MATH 454 - BVPs - Lecture notes

Lecture 1

$$\partial_{x_j} f(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon e_j) - f(x)}{\epsilon}$$

Partial differential operator (PDO)

$$P(x, \partial_x) = \sum_{\alpha} C_\alpha(x)\, \partial_x^\alpha \qquad \partial_x^\alpha = \prod_{1 \le j \le d} \partial_{x_j}^{\alpha_j}$$

↳ degree

eg: ① $x\partial_x + y\partial_y$

$\sin(x+y+z)\, \partial_x \partial_y \partial_z + \partial_x^2 + \partial_y \partial_z$.

$$P(x, \partial_x) f(x) = \sum_{\alpha} C_\alpha(x)\, \partial_x^\alpha f(x)$$

Lec 2

Linear PDO

$$\mathcal{L} = P(t, x, \partial_t, \partial_x)$$

$$\mathcal{L}(\alpha f + \beta g) = \alpha \mathcal{L}(f) + \beta \mathcal{L}(g)$$

Linear PDE

$$\underbrace{P(t, x, \partial_t, \partial_x)}_{\text{Linear PDO}}(\overbrace{u}^{\text{unknown. } u(t,x)}) = \underbrace{f}_{\text{known}}$$

Lec 3

Linear PDEs

$f = 0 \Rightarrow$ homogeneous

$f \neq 0 \Rightarrow$ inhomogeneous.

Laplace: $\Delta u = 0$

Poisson: $\Delta u = f$

Helmholtz: $\Delta u = -k^2 u \Rightarrow (\Delta + k^2) u = 0$

Heat: $(\partial_t - k\Delta) u = 0$

Wave: $(\partial_t^2 - c^2 \Delta) u = 0$

Schrödinger: $\left(i\hbar \partial_t + \frac{\hbar^2}{2m} \Delta \mp V\right) u = 0$

Lec 4 Exponential Ansatz $\rightarrow$ a special form of solution.

$f(x) = e^{\lambda x} \rightarrow$ trial solution. where $\lambda \in \mathbb{C}$.

or $f(x,t) = e^{\lambda t + \mu x}$

Lec 5 **Seperation of Variables**

$$u(t,x) = \phi(t)\, A(x)$$

Eg: Heat equation: $\partial_t u = k\, \partial_x^2 u$

$$\partial_t u = \phi'(t)\, A(x)$$

$$k\, \partial_x^2 u = k A''(x)\, \phi(t)$$

$$\Rightarrow \quad \frac{\phi'(t)}{\phi(t)} = \frac{k A''(x)}{A(x)}$$

But each is independant so it must equal constant λ.

$$\Rightarrow \quad \phi'(t) = \lambda \phi(t) \quad \Rightarrow \quad \phi(t) = e^{\lambda t}$$

$$A''(x) = \frac{1}{k} \lambda A(x) \Rightarrow A(x) = \alpha + \beta x \quad \text{if } \lambda = 0$$

and $A(x) = \alpha e^{\sqrt{\lambda/k}\, x} + \beta e^{-\sqrt{\lambda/k}\, x}$ if $\lambda > 0$

blows up (unphysical)

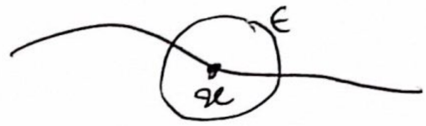
$$\Rightarrow \quad A(x) = \alpha e^{i\sqrt{-\lambda/k}\, x} + \beta e^{-i\sqrt{\frac{-\lambda}{k}}\, x} \quad \text{if } \lambda < 0$$

$$= \tilde{\alpha} \cos\left(\sqrt{\tfrac{-\lambda}{k}}\, x\right) + \tilde{\beta} \sin\left(\sqrt{\tfrac{-\lambda}{k}}\, x\right).$$

$$\Rightarrow \lambda < 0: \quad u(t,x) = e^{\lambda t}\left(\tilde{\alpha} \cos\left(\sqrt{\tfrac{-\lambda}{k}}\, x\right) + \tilde{\beta} \sin\left(\sqrt{\tfrac{-\lambda}{k}}\, x\right)\right)$$

Boundary Value Problems (BVP).

<u>D</u>: <u>Boundary</u>: $\Omega \in \mathbb{R}^d$; $\partial\Omega = \{x \mid x \in \mathbb{R}^d$ & $\forall \epsilon > 0$
$\exists\, x_{int} \in \Omega$ & $x_{ext} \in \Omega^c = \mathbb{R}^d \setminus \Omega$, $dist(x, x_{int}) < \epsilon$ &
$dist(x, x_{ext}) < \epsilon \}$



<u>D</u>: <u>Closure</u>: $\bar{\Omega} = \Omega \cup \partial\Omega$

<u>Interior</u>: $\Omega^\circ = \Omega \setminus \partial\Omega$.

<u>D</u>: <u>Open</u>: $\Omega = \Omega^\circ$ or every pt is an int. pt.

<u>closed</u>: if Ω^c is open. or $\Omega = \bar{\Omega}$ or set of all limit pts.

Lec 6 <u>Boundary Conditions</u>

<u>Rmk</u>: For PDEs, Ω must be open since we need to take derivatives.

Analytic Characterization of Boundaries

Defined by $F: \mathbb{R}^d \to \mathbb{R}$ as

$$\Omega = \{x \in \mathbb{R}^d : F(x) < 0\}$$

$$\partial\Omega = \{x \in \mathbb{R}^d : F(x) = 0\}$$

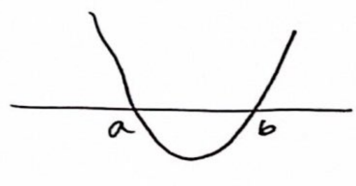
$$\Omega^c = \{x \in \mathbb{R}^d : F(x) \geq 0\}$$

eg: ① $F(x,y,z) = x^2 + y^2 + z^2 - R^2$

$\Rightarrow \Omega =$ open ball in $\mathbb{R}^3$.

② $\Omega = (a,b) \qquad -\infty < a < b < \infty$

$\Rightarrow F(x) = (x-a)(x-b)$



Rmk: If F smooth, $\nabla F \perp \partial\Omega$

In general ∇F is $\perp$ to it's level sets (by defn) & $\partial\Omega$ is a level set.

$$\vec{n} = \frac{\nabla F}{|\nabla F|}$$ if ∇F is nonvanishing everywhere.

& $\vec{n} \perp \partial\Omega$. & is the exterior unit normal.

Lec 7 Boundary Conditions

$\Omega \ni (t', x')$ s.T $P(t, x, \partial_t, \partial_x)\, u(t', x') = g(t, x)$

I Dirichlet Boundary Condition

$u(t, x) = g(t, x)$. Usually $u(t, x) = 0$

II Neumann Boundary Condition

$\partial_{\vec{n}} u(t, x) = g(t, x)$ Usually $\partial_{\vec{n}} u(t, x) = 0$

$\hookrightarrow \vec{n} \cdot \nabla u$

III Robin Boundary Condition

$(\alpha u + \beta \partial_{\vec{n}} u)(x) = g(x)$ where $\alpha, \beta \in \mathbb{C}$ & $\neq 0$.

IV Cauchy Boundary condition (a.k.a initial condition)

$u(t_0, x) = g(x)$ Usually $u(0, x) = u_0(x) = g(x)$.

If there are l time derivatives,

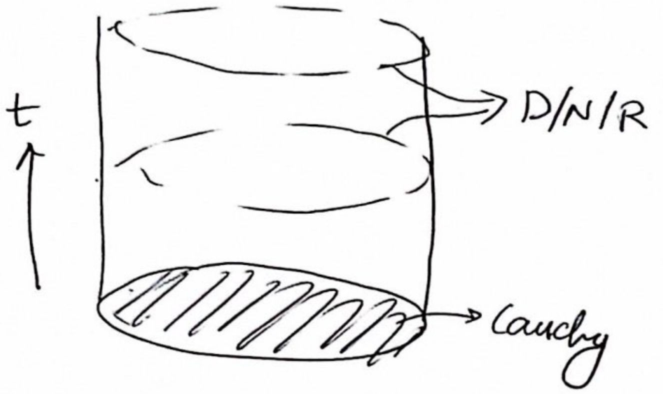
$u(t_0, x)$; $\partial_t u(t_0, x)$; $\partial_t^2 u(t_0, x)$; .... $\partial_t^{l-1} u(t_0, x)$ are specified

Lec 8 Mixed B-C

For circle $x^2+y^2=a^2$, the outward normal DBC is

$$(x\partial_x + y\partial_y)\,u(x,y)=0.$$

Cauchy & D/N/R



Seperation of Variables with Boundary Conditions

$$\partial_t u = P(x,\partial_x)\,u$$

$$B(x,\partial_x)\,u(x)=0 \quad \forall x\in\partial\Omega$$

Look for solutions $u(t,x)=\phi(t)A(x)$

$$\partial_t u = \phi'(t)A(x)$$

$$P(x,\partial_x)u = \phi(t)\,P(x,\partial_x)A(x)$$

$$\phi'(t)A(x) = \phi(t)\,P(x,\partial_x)\,A(x)$$

$$\Rightarrow \frac{\phi'(t)}{\phi(t)} = \frac{P(x,\partial_x)A(x)}{A(x)} = \lambda$$

$\Rightarrow \phi'(t) = \lambda \phi(t) \quad \Rightarrow \quad \phi(t) = ce^{\lambda t}$

and $P(x, \partial_x) A(x) = \lambda A(x)$

<u>B.C</u>: $\phi(t) \; B(x, \partial_x) A(x) = 0$

$\phi(t) \neq 0 \quad \Rightarrow \quad B(x, \partial_x) A(x) = 0$

$$\Rightarrow \boxed{\begin{aligned} &P(x, \partial_x) A(x) = \lambda A(x) \\ \& \quad &B(x, \partial_x) A(x) = 0 \end{aligned}}$$ "Eigenvalue problem".

<u>D: Eigenmode</u>

A solution to the BVP $\begin{cases} \partial_t u = P(x, \partial_x) u \\ B(x, \partial_x) u = 0 \end{cases}$

is an eigenmode if one of the following conditions hold.

① At some time t_0, $u(t_0)$ is an eigen function,

$$\begin{cases} P(x, \partial_x) u(t_0) = \lambda(t_0) u(t_0) & \Omega \\ B(x, \partial_x) u(t_0) = 0 & \partial\Omega \end{cases}$$

for some eigenvalue $\lambda(t_0)$.

② Same as above but for all time t_0

If at time t_0,

$$P(x, \partial_x)\, u(t_0) = \lambda\, u(t_0)$$

& $\quad B(x, \partial_x)\, u(t_0) = 0$

Then,

$$u(t, x) = \phi(t)\, u(t_0)$$

$$\Rightarrow \cancel{\partial_t u \big|_{t_0} = P(x, \partial_x) u \big|_{t_0}}$$

$$\Rightarrow \cancel{(\partial_t - P(x, \partial_x))\, u(t_0) = 0}$$

$$\Rightarrow \partial_t u(t, x) = P(x, \partial_x)\, u(t, x)$$

$$\Rightarrow 0 = (\partial_t - P(x, \partial_x))\, u(t, x)$$

$$= \phi'(t)\, u(t_0) - \phi(t) \lambda\, u(t_0)$$

$$\Rightarrow \phi'(t) = \lambda \phi(t) \quad \& \quad \phi(t_0) = 1 \quad \text{since } u(t_0, x) = u(t_0)$$

$$\Rightarrow \phi(t) = c\, e^{\lambda(t - t_0)} \quad \& \quad \boxed{u(t, x) = e^{\lambda(t - t_0)}\, u(t_0)}$$

$\Rightarrow$ If u is an eigenmode, the $\exists\, \lambda$ s.t.

$$u(t_1, x)\, e^{-\lambda t_1} = u(t_2, x)\, e^{-\lambda t_2}.$$

Linearity of eigenmodes

Let (A_j) be eigenfns. with e-vals λ_j, then

$$u(t,x) = \sum_j \alpha_j e^{\lambda_j t} A_j(x) \quad \text{are solutions.}$$

Lec 9 Fourier modes

I Dirichlet condition

$$\partial_x^2 u = \lambda u$$

& $u(0) = u(\ell) = 0$

If u is a non-trivial ~~condition~~ solution –

$$\int_0^\ell \bar{u}\, \partial_x^2 u\, dx = \int_0^\ell \bar{u} \lambda u\, dx = \lambda \int_0^\ell |u|^2 dx$$

$$\underset{\text{(IBP Int by parts)}}{=} \underbrace{\cancel{\left[\bar{u}\,\partial_x u\right]_0^\ell}}_{\substack{0 \\ \text{from B.C}}} - \int_0^\ell \partial_x \bar{u}\, \partial_x u\, dx$$

$$\Rightarrow -\int_0^\ell |\partial_x u|^2 dx = \lambda \int_0^\ell |u|^2 dx$$

$$\Rightarrow \lambda = \frac{-\int_0^\ell |\partial_x u|^2 dx}{\int_0^\ell |u|^2 dx}$$

$\partial_x u \neq 0$ in order to satisfy B.C nontrivially.

$\Rightarrow \lambda < 0$

Let $\lambda = -k^2$

$$\boxed{\Rightarrow \partial_x^2 u = -k^2 u \quad \& \quad u(0) = u(\ell) = 0. \qquad \text{General soln. is} \quad u = \alpha \cos(kx) + \beta \sin(kx)}$$

Apply B.C $\Rightarrow$ $u(0) = \alpha = 0$

$u(\ell) = \beta \sin(k\ell) = 0$

$\Rightarrow k\ell = n\pi \;,\; n \in \mathbb{Z}$

$$\Rightarrow \lambda = -\left(\frac{n\pi}{\ell}\right)^2 \; \& \; u_\lambda(x) = \beta \sin\left(\frac{n\pi}{\ell}x\right) \; //$$

II Neumann condition

$$\partial_x^2 u = \lambda u$$

$$u'(0) = u'(\ell) = 0$$

$$\lambda = -\frac{\int_0^\ell |\partial_x u|^2 \, dx}{\int_0^\ell |u|^2 \, dx}$$

$\Rightarrow \lambda \leq 0$, $\lambda = 0$ is a valid case.

$\Rightarrow \quad \partial_x^2 u = -k^2 u \quad \& \quad u'(0) = u'(\ell) = 0$

$\Rightarrow \quad u = \alpha \sin(kx) + \beta \cos(kx)$

$\Rightarrow \quad \cancel{u_2 = \alpha \sin kx \quad k\ell = \frac{2n+1}{2}\pi}$

$$\lambda = \frac{-n^2\pi^2}{\ell^2} \quad n \in \mathbb{N}$$

$$u_\lambda(x) = \cos(kx) = \cos\left(\frac{n\pi x}{\ell}\right)$$

Periodic condition

$$\partial_x^2 u = \lambda u \qquad (-p, p)$$

$$u(-p) = u(p) \quad \& \quad \partial_x u(-p) = \partial_x u(p).$$

$$\lambda \leq 0 \text{ from } \lambda = \frac{-\int_{-p}^{p} |\partial_x u|^2 \, dx}{\int_{-p}^{p} |u|^2 dx}$$

$$u = \alpha \cos(kx) + \beta \sin(kx).$$

$$u(-p) = u(p) \Rightarrow 2\beta \sin(kp) = 0$$

$$\Big[\;\Rightarrow \beta = 0 \text{ or } kp = n\pi \quad n \in \mathbb{N} \qquad \text{if } k > 0, n > 0$$

$$\text{or } u = \alpha \qquad \text{if } k = 0 \;\Big]$$

$$\partial_x u(-p) = \partial_x u(p) \Rightarrow -\alpha \sin(kp) + \beta \cancel{\cos}(kp) = +\alpha \sin(kp) + \beta \cos.$$

$$\Rightarrow \quad \alpha = 0 \text{ or } kp = n\pi$$

$$\Rightarrow \left\{ \begin{array}{ll} u = \alpha \cos\left(\frac{n\pi}{p} x\right) + \beta \sin\left(\frac{n\pi}{p} x\right) & k > 0, n > 0 \\ \text{or } u = \alpha & k = 0, \end{array} \right\}$$

Lec 10 Fourier analysis

$$\partial_t u = k \partial_x^2 u \quad \text{—— (1)}$$

$$u(t,0) = u(t,\pi) = 0$$

$$u(0,x) = u_0(x)$$

where $u_0(x) = \sum_{n=1}^{N} \alpha_n \sin(nx)$

Look for solution of the form $\sum_{n=1}^{N} \phi_n(t) \sin(nx)$.

Plugging into (1)

$$0 = (\partial_t - k\partial_x^2) \sum_{n=1}^{N} \phi_n(t) \sin(nx)$$

$$= \sum_{n=1}^{N} (\partial_t - k\partial_x^2)(\phi_n(t) \sin(nx))$$

$$= \sum_{n=1}^{N} \phi_n'(t) \sin(nx) + k n^2 \phi_n(t) \sin(nx)$$

$$= \sum_{n=1}^{N} [\phi_n'(t) + k n^2 \phi_n(t)] \sin(nx)$$

$$\Rightarrow \phi_n'(t) = -k n^2 \phi_n(t) \Rightarrow \phi_n(t) = C_n e^{-kn^2 t}$$

$$u_0(x) = \sum_{n=1}^{N} \phi_n(0) \sin(nx) = \sum_{n=1}^{N} \alpha_n \sin(nx) = \sum_{n=1}^{N} C_n \sin(nx)$$

$$\Rightarrow \alpha_n = C_n$$

$$\Rightarrow \phi_n(t) = \alpha_n e^{-kn^2 t}$$

$$\Rightarrow \boxed{u(t,x) = \sum_{n=1}^{N} \alpha_n e^{-kn^2 t} \sin(nx)}$$

↳ From uniqueness, it is unique.

