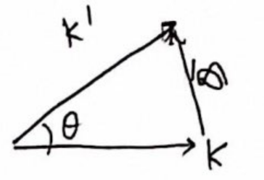
binomial expansion $(1-x)^{+1/2} = 1-\frac{x}{2}$

$$\simeq r\left(1-\frac{\bar r\cdot\bar r'}{r^2}\right) \longrightarrow \text{Same as antenna far field approximation.}$$

$$\Rightarrow k|\bar r-\bar r'| = kr - k\,\hat r\cdot\bar r'$$

$k\hat r = \bar k'$ → wave vector of scattered wave along observation direction.

$$\Rightarrow \psi(\bar r) = e^{i\bar k\cdot\bar r} + \frac{e^{ikr}}{r}\left\{\underbrace{-\frac{\lambda}{4\pi}\int d^3r'\, e^{-i\bar k'\cdot\bar r'}\, U(r')\,\psi(r')}_{f(k,\theta)\,\longrightarrow\,\text{scattering amplitude}}\right\}$$



$|\bar k| = |\bar k'|$, $\bar k' - \bar k$ = momentum transfer; $\bar Q$

$$f(k,\theta) = \frac{-\lambda}{4\pi}\int d^3r'\, e^{-i\bar{k}'\cdot\bar{r}'}\, U(\bar{r}')\,\psi(\bar{r}') \longrightarrow \text{exact not approximate.}$$

$$q^2 = k^2 + k'^2 - 2kk'\cos\theta$$

$$\boxed{q^2 = 2k^2(1-\cos\theta)} \qquad \frac{2mE}{\hbar^2} = k^2 \Rightarrow \boxed{q = 2k\sin\frac{\theta}{2}}$$

Going back to Fredholm

$$|f\rangle = |g\rangle + \lambda K|f\rangle$$

$$\Rightarrow |f\rangle = (1-\lambda K)^{-1}|g\rangle$$

Using binomial $\Rightarrow \underbrace{(\mathbb{1} + \lambda K + \lambda^2 K^2 + \lambda^3 K^3 + \dots)}_{\text{Neumann series.}}|g\rangle$ this is convergent when norm of $\lambda K < 1$

$$\Rightarrow |\lambda|\ \|K\| < 1$$

Norm of $\|K\|^2 \equiv \underset{|\psi\rangle\in V}{\lim\sup} \frac{\langle K\psi|K\psi\rangle}{\langle\psi|\psi\rangle}$ least upper bound.

$$\langle x|K^2|f\rangle \longrightarrow \int dz\, K(x,z)\int dy\, K(z,y)\, f(y)$$

$\therefore$ <u>Formal solution.</u>

$$\therefore\ \psi(\bar{r}) = e^{i\bar{k}\cdot\bar{r}} - \frac{\lambda}{4\pi}\int d^3r' \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}U(\bar{r}')\, e^{i\bar{k}\cdot\bar{r}'}$$

$$+\left(\frac{-\lambda}{4\pi}\right)^2\int d^3r'\int d^3r'' \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}\frac{e^{ik|\bar{r}'-\bar{r}''|}}{|\bar{r}'-\bar{r}''|}U(r')U(r'')\, e^{i\bar{k}\cdot\bar{r}''} + \dots$$

Born series

Converges when $|\lambda| < \frac{1}{\|K\|}$

1st Born approximation $\Rightarrow$ retaining first 2 terms.

$$f(k,\theta) = \frac{-\lambda}{4\pi}\int d^3r'\, e^{-i\bar{k}'\cdot\bar{r}'}\, U(r')\,\psi(\bar{r}')$$

Using Born approximation. (1st)

$$f_B(k,\theta) = -\frac{\lambda}{4\pi}\int d^3r'\, e^{-i\bar{k}'\cdot\bar{r}'}\, U(r')e^{i\bar{k}\cdot\bar{r}'}$$

$$= \frac{-\lambda}{4\pi}\int d^3r'\, e^{-i\bar{Q}\cdot\bar{r}'}\, U(r') \qquad \hbar\bar{Q} = \hbar(\bar{k}'-\bar{k})$$

↳ Looks like Fourier Transform of $U(\bar{r}')$

What is $\langle \bar{p}' | V | \bar{p}\rangle$? Inserting a couple of complete set of states.

