Antennas - John D. Kraus

Chapter 2 - Basic Concepts.

Normalised Electric Field

$$E_\theta(\theta,\phi)_n = \frac{E_\theta(\theta,\phi)}{E_\theta(\theta,\phi)_{max}}$$

Normalised Power Pattern

$$P_n(\theta,\phi) = \frac{S(\theta,\phi)}{S(\theta,\phi)_{max}}$$ → mag. of Poynting Vector $\vec{S}(\theta,\phi)$

$$S(\theta,\phi) = \frac{[E_\theta^2(\theta,\phi) + E_\phi^2(\theta,\phi)]}{Z_0} \quad W/m^2$$

Solid angle Ω in steradian (sr)

1 steradian = 1 radian^2 = 3282.8064 degree^2

4π steradian = 41253 sr → solid angle in a sphere.

$$\boxed{d\Omega = \sin\theta\, d\theta\, d\phi}$$

Beam area

$$\Omega_A = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} P_n(\theta,\phi)\, d\Omega$$

$$\boxed{\Omega_A \simeq \theta_{HP}\, \phi_{HP}}$$

Radiation Intensity (U) :- Power radiated per unit solid angle.

$$\therefore\ P_n(\theta,\phi) = \underbrace{\frac{U(\theta,\phi)}{U(\theta,\phi)_{max}}}_{\text{Independant of distance}} = \underbrace{\frac{S(\theta,\phi)}{S(\theta,\phi)_{max}}}_{\text{Depends on distance}}$$

} Assuming we are in far field.

Beam Efficiency

$$\underbrace{\Omega_A}_{\text{total beam area}} = \underbrace{\Omega_M}_{\text{main beam area}} + \underbrace{\Omega_m}_{\text{minor lobe area.}}$$

$$\underbrace{\epsilon_M}_{\text{main beam efficiency}} = \frac{\Omega_M}{\Omega_A}\ ; \quad \underbrace{\epsilon_m}_{\text{Stray factor}} = \frac{\Omega_m}{\Omega_A}$$

$\Rightarrow \epsilon_M + \epsilon_m = 1$

Directivity (Ratio of max power/solid angle to power/4π sr)

$$D = \frac{U(\theta,\phi)_{max}}{U_{avg}} = \frac{S(\theta,\phi)_{max}}{S_{avg.}} \qquad \frac{\text{max radiation intensity}}{\text{avg. radiation intensity.}}$$

$\uparrow D \Rightarrow$ most of power is concentrated at the maximum.

$$\boxed{D \cong \frac{4\pi}{\Omega_A}} \qquad D = \frac{1}{\frac{1}{4\pi}\iint P_n(\theta,\phi)\,d\Omega} = \boxed{\frac{1}{\frac{1}{4\pi}\iint P_n(\theta,\phi)\sin\theta\,d\theta\,d\phi}}$$

Isotropic $\Rightarrow P_n(\theta,\phi) = 1 \Rightarrow$ Equal power everywhere.

$\Rightarrow \Omega_A = 4\pi\ ;\ D = 1 = 0\,dB.$

$$D \cong \frac{4\pi}{\theta_{HP}\,\phi_{HP}} \simeq \frac{41253}{\theta^\circ_{HP}\,\phi^\circ_{HP}} \simeq \frac{41000}{\theta^\circ_{HP}\,\phi^\circ_{HP}}$$

Eg:- If $\theta_{HP}^\circ = \phi_{HP}^\circ = 20^\circ \Rightarrow D = \frac{41000}{400} \simeq 103 \simeq 20\,dBi$

↳ dB above isotropic

$\Rightarrow$ Antenna radiates 100 times more power in the main lobe direction than that radiated by an isotropic antenna for same input power.

$\underline{\text{Gain}} = G = KD$

Efficiency → Depends only on ohmic losses of antenna.

TX ⇒ power delivered to the antenna vs. power radiated by antenna.

$\underline{\text{Resolution}}$ (Rayleigh resolution) $= \dfrac{FNBW}{2}$. How close can two beams be?

$FNBW = 2° \Rightarrow$ max beams are $1°$ apart.

$$HPBW \simeq \frac{FNBW}{2} \Rightarrow \Omega_A = \frac{BWFN}{2}\theta \; \frac{BWFN}{2}\phi$$

$N = \dfrac{4\pi}{\Omega_A}$ → no. of pt. sources an antenna can resolve.