Linear function Spaces

Let $\mathbb{F}$ be a field & J be a set. Let V be a set of functions defined on J i.e $f \in V$ if $f: J \to \mathbb{C}$. Then we say V is $\mathbb{F}$-linear if $\forall$ $f, g \in V$ & $\alpha, \beta \in \mathbb{F}$ we have

$$\alpha f + \beta g \in V.$$

Lec 11

Lineae independence of $\sin(nx)$

Lemma: $\forall\, N \in \mathbb{N}$, & $\forall\, b_n \in \mathbb{R}$.

if $b_1 \sin(x) + \dots + b_N \sin(Nx) = 0$

$\Rightarrow \quad b_1 = b_2 = \dots = b_N = 0.$

Pf ①: Apply ∂_x^2 again & again. $(N-1)$ times.

$$\Rightarrow \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 2^2 & 3^2 \cdots & (N-1)^2 & N^2 \\ \vdots & & & & \\ 1 & (2^2)^{N-1} & \cdots & & (N^2)^{N-1} \end{pmatrix}}_{\text{Vandermonde matrix} \Rightarrow \text{invertible}} \begin{pmatrix} b_1 \sin x \\ \vdots \\ \vdots \\ b_N \sin(Nx) \end{pmatrix} = 0$$

$\Rightarrow \quad b_1 \sin x = \dots = b_N \sin(Nx) = 0$

$\Rightarrow \quad b_1 = b_2 = \dots = b_N = 0.$

Pf ② $\quad -\partial_x^2 \sin mx = m^2 \sin(mx)$

$\Rightarrow \cancel{-\partial_x^2 f}$

$$\Rightarrow \quad n^2 \int_0^{\pi} \sin(mx)\,\sin(nx)\,dx$$

$$= \int_0^{\pi} \sin(mx)\; -\partial_x^2 \sin(nx)\; dx$$

$$= m^2 \int_0^{\pi} \sin(mx)\,\sin(nx)\,dx$$

$$\Rightarrow \quad (m^2-n^2) \int_0^{\pi} \sin(mx)\,\sin(nx)\,dx = 0$$

$$\Rightarrow \quad \int_0^{\pi} \sin(mx)\,\sin(nx)\,dx = 0 \quad \text{if} \quad m \neq n$$

$$\& \quad \int_0^{\pi} |\sin(nx)|^2\,dx \neq 0 \quad \text{if} \quad m = n.$$

$$\Rightarrow \int_0^{\pi} \sin(mx) \Big[\underbrace{b_1 \sin(nx) + \cdots + b_N \sin(Nx)}_{0} \Big] dx$$

$$= 0$$

$$\Rightarrow \quad b_m \int \underbrace{|\sin(mx)|^2 dx}_{\neq 0} = 0 \quad \Rightarrow \quad b_m = 0$$

Note,

$$b_n = \frac{\int_0^{\pi} u(x) \sin(nx)\, dx}{\int_0^{\pi} (\sin(nx))^2\, dx}$$

D: Inner Product

$V \times V \longrightarrow \mathbb{C}$ S.T

① $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$

② $\langle f, g \rangle = \overline{\langle g, f \rangle}$

③ $\langle f, f \rangle \geq 0$ & $\langle f, f \rangle = 0 \iff f = 0$.

Eg: $\mathbb{C}^d$ $\langle x, y \rangle = [x]^* [y]$

Eg: $f, g \Rightarrow \langle f, g \rangle = \int_a^b f(x) \overline{g(x)}\, dx$

Lec 12

D: l^2-space

① Let $J \subseteq \mathbb{Z}$, $l^2(J, \mathbb{R})$ is the set of all sequences

$$a = (a_j)_{j \in J} \quad \text{s.t} \quad a_j \in \mathbb{R} \quad \& \quad \sum_{j \in J} a_j^2 < \infty.$$

② $l^2(J, \mathbb{R})$ is the set of all sequences $a = (a_j)_{j \in J}$

s.t $a_j \in \mathbb{C}$ $\forall j$ and $\sum_j a_j \overline{a_j} = \sum_j |a_j|^2 < \infty$.

Cauchy Schwartz

$$|\langle a, b \rangle| \leq \sqrt{\langle a, a \rangle \cdot \langle b, b \rangle}$$

On l^2, define the following inner products.

- $l^2(J; \mathbb{R}) \rightarrow \langle a, b \rangle = \sum_{j \in J} a_j b_j$

$l^2(J, \mathbb{C}) \rightarrow \langle a, b \rangle = \sum_{j \in J} a_j \overline{b_j}$

D: L^2-space (square integrable fns.)

If $(a,b) \subseteq \mathbb{R}$ then $L^2(a,b)$ is the set of all functions $f: (a,b) \rightarrow \mathbb{C}$ s.t

$$\int_a^b |f(x)|^2 \, dx < \infty.$$

Inner product: $\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} \, dx$

> L^2 is an inner product space, under the condition that $|f-g| = 0 \Rightarrow f = g$ (in the L^2 sense).

> f, g orthogonal if $\langle f, g \rangle = 0$

On $L^2(0, \ell)$, the following sets are orthogonal,

$$\left\{ \sin\left(\frac{k\pi x}{\ell}\right) \right\}_{k \in \mathbb{N} - \{0\}} \Rightarrow \int_0^\ell \sin\left(\frac{m\pi x}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right) dx = \begin{cases} 0 & m \\ \neq 0 & m= \end{cases}$$

$$\left\{ \cos\left(\frac{k\pi x}{\ell}\right) \right\}_{k \in \mathbb{N}}$$

//

- $\left\{ e^{\frac{ik\pi x}{P}} \right\}_{k \in \mathbb{Z}}$ is orthogonal on $L^2(-P, P)$.

<u>Thm</u>: If V ips & $S \subseteq V$ orthogonal,

& $u = \sum c_j e_j$ where $c_j \in \mathbb{C}$ & $e_j \in S$.

$$\Rightarrow \quad \boxed{c_j = \frac{\langle u, e_j \rangle}{\langle e_j, e_j \rangle}} \quad \forall j$$

<u>Pf</u>:

$$\langle u, e_k \rangle = \langle \sum_j c_j e_j, e_k \rangle$$

$$= \sum_j \langle c_j e_j, e_k \rangle$$

$$= c_k \langle e_k, e_k \rangle$$

$$\Rightarrow \quad c_k = \frac{\langle u, e_k \rangle}{\langle e_k, e_k \rangle} \; //$$

Lec 13 <u>Hilbert space</u> (IPS with Limit)

An IPS V is a Hilbert space if $\forall$ sequences

$(f_n)_{n \geq 0}$ that satisfies $\lim\limits_{n \to \infty} \sup\limits_{m \geq n} |f_m - f_n| = 0$

there exists a function $f \in V$ s.t.

$$\lim_{n \to \infty} |f_n - f| = 0 .$$

<u>Norms</u>

$$\|f\|_{\ell^2} = \sqrt{\langle f, f \rangle} = \sqrt{\sum_{j \in J} f_j \overline{f_j}}$$

$$\|f\|_{L^2} = \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b f \, \overline{f} \, dx} = \sqrt{\int_a^b |f(x)|^2 dx}$$

D: <u>Orthogonal projection</u>

$$g = \underbrace{\frac{\langle f, e \rangle}{\langle e, e \rangle}}_{\lambda} e \quad \& \quad (f-g) \perp e .$$

D: Orthogonal projection.

Let $S = \{e_1, e_2, \ldots, e_n\}$ be a ~~basis~~ set of fns.

orthogonal projection of f onto span(S) is

$$g = \lambda_1 e_1 + \cdots + \lambda_n e_n \quad \text{where}$$

$$\lambda_j = \frac{\langle f, e_j \rangle}{\langle e_j, e_j \rangle}$$

D: Hilbert space (again)

V IPS is Hilbert if $\forall$ sequences $(u_n)_{n \geq 0} \subset V$ s.t.

$$\lim_{n\to\infty} \sup_{m \geq n} |u_n - u_m| = 0 \;, \; \exists \; u \in V \text{ s.t.}$$

$$\lim_{n\to\infty} |u_n - u| = 0 \quad \text{or} \quad \lim_{n\to\infty} u_n = u .$$

Eg: l^2 & L^2 are Hilbert spaces.

Lec 14

Fourier Series.

Let $S = (e_j)_{j\in J}$ be an orthogonal subset of a Hilbert space V & $(J \in \mathbb{Z})$. Then for all $f \in V$, the Fourier Series of f is the series

$$\tilde{f} = \sum_{j\in J} \underbrace{\frac{\langle f, e_j\rangle}{\langle e_j, e_j\rangle}}_{\text{Fourier coeffs.}} e_j .$$

Also $\lim_{n\to\infty} \left| \sum_{\substack{j \\ |j|\leq n}} \frac{\langle f, e_j\rangle}{\langle e_j, e_j\rangle} e_j - \tilde{f} \right| = 0$.

Eg: On $L^2(0,\ell)$, $\tilde{f} = \sum_{n\geq 1} b_n \sin\left(\frac{n\pi x}{\ell}\right)$,

$$b_n = \frac{\int_0^\ell f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx}{\int_0^\ell \sin\left(\frac{n\pi x}{\ell}\right)^2 dx} = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

(Same for cosine.)

Lec 15 (Reviews mid-term solutions)

Lec 16

Review

Linear IPS with limits.

D: Complex Hilbert Space H. (3 properties)

1. H is a linear function space.
2. H is an inner product space.
3. In H we can take limits.

Eg: $\ell^2(\mathbb{N})$ or $\ell^2(\mathbb{C}) = \{ (a_n)_{n=0}^{\infty} \text{ s.t } \sum_{n=0}^{\infty} |a_n|^2 < \infty \}$

also $\hookrightarrow$ $\ell^2(\mathbb{N}^*)$, $\mathbb{N}^* \to \mathbb{N}/\{0\}$

$$L^2(a,b) = \{ f : (a,b) \to \mathbb{C} \text{ s.t } \int_a^b |f(x)|^2 dx < \infty \}$$

D: Fourier Series

H: Hilbert space

$(e_j)_{j \in J}$: complete orthogonal set of functions.

If $f \in H$, then: $f = \sum_{j \in J} \gamma_j e_j$ where $\gamma_j = \dfrac{\langle f, e_j \rangle}{\langle e_j, e_j \rangle}$

Parseval's identity (Pythagorean Thm in ∞dim space)

$$\langle f, f \rangle = \|f\|^2 = \sum_{j\in J} \langle \gamma_j e_j, \gamma_j e_j \rangle = \sum_{j \in J} |\gamma_j|^2 \|e_j\|^2$$

Summary:

	(Sine)	(Cosine)	(Exponential)
Hilbert space	$L^2(0,\rho)$	$L^2(0,\rho)$	$L^2(-\rho,\rho)$
Orthogonal basis	$\left\{\sin\left(\frac{n\pi x}{\rho}\right)\right\}_{n\in\mathbb{N}^*}$	$\left\{\cos\left(\frac{n\pi x}{\rho}\right)\right\}_{n\in\mathbb{N}}$	$\left\{e^{\frac{i\pi k x}{\rho}}\right\}_{k\in\mathbb{Z}}$
Fourier coeff.	$b_n = \frac{2}{\rho}\int_0^\rho f(x)\sin\left(\frac{n\pi x}{\rho}\right)dx$	$a_n = \frac{2}{\rho}\int_0^\rho f(x)\cos\left(\frac{n\pi x}{\rho}\right)dx$	$c_n = \frac{1}{2\rho}\int_{-\rho}^{\rho} f(x)\, e^{\frac{-i\pi k x}{\rho}}dx$
Fourier series	$f(x) = \sum_{n=1}^\infty b_n \sin\left(\frac{n\pi x}{\rho}\right)$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos\left(\frac{n\pi x}{\rho}\right)$	$f(x) = \sum_{k=-\infty}^\infty c_k e^{\frac{i\pi k x}{\rho}}$
Parsevals identity	$\int_0^\rho \lvert f(x)\rvert^2 dx = \sum_{n=1}^\infty \int_0^\rho \lvert b_n \sin\left(\frac{n\pi x}{\rho}\right)\rvert^2 dx$ $= \sum_{n=1}^\infty \lvert b_n\rvert^2 \frac{\rho}{2}$ $\boxed{\Rightarrow \frac{2}{\rho}\int_0^\rho \lvert f(x)\rvert^2 dx = \sum_{n=1}^\infty \lvert b_n\rvert^2}$	$\int_0^\rho \lvert f(x)\rvert^2 dx$ $= \sum_{n=1}^\infty \int_0^\rho \lvert a_n \cos\left(\frac{n\pi x}{\rho}\right)\rvert^2 dx$ $+ \int_0^\rho \lvert \frac{a_0}{2}\rvert^2 dx$ $= \sum_{n=1}^\infty \lvert a_n\rvert^2 \frac{\rho}{2} + \frac{\rho}{4}\lvert a_0\rvert^2$ $\Rightarrow \boxed{\frac{2}{\rho}\int_0^\rho \lvert f(x)\rvert^2 dx = \frac{1}{2}\lvert a_0\rvert^2 + \sum_{n=1}^\infty \lvert a_n\rvert^2}$	$\int_{-\rho}^\rho \lvert f(x)\rvert^2 dx$ $= \sum_{k=-\infty}^\infty \int_{-\rho}^\rho \lvert c_k e^{\frac{\pi i k x}{\rho}}\rvert^2 dx$ $= \sum_{k=-\infty}^\infty \lvert c_k\rvert^2 2\rho$ $\Rightarrow \boxed{\frac{1}{2\rho}\int_{-\rho}^\rho \lvert f(x)\rvert^2 dx = \sum_{k=-\infty}^\infty \lvert c_k\rvert^2}$

§2.4 Pointwise Convergence.

Let us only consider $L^2(-p,p)$. We know if $f \in L^2(-p,p)$. Then $f(x) = \sum_{k=-\infty}^{\infty} C_k e^{\pi i k x/p}$

where $C_k = \frac{1}{2p}\int_{-p}^{p} f(x) e^{-\pi i k x/p} dx$

The equality holds in the L^2-sense

i.e. $\int_{-p}^{p} |f(x) - g(x)|^2 dx = 0.$

Thm (Lebesgue)

If $f = g$ in L^2-sense, then for almost every x you have $f(x) = g(x)$.

↑ (measure theory) (real analysis)

(i.e. measure of points where $f(x) \neq g(x)$ has measure 0).

(Probability $(x : f(x) \neq g(x)) = 0$.

Thm (Carleson, Luzin)

If $f \in L^2(-p,p)$, then for a.e $x \in (-p,p)$ we have

$f(x) = \sum_{k=-\infty}^{\infty} C_k e^{\pi i k x/p}$; $f \in L^2 \Leftrightarrow (C_k) \in \ell^2(\mathbb{Z}) \Leftrightarrow |C_k|^2 < \infty$

Thm

If $\sum_{k=-\infty}^{\infty} |c_k| < \infty$, then $\forall x \in (-p,p)$ the series

$\sum_{k=-\infty}^{\infty} c_k e^{\pi i k x/p}$ is convergent. ⇝ pointwise convergence.

and it converges to a function f with this prescribed Fourier series. } convergence of series of functions

$$\Rightarrow \left[\begin{array}{l} \text{Summable Fourier coeff} \Rightarrow \text{pointwise convergence} \\ \text{Square summable Fourier coeff} \Rightarrow \text{a.e. convergence.} \end{array} \right] \bigstar$$

Summary Let $f \in L^2(-p,p)$

① if $\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty$ then for a.e. x, $f(x) \stackrel{a.e.}{=} \sum_{k=-\infty}^{\infty} c_k e^{\pi i k x/p}$

② if $\sum_{k=-\infty}^{\infty} |c_k| < \infty$, then $\forall x$, $g(x) \stackrel{\forall}{=} \sum_{k=-\infty}^{\infty} c_k e^{\pi i k x/p}$

(i) $f = g$ in L^2 sense

(ii) $f(x) = g(x)$ almost everywhere

(iii) g is continuous! & f is almost continuous.

↳ (Proof in notes)

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Theorem ① Convergence in $L^2(-p,p) \Leftrightarrow$ square summability of Fourier coefficients.

$\Rightarrow$ almost everywhere convergence of Fourier series.

i.e. with probability 1, the series $\sum_{k=-\infty}^{\infty} c_k e^{\pi i k x/p}$ is convergent.

(i) Since each term in $\sum_{k=-\infty}^{\infty} c_k e^{\frac{\pi i k x}{p}}$ is $2p$-periodic,

$\Rightarrow$ the Fourier series is $2p$- periodic.

(ii) $\Rightarrow$ if the series is convergent at x, it is also convergent at $x + 2mp$.

(iii) Therefore, the Fourier series of a function $f \in L^2(-p,p)$ gives a periodic extension of f from $(-p,p)$ to $\mathbb{R}$.

(iv) The limit/infinite sum of continuous functions may cease to be continuous.

Eg: Suppose $\lim_{x \to x_0} u(x) = u(x_0)$ and $\lim_{n\to\infty} u_n(x) = u(x)$

In order for u to be continuous, the two limits must commute.

$$u(x_0) = \lim_{x\to x_0} u(x) = \lim_{x\to x_0} \lim_{n\to\infty} u_n(x) \stackrel{?}{=} \lim_{n\to\infty} \lim_{x\to x_0} u_n(x) = \lim_{n\to\infty} u_n(x_0) = u(x_0)$$

//

Thm: If $(c_k)_{k=-\infty}^{\infty} \in \ell^1(\mathbb{Z})$, i.e. $\sum_{k=-\infty}^{\infty} |c_k| < \infty$.