$$\langle \bar{p}' | V | \bar{p}\rangle = \langle \bar{p}' | \underbrace{\int d^3r_1\, |\bar{r}_1\rangle\langle \bar{r}_1|}_{\mathbb{1}}\, V \underbrace{\int d^3r_2\, |\bar{r}_2\rangle\langle \bar{r}_2|}_{\mathbb{1}} \bar{p}\rangle$$

$$= \int d^3r_1 \int d^3r_2\, \underbrace{\langle \bar{p}' | \bar{r}_1\rangle}_{e^{-i\bar{k}'\cdot\bar{r}_1}} \underbrace{\langle \bar{r}_1 | V | \bar{r}_2\rangle}_{V(r_2)\,\delta^{(3)}(\bar{r}_1-\bar{r}_2)} \underbrace{\langle \bar{r}_2 | \bar{p}\rangle}_{e^{i\frac{\bar{p}\cdot\bar{r}_2}{\hbar}} = e^{i\bar{k}\cdot\bar{r}_2}}$$

$$\Rightarrow \int d^3r_1\, e^{-i\bar{Q}\cdot\bar{r}_1}\, V(r_1)$$

$$\Rightarrow f_B(k,\theta) = \frac{-\lambda}{4\pi}\frac{2m}{\hbar^2}\langle \bar{p}' | V(r) | \bar{p}\rangle$$

Lets choose $\lambda V(r) = \dfrac{\lambda e^{-r/\xi}}{r}$ where $\xi > 0$

Called Yukawa potential

Let's first try to simplify f_B.

$$f_B(k,\theta) = \frac{-m\lambda}{2\pi\hbar^2}\int d^3r\; e^{-i\bar{Q}\cdot\bar{r}}\, V(r)$$

Choose polar along $\bar{Q}$

$$= \frac{-m\lambda}{\hbar^2}\int_0^\infty dr\, r^2 \int_{-1}^{1} e^{-iQr\cos\theta}\, d\cos\theta \cdot V(r)$$

$$= \frac{-m\lambda}{\hbar^2}\int_0^\infty dr\, r^2\, \frac{\cancel{+2i}\sin Qr}{\cancel{+i}Qr}\, V(r)$$

$$= \frac{-2m\lambda}{\hbar^2 Q}\int_0^\infty dr\; r\, V(r)\sin Qr$$

$$\lambda V(r) = \frac{\lambda e^{-r/\xi}}{r}$$

$$\Rightarrow f_B(k,\theta) = -\frac{2m\lambda}{\hbar^2 Q}\int_0^\infty dr \sin(Qr)\, e^{-r/\xi}$$

$$= -\frac{2m\lambda}{\hbar^2 \cancel{Q}}\cdot\frac{\cancel{Q}}{Q^2 + 1/\xi^2} = \frac{-2m\lambda}{\hbar^2}\cdot\frac{1}{Q^2 + 1/\xi^2}$$

$$\int_0^\infty dx\, e^{-ax}\begin{cases}\sin bx \longrightarrow \frac{b}{a^2+b^2}\\ \cos bx \longrightarrow \frac{a}{a^2+b^2}\end{cases}$$

$$\operatorname{Im}\int_0^\infty dx\, e^{-ax} e^{ibx}$$

$$= \operatorname{Im}\int_0^\infty dx\cdot e^{-(a-ib)x}$$

$$= \operatorname{Im}\left(\left.\frac{-e^{-(a-ib)x}}{a-ib}\right|_0^\infty\right) = \operatorname{Im}\left(\frac{1}{a-ib}\right)$$

$$= \frac{b}{a^2+b^2}$$

$$\boxed{f_B(k,\theta) = -\frac{2m\lambda}{\hbar^2} \cdot \frac{1}{\left(2k^2(1-\cos\theta) + \frac{1}{\xi^2}\right)}}$$

Calculate $\frac{d\sigma}{d\Omega}$ & σ

$\xi \to \infty$ gives the Coulomb potential solution. This cannot be directly derived since $\int_0^\infty \sin x\, dx$ is not solvable (oscillatory)

Could solve as $\lim_{a\to\infty} \int_0^\infty \sin x\, e^{-ax} dx \to \lim_{a\to\infty} \operatorname{Im} \int_0^\infty e^{x(-a+i)} dx$..

The Wave Equation.