⇒ $\boxed{N = D}$ → assuming ideal sources that are uniformly distributed.

Aperture

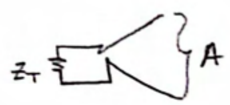
Power absorbed by Rx antenna $= S \times A$ → Aperture area.

S: Poynting vector; A: area.

1) Effective aperture

$A = \dfrac{P}{S}$ P → power delivered to load connected to antenna.

S → poynting vector or incident power density.

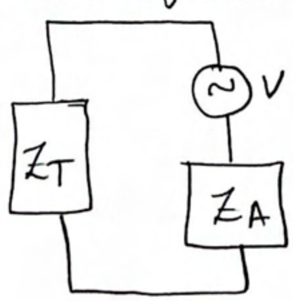


Matched load ⇒ $A_e = \dfrac{V^2}{4S(R_r + R_L)} = \dfrac{V^2}{4SR_r}$ for loss less ⇒ max $\underline{A_e}$.

A_e: Effective Aperture; R_r: rad res; R_L: loss res.

$\dfrac{V^2}{4SR_r}$ → $\underline{A_{em}}$

Thevenin Equivalent



$Z_T = R_T + jX_T$ $Z_A = R_A + jX_A$

$R_A = R_r + R_L$; R_r → radiation resistance

R_L → loss → heat.

2) Scattering Aperture

$$P_{received} = \underbrace{P_{dissipated\ as\ heat}}_{R_L}^{P'} + \underbrace{P_{delivered\ to\ Z_T}}_{R_T} + \underbrace{P_{reradiated\ by\ antenna}}_{R_r}^{P''}$$

$P' = I^2 R_L$

$P'' = I^2 R_r \longrightarrow$ Re radiated or Scattered power.

$$\boxed{A_S = \frac{P''}{S}} \quad \frac{\text{Reradiated power}}{\text{Incident power density}} \qquad P'' = I^2 R_r = \frac{V^2 R_r}{(R_r + R_L + R_T)^2 + (X_A + X_T)^2}$$

$$= \frac{V^2}{4 S R_r} \text{ when matched \& lossless.} \Rightarrow \boxed{A_S = A_{em}}$$

$\Rightarrow$ If antenna is lossless, equal power is radiated back as that delivered to load.

$\rightarrow$ If $R_L = 0$ & $X_T = -X_A \Rightarrow \boxed{A_S = 4A_{em}} \Rightarrow$ 4 times as much power is reradiated w.r.t matched case. Based on the phase of the receiver antenna current it could be a reflector or director.

$\rightarrow$ If $R_L = \infty \Rightarrow \boxed{A_S = A_e = 0}$

$\rightarrow \boxed{\beta = \frac{A_S}{A_e}} \rightarrow$ Scattering ratio.

$\rightarrow$ Incident wave & scattered wave could interfere & lead to nulls!

3) Loss Aperture

Power lost to heat P_L

$$\Rightarrow A_L = \frac{I^2 R_L}{S} = \frac{V^2 R_L}{S[(R_r + R_L + R_T)^2 + (X_A + X_T)^2]}$$

4) Collecting Aperture

$$A_c = \frac{V^2 (R_r + R_L + R_T)}{S[(R_r + R_L + R_T)^2 + (X_A + X_T)^2]} = A_s + A_L + A_e.$$

5) Physical Aperture

> Dimensions. → Arbitrary definition. → A_p

$$\epsilon_{ap} = \frac{A_e}{A_p} \begin{array}{l} \rightarrow \text{Effective.} \\ \rightarrow \text{Physical.} \end{array}$$

Aperture Efficiency.

Effective aperture :⇒ power delivered to load

Scattering aperture ⇒ power reradiated.

Loss aperture ⇒ heat.

Q: Matched ⇒ $P'' = \frac{V^2}{4R_r}$ is reradiated & $\frac{V^2}{4R_r}$ is delivered to load.

Short ⇒ $P'' = \frac{V^2}{R_r}$ is reradiated & 0 is delivered.