Then the Fourier series

$$\sum_{k=-\infty}^{\infty} c_k e^{i\pi k x/p}$$ converges absolutely & uniformly to a continuous function.

i.e. $\exists$ a continuous $2p$-periodic function g s.t

$$\lim_{N\to\infty} \sup_{x\in\mathbb{R}} \left| g(x) - \sum_{k=-N}^{N} c_k e^{\pi i k x/p} \right| = 0$$

Let $f \in L^2(-p,p)$ & let $x_0 \in \mathbb{R}$. What can we say about $\sum_{k=-\infty}^{\infty} c_k e^{\frac{\pi i k x_0}{p}}$?

Thm: (Dini): Let f be $2p$-periodic, square integrable $\forall N \in \mathbb{N}$. Let $S_n[f](x) = \sum_{k=-N}^{N} c_k e^{\pi i k x/p}$. Let $x_0 \in \mathbb{R}$.

Suppose that, $\exists$ $A \in \mathbb{C}$, $\delta > 0$, s.t

$$\int_0^{\delta} \left| \frac{f(x_0+z) + f(x_0-z)}{2} - A \right| \frac{dz}{z} < \infty,$$ then

$$\lim_{N\to\infty} S_N[f](x_0) = A.$$

Corollary: If f is differentiable at x_0 then

$$\sum_{k=-\infty}^{\infty} C_k e^{\pi i k x/\ell} = f(x_0).$$

Pf, i.e., $\big|\, |f(x_0+z) - f(x_0) - f'(x_0) z| \leq C(z)|z|$. $\lim_{z\to 0} C(z) = 0$

Particularly,

$$|f(x_0+z) - f(x_0)| \leq C|z|$$

Let $A = f(x_0)$

$$\left| \frac{f(x_0+z) + f(x_0-z)}{2} - A \right| = \frac{1}{2} \Big| (f(x_0+z) - f(x_0)) + |f(x_0-z) - f(x_0)| $$

$$\leq \frac{1}{2} |f(x_0+z) - f(x_0)| + \frac{1}{2} |f(x_0-z) - f(x_0)|$$

$$\leq \frac{C}{2}|z| + \frac{C}{2}|z| = C|z| \quad //$$

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Theorem (Dini)

f is $2p$-periodic, square integrable on $(-p, p)$ and suppose that, for some $A \in \mathbb{C}$, $\delta > 0$, we have

$$\int_0^\delta \left| \frac{f(x_0+z) + f(x_0-z)}{2} - A \right| \frac{dz}{z} < \infty$$

then $\sum_{k=-\infty}^{\infty} C_k e^{\pi i k x_0/p} = A.$

Cor: If in addition ~~f~~ f is differentiable at x_0, then

$$\sum_{k=-\infty}^{\infty} C_k e^{\pi i k x_0/p} = f(x_0)$$

Def: A function f is called Hölder α-continuous (where $0 < \alpha < \infty$) at x if for some $\begin{cases} C > 0 \\ \delta > 0 \end{cases}$, we have

$|f(x+z) - f(x)| \leq C|z|^\alpha, \ \forall \ |z| < \delta.$

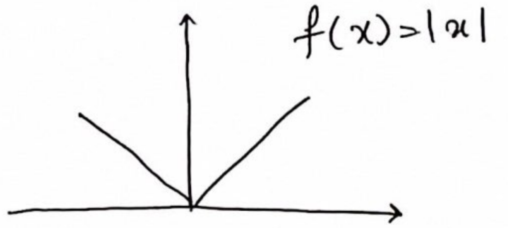
① We know that if f is differentiable at x_0, then f is Hölder 1-continuous at x_0.

$$|f(x_0+z) - f(x_0)| \leq C|z|$$

(2) The converse is not true,

a function can be Hölder 1-continuous (aka Lipschitz) without being differentiable at a point.

<u>counter example</u>:



(1) not differentiable at 0.

(2) Lipschitz at 0.

$$|f(x)-f(0)| = \big||x|-|0|\big| = |x| \le 1\,|x|'$$

The Corollary that

Differentiability $\Rightarrow$ convergence of Fourier series

can be strengthened to

Lipschitz continuity $\Rightarrow$ convergence of Fourier series.

Proof (Exercise)

<u>Theorem</u>: If f is $2p$-periodic, square integrable over $(-p,p)$ and if f is α-continuous at x_0 $(\alpha>0)$.

Then
$$\sum_{k=-\infty}^{\infty} C_k\, e^{\pi i k x_0/p} = f(x_0)$$

<u>Proof</u>:
$$\frac{f(x_0+z)+f(x_0-z)}{2} - f(x_0) = \frac{1}{2}\big(f(x_0+z)-f(x_0)\big) + \frac{1}{2}\big(f(x_0-z)-f(x_0)\big)$$

Triangle inequality,

$$\Rightarrow \left| \frac{f(x_0+z) + f(x_0-z)}{2} - f(x_0) \right| \leq \frac{1}{2} |f(x_0+z) - f(x_0)| + \frac{1}{2} |f(x_0-z) - f(x_0)|$$

$$\leq \frac{1}{2} C|z|^\alpha + \frac{1}{2} C|z|^\alpha = C|z|^\alpha \quad \text{when } |z| < \delta.$$

We have,

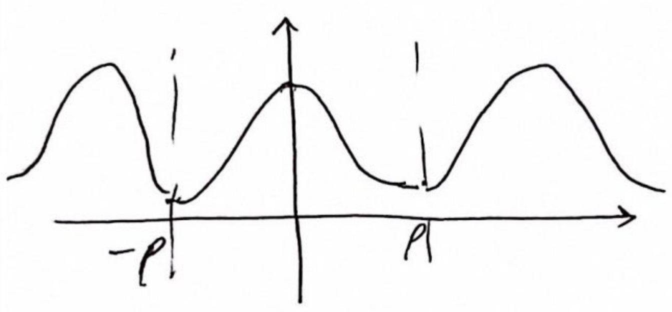
$$\int_0^\delta \left| \frac{f(x_0+z) + f(x_0-z)}{2} - f(x_0) \right| \frac{dz}{z} \leq \int_0^\delta C|z|^\alpha \frac{dz}{z}$$

$$= \int_0^\delta C z^{\alpha-1} dz = \left[C \frac{z^\alpha}{\alpha} \right]_0^\delta < \infty \quad \text{when } \alpha > 0$$

//

Q: What if f is not continuous at a point x_0?

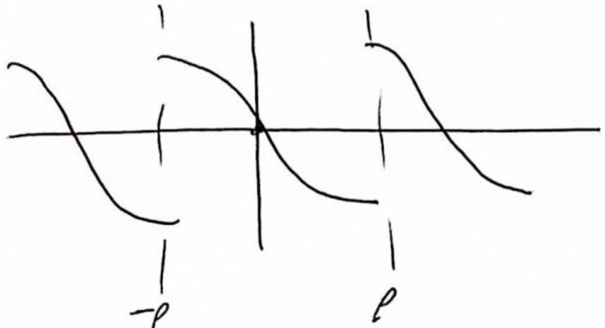
① Suppose that f is even. (very continuous).



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{P}\right)$$

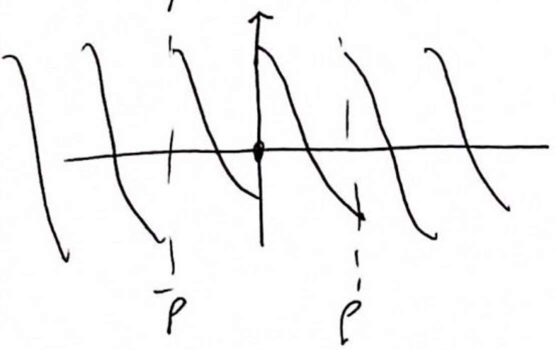
pointwisely at $x=0$.

② Suppose f is odd. (assuming $\lim_{x\to 0^{\pm}} f(x)$ exists)



$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) \quad \text{almost everywhere.}$$

when $x=0$, $\text{RHS} = 0 = \dfrac{f(0^+)+f(0^-)}{2}$



③ General case

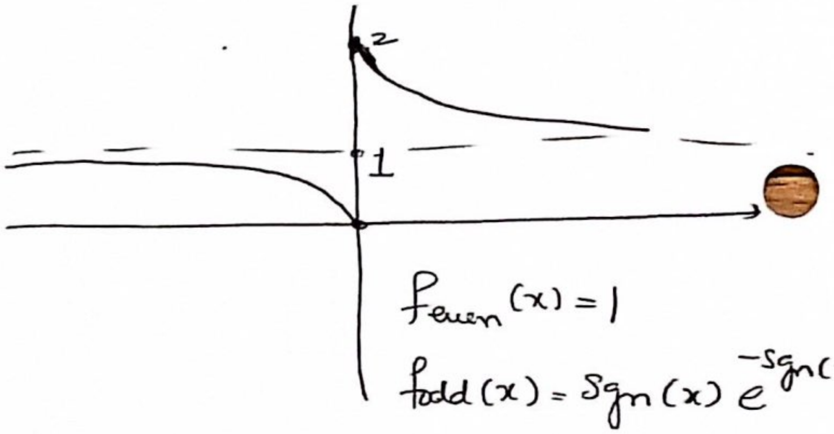
$$f(x) = \underbrace{f_{even}(x)}_{\frac{1}{2}(f(x)+f(-x))} + f_{odd}(x) \longrightarrow \frac{1}{2}(f(x)-f(-x))$$

$$f(x) = \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right)}_{\text{even}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)}_{\text{odd}}$$

At $x=0$, Fourier series at 0 = $\underbrace{\dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n}_{f_{even}(0) = \frac{1}{2}(f(0^+)+f(0^-))}$

$$f(x) = 1 + \operatorname{sgn}(x)\, e^{-\operatorname{sgn}(x)}$$

$$= \begin{cases} 1 + e^{-x} & x > 0 \\ 1 & x = 0 \\ 1 - e^{x} & x < 0 \end{cases}$$



$f_{even}(x) = 1$

$f_{odd}(x) = \operatorname{sgn}(x)\, e^{-\operatorname{sgn}(x)}$

i.e. $$\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{P}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{P}\right)\right]\Bigg|_{x=0}$$

$$= \frac{1}{2}\left(f(0^+) + f(0^-)\right)$$

$$= \frac{1}{2}\left(\lim_{x\to 0^+} f(x) + \lim_{x\to 0^-} f(x)\right)$$

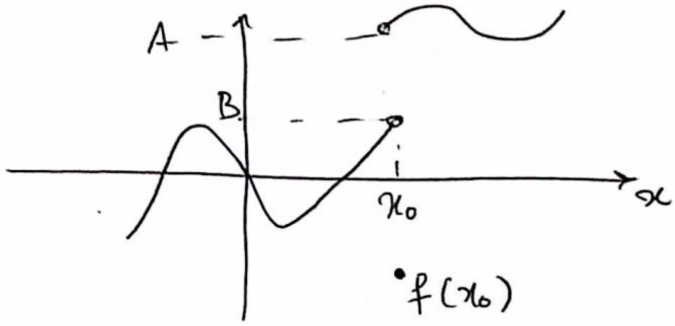
Def: ① A function is called Hölder α continuous from right at a pt. x_0, if $\exists$ $\begin{cases} A \in \mathbb{C} \\ \delta > 0 \\ C > 0 \end{cases}$ s.t

$$|f(x_0 + z) - A| \le C z^{\alpha}, \ \forall\ 0 < z < \delta.$$

② f is α- continuous from left at x_0, if $\exists$ $B \in \mathbb{C}$, $\delta > 0$, $C > 0$ s.t

$$|f(x_0 - z) - B| \le C z^{\alpha}, \ \forall\ 0 < z < \delta.$$

eg:



Thm: Let f be $2p$-periodic, square integrable over $(-p,p)$. If f is α-continuous from right at x_0 & β-continuous from left at x_0.

Then, $\displaystyle\sum_{k=-\infty}^{\infty} c_k e^{\pi i k x_0/p}$ is convergent.

In fact, if
$$\begin{cases} |f(x_0+z)-A| \le C_1 z^{\alpha} & \text{where } z>0 \text{ is small} \\ |f(x_0-z)-B| < C_2 z^{\beta} \end{cases}$$

then $\displaystyle\sum_{k=-\infty}^{\infty} c_k e^{\pi i k x_0/p} = \frac{1}{2}(A+B)$

Proof (will come in exam) Dini again

$$\left| \frac{f(x_0+z)+f(x_0-z)}{2} - \frac{1}{2}(A+B) \right| = \frac{1}{2}\left| (f(x_0+z)-A)+(f(x_0-z)-B) \right|$$

$$\le \frac{1}{2}|f(x_0+z)-A| + \frac{1}{2}|f(x_0-z)-B| \le \frac{1}{2}C_1 z^{\alpha} + \frac{1}{2}C_2 z^{\beta}$$
$(0<z<\delta)$

$$\int_0^\delta \left| \frac{f(x_0+z)+f(x_0-z)}{2} - \frac{1}{2}(A+B)\right| \frac{dz}{z}$$

$$\le \int_0^\delta \left(\frac{1}{2} c_1 z^\alpha + \frac{1}{2} c_2 z^\beta\right)\frac{dz}{z} = \frac{1}{2}\left[c_1\int_0^\delta z^{\alpha-1}\,dz + c_2\int_0^\delta z^{\beta-1}\,dz\right] < \infty$$

§2.5 Periodic extensions, Dirichlet/Neumann boundary conditions

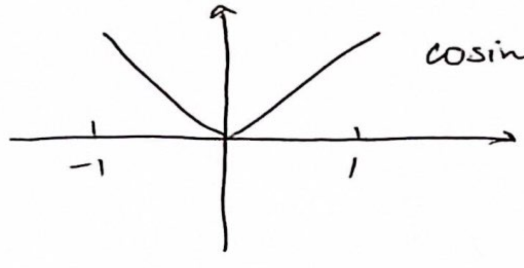
Given a function $f \in L^2(0,\rho)$

You have,

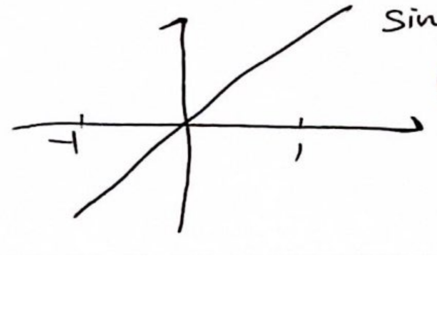
$$\begin{cases} f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\rho}\right) \\ f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\rho}\right) \end{cases} \quad \text{in } L^2$$

The RHS in both expressions are defined for (almost everywhere) $x \in \mathbb{R}$.

<u>Difference</u> : $\begin{cases} \text{cosine series is even} \\ \text{sine series is odd.} \end{cases}$



cosine



Sine.

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§2.5 Periodic Even-Odd extensions

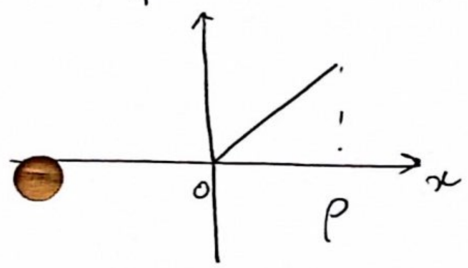
Let $f \in L^2(0,p)$ → cosine series

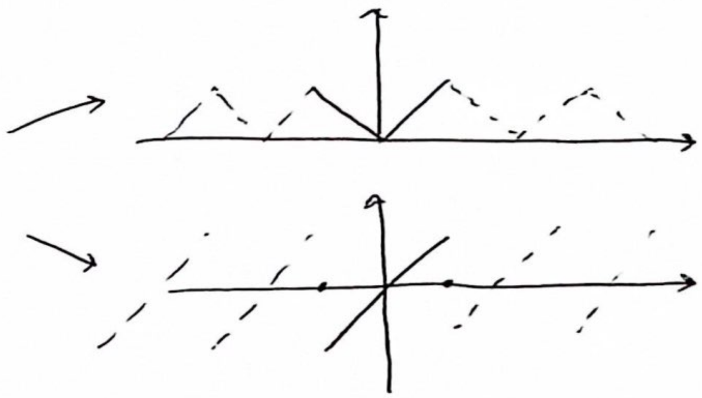
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right) \quad 2p\text{-periodic (even)}$$

→ sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right) \quad 2p\text{-periodic (odd)}$$

$f(x) = x$ on $(0,p)$





even ext.

odd ext.

Question: Let $f \in L^2(0,p)$

Sine series: $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$ (defined a.e on $\mathbb{R}$)

Particularly, if $x = kp$ where $k \in \mathbb{Z}$

then $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi kp}{p}\right) = 0$

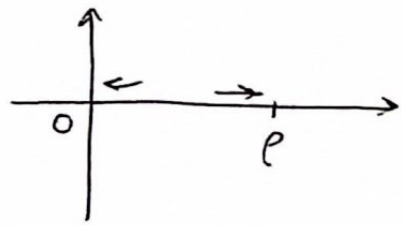
Conclusion: f satisfies Dirichlet? False since only the periodic ext is 0 but not the function itself, we don't have pointwise convergence. Even under pointwise convergence, DBC is a "limit condition" & does not consider the actual value at a pt.

Therefore, in order for the Fourier series

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\rho}\right)$$ to satisfy the Dirichlet condition at

$0, \rho$, we need

$$\begin{cases} \lim\limits_{x \to 0^+} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\rho}\right) = 0 \\ \lim\limits_{x \to \rho^-} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\rho}\right) = 0 \end{cases}$$



If continuity is not guaranteed, the limit condition may not be satisfied.