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) f(\bar{r},t) = g(\bar{r},t)$$

Green's function is chosen s.t there is no f before g turns on.
AKA principle of causality.

$$f(\bar{r},t) = \int d^d r' \int dt' \underbrace{G(\bar{r},t;\bar{r}',t')}_{\text{Green Function.}} g(\bar{r}',t')$$

$$G(\bar{r},t,\bar{r}',t') \rightarrow \underset{\text{(step fn)}}{\theta(t-t')}\, K(\bar{r}-\bar{r}', t-t')$$

↳ ∴ invariance

$$\Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(\bar{r},t,\bar{r}',t') = \delta^{(d)}(\bar{r}-\bar{r}')\,\delta(t-t')$$

Also $G \longrightarrow 0$ as $|\bar{r}-\bar{r}'| \longrightarrow \infty$

Also $G = 0$ for $t < t'$. (Causality)

Let $\bar{R} = \bar{r}-\bar{r}'$, $\tau = t-t'$

$$\Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial \tau^2} - \nabla_R^2\right) G(\bar{R},\tau) = \delta^{(d)}(\bar{R})\,\delta(\tau)$$

> F.T w.r.t space & time.

deliberated inverted in time since it looks like a wave.

$$G(\bar{R},\tau) = \frac{1}{(2\pi)^d}\int d^d k \frac{1}{2\pi}\int d\omega\, e^{i(\bar{k}\cdot\bar{R} - \omega\tau)}\, \tilde{G}(\bar{k},\omega)$$

$$\delta^{(d)}(\bar{R})\,\delta(\tau) = \frac{1}{(2\pi)^d}\int d^d k \frac{1}{2\pi}\int d\omega\, e^{i(\bar{k}\cdot\bar{R} - \omega\tau)}$$

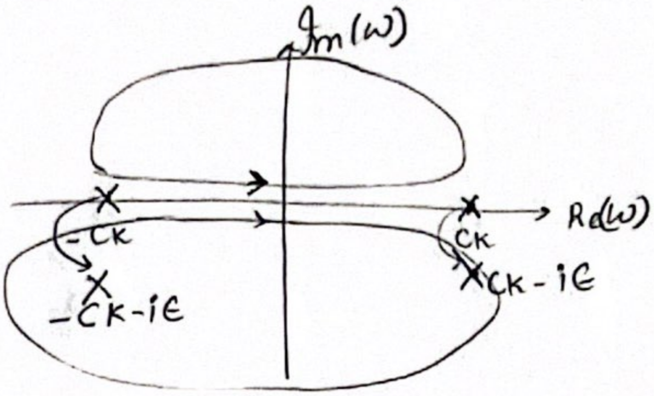
Subs. in wave eqn & equate coefficients. of $e^{i(\bar{k}\cdot\bar{R} - \omega\tau)}$

$$\left(\frac{-\omega^2}{c^2}+k^2\right)\tilde{G}(\bar{k},\omega)=1$$

$$\Rightarrow \tilde{G}(\bar{k},\omega)=\frac{-c^2}{\omega^2-k^2c^2}$$

$$\Rightarrow G(\bar{R},\tau)=\frac{-c^2}{(2\pi)^d}\int d^dk\, e^{i\bar{k}\cdot\bar{R}}\,\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\,\frac{e^{-i\omega\tau}}{\omega^2-c^2k^2}$$

Poles on real axis ⇒ use $i\epsilon$ prescription to satisfy the causal B.C.



$\tau<0$ we want $G\to 0$ ⇒ need to close in UHP since $e^{-i\omega\tau}\to e^{i\omega\tau}$

But we want no poles in UHP ⇒ move both poles down by ϵ ⇒ $-ck-i\epsilon$ & $ck-i\epsilon$.

$$\Rightarrow \int_{-\infty}^{\infty}d\omega\,\frac{e^{-i\omega\tau}}{(\omega+i\epsilon)^2-c^2k^2}$$

⇒ poles now at $ck-i\epsilon$ & $-ck-i\epsilon$.

$$\Rightarrow G(\bar{R},\tau)=\frac{-c^2\theta(\tau)}{(2\pi)^d}\int d^dk\, e^{i\bar{k}\cdot\bar{R}}\cdot\frac{2\pi i}{2\pi}\left[\frac{e^{-ick\tau}}{2ck}+\frac{e^{ick\tau}}{-2ck}\right]$$

$$=\frac{c\,\underbrace{\theta(\tau)}_{\text{for causality}}}{(2\pi)^d}\int d^dk\, e^{i\bar{k}\cdot\bar{R}}\,\frac{\sin ck\tau}{k}.$$

In reality $t\geq t'$ is not enough, we need $t\geq t'+\frac{|\bar{r}-\bar{r}'|}{c}$ → Can we get this?

1 dimension.

$k = |k|$

$X = x - x'$

$$G^{(1)}(X, \tau) = \frac{c\,\theta(\tau)}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikX}\,\frac{\sin(c|k|\tau)}{|k|}$$

→ only a fn of R due to symmetry.