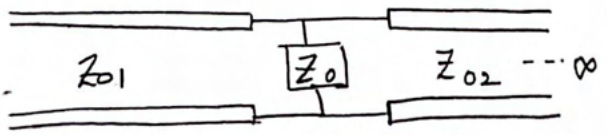
matched ⇒ Power received (reradiated + delivered) = $\frac{V^2}{2R_r}$

Short ⇒ Power received = $\frac{V^2}{R_r}$ → Is more power received when load is short? 2× more power? Yes!

→ When the load is short the antenna receives more power since more current flows through the circuit. It is not really useful since no power can be delivered to the load (if $R_L \downarrow$ I^2R_L also $\downarrow$)

Reflector.

TL Analogy

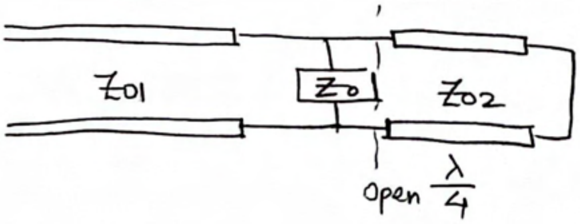


Z_{01} & Z_{02} are air = 377 Ω

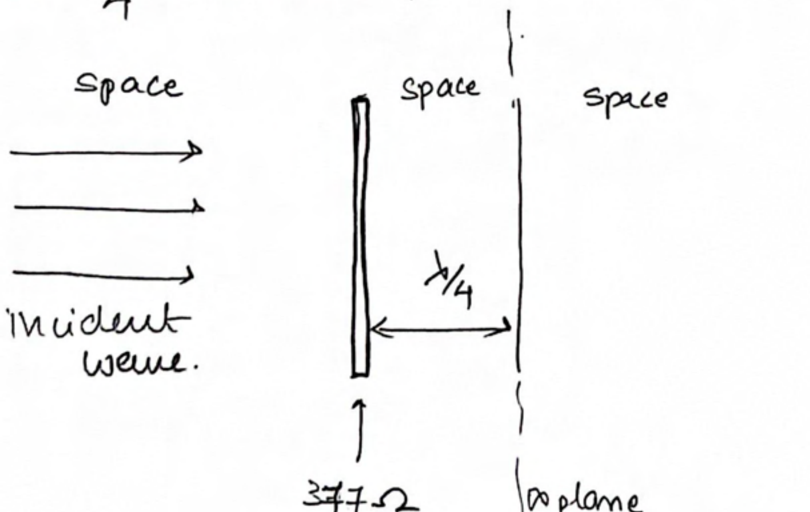
Z_0 is the antenna.

⇒ No way you can match Z_{01} to Z_0 because Z_{01} & Z_{02} are equal.

Solution?



> If $Z_0 = Z_{01}$, it is matched! ⇒ use an infinite ideal conductor at $\frac{\lambda}{4}$ distance from antenna.



Q: Would an open circuit instead of Z_{02} work? No because space is always there.

> If antenna ≠ 377Ω, change $d \neq \lambda/4$ to match?

→ For any antenna,

$$\lambda^2 = \underbrace{A_{em}}_{\text{effective aperture when } R_L = 0.} \Omega_A$$

$\Omega_A \simeq \theta_{HP} \phi_{HP}$ (total beam area)

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} P_n(\theta, \phi)\, d\Omega$$

$$\Rightarrow \boxed{D = \frac{4\pi}{\lambda^2} A_{em}} \quad \text{since } D = \frac{4\pi}{\Omega_A}$$

Lossy case $\Rightarrow \boxed{G = \frac{4\pi}{\lambda^2} A_e}$

Directivity formulae.

$$D = \frac{U(\theta, \phi)_{max}}{U_{av}} = \frac{S(\theta, \phi)_{max}}{S_{av}}$$

$$D = \frac{4\pi}{\Omega_A}$$

$$D = \frac{4\pi}{\lambda^2} A_{em}$$

Friis's Transmission formula.

$$P_r = P_t \frac{A_{et} A_{er}}{r^2 \lambda^2}$$ only valid in far field.