We need the sine series to converge to continuous function (at least continuous at the points 0 & ρ) to be able to talk about the Dirichlet condition.

Recall that, if $\sum_{n=1}^{\infty} |b_n| < \infty$, then the sine series defines a continuous function, this continuous function must satisfy the Dirichlet boundary condition.

What about the Neumann condition?

Formally, write $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$

$$f'(x) = \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)\right)'$$

maybe illegal.

$$\stackrel{?}{=} \left(\frac{a_0}{2}\right)' + \sum_{n\geq 1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{p}\right)\right)'$$

$$= -\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{p}\right) \sin\left(\frac{n\pi x}{p}\right)$$

$$= -\frac{\pi}{p} \sum_{n=1}^{\infty} (n a_n) \sin\left(\frac{n\pi x}{p}\right)$$

if $\sum_{n=1}^{\infty} |n a_n| < \infty$, then f' is continuous $\Rightarrow$ satisfies Dirichlet.

$\Rightarrow f$ satisfies the Neumann condition.

For this to make sense, we need to justify termwise differentiation of Fourier series.

§ 2.6 Termwise differentiation of Fourier series.

Lemma: Let f be $2p$-periodic such that $f' \in L^2(-p,p)$. (f is a.e. differentiable).

...en $\displaystyle \frac{1}{2\rho}\int_{-\rho}^{\rho} f'(x)\, e^{-\pi i k x/\rho}\, dx = \frac{\pi i k}{\rho}\,\frac{1}{2\rho}\int_{-\rho}^{\rho} f(x)\, e^{-\pi i k x/\rho}\, dx.$

In other words, if $\displaystyle f(x) = \sum_{k=-\infty}^{\infty} c_k\, e^{\pi i k x/\rho}$, then

$$f'(x) = \frac{\pi i}{\rho}\sum_{k=-\infty}^{\infty} k\, c_k\, e^{\pi i k x/\rho}$$ Equality in L^2 space.

<u>Lemma</u>: if f is 2ρ-periodic, $f' \in L^2(-\rho,\rho)$, then

f is everywhere $\frac{1}{2}$-Hölder continuous!

$$f(y) - f(x) = \int_x^y f'(z)\, dz$$

$$|f(y) - f(x)| = \left|\int_x^y f'(z)\,dz\right| = \left|\int_x^y f'(z)\cdot 1\, dz\right| = |\langle f', 1\rangle|$$

$$\le \|f'\|\cdot\|1\| = \sqrt{\int_x^y |f'(z)|^2 dz}\cdot\sqrt{\int_x^y 1^2 dz} \le \underbrace{\sqrt{\int_{-\rho}^{\rho} |f'(z)|^2 dz}}_{\text{constant}}\,|y-x|^{\frac{1}{2}}$$

$\Rightarrow$ f is $\frac{1}{2}$-Hölder continuous everywhere! //

<u>Q</u>. What about the pointwise convergence of $f'(x) = \frac{\pi i}{\rho}\sum_k k\, c_k\, e^{\frac{\pi i k x}{\rho}}$?

1) <u>if f' satisfies</u> $\begin{cases} |f'(x_0+z) - A| \le M z^{\alpha} \\ |f'(x_0-z) - B| \le N z^{\beta} \end{cases}$ $\forall\, z \in (0,\delta)$ $\{\alpha,\beta > 0\}$

Then at $x = x_0$,

$$\frac{\pi i}{p} \sum_{k=-\infty}^{\infty} k\, c_k\, e^{\pi i k x_0/p} = \frac{1}{2}(A+B).$$

2) If f' is differentiable at x_0 (i.e) $f''(x_0)$ exists,

then $$\frac{\pi i}{p} \sum_{k=-\infty}^{\infty} k\, c_k\, e^{\pi i k x_0/p} = f'(x_0).$$

Observe that, ① $f \in L^2(-p,p) \iff \sum_{k=-\infty}^{\infty} |c_k|^2 < \infty$

② $f' \in L^2(-p,p) \iff \sum_{k=-\infty}^{\infty} |k|^2 |c_k|^2 < \infty$

$\Rightarrow k c_k \to 0$ (as $k \to \infty$)

<u>Generalization to Higher order</u>

<u>Def</u>: Let (a,b) be an open interval. Let $m \in \mathbb{N}$, the space of functions $H^m(a,b)$ is the set of all functions $f: (a,b) \to \mathbb{C}$ s.t $f^{(m)} \in L^2(a,b)$.

We have shown that, if $f^{(m)} \in L^2(a,b)$, then $f^{(m-1)}$ $\frac{1}{2}$- continuous. $\Rightarrow f^{(m-1)} \in L^2$. Therefore,

$$H^m \subset H^{m-1} \subset H^{m-2} \subset \dots \subset H^0 = L^2$$

<u>Theorem:</u> If f is $2p$-periodic, $f \in H^m(-p,p)$ and write

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{\pi i k x/p}. \quad \text{Then } f^{(m)}(x) = \sum_{k=-\infty}^{\infty} \left(\frac{\pi i k}{p}\right)^m c_k e^{\pi i k x/p}$$

(in L^2 a.e.).

<u>Corollary:</u> (under the same condition)

$$\sum_{k=-\infty}^{\infty} |k^m c_k|^2 < \infty$$

$$\sum_{k=-\infty}^{\infty} |k|^{2m} |c_k|^2 < \infty$$

Particularly,
$\lim_{|k| \to \infty} k^m c_k = 0.$

c_k goes to 0 faster than $\frac{1}{k^m}$

Lec 20 Quick review of Sobolev Spaces & their relations with Fourier series

Def :- Let $m \in \mathbb{N}$, then the "Sobolev Space" $H^m(a,b)$ is the set of all functions $f : (a,b) \longrightarrow \mathbb{C}$, such that f is a.e m-times differentiable.

$$f^{(m)} \in L^2(a,b); \quad \text{i.e} \int_a^b |f^{(m)}(x)|^2 dx < \infty .$$

From the def :

$$L^2 = H^0 \supset H^1 \supset H^2 \cdots \supset \bigcap_{n=0}^{\infty} H^n = H^\infty \qquad (*)$$

(in fact $H^\infty \subset C^\infty$, space of smooth functions)

Thm: Let $f \in L^2(-\rho,\rho)$ (& 2ρ-periodic) and write $f(x) = \sum_K C_K e^{\frac{+\pi i k x}{\rho}}$

$\Rightarrow \forall m \in \mathbb{N}$, $f \in H^m(-\rho,\rho)$ iff $(k^m C_K) \in \ell^2(\mathbb{Z})$.

in other words,

$$\int_{-\rho}^{\rho} |f^{(m)}(x)|^2 dx < \infty \iff \sum_{K=-\infty}^{\infty} |K|^{2m} |C_K|^2 < \infty \qquad (**)$$

Rmk : $(*)$ is a direct consequence of $(**)$

$$\Rightarrow \quad f^{(m)}(x) = \sum_{K=-\infty}^{\infty} \left(\frac{\pi i k}{\rho}\right)^m C_K \, e^{\pi i k x/\rho}$$

More precisely, (repeating)

$$f^{(m)}(x) = \sum_{k=-\infty}^{\infty} \left(\frac{\pi i k}{\rho}\right)^m c_k \, e^{\pi i k x/\rho}$$

To prove this Thm, we use,

<u>Lemma</u>: If $f \in H^1(-\rho,\rho)$ then $\int_{-\rho}^{\rho} f'(x) e^{+\pi i k x/\rho} dx = \frac{\pi i k}{\rho} \int_{-\rho}^{\rho} f(x) e^{\frac{\pi i k x}{\rho}} dx$.

<u>Proof</u>:- Integrate by parts.

$$\int_{-\rho}^{\rho} f'(x) e^{+\frac{\pi i k x}{\rho}} dx = \left[f(x) e^{\frac{+\pi i k x}{\rho}}\right]_{-\rho}^{\rho} - \int f(x) \left(e^{\frac{\pi i k x}{\rho}}\right)' dx$$

How do we talk about $f(x)$ at $x = \pm\rho$?

$f \in H^1 \Rightarrow f$ is $\frac{1}{2}$ Hölder continuous.

$\Rightarrow$ in fact it is uniformly continuous

since $|f(x) - f(y)| \leq C |y-x|^{1/2}$

$\Rightarrow$ we can talk about $f(\pm\rho)$ when f is continuous.

Consequently, the boundary value vanishes,

$$\int_{-\rho}^{\rho} f'(x) e^{-\pi i k x/\rho} dx = -\int_{-\rho}^{\rho} f(x) \left(e^{-\pi i k x/\rho}\right)' dx = \frac{\pi i k}{\rho} \int_{-\rho}^{\rho} f(x) \, e^{-\pi i k x/\rho} dx.$$

Section 3 Fourier method.

3.1 1D - Laplace equation.

$$u'' = f \qquad -\rho < x < \rho$$
$$u(-\rho) = u(\rho) \quad , u'(-\rho) = u'(\rho)$$

Formally write $\quad u(x) = \sum_{k=-\infty}^{\infty} c_k e^{\pi i k x/\rho}$

$$f(x) = \sum_{k=-\infty}^{\infty} \gamma_k e^{\pi i k x/\rho}$$

$$u'' = \sum_{k=-\infty}^{\infty} c_k \left(\frac{\pi i k}{\rho}\right)^2 e^{\pi i k x/\rho}$$

Equating Fourier coefficients

$$c_k \left(\frac{\pi i k}{\rho}\right)^2 = \gamma_k \quad \forall k \Rightarrow c_k = -\frac{\gamma_k \rho^2}{\pi^2 k^2}, \quad k \neq 0$$

When $k = 0$, $\gamma_0 = 0$ in order to obtain a valid solution & c_0 can be arbitrary.

Conclusion:

① If $\gamma_0 = \frac{1}{2\rho} \int_{-\rho}^{\rho} f(x)\,dx \neq 0$ then no solution.

② If $\gamma_0 = 0$, then ∞-number of solns. $\Big\}$ $u(x) = c_0 - \frac{\rho^2}{\pi^2} \sum_{k \neq 0} \frac{\gamma_k}{k^2} e^{\pi i k x/\rho}$

//

Rmk on ①

$$\int_{-p}^{p} f(x)\,dx = \int_{-p}^{p} u''(x)\,dx = \int_{-p}^{p} (u'(x))'\,dx = u'(p) - u'(-p) = 0$$

② If u is a solution,

then $u + C$ is also a solution.

In order to make these operations legal,

we need $f \in L^2(-p,p)$

We look for solutions $u \in H^2(-p,p)$.

Let $f \in L^2(-p,p)$. We would like to find $u \in H^2(-p,p)$ s.t

$u'' = f$ and periodic conditions.

Write
$$f(x) = \sum_{k=-\infty}^{\infty} \gamma_k\, e^{\pi i k x/p}$$

$$u(x) = \sum_{k=-\infty}^{\infty} C_k\, e^{\pi i k x/p}$$

$$u \in H^2 \Rightarrow \quad u' = \sum_{k=-\infty}^{\infty} C_k \left(\frac{\pi i k}{p}\right)^2 e^{\pi i k x/p}$$

We find that if $\gamma_0 = 0$ then

$$u(x) = C_0 - \frac{p^2}{\pi^2} \sum_{k} \frac{\gamma_k}{k^2}\, e^{\pi i k x/p}$$ is a soln. Why?

it suffices to verify $u \in H^2$

Why $u \in H^2$?

Because $\sum_{k=-\infty}^{\infty} |k^2 c_k|^2 = \sum_{k \neq 0} \left| k^2 \frac{\gamma_k}{k^2} \frac{p^2}{\pi^2} \right|^2$

$$= \frac{p^4}{\pi^4} \sum_{k \neq 0} |\gamma_k|^2 < \infty \quad (f \in L^2).$$

$\Rightarrow u \in H^2$.

To verify the boundary condition,

$u(-p) = u(+p)$

$u'(-p) = u'(+p)$

Need to show that u, u' are $2p$ periodic & continuous.

$u \in H^2 \Rightarrow u' \in H^1 \Rightarrow u'$ $\frac{1}{2}$ Hölder continuous

$\Rightarrow u'(+p) = u'(-p)$ //

Lec 21 (Correcting errors from previous lecture)

Lemma: f: 2ρ-periodic, $^{(*)} f \in H^1(-\rho,\rho)$ then $\int_{-\rho}^{\rho} f'(x) e^{-\pi i k x/\rho} dx$

$$= \frac{\pi i k}{\rho} \int_{-\rho}^{\rho} f(x)\, e^{-\pi i k x/\rho} dx$$

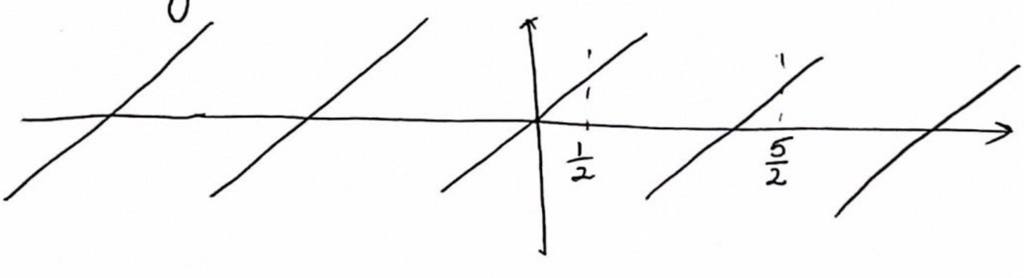
(*) Has a problem which makes this lemma incorrect. It has to be replaced by a stronger condition.

(**) $f \in H^1(x_0-\rho, x_0+\rho)$ $\forall x_0 \in \mathbb{R}$ { In HW, (**) $\Rightarrow f \in H^1_{loc}$ i.e $f \in H^1(a,b)$ $\forall (a,b)$

The conditions (*) & (**) are not equivalent.

Counter example : f : 2-periodic $f(x) = x$ $\quad -1 < x < 1$

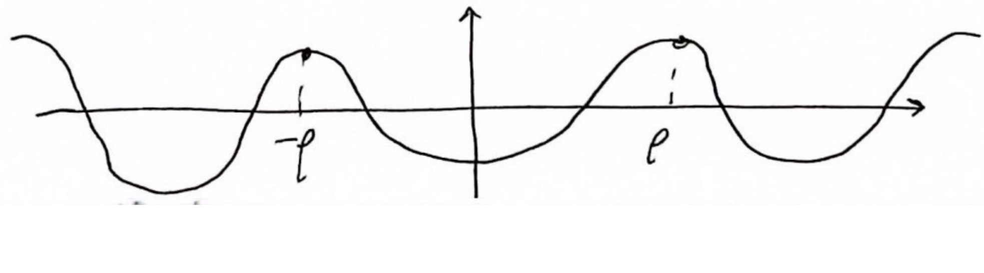
Clearly $f \in H^1(-1,1)$ but $f \notin H^1(\frac{1}{2}, \frac{5}{2})$



If $f \in H^1(\frac{1}{2}, \frac{5}{2})$ then f is continuous over $(\frac{1}{2}, \frac{5}{2})$. Which is clearly untrue.

Rmk : (**) $\Rightarrow$ (*) + $\lim_{x \to \rho^+} f(x) = \lim_{x \to \rho^-} f(x)$ These 2 limits are equal & finite.

f: 2ρ-periodic. $f \in H^1(-\rho,\rho)$



Rmk: f 2ρ-periodic, $f \in H^2(-\rho,\rho)$ $(*)$

$f \in H^2(x_0-\rho, x_0+\rho)\ \forall x_0$ $(**)$

$(**) \Longleftrightarrow (*) + \boxed{\begin{array}{l} \lim\limits_{x\to -\rho^+} f(x) = \lim\limits_{x\to \rho^-} f(x) \\ \lim\limits_{x\to -\rho^+} f'(x) = \lim\limits_{x\to \rho^-} f'(x) \end{array}}$ ← periodic boundary condition for 1D (Laplacian $\Delta = \partial_x^2$

$u(\rho) = u(-\rho)$;

$u'(\rho) = u'(-\rho)$.

Recall that,

if we use the Fourier method to solve.

$\partial_x^2 u = f \in L^2(-\rho,\rho)$

$u(\rho) = u(-\rho)$, $u'(\rho) = u'(-\rho)$.

$\Rightarrow u \in H^2(-\rho,\rho)$] In fact, solutions obtained this way belongs to a stronger class of functions.

$u \in H^2_{loc}$, i.e. $u \in H^2(x_0-\rho, x_0+\rho)\ \forall x_0$

i.e. By using the Fourier method, you assume (implicitly) the periodic boundary condition.

---

eg: $\begin{cases} u'' = 0 \\ u(-1) = u(1)\ ;\ u'(-1) = u'(1) \end{cases}$

Look for solutions $u \in H^2_{loc}$. Write $u(x) = \sum_{k=-\infty}^{\infty} c_k e^{\pi i k x}$

$u''(x) = \sum_{k=-\infty}^{\infty} (\pi i k)^2 c_k e^{\pi i k x} = 0 \Rightarrow (\pi i k)^2 c_k = 0 \Rightarrow c_k = 0\ \forall k \neq 0$

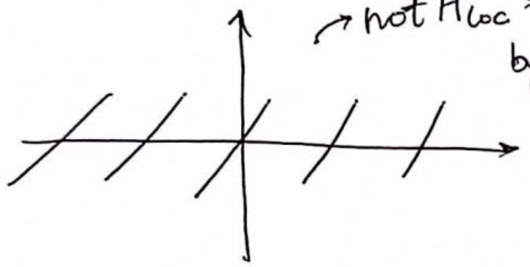
$\Rightarrow u(x) = c_0\ \forall c_0$.