$\frac{\sin c|k|\tau}{|k|} \rightarrow \frac{\sin ck\tau}{k}$ since it is an even fn.

sin part of e^{ikx} vanishes since it is an odd fn.

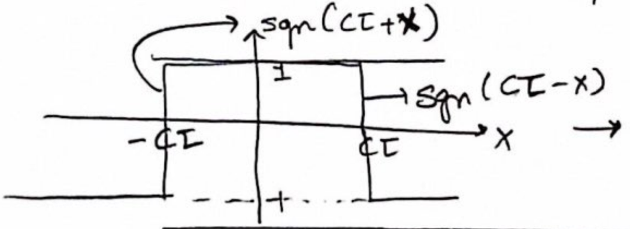
$$\Rightarrow G^{(1)}(X, \tau) = \frac{c\,\theta(\tau)}{2(2\pi)}\int_{-\infty}^{\infty} dk\, 2\frac{\sin(ck\tau)}{k}\cos kx.$$

$$= \frac{2(c\,\theta(\tau))}{4\pi}\int_{0}^{\infty} dk\left[\frac{\sin((c\tau+x)k)}{k} + \frac{\sin((c\tau-x)k)}{k}\right]$$

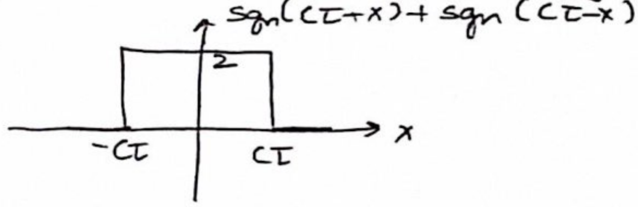
Symmetric fn (even)

$$\Rightarrow G^{(1)}(X, \tau) = \frac{c\,\theta(\tau)}{2\pi}\cdot\frac{\pi}{2}\left[\overset{\text{sign}}{\mathrm{sgn}}(c\tau + X) + \mathrm{sgn}(c\tau - x)\right]$$

$$= \frac{c\,\theta(\tau)}{4}\left[\mathrm{sgn}(c\tau + X) + \mathrm{sgn}(c\tau - X)\right]$$



→



$$\Rightarrow \boxed{G^{(1)}(X, \tau) = \frac{c\,\theta(\tau)}{2}\,\theta(c\tau - |X|)}$$

⇒ signal only exists when $|X| < c\tau$

$\Rightarrow \frac{|X|}{c} < t - t'$

$\Rightarrow t > \frac{|x-x'|}{c} + t'$

⇒ Causality.
Did not impose, it is inherent.

⇒ $c\tau > |X|$ ⇒ after the response reaches a point the value of G is constant ⇒ the signal received remains for all time. Weird & only exists in 1D.

2-dimensions.

$$G^{(2)}(R,\tau) = \frac{c\,\theta(\tau)}{(2\pi)^2}\int d^2k\, e^{i\bar{k}\cdot\bar{R}} \sin\frac{c\tau k}{k}.$$ choose k_z along $\bar{R}$.

$$= \frac{c\,\theta(\tau)}{(2\pi)^2}\int_0^\infty k\,dk\, \sin\frac{c\tau k}{k} \underbrace{\int_0^{2\pi} d\phi\, e^{ikR\cos\phi}}_{\to\, 2\pi J_0(kR)}$$

→ not an elementary integral.

→ Bessel of First kind order 0.

$$= \frac{c\,\theta(\tau)}{(2\pi)}\int_0^\infty dk\, \underbrace{\sin c\tau k \cdot J_0(kR)}_{\text{Not easy but can be done.}}$$

$$= \frac{c\,\theta(\tau)}{(\ldots)}\begin{cases} \dfrac{1}{\sqrt{c^2\tau^2-R^2}} & c^2\tau^2 > R^2 \\ 0 & c^2\tau^2 \le R^2 \end{cases}$$

$c^2\tau^2 > R^2$ → again ensures retarded.