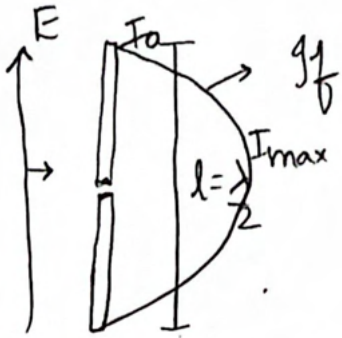
<u>Far field</u>: $R > \frac{2D^2}{\lambda}$

- Fields are TEM
- power flow is radially outward.
- Field pattern shape is independant of distance.

Effective Height

$h = \frac{V}{E}$ $V \rightarrow$ voltage induced, $E \rightarrow$ E field incident.

$\rightarrow \frac{\lambda}{2}$ dipole.



If field distribution was uniform, $h = l$. But since it is sinusoidal

$$\boxed{h = 0.64\,l}$$

$\rightarrow l \ll \lambda \Rightarrow \boxed{h = 0.5\,l} \rightarrow$ Triangular distribution.

$$\Rightarrow h_e = \frac{1}{I_0}\int_0^{h_p} I(z)\,dz = \frac{I_{av}}{I_0}\cdot h_p$$

$h_e \rightarrow$ effective. $h_p \rightarrow$ physical

If Polarization is accounted for h is a vector & $V = \vec{h_e}\cdot\vec{E}$

$$\Rightarrow P = \frac{V^2}{4R_r} = \frac{h^2E^2}{4R_r} \rightarrow \text{matched load power delivered.}$$

But $P = S\cdot A_e = \frac{E^2}{Z_0}\cdot A_e$ $\quad Z_0 \rightarrow 377\,\Omega$

$$\Rightarrow \boxed{h_e = 2\sqrt{\frac{R_r A_e}{Z_0}}} \quad \& \quad \boxed{A_e = \frac{h_e^2 Z_0}{4R_r}}$$

Poynting Vector.

$$S = \frac{1}{2}\vec{E}\times\vec{H}^*$$

Avg. poynting vector : $S_{av} = Re(S) = \frac{1}{2}Re\{E \times H^*\}$

Let $E_x = E_1 e^{j(\omega t - \beta z)}$ $\quad | \quad H_x = -H_2 e^{j(\omega t - \beta z + \delta - \xi)}$

$E_y = E_2 e^{j(\omega t - \beta z + \delta)}$ $\quad | \quad H_y = H_1 e^{j(\omega t - \beta z - \xi)}$

$\xi \rightarrow$ Phase lag of H vs. E : In a lossless medium $\xi = 0$

Avg Poynting Vector.

$$S_{avg} = \frac{1}{2} \hat{z} \left(E_1 H_1 + E_2 H_2 \right) \cos \xi.$$

Lossless $\Rightarrow$ $\xi = 0$ and $\frac{E}{H} = Z_0$

$$\Rightarrow S_{avg} = \frac{1}{2} \hat{z} H^2 Z_0 = \frac{1}{2} \hat{z} \frac{E^2}{Z_0}$$

$$\left.\begin{aligned} H &= \sqrt{H_1^2 + H_2^2} \\ E &= \sqrt{E_1^2 + E_2^2} \end{aligned}\right\}$$ Amplitude of the total E or H field.

> Polarization

⊙ Wave travelling out of the page & ↻ rotating clockwise is Left hand circularly polarized according to IEEE. This is opposite in classical optics.

An in depth view of Polarization. → Youtube : Polarization by Sander Konijnenberg.

↳ The signs of j terms maybe inconsistent.

$$E(x,t) = \begin{bmatrix} E_x(x,t) \\ E_y(x,t) \end{bmatrix}$$

Intensity $I(x) = \langle |E(x,t)|^2 \rangle = \langle |E_x(x,t)|^2 \rangle + \langle |E_y(x,t)|^2 \rangle$

Vectorial wave equation

$$\nabla^2 \vec{E}(x,t) - \frac{1}{c^2} \frac{\partial^2 \vec{E}(x,t)}{\partial t^2} = 0$$

Where $\vec{E} = \vec{E}_x \hat{x} + \vec{E}_y \hat{y} + \vec{E}_z \hat{z}$.

Plane wave solution

$$\vec{E}(x,t) = \vec{A} e^{j(\omega t - kx)},$$

$\vec{k}$ is always perpendicular to $\vec{A}$ from subs. this in