Another method

$u'' = 0 \Rightarrow u(x) = Ax + B$ or $\alpha + \beta x$.

boundary condition $\Rightarrow \beta = 0$.

(i) $\beta \neq 0$

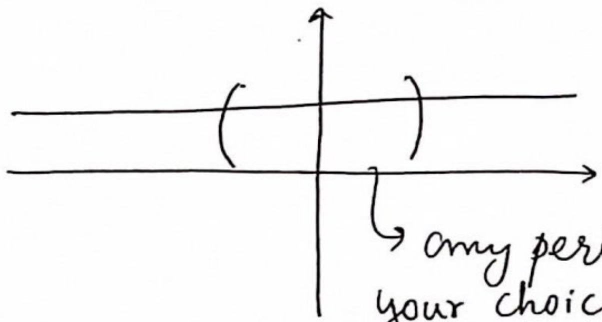


$\to$ not $H^2_{loc} \Rightarrow$ discarded by Fourier method.

$\downarrow$

Fourier method implicitly enforces (periodic) boundary condition

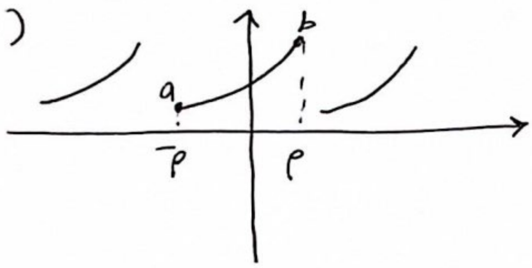
(ii)



$\to$ any period of your choice $u \in H^1(\text{—})$.

Q: Why are periodic conditions required to extend $f \in H^1(-\rho, \rho)$ to $f \in H^1_{loc}$.

(i)



$$\begin{cases} f \in H^1(-\rho,\rho) & \xrightarrow{\text{periodize}} \text{new } f \notin H^1_{loc}. \\ g = f' \in L^2(-\rho,\rho) & \text{new } g \in L^2_{loc} = H^0_{loc} \end{cases}$$

$\text{new} f' \neq \text{new} g$

$$f' = g + \sum_{k \in \mathbb{Z}} (a-b)\, \delta_{(2k+1)\rho}$$

Rmk: f 2ρ periodic,

(*) $f \in H^m_{loc}$, i.e. $f \in H^m(x_0 - \rho, x_0 + \rho)$ $\forall x_0$

$\Updownarrow$

(**) $f \in H^m(-\rho, \rho)$ + $\forall j = 0, 1, 2, \dots, m-1$, the 2 limits are equal & finite.

i.e. $f^{(j)}(x) = \lim f^{(j)}(x)$

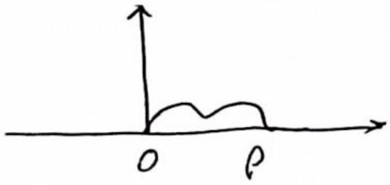
1D Laplacian with Dirichlet Condition.

- $u'' = f \in L^2(0,\rho)$

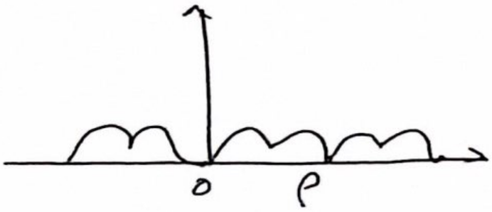
  $u(0) = u(\rho) = 0.$

Fourier method $\longleftarrow$ periodization, how?

Given



$\xrightarrow{\text{Try}}$

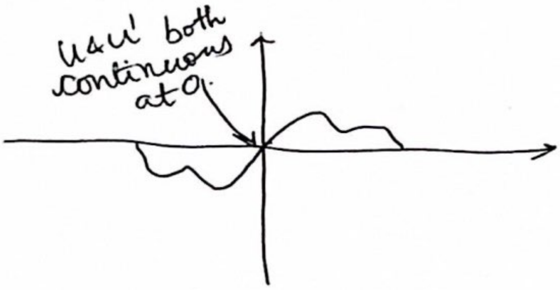


> This give a continuous function with possibly discontinuous derivatives

- In general such functions do not belong to H^2_{loc}.

Try another method

Odd periodization.



$$\begin{cases} u'' = f & 0 < x < \rho \\ u(0) = u(\rho) = 0 \end{cases}$$

$$\Downarrow$$

$$\tilde{u}(x) = \begin{cases} u(x) & 0 < x < \rho \\ -u(-x) & -\rho < x < 0 \end{cases}$$

$$\tilde{f}(x) = \begin{cases} f(x) & 0 < x < \rho \\ -f(-x) & -\rho < x < 0 \end{cases}$$

Do we still have $\tilde{u}'' = \tilde{f}$?

- For $-\rho < x < 0$ $\quad \tilde{u}'' = -u(-x)'' = -u''(-x) = -f(-x) = \tilde{f}(x)$

Dirichlet Laplacian

$$\begin{cases} u'' = f \in L^2(0,p) \\ u(0) = u(p) = 0 \end{cases}$$

$$\Updownarrow$$

Periodic Laplacian / odd data

$\tilde{u}'' = \tilde{f} \in L^2_{loc}$ & $2p$-periodic.

$\tilde{u}, \tilde{f}$ odd.

odd + $2p$-periodic $\Rightarrow \tilde{u}(0) = \tilde{u}(p) = 0$

Looking for solutions $\begin{cases} \tilde{u} \in H^2_{loc} \\ \text{odd} \\ 2p\text{-periodic} \end{cases}$

$$\tilde{u}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

Then write

$$\tilde{f}(x) = \sum_{n=1}^{\infty} \gamma_n \sin\left(\frac{n\pi x}{p}\right)$$

$$\tilde{u}''(x) = -\sum_{n=1}^{\infty} b_n \left(\frac{n\pi}{p}\right)^2 \sin\left(\frac{n\pi x}{p}\right) = \tilde{f}(x)$$

$$\Rightarrow \forall n, \quad -b_n \left(\frac{n\pi}{p}\right)^2 = \gamma_n \quad \Rightarrow \quad b_n = \frac{-\gamma_n}{\left(n^2\pi^2/p^2\right)}$$

$$\tilde{u}(x) = -\sum_{n=1}^{\infty} \frac{\gamma_n}{n^2\pi^2/p^2} \sin\left(\frac{n\pi x}{p}\right)$$

$$= \frac{-p^2}{\pi^2} \sum_{n=1}^{\infty} \frac{\gamma_n}{n^2} \sin\left(\frac{n\pi x}{p}\right)$$

because $\sum |\gamma_n|^2 < \infty \Rightarrow \tilde{u} \in H^2_{loc} \Rightarrow \tilde{u}, \tilde{u}'$ continuous

because $\tilde{u}(0) = \tilde{u}(p) = 0 \Rightarrow$ Dirichlet condition.

Q: 1D Neumann Laplacian

- Make even, then periodize
- Involves cosine series.

Lec 22 Applications of Fourier method.

Laplace equation (1D) ✓ done.

Heat equation $\partial_t u = \partial_x^2 u \qquad x \in (0,\rho) \qquad t > 0$

$$u(0,x) = u_0$$

$$u(t,0) = u(t,\rho) = 0 \qquad t > 0$$

Formally, $u(t,x) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi x}{\rho}\right)$.

$$\partial_t u(t,x) = \sum_{n=1}^{\infty} b_n'(t) \sin\left(\frac{n\pi x}{\rho}\right)$$

$$\partial_x^2 u(t,x) = \sum_{n=1}^{\infty} b_n(t) \left[-\left(\frac{n\pi}{\rho}\right)^2 \sin\left(\frac{n\pi x}{\rho}\right)\right]$$

At $t=0$,

$$u(0,x) = \sum_{n=1}^{\infty} b_n(0) \sin\left(\frac{n\pi x}{\rho}\right)$$

$$\parallel$$

$$u_0(x) = \sum_{n=1}^{\infty} \gamma_n \sin\left(\frac{n\pi x}{\rho}\right) \qquad \Rightarrow b_n(0) = \gamma_n.$$

In conclusion,

$$b_n'(t) = -\frac{n^2\pi^2}{\rho^2} b_n(t) \qquad \forall n \in \mathbb{N}^*$$

$$b_n(0) = \gamma_n$$

Solution: $b_n(t) = \gamma_n e^{-\frac{n^2\pi^2}{\rho^2}t}$

$$u(t,x) = \sum_{n=1}^{\infty} \gamma_n e^{-\frac{n^2\pi^2}{\rho^2}t} \sin\left(\frac{n\pi x}{\rho}\right)$$

$$u_0 \in L^2(0,\rho) \Rightarrow \sum_{n=1}^{\infty} |\gamma_n|^2 < \infty \Rightarrow \sum_{n=1}^{\infty} n^{2m} \left|\gamma_n e^{-\frac{n^2\pi^2}{\rho^2}t}\right|^2 < \infty \quad \forall m \in \mathbb{N}, \ \forall t > 0$$

$\hookrightarrow$ exp. decay

$$\Rightarrow u(t,\cdot) \in H^{\infty}(0,\rho)$$

Q: What happens when $t<0$?

The factor $e^{-\frac{n^2\pi^2}{\rho^2}t}$ grows exponentially $\uparrow \infty$ as $n \uparrow \infty$.

The solution $u(t,x) = \sum_{n=1}^{\infty} \gamma_n e^{-\frac{n^2\pi^2}{\rho^2}t} \sin\left(\frac{n\pi x}{\rho}\right)$

is generally not defined for $u_0 \in L^2$

As in general $\left(\gamma_n e^{-\frac{n^2\pi^2}{\rho^2}t}\right)_{n=1}^{\infty} \notin \ell^2(\mathbb{N}^*)$

for fixed $t<0$

i.e. $u(t,\cdot) \notin L^2(0,\rho)$.

[of course, in some special cases, the solution is still defined]
e.g. when u_0 has a finite Fourier series.

Boundary condition? Dirichlet condition satisfied when $t>0$.
u_0 does not need to satisfy the Dirichlet condition

Another perspective

$$\partial_t u = \partial_x^2 u \quad x \in (0,\rho), \ t>0$$

$\forall$ time fixed

$$\partial_x^2 u \in L^2(0,\rho) \Leftrightarrow u \in H^2(0,\rho)$$

$$\parallel$$

$$\partial_t u \in L^2(0,\rho)$$

Assuming that u continuous in t.

$\partial_t u$ continuous in t.

Let $b_n(t) = \frac{2}{\ell}\int_0^\ell u(t,x)\sin\left(\frac{n\pi x}{\ell}\right)dx$

$$\frac{d}{dt} b_n(t) = \frac{2}{\ell}\int_0^\ell \partial_t u(t,x)\sin\left(\frac{n\pi x}{\ell}\right)dx.$$

$$= \frac{2}{\ell}\int_0^\ell \partial_x^2 u(t,x)\sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{-2}{\ell}\left(\frac{n\pi}{\ell}\right)^2\int_0^\ell u(t,x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$\Rightarrow \frac{d}{dt} b_n(t) = -\left(\frac{n\pi}{\ell}\right)^2 b_n(t).$$

<u>Conclusion</u>: In the function space,

① $u \in H^2$ w.r.t x

② $\partial_t u$ continuous w.r.t t.

There exists a unique solution to the BVP.

$$u(t,x) = \sum_{n=1}^{\infty} \gamma_n e^{\frac{-n^2\pi^2}{\ell^2}t}\sin\left(\frac{n\pi x}{\ell}\right)$$

<u>Ex</u>: Show that u is smooth in t.

i.e. $\partial_t^m u$ is continuous in t $(\forall m)$

<u>Rmk</u>: Therefore u is smooth in $(t,x) \in (0,\infty)\times(0,\ell)$

Schrödinger eqn: Exercise.

Wave eqn: $\partial_t^2 u = c^2 \partial_x^2 u \qquad (c>0)$

$\partial_x u(t,0) = \partial_x u(t,\rho) = 0 \qquad 0<x<\rho$

$\qquad -\infty < t < \infty.$

$u(0,x) = u_0(x) \qquad \partial_t u(0,x) = u_1(x)$

$$u(t,x) = \frac{a_0(t)}{2} + \sum_{n\geq 1}^{\infty} a_n(t) \cos\left(\frac{n\pi x}{\rho}\right)$$

$$\partial_t^2 u(t,x) = \frac{a_0''(t)}{2} + \sum_{n\geq 1}^{\infty} a_n''(t) \cos\left(\frac{n\pi x}{\rho}\right)$$

$$\partial_x^2 u(t,x) = 0 + \sum_{n\geq 1}^{\infty} a_n(t) \left(-\frac{n^2\pi^2}{\rho^2} \cos\left(\frac{n\pi x}{\rho}\right)\right)$$

$$\begin{cases} a_0''(t) = 0 \\ a_n''(t) = \dfrac{-n^2\pi^2c^2}{\rho^2} a_n(t) \qquad n\geq 1 \end{cases}$$

$$\begin{cases} a_0(t) = P_0 + q_0 t \\ a_n(t) = P_n \cos\left(\dfrac{n\pi ct}{\rho}\right) + q_n \sin\left(\dfrac{n\pi ct}{\rho}\right) \end{cases}$$

$$u(t,x) = \frac{P_0 + q_0 t}{2} + \sum_{n=1}^{\infty} \left\{ P_n \cos\left(\frac{n\pi ct}{\rho}\right) + q_n \sin\left(\frac{n\pi ct}{\rho}\right)\right\} \times \cos\left(\frac{n\pi x}{\rho}\right)$$

$$\partial_t u(t,x) = \frac{q_0}{2} + \sum_{n\geq 1}^{\infty} \left\{ -P_n \frac{n\pi c}{\rho} \sin\left(\frac{n\pi ct}{\rho}\right) + q_n \cos\left(\frac{n\pi ct}{\rho}\right)\right\} \cos\left(\frac{n\pi x}{\rho}\right)$$

$$\left.\begin{array}{l} u(0,x) = \dfrac{p_0}{2} + \displaystyle\sum_{n=1}^{\infty} p_n \cos\left(\frac{n\pi x}{\rho}\right) \\ \quad \| \\ u_0(x) = \dfrac{\gamma_0}{2} + \displaystyle\sum_{n=1}^{\infty} \gamma_n \cos\left(\frac{n\pi x}{\rho}\right) \end{array}\right\} \Rightarrow p_n = \gamma_n$$

$$\left.\begin{array}{l} \partial_t u(0,x) = \dfrac{q_0}{2} + \displaystyle\sum_{n=1}^{\infty} q_n \frac{n\pi c}{\rho} \cos\left(\frac{n\pi x}{\rho}\right) \\ u_1(x) = \dfrac{w_0}{2} + \displaystyle\sum_{n=1}^{\infty} w_n \cos\left(\frac{n\pi x}{\rho}\right) \end{array}\right\} \Rightarrow \begin{array}{l} q_0 = w_0 \\ q_n = \dfrac{w_n \rho}{n\pi c} \end{array}$$

$$u(t,x) = \frac{\gamma_0 + w_0 t}{2} + \sum_{n=1}^{\infty} \left\{ \gamma_n \cos\left(\frac{n\pi c t}{\rho}\right) + \frac{w_n \rho}{n\pi c} \sin\left(\frac{n\pi c t}{\rho}\right) \right\} \cos\left(\frac{n\pi x}{\rho}\right)$$

: Boundary condition? i.e. $\partial_x u$ vanishes at $x \to 0$ or ρ?

Clearly $\partial_x u$ vanishes at $x = 0$ or $x = \rho$; it remains to show continuity in x.

We need summability of the Fourier coefficients of $\partial_x u$.

$$\partial_x u(t,x) = \sum_{n=1}^{\infty} \left\{ \gamma_n \cos\left(\frac{n\pi c t}{\rho}\right) + \frac{w_n \rho}{n\pi c} \sin\left(\frac{n\pi c t}{\rho}\right) \right\} \left(-\frac{n\pi}{\rho} \sin\left(\frac{n\pi x}{\rho}\right) \right)$$

We would like,

$$(*) \quad \begin{cases} \displaystyle\sum_{n=1}^{\infty} |\gamma_n|\, n < \infty \\ \sum |w_n| < \infty \end{cases} \qquad \left(\begin{array}{l} \Rightarrow \text{absolute convergence of } \partial_x u \\ \Rightarrow \text{continuity of } \partial_x u \end{array} \right.$$

We can guarantee (*) by requiring $\begin{cases} \sum n^2 |\omega_n|^2 < \infty \\ \sum n^4 |\gamma_n|^2 < \infty \end{cases}$ (By Cauchy's Inequality)

$\hookrightarrow (**)$

$$\begin{cases} \sum |\omega_n| \leq \sqrt{\sum (n^2 |\omega_n|^2)} \sqrt{\sum \frac{1}{n^2}} \\ \sum n |\gamma_n| \leq \sqrt{\sum (n^4 |\gamma_n|^2} \sqrt{\sum \frac{1}{n^2}} \end{cases}$$

$(**) \Leftrightarrow$ the even periodic extension of $u_0 \in H^2_{loc}$ & the even periodic extension of $u_1 \in H^1_{loc}$.