> Here note that the G decreases with time at any point due to $\propto \frac{1}{\sqrt{\tau^2}}$
> The signal also decays with R as $\propto \frac{1}{R}$ since $c\tau$ always $> R$.
> Delta pulse does not travel as a delta but as some decaying fn

> In 3D we will see that a δ at the source indeed propagates as a δ fn & therefore does not undergo distortion. This has made communication possible.

$$\to \int_0^\infty dk \sin(c\tau k) J_0(kR) = \mathrm{Im}\underbrace{\int_0^\infty dk\, e^{ic\tau k} J_0(kR)}.$$

Regard this as the Laplace transform of $s = -ic\tau$ of $J_0(kR)$

$$\int_0^\infty dk\, e^{-sk} J_0(kR) = \frac{1}{\sqrt{s^2+R^2}}$$

Recall wave equation Green's function.

$$G^{(d)}(R,\tau) = \frac{c\,\theta(\tau)}{(2\pi)^d}\int d^dk\; e^{i\bar{k}\cdot\bar{R}}\,\frac{\sin c\tau k}{k}.$$

$\hookrightarrow \hat{k}_z$ along $\bar{R}$

3 dimensions.

$$G^{(3)}(R,\tau) = \frac{c\,\theta(\tau)}{(2\pi)^{\cancel{3}2}}\int_0^\infty k^2 dk\,\sin c\tau k\;\cancel{2\pi}\int_{-1}^{1} d(\cos\theta)\, e^{ikR\cos\theta}.$$

$$= \frac{c\,\theta(\tau)}{(2\pi)^2}\int_0^\infty \cancel{k^2}\,dk\,\frac{\sin c\tau k}{\cancel{k}}\,\frac{\cancel{2i}\sin kR}{\cancel{ik}R}.$$

$$= \frac{c\,\theta(\tau)}{4\pi^2 R}\int_0^\infty dk\,2\sin c\tau k\,\sin kR.$$

$$= \frac{1}{2}\,\frac{c\,\theta(\tau)}{4\pi^2 R}\int_{-\infty}^{\infty} dk\,\left[\cos k(c\tau - R) - \cos k(c\tau + R)\right]$$

Since cos is even.

Add sine since it is an odd fn.

$$= \frac{c\,\theta(\tau)}{8\pi^2 R}\underbrace{\int_{-\infty}^{\infty} dk\left[e^{ik(c\tau-R)} - e^{ik(c\tau+R)}\right]}_{2\pi\times\text{ delta fns}}$$

$$= \frac{c\,\theta(\tau)}{4\pi R}\left[\delta(c\tau - R) - \delta(c\tau + R)\right]$$

$$= \frac{\cancel{c}\,\theta(\tau)}{\cancel{4}\pi R}\left[\frac{\delta(\tau - \frac{R}{c})}{\cancel{c}} - \frac{\delta(\tau + \frac{R}{c})}{\cancel{c}}\right]$$

$$= \frac{\theta(\tau)}{4\pi R}\left[\delta(\tau - \tfrac{R}{c}) - \underbrace{\cancel{\delta(\tau + \tfrac{R}{c})}}_{\to 0 \text{ since } \tau>0\ \&\ R>0}\right]$$

$$\boxed{\Rightarrow G^{(3)}(R,\tau) = \frac{\theta(\tau)}{4\pi R}\,\delta\left(t - t' - \frac{|\bar{r}-\bar{r}'|}{c}\right)}$$

> In the wave equation as $c \to \infty$ we have $\frac{1}{c^2}\cancel{\frac{\partial^2}{\partial t^2}}^{0} - \nabla^2 \Rightarrow -\nabla^2$.
The Green's fn of $-\nabla^2$ is $\frac{1}{4\pi R}$ which is what the result here also gives. This happens (only?) in 3D.

<u>Solutions of the wave equation.</u>

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) f(\bar{r},t) = g(\bar{r},t)$$

$$\Rightarrow f(\bar{r},t) = \int \frac{d^3r' \, g(\bar{r}',t')}{4\pi|\bar{r}-\bar{r}'|}$$

the $\int dt \, g(\bar{r},t)$ gets evaluated only at t' where

$$t' = t + \frac{|\bar{r}-\bar{r}'|}{c}.$$

Therefore source at time t' produces signal at time t. Causality!

Note that the wave equation applied to time harmonic quantities becomes the Helmholtz equation in phasor domain.

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) A(\bar{r},t)$$

phasors

$$= \left(\frac{1}{c^2}(i\omega)^2 - \nabla^2\right) A(\bar{r},\omega)$$

$$= \left(-\nabla^2 - \frac{\omega^2}{c^2}\right) A(\bar{r},\omega)$$

$$= -\left(\nabla^2 + k^2\right) A(\bar{r},\omega)$$

Let $A(\bar{r},t) = A_0(\cos(\omega t - kr))$

phasors $A(\bar{r},t) = A_0 e^{i(kr-\omega t)}$

$$\Rightarrow \frac{\partial^2}{\partial t^2} \rightarrow \times(-i\omega)^2$$