Lec 23

Wave eqn.
$$\begin{cases} \partial_t^2 u = c^2 \partial_x^2 u & (0,\ell)\times\mathbb{R} \\ \partial_x u(t,0) = \partial_x u(t,\ell) = 0 & \text{Neumann.} \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x). \end{cases}$$

Fourier method: $u(t,x) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cos\left(\frac{n\pi x}{\ell}\right)$

ODEs for Fourier coefficients.

$$a_n'' = -\frac{n^2\pi^2c^2}{\ell^2} a_n$$

Solutions to ODEs

$$a_n(t) = \begin{cases} p_0 + q_0 t \\ p_n \cos\left(\frac{n\pi ct}{\ell}\right) + q_n \sin\left(\frac{n\pi ct}{\ell}\right), & n \geq 1 \end{cases}$$

$$\Rightarrow u(t,x) = \frac{p_0+q_0t}{2} + \sum_{n=1}^{\infty}\left[p_n \cos\left(\frac{n\pi ct}{\ell}\right) + q_n \sin\left(\frac{n\pi ct}{\ell}\right)\right]\cos\left(\frac{n\pi x}{\ell}\right)$$

$$\partial_t u(t,x) = \frac{q_0}{2} + \sum_{n=1}^{\infty}\left[-\frac{n\pi c}{\ell} p_n \sin\left(\frac{n\pi ct}{\ell}\right) + \frac{n\pi c}{\ell} q_n \cos\left(\frac{n\pi ct}{\ell}\right)\right] \times \cos\left(\frac{n\pi x}{\ell}\right)$$

$t=0 \Rightarrow$

$$u(0,x) = \frac{p_0}{2} + \sum_{n=1}^{\infty} p_n \cos\left(\frac{n\pi x}{\ell}\right) = u_0(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos\left(\frac{n\pi x}{\ell}\right)$$

$$\partial_t u(0,x) = \frac{q_0}{2} + \sum_{n=1}^{\infty} \frac{n\pi c}{\ell} q_n \cos\left(\frac{n\pi x}{\ell}\right) = u_1(x) = \frac{\beta_0}{2} + \sum_{n=1}^{\infty} \beta_n \cos\left(\frac{n\pi x}{\ell}\right)$$

$\Rightarrow \quad p_n = \alpha_n \quad ; \quad n = 0, 1, 2 \ldots$

$q_0 = \beta_0 \;, \quad q_n = \frac{\rho}{n\pi c} \beta_n . \quad n = 1, 2, \ldots$

$$\Rightarrow \quad u(t,x) = \frac{\alpha_0 + \beta_0 t}{2} + \sum_{n=1}^{\infty} \left[\alpha_n \cos\left(\frac{n\pi c t}{\rho}\right) + \frac{\rho}{n\pi c} \beta_n \sin\left(\frac{n\pi c t}{\rho}\right) \right] \cos\left(\frac{n\pi x}{\rho}\right)$$

Question : Does u satisfy the Neumann condition?

Neumann : $\Leftarrow \partial_x u$ continuous in x. (Need to show continuity at boundaries).

Not possible? to show continuity at a pt. for Fourier series, so must show continuity everywhere.

$\Leftarrow \quad \partial_x u \in H^1(0,\rho) \Rightarrow$ it is continuous

$\Leftrightarrow \quad u \in H^2(0,\rho) \leftarrow$ Note that by Fourier method u is even & 2ρ-periodic.

suffices to show
$\Leftarrow \quad u \in H^2_{loc}$ (i.e. the 2ρ periodic & even extension of u belongs to H^2_{loc}).

$\Leftrightarrow$ if $u(t,x) = \sum c_k e^{\pi i k x/\rho}$

then $(k^2 c_k) \in \ell^2 \Leftrightarrow \displaystyle\sum_{k=-\infty}^{\infty} k^4 |c_k|^2 < \infty .$

Or
$$u(t,x) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cos\left(\frac{n\pi x}{\rho}\right)$$

then $\displaystyle\sum_{k=1}^{\infty} n^4 |a_n(t)|^2 < \infty .$

$$a_n(t) = \alpha_n \cos\left(\frac{n\pi ct}{\rho}\right) + \frac{\rho}{n\pi c}\beta_n \sin\left(\frac{n\pi ct}{\rho}\right)$$

$$= A_n \cos\left(\frac{n\pi ct}{\rho} - \phi_n\right) \quad \text{where} \quad A_n = \sqrt{|\alpha_n|^2 + |\beta_n|^2 \frac{\rho^2}{n^2\pi^2c^2}}$$

We want $\sum_{n=1}^{\infty} n^4 |A_n|^2 < \infty$?? (at the minimum)

$$\Leftrightarrow \sum_{n=1}^{\infty} n^4 |\alpha_n|^2 < \infty \quad \& \quad \sum_{n=1}^{\infty} n^4 \frac{|\beta_n|^2}{n^2} < \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} n^2 |\beta_n|^2 < \infty \,//$$

$\Leftrightarrow$ The 2ρ-periodic & even extension of $u_0 \in H^2_{loc}$.

and the 2ρ-periodic even extension of $u_1 \in H^1_{loc}$.

Stationary Waves

By Fourier method, we express the solution to the wave eq. as a superposition of Fourier method.

$$u(t,x) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} \underbrace{a_n(t) \cos\left(\frac{n\pi x}{\rho}\right)}_{\text{each of these solves the wave eqn.}}$$

$$A_n \cos\left(\frac{n\pi ct}{\rho} - \phi_n\right)$$

$\Rightarrow$ We get a stationary wave.

* The Fourier expansion is thus a sum of stationary waves, which are in turn a sum of propagating waves in 2 directions

<u>Another method</u>. (will come in exam).

$$(\partial_t^2 - c^2\partial_x^2)\, u = 0 \qquad (x,t) \in \mathbb{R}\times\mathbb{R}\,. \quad ; \quad u(0) = u_0 \,;\, \partial_t u(0) = u_1\,.$$

$$\Rightarrow \quad (\partial_t - c\partial_x)(\partial_t + c\partial_x) = 0$$

Let $\quad w = (\partial_t - c\partial_x)u$.

$$\Rightarrow \quad (\partial_t + c\partial_x)\, w = 0$$

Now solve,

$$(\partial_t - c\partial_x)\, u = w \quad \& \quad (\partial_t + c\partial_x)\, w = 0$$

$$\left|\begin{aligned} u(0,x) &= u_0(x) \\ w(0,x) &= \partial_t u(0,x) - c\partial_x u(0,x) \\ &= u_1(x) - c\partial_x u_0(x) \end{aligned}\right.$$

It suffices to solve,

$$\left.\begin{aligned} (\partial_t + v\partial_x)\, u(t,x) &= f(t,x) \\ u(0,x) &= u_0(x). \end{aligned}\right\} \text{ Transport equation.}$$

① Homogeneous setting, $f = 0 \quad (\partial_t + \overset{\text{velocity} \in \mathbb{R}.}{v}\partial_x)\, u = 0$

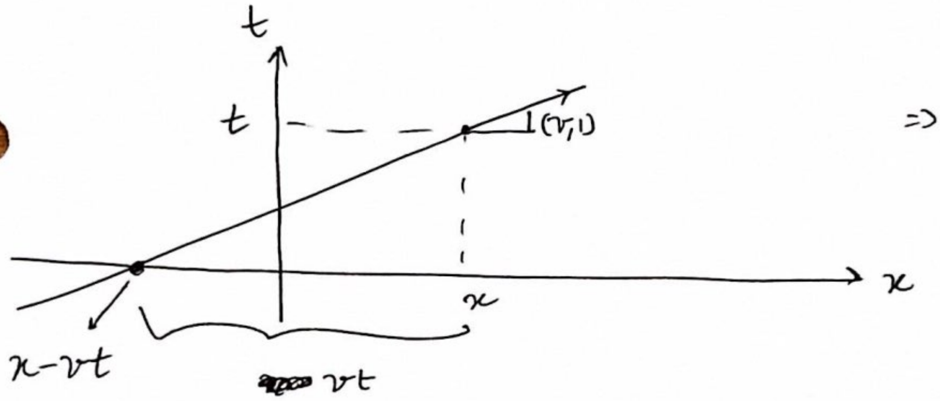
method 1

Let $\nabla_{xt} u = (\partial_x u, \partial_t u)$

Write $(\partial_t + v\partial_x) u = (v, 1) . \nabla_{xt} u . = 0$

$\Rightarrow$ $(\partial_t + v\partial_x) u$ is the directional derivative in space time $\mathbb{R} \times \mathbb{R}$ in $(v,1)$ direction.

$\Rightarrow$ u is constant along $(v,1)$ direction.



$\Rightarrow u(x,t) = u(x - vt, 0)$

$= u_0(x - vt)$ //

method 2   Let $w(t, x) = u(t, x + vt)$

$$\partial_t w(t,x) = (\partial_t u)(t, x+vt) + (\partial_x u)(t, x+vt)\, \partial_t(x+vt)$$

$$= (\partial_t u)(t, x+vt) + (v\partial_x u)(x + vt)$$

$$= (\partial_t u + v\partial_x u)(t, \cancel{x} + vt) = 0$$

$\Rightarrow$ w is time independant.

$w(t,x) = w(0,x) \Leftrightarrow u(t, x+vt) = u(0, x + v0)$

$\Leftrightarrow \quad u(t,x) = u_0(x - vt)$ //

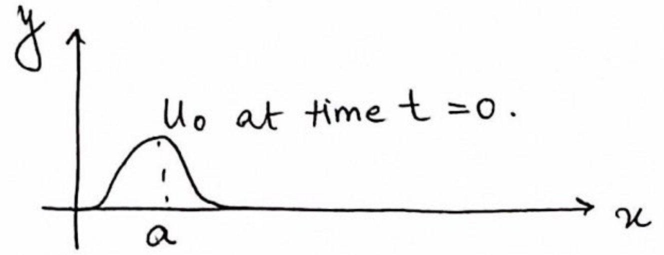
Lec 24 Transport Equation

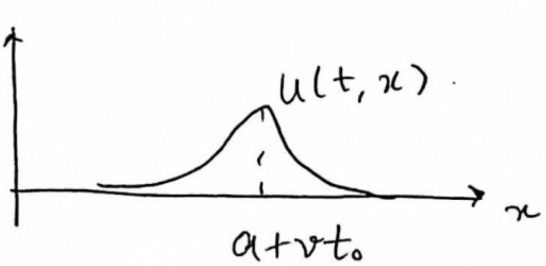
$$\begin{cases} (\partial_t u + v\partial_x) u = f \\ u(0,x) = u_0(x) \end{cases} \qquad \begin{aligned} u &= u(t,x) \\ f &= f(t,x) \end{aligned} \qquad (t,x) \in \mathbb{R} \times \mathbb{R}$$

idea is to study behaviours of u in direction v.

- Homogeneous case, $f \equiv 0$.

$$u(t,x) = u_0(x - vt)$$





- Inhomogeneous case,

Let $w(t,x) = u(t, x+vt)$.

By chain rule $\Rightarrow$

$$\partial_t w(t,x) = (\partial_t u)(t, x+vt) + (v\partial_x u)(t, x+vt)$$

$$= (\partial_t u + v\partial_x u)(t, x+vt)$$

$$= f(t, x+vt).$$

$$\Rightarrow \partial_t w(t,x) = f(t, x+vt)$$

$$\Rightarrow w(t,x) = w(0,x) + \int_0^t f(s, x+vs)\,ds.$$

$$u(t, x+vt) = u(0, x+v0) + \int_0^t f(s, x+vs)\,ds$$

$$= u_0(x) + \int_0^t f(s, x+vs)\,ds.$$

replace x by $x - vt$.

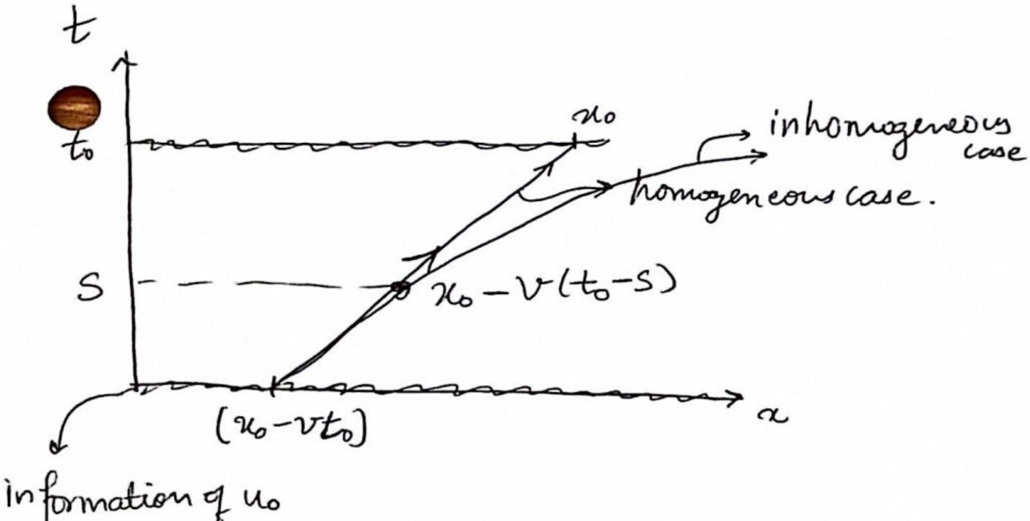
$\Rightarrow$

$$u(t,x) = u_0(x-vt) + \int_0^t f(s, x - vt + vs)\,ds$$

$$= u_0(x-vt) + \int_0^t f(s, x - v(t-s))\,ds.$$

Rmk : This solution makes sense if

1. u is differentiable. $\Rightarrow$ u_0 has a continuous derivative.
2. f is integrable. $\Rightarrow$ (piecewise continuous)



Duhamel Principle

At each pt. in time f changes the soln a little & the integral of these affects accumulate

Duhamel Principle.

$$\begin{cases} \partial_t u = \mathcal{L} u \\ u(0,x) = u_0(x) \\ \text{Boundary conditions (or not)} \end{cases}$$

$\mathcal{L}$: PDO

$\mathcal{L} = v\partial_x$ transport
$= \partial_x^2$ heat
$= i\partial_x^2$ Schrödinger.

$\forall t$, Define the operator $\Phi(t) : u_0 \longrightarrow u(t)$

$\Phi(t)$ is a mapping that maps a function to another.

Properties of $\Phi(t)$ → called solution flow or solution group.

① $\Phi(0) = id$, $\Phi(0)\, u_0 = u_0$

② $u : t \to \Phi(t)\, u_0$ solves the PDE $\begin{cases} \partial_t u = Lu \\ u(0) = u_0 \end{cases}$

③ $\Phi(t)$ is linear

$$\Phi(t)(\alpha u_0 + \beta v_0) = \alpha \underbrace{\Phi(t)\, u_0}_{\text{initial } u_0} + \beta \underbrace{\Phi(t)\, v_0}_{\text{initial } v_0}.$$

Eg: Transport eqn.

$$\Phi(t)\, u_0(x) = u_0(x - vt)$$

$$\Rightarrow \quad u(t, x) = \Phi(t)\, u_0(x)$$

Eg: Heat eqn.

$$\left.\begin{aligned} \partial_t u &= \partial_x^2 u \\ u(0, x) &= u_0(x) \end{aligned}\right\} \begin{aligned} &\text{Dirichlet} \\ &(0, \pi) \times (0, \infty) \end{aligned}$$

Define, $\Phi(t)$ as follows,

if $u_0(x) = \sum_{n=1}^{\infty} \gamma_n \sin(nx)$

then $\Phi(t)\, u_0(x) = \sum_{n=1}^{\infty} \gamma_n e^{-n^2 t} \sin(nx)$. → from soln. earlier.

$\Rightarrow$ $u(t) = \Phi(t)\, u_0$ solves the BVP.

Inhomogeneous BVP

$$\partial_t u = \mathcal{L} u + f \quad \& \quad u(0) = u_0$$

Solution formula,

$$u(t) = \Phi(t)\, u_0 + \int_0^t \Phi(t-s) f(s)\, ds.$$

i.e. $$\boxed{u(t,x) = \Phi(t)\, u_0(x) + \int_0^t \Phi(t-s) f(s,x)\, ds.} \quad (*)$$

Want to show that $(*)$ solves the BVP.

① Initial condition,

$$u(0) = \Phi(0)\, u_0 + 0 = u_0$$

② $$\partial_t u(t) = \partial_t \Phi(t)\, u_0 + \partial_t \int_0^t \phi(t-s) f(s)\, ds$$

(i) $\partial_t \Phi(t)\, u_0 = \mathcal{L}\, \Phi(t)\, u_0. \leftarrow$ homogeneous PDE.

(ii) $$\partial_t \int_0^t \Phi(t-s) f(s)\, ds = \Phi(t-t) f(t) + \int_0^t \partial_t \Phi(t-s) f(s)\, ds$$

$$= f(t) + \int_0^t \mathcal{L} \Phi(t-s) f(s)\, ds.$$

$$= f(t) + \mathcal{L} \int_0^t \Phi(t-s) f(s)\, ds$$

Summing (i) & (ii)

$$\partial_t u(t) = \mathcal{L}\Big(\underbrace{\Phi(t)\, u_0 + \int_0^t \Phi(t-s) f(s)\, ds}_{u(t)}\Big) + f(t) = \mathcal{L}\, u(t) + f(t)$$

//

Back to wave equation.

$$(\partial_t^2 - c^2\partial_x^2)\, u = 0.$$

Let $w = (\partial_t - c\partial_x) u$ then

$$(\partial_t - c\,\partial_x)\, u = w \qquad u(0) = u_0$$

and $$(\partial_t + c\partial_x)\, w = 0 \qquad w(0) = u_1 - c\,\partial_x u_0$$

By the solution formula for transport eqn.

$$w(t,x) = w(0, x-ct) = u_1(x-ct) - c\,\partial_x u_0(x-ct)$$

$$\Rightarrow \quad u(t,x) = u(0, x+ct) + \int_0^t w(s, x+c(t-s))\,ds$$

$$= u_0(x+ct) + \int_0^t u_1(x+c(t-s)-cs)\,ds$$

$$- c\int_0^t \partial_x u_0(x+c(t-s)-cs)\,ds$$

Let $z = x + c(t-s) - cs = x + ct - 2cs$

$s=0 \Rightarrow z = x+ct$ $\qquad dz = d(x+ct-2cs) = -2c\,ds.$

$s=t \Rightarrow z = x-ct$

$$\Rightarrow u(t,x) = u_0(x+ct) + -\frac{1}{2c}\int_{x+ct}^{x-ct} u_1(z)\,dz - c\,\frac{(-1)}{2c}\int_{x+ct}^{x-ct} \partial_x u_0(z)\,dz.$$

$$= u_0(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(z)\,dz - \frac{1}{2}\int_{x-ct}^{x+ct} u_0'(z)\,dz.$$

$$= u_0(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(z)\,dz - \frac{1}{2}\left(u_0(x+ct) - u_0(x-ct)\right)$$

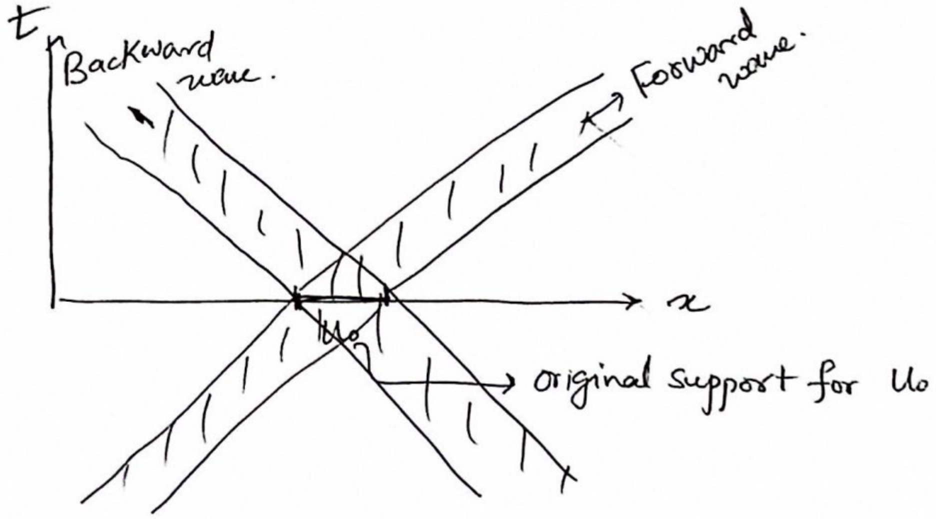
$$\boxed{u(t,x) = \frac{1}{2}\left[u_0(x+ct) + u_0(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(z)\,dz.}$$

This derivation will come up in Final.

d'Alembert Formula.

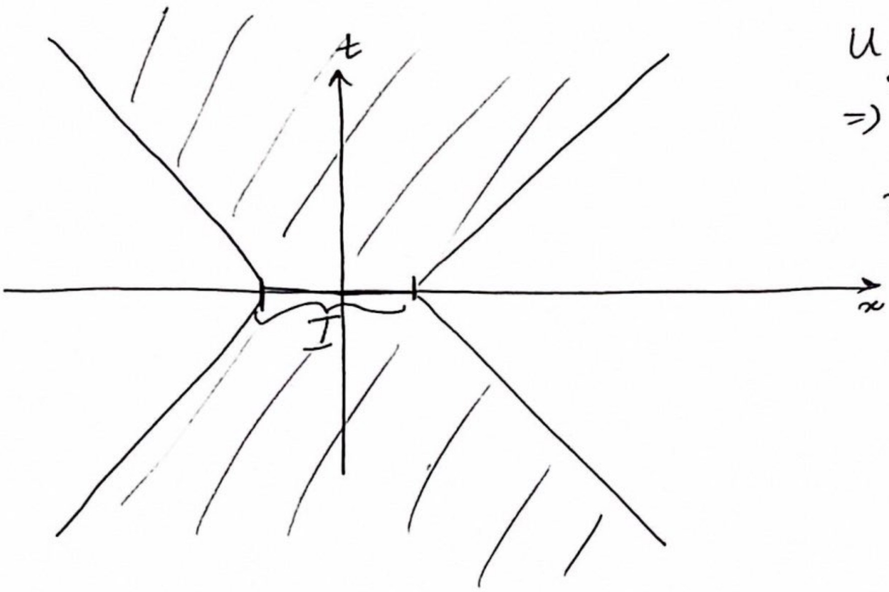
Implication from d'Alembert

if $u_1 = 0$; $u(t,x) = \frac{1}{2}\left[u_0(x+ct) + u_0(x-ct)\right]$



if $u_0 = 0$,

$$u(t,x) = \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(z)\,dz.$$



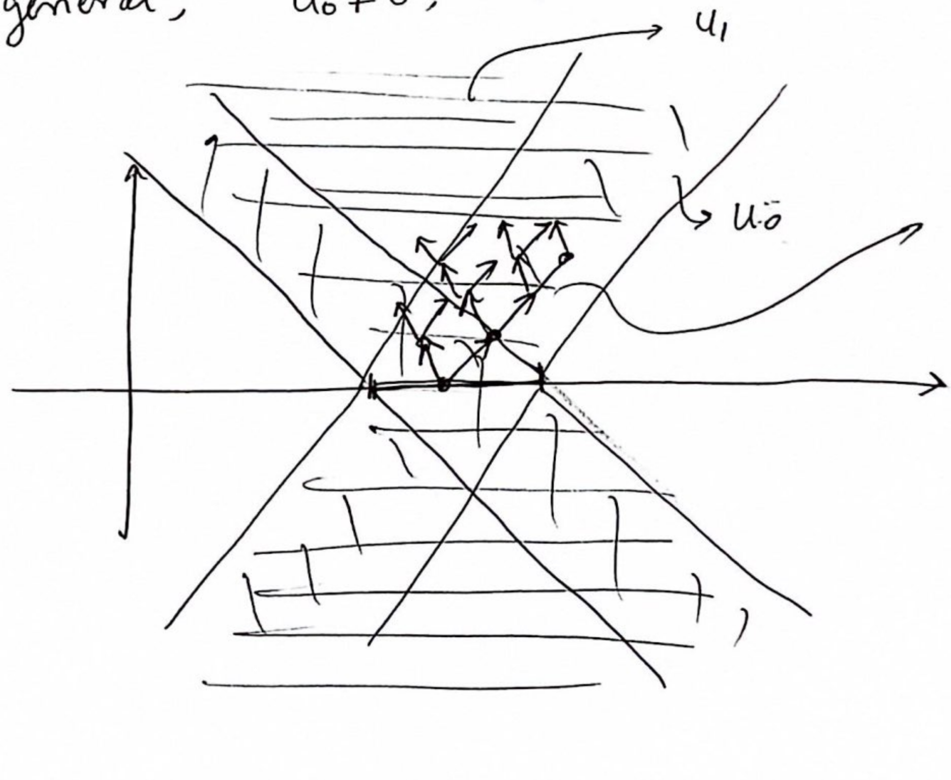
$u(t,x) \neq 0$

$\Rightarrow \int_{x-ct}^{x+ct} u(z)dz \neq 0$

$\Rightarrow \exists\, z \in [x-ct, x+ct]$ s.t $u_1(z) \neq 0$

$\Rightarrow [x-ct, x+ct] \cap I \neq 0$

In general, $u_0 \neq 0$, $u_1 \neq 0$



Huygen's Principle.

 Rmk: Duhamels Principal.

$$\begin{cases} \partial_t u = Lu \\ u(0,x) = u_0(x) \end{cases} \rightsquigarrow$$

Solution flow $t \to \Phi(t)$

$\Phi(t)$ is an operator, it maps a function to another.

Solution formula.

$U(t) = \Phi(t)\, u_0$.

eg: Heat, Schrödinger eq., bounded interval, boundary conditions.

$\Phi(t)$ changes the Fourier coefficients of u_0 to define a new function $\Phi(t)\,u_0$.

Precisely, via the Fourier method,

$$u(t,x) = \sum_{k=-\infty}^{\infty} c_k(t)\, e^{\pi i k x/\ell} \quad ; \quad c_k(0) = \gamma_k \quad \text{where } u_0 = \sum_k \gamma_k e^{\pi i k x/\ell}$$

The solution $\Phi(t)$ changes the coordinates γ_k to the new coordinates at time t, $\left(c_k(t)\right)_{k=-\infty}^{\infty}$.

$$\Phi(t)\left(\sum_{k=-\infty}^{\infty} \gamma_k\, e^{\pi i k x/\ell}\right) = \sum_{k=-\infty}^{\infty} c_k(t)\, e^{\pi i k x/\ell}$$

Once you know $\Phi(t)$ you can solve

$$\begin{cases} \partial_t u = Lu + f \\ u(0,x) = u_0(x) \end{cases} \quad ; \quad f = f(t,x)$$

via the solution formula.

$$u(t,x) = \underbrace{\Phi(t)\, u_0(x)}_{\text{homogeneous part}} + \underbrace{\int_0^t \Phi(t-s)\, f(s,x)\, dx}_{\text{inhomogeneous part. (correction)}}$$

Q: $\partial_t^2 u = c^2 \partial_x^2 u + f$

$u(t,-\pi) = u(t,\pi)$; $\partial_x u(t,-\pi) = \partial_x u(t,\pi)$ Periodic B.C.

$u(0,x) = u_0(x)$; $d_t u(0,x) = u_1(x)$.

~~take~~ ~~the~~ Let $W = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}$

$$\partial_t W = \begin{pmatrix} \partial_t u \\ \partial_t^2 u \end{pmatrix} = \begin{pmatrix} \partial_t u \\ c^2 \partial_x^2 u + f \end{pmatrix} = \begin{pmatrix} \partial_t u \\ c^2 \partial_x^2 u \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ c^2 \partial_x^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$$\boxed{\partial_t W = \begin{pmatrix} 0 & 1 \\ c^2 \partial_x^2 & 0 \end{pmatrix} W + \begin{pmatrix} 0 \\ f \end{pmatrix}}$$

$$\Leftrightarrow \quad \partial_t W_0 = W_1$$

$$\underbrace{\partial_t W_1 = c^2 \partial_x^2 W_0}_{\text{homogeneous}} + \underbrace{f}_{\text{inhomo}}$$

Solution flow $\Phi(t)$

$$\Phi(t) : \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \to W(t) = \begin{pmatrix} W_0(t) \\ W_1(t) \end{pmatrix}$$

$\Downarrow$

Duhamel's Principle

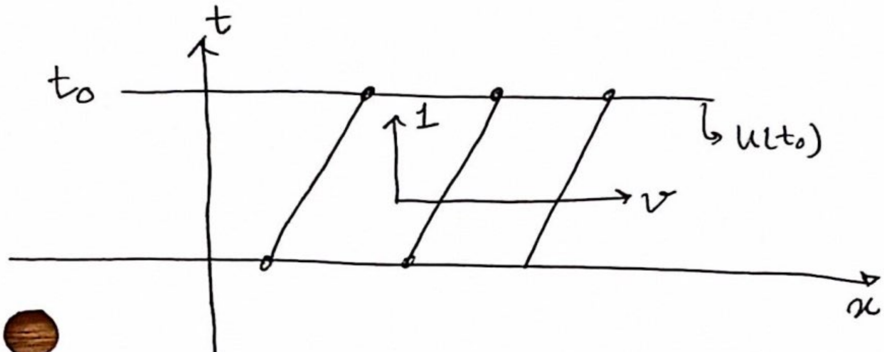
Exercise.

Characteristic Method

- Transport equation.

$$\begin{cases} \partial_t u + v \partial_x u = 0 \\ u(0,x) = u_0(x) \end{cases} \qquad v \in \mathbb{R}.$$

There is a special direction in $\mathbb{R}^{1+1} = \mathbb{R} \times \mathbb{R}$, along which the solution is constant.



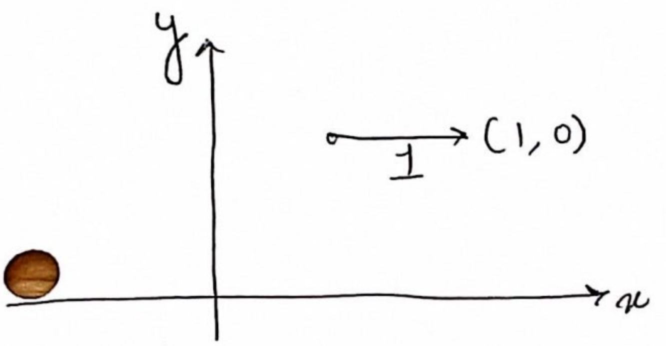
$$\Rightarrow u(t,x) = u_0(x - vt)$$

② $a \partial_x u + b \partial_y u = 0 \qquad (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}.$

- if $b \neq 0 \Rightarrow \partial_y u + \frac{a}{b} \partial_x u = 0 \Rightarrow$ transport eq.
- $(a,b) \cdot \nabla u = 0 \Leftrightarrow D_{(a,b)} u \Leftrightarrow 0$

$\Leftrightarrow u$ is constant along (a,b) direction.

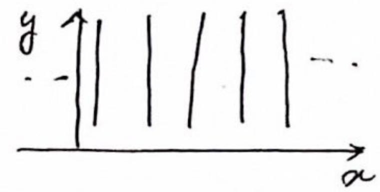
eg 1 $a = 1,\ b = 0 \Rightarrow \partial_x u = 0 \rightarrow u$ independent of x.



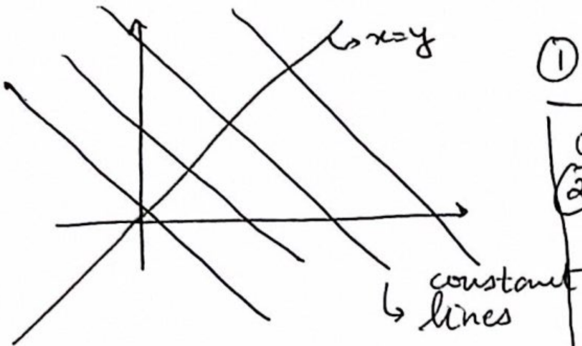
$u(x,y)$ Constant along x direction.

eg: $a=0\ ;\ b=1$

$u(x,y)$ constant along y direction



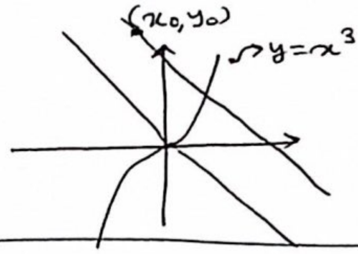
eg: $a=1,\ b=-1$ $\Rightarrow$ $\partial_x u - \partial_y u = 0$



① $u(x,y) = x^2+y^2$ when $x=y$

or

② $u(x,y) = x-y$ when $y=x^3$



<u>Variable coefficients case</u>.

$$a(x,y)\,\partial_x u + b(x,y)\,\partial_y u = 0 \quad \text{S.T} \quad \underbrace{(a(x,y), b(x,y))}_{\in \mathbb{R}^2} \neq (0,0) \quad \forall (x,y)$$

<u>Def</u>: A vector field on $\mathbb{R}^2$ is a mapping

$$V : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \underset{\text{point}}{(x,y)} \longmapsto \underset{\text{vector}}{V(x,y)} = (a(x,y), b(x,y))$$

eg: $V(x,y) = (a,b)$ → constant field

$V(x,y) = (x,y)$ → radial field

$V(x,y) = (y,x)$ → rotational field

<u>Def</u>: Let $V: \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field, a trajectory of V is a mapping $\phi : I \to \mathbb{R}^2$ S.T $\frac{d}{dt}\phi(t) = V(\phi(t))$

↳ any interval $(a,b) \in \mathbb{R}$

Thm: If u satisfies the equation

$$(a(x,y)\,\partial_x + b(x,y)\,\partial_y)\,u = 0 \quad \text{i.e.} \quad V(x,y)\cdot\nabla u = 0$$

& if $\phi : I \to \mathbb{R}^2$ is a trajectory of V, then u is constant along ϕ.

i.e. $\dfrac{d}{dt}\,u(\phi(t)) = 0$

Proof :- Chain rule

$$\frac{d}{dt}\,u(\phi(t)) = \nabla u(\phi(t)) \cdot \frac{d}{dt}\,\phi(t)$$

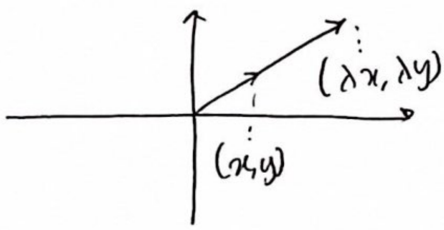
$$= \nabla u(\phi(t)) \cdot V(\phi(t))$$

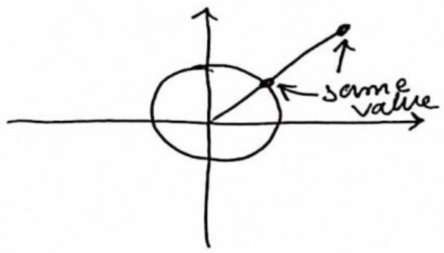
$$= (V\cdot\nabla u)(\phi(t)) = 0 \;\; //$$

e.g. $(x\,\partial_x + y\,\partial_y)\,u = 0$

then $u(x,y) = u(\lambda x, \lambda y) \quad \forall\,(x,y) \neq (0,0)$

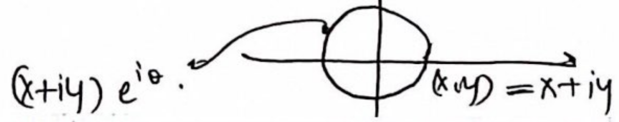
$\forall\,\lambda > 0$





eg: $(y\,\partial_x - x\,\partial_y)\,u = 0$; $u(x,y) = u(x+iy) = u((x+iy)\,e^{i\theta})$

$= u((x+iy)(\cos\theta + i\sin\theta)) = u(x\cos\theta - y\sin\theta\,,\, y\cos\theta + x\sin\theta)$



Lec 26 <u>Sturm - Liousville Theory.</u>

Consider the following second order ODE for $u = u(x)$ on a finite interval (a,b).

$$(ru')' + (q + \lambda p)\, u = 0$$

Here r, p, q are real valued smooth functions of $x \in [a,b]$ and we require that

- $p(x) \neq 0$ for all $x \in [a,b]$
- $r(x) \neq 0$ for all $x \in [a,b]$

(Robin) Boundary conditions.

Let $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that $(a_1, a_2) \neq (0,0)$ and $(b_1, b_2) \neq 0$.

Boundary conditions at the end pts,

- At $x = a$, we have $a_1 u(a) + a_2 u'(a) = 0$
- At $x = b$, we have $b_1 u(b) + b_2 u'(b) = 0$.

Sturm Liouville as an eigenvalue problem.

$\lambda \to$ eigenvalue ; $u \to$ eigenfunction.

In fact, let L be a differential operator given by

$$L = \frac{d}{dx} r \frac{d}{dx} + q,$$

Then SL equation is given by

$$Lu = -\lambda p u \; ; \quad p^{-1} L u = -\lambda u .$$

Eg:

Laplace equation $\quad u'' = -\lambda u$.

is SL with $r = 1$, $q = 0$, $p = 1$.

> Boundary conditions:

Neumann : $a_1 = b_1 = 0$

Dirichlet : $a_2 = b_2 = 0$

> 2D Laplace equation in polar coordinates.

$$\Delta = \partial_x^2 + \partial_y^2 = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2 .$$

where $x = r\cos\theta$; $y = r\sin\theta$.

> Sometimes we need to find eigenfunctions of this 2D Laplace equation with prescribed angular momentum.

> Note that if u is a function of x, y then under polar coordinates, u is 2π periodic with respect to θ.

That is, $\quad u(r,\theta) = u(r, \theta + 2\pi)$.

> Consequently, for each r you can do Fourier analysis w.r.t θ.

Each Fourier mode is of the form,

$$u(r,\theta) = A(r)\, e^{ik\theta},$$

where $k \in \mathbb{Z}$ and A is a function of r (which is called the amplitude).

Now assume that u is an eigenfunction of the Laplace equation : $\Delta u = -\lambda u$. Plugging the ansatz:

$$\Delta u = e^{ik\theta}(\partial_r^2 + r^{-1}\partial_r)\, A(r) + r^{-2} A(r)\, \partial_\theta^2 (e^{ik\theta})$$

$$= (A''(r) + r^{-1} A'(r))\, e^{ik\theta} - \frac{k^2}{r^2} A(r)\, e^{ik\theta} = -\lambda u$$

$$= -\lambda\, A(r)\, e^{ik\theta}.$$

Dividing both sides by $e^{ik\theta}$, we get

$$A'' + r^{-1}A' - k^2 r^{-2} A = -\lambda A.$$

Multiplying both sides by r,

$$rA'' + A' - k^2 r^{-1} A = -\lambda r A.$$

Note that $rA'' + A' = (rA')'$, so

$$(rA')' + (-k^2 r^{-1} + \lambda r) A = 0$$

Soln. is non-singular (regular) when $r \neq 0$.

Functional Analysis of SL equation.

On the finite interval (a,b) :

$$(ru')' + (q+\lambda p)u = 0.$$

Boundary conditions at the end points:

- At $x=a$, we have $a_1 u(a) + a_2 u'(a) = 0$
- At $x=b$, we have $b_1 u(b) + b_2 u'(b) = 0$

Assume that,

- $r(x) > 0 \quad \forall x \in [a,b]$
- $p(x) > 0 \quad \forall x \in [a,b]$

Weighted L^2 spaces

The weighted L^2 space over (a,b) with ~~the~~ weight p is denoted by $L_p^2(a,b)$. It is the set of all functions $f : (a,b) \to \mathbb{C}$ such that

$$\int_a^b |f(x)|^2 p(x)\, dx < \infty.$$

If $f \in L_p^2(a,b)$, then it's L_p^2-norm is defined by

$$\|f\|_{L_p^2} = \sqrt{\int_a^b |f(x)|^2 p(x)\, dx}$$

Thm: $L_p^2(a,b)$ is a Hilbert space.

Our goal is to show that there exists a sequence of eigen functions w.r.t the SL eqn. that forms a complete orthogonal set for $L_p^2(a,b)$.

Lemma: If (λ, u) and (γ, v) are solutions to the SL eqn. (satisfying the BC) and if $\lambda \neq \gamma$, then $u \perp v$ w.r.t $L_p^2(a,b)$ space.

Rmk: If p is constant, then the above lemma holds in the usual $L^2(a,b)$ space.

Proof: For simplicity, let us denote

$$\langle u, v\rangle_p = \int_a^b u(x)\,\overline{v(x)}\,p(x)\,dx$$

$$\langle u, v\rangle = \int_a^b u(x)\,\overline{v(x)}\,dx$$

We have

$$\lambda\langle u,v\rangle_p = \langle \lambda u, v\rangle_p = \langle \lambda p u, v\rangle$$

$$= -\langle (ru')' + qu, v\rangle = -\langle (ru')', v\rangle - \langle qu, v\rangle$$

Integration by parts,

$$\langle (ru')', v\rangle = -\langle ru', v'\rangle + [ru'\bar{v}]_a^b = -\langle u', rv'\rangle + [ru'\bar{v}]_a^b$$

$$= \langle u, (rv')'\rangle + [r u'\bar{v}]_a^b - [ru\bar{v}']_a^b$$

Summing up, we have

$$\lambda \langle u, v\rangle_p = -\langle u, (rv')'\rangle - \langle qu, v\rangle - [r(u'\bar{v} - u\bar{v}')]_a^b$$

$$= -\langle u, (rv')' + qv\rangle - [r(u'\bar{v} - u\bar{v}')]_a^b$$

$$= -\langle u, -\gamma p v\rangle - [r(u'\bar{v} - u\bar{v}')]_a^b$$

$$= \bar{\gamma}\langle u, v\rangle_p - [r(u'\bar{v} - u\bar{v}')]_a^b$$

Rearranging,

$$(\lambda - \bar{\gamma})\langle u, v\rangle_p = -[r(u'\bar{v} - u\bar{v}')]_a^b$$

We claim the boundary term vanishes:

- At $x = a$, we have
  - $a_1 u(a) + a_2 u'(a) = 0$
  - $a_1 v(a) + a_2 v'(a) = 0$
  - $a_1 \bar{v}(a) + a_2 \bar{v}'(a) = 0$; & similarly at $x = b$.

$$\Rightarrow \begin{pmatrix} u(a) & u'(a) \\ \bar{v}(a) & \bar{v}'(a) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow \quad \det \left(\quad '' \quad \right) = u(a)\,\bar{v}'(a) - u'(a)\,\bar{v}(a) = 0.$$

Conclusion: the boundary term vanishes and we have

$$(\lambda - \bar{\gamma}) \langle u, v \rangle_p = 0$$

Other conclusions:

- Let $u = v$, then: $(\lambda - \bar{\lambda}) \langle u, u \rangle_p = 0$

Because $\langle u, u \rangle_p \neq 0$ (Since $u \neq 0$) we must have $\lambda = \bar{\lambda} \Rightarrow \lambda$ real!

$\Rightarrow \lambda, \gamma$ are real $\Rightarrow (\lambda - \gamma) \langle u, v \rangle_p = 0$.

If $\lambda \neq \gamma$, then $\langle u, v \rangle_p = 0 \Rightarrow u \perp v$.

Sturm Liousville theory.

$$(BVP)\begin{cases} (ru')' + (q+\lambda p)u = 0 & (S-L) \qquad x \in (a,b) \\ a_1 u(a) + a_2 u'(a) = 0 \\ b_1 u(b) + b_2 u'(b) = 0 \end{cases} \Big\} (B-C) \qquad \begin{array}{l}(a_1, a_2) \neq (0,0) \\ (b_1, b_2) \neq (0,0)\end{array}$$

<u>assumptions</u> ① p, q, r smooth / real-valued

② $r(x) > 0$ & $p(x) > 0 \quad \forall x \in [a,b]$.

<u>Thm</u>: ① If λ is an eigenvalue, then $\lambda \in \mathbb{R}$

② If u, v are eigen functions corresponding to distinct eigenvalues, then $u \perp v$ in $L^2_p(a,b)$;

i.e. $\displaystyle\int_a^b p(x)\, u(x)\, \overline{v(x)}\, dx = 0.$

<u>Pf</u>: Integration by parts.

<u>Alternative view</u>

$$L = \frac{d}{dx} r \frac{d}{dx} + q$$

$(S-L) \Leftrightarrow Lu = -\lambda p u.$

In a previous lecture, we proved that if u, v satisfy B-Cs [do not need to be solutions], then integrating by parts gives

$$\boxed{\langle u, \mathcal{L}v\rangle_{L^2} = \langle \mathcal{L}u, v\rangle_{L^2}}$$

i.e. $\displaystyle\int_a^b u(x)\,\overline{\mathcal{L}v(x)}\,dx = \int_a^b \mathcal{L}u(x)\,\overline{v(x)}\,dx.$

Generally: If $\mathcal{L}$ is an arbitrary diff. operator then

$$\langle \mathcal{L}u, v\rangle_{L^2} = \langle u, \mathcal{L}^* v\rangle_{L^2} + \text{boundary terms.}$$

① $L^2 = L^2(a,b)$

② Boundary terms donot always vanish,

however, if u, v both vanish near the boundary

For such functions,

$u^{(m)}(a) = u^{(m)}(b) = v^{(m)}(a) = v^{(m)}(b) = 0.$

Then the boundary terms vanish.

③ $\mathcal{L}^*$ is another differential operator (adjoint operator of $\mathcal{L}$).

it can be computed explicitly via IBP.
↳ int by parts.

If $\mathcal{L} = \mathcal{L}^*$ then $\mathcal{L}$ is (formally) self-adjoint.

Proof of Thm (again)

Suppose that $\begin{cases} \mathcal{L}u = -\lambda p u \\ \mathcal{L}v = -\gamma p v \end{cases}$

$$-\lambda \langle u, v\rangle_{L^2_p} = -\lambda \langle pu, v\rangle_{L^2} = \langle -\lambda pu, v\rangle_{L^2}$$

$$= \langle Lu, v\rangle_{L^2} = \langle u, Lv\rangle_{L^2}$$

$$= \langle u, -\bar{\gamma} p v\rangle_{L^2} = -\bar{\gamma}\langle u, pv\rangle_{L^2} = -\bar{\gamma}\langle u, v\rangle_{L^2_p}.$$

$$\Rightarrow (\lambda - \bar{\gamma})\langle u, v\rangle_{L^2_p} = 0.$$

What if u, v are eigenfunctions corresponding to the same eigenvalue $\lambda \in \mathbb{R}$?

Conjecture: They are colinear, i.e $u = cv$ for some $c \in \mathbb{C}$.

Consider the 2nd ODE

$$(*) \begin{cases} u'' + P(x)u' + Q(x)u = 0 \\ \alpha u(x_0) + \beta u'(x_0) = 0 \end{cases}$$

Wronskian.

Thm: If u, v are 2 solutions to $(*)$ then $W[u, v] = \det\begin{pmatrix} u & v \\ u' & v' \end{pmatrix}$

$= uv' - u'v = 0.$

Proof: At x_0, $W = W[u, v]$ vanishes. because

$$\begin{pmatrix} u(x_0) & u'(x_0) \\ v(x_0) & v'(x_0) \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0. \Rightarrow \underbrace{\det\begin{pmatrix} u(x_0) & u'(x_0) \\ v(x_0) & v'(x_0) \end{pmatrix}}_{= W[u,v](x_0)} = 0$$

How to show it varies everywhere? Show it is zero at x_0 & derivative is 0.

$$W = W[u,v] = uv' - u'v$$

$u'' + pu' + Qu = 0$

$u'' = -pu' - Qu$

$$W' = (uv' - u'v)'$$

$$= u'v' + uv'' - u'v' - u''v$$

$$= uv'' - u''v$$

$$= u(-pv' - Qv) - (-pu' - Qu)v$$

$$= -p(uv' - u'v) = -pW$$

$$\Rightarrow W' = -pW$$

$$\Rightarrow W(x) = \underbrace{W(x_0)}_{0} e^{-\int_{x_0}^{x} p(y)\,dy} \equiv 0$$

$\Rightarrow$ Conclusion.

2 eigenfunctions corresponding to the same eigenvalue must be colinear!

Cor: If u is an eigenfunction, then $\exists\, c \in \mathbb{C}$ s.t cu is real valued.

Proof: If $Lu = -\lambda p u$ then $\begin{cases} L \operatorname{Re} u = -\lambda p \operatorname{Re} u \\ L \operatorname{Im} u = -\lambda p \operatorname{Im} u. \end{cases}$

u is an eigenfunction $\Rightarrow u \neq 0 \Rightarrow$ either $\operatorname{Re} u \neq 0$ or $\operatorname{Im} u \neq 0$.

Say that $\operatorname{Re} u \neq 0$. $\operatorname{Re} u$ is an eigenfunction. $\Rightarrow \exists\, c \in \mathbb{C}$ s.t $\operatorname{Re}(u) = cu$

Recall that, $\begin{cases} u'' = -\lambda u \\ \text{Dirichlet} \end{cases}$ or $\begin{cases} u'' = -\lambda u \\ \text{Neumann.} \end{cases}$

then $\lambda \geq 0$.

Q: True for S-L?

$$\lambda \langle u, u \rangle_{L^2_p} = \lambda \langle pu, u \rangle_{L^2} = \langle \lambda pu, u \rangle_{L^2}$$

$$= \langle -Lu, u \rangle_{L^2} = \langle -(ru')' - qu, u \rangle_{L^2}$$

$$= -\langle (ru')', u \rangle_{L^2} - \langle qu, u \rangle_{L^2}$$

$$= \underbrace{\langle ru', u' \rangle_{L^2}}_{\geq 0} - \underbrace{\langle qu, u \rangle_{L^2}}_{\geq 0 \text{ if } q \leq 0} - \underbrace{[ru'u]_a^b}_{\substack{\| \\ r(a)u'(a)u(a) \\ -r(b)u'(b)u(b) \\ \geq 0?}}$$

$$\Rightarrow \lambda \langle u, u \rangle_{L^2_p} \geq -\langle qu, u \rangle_{L^2} + (\underbrace{r(a)}_{\geq 0} u'(a)u(a) - r(b)u'(b)u(b))$$

At a, $a_1 u(a) + a_2 u'(a) = 0$

$$\begin{cases} \times u(a) \Rightarrow & a_1 \underbrace{u(a)^2}_{\geq 0} + a_2 u(a)u'(a) = 0 \quad ① \\ \times u'(a) \Rightarrow & a_1 u(a)u'(a) + a_2 \underbrace{u'(a)^2}_{\geq 0} = 0 \quad ② \end{cases}$$

$(a_1, a_2) \neq 0 \Rightarrow a_1^2 + a_2^2 > 0$

$① \times a_2 \Rightarrow a_1 a_2\, u(a)^2 + a_2^2\, u(a)\, u'(a) = 0$

$② \times a_1 \rightsquigarrow a_1^2\, u(a)\, u'(a) + a_1 a_2\, u'(a)^2 = 0.$

Sum up, $$(a_1^2 + a_2^2)\, u(a)\, u'(a) = -a_1 a_2 \left(u(a)^2 + u'(a)^2 \right)$$

$$u(a)\, u'(a) = -\frac{a_1 a_2}{\underset{\geq 0}{a_1^2 + a_2^2}} \left(\underbrace{u(a)^2 + u'(a)^2}_{\geq 0} \right)$$

if $a_1 a_2 \leq 0$ then $u(a)\, u'(a) \geq 0$

Similarly, if $b_1 b_2 \geq 0$ then $u(b)\, u'(b) \leq 0$

Thm: If $q \leq 0$, $\begin{cases} a_1 a_2 \leq 0 \\ b_1 b_2 \geq 0 \end{cases}$

Then $\lambda \langle v, u \rangle_{L^2_p} \geq 0$; consequently $\lambda \geq 0$ //

S-L version of Fourier method.

$$\begin{cases} L\, \phi_n = -\lambda_n\, \phi_n \\ n = 0, 1, 2, \dots \end{cases}$$

(λ_n, ϕ_n) are (eigenvalues, eigen fns.)

$$0 \leq \lambda_0 \leq \lambda_1 < \lambda_2 \dots < \lambda_n < \dots$$
$$\begin{array}{ccc} \Updownarrow & \Updownarrow & \Updownarrow \\ \phi_0 & \phi_1 & \phi_2 \quad \dots \end{array}$$

$(\phi_n)_{n=0}^{\infty}$ is a complete orthogonal set of $L_p^2(a,b)$.

If $u \in L_p^2(a,b)$ then

$$u(x) = \sum_{n=0}^{\infty} \alpha_n\, \phi_n(x).$$

To compute α_n,

$$\langle u, \phi_n \rangle_{L_\rho^2} = \alpha_n \langle \phi_n, \phi_n \rangle_{L_\rho^2}$$

$$\Rightarrow \boxed{\alpha_n = \frac{\langle u, \phi_n \rangle_{L_\rho^2}}{\langle \phi_n, \phi_n \rangle_{L_\rho^2}}}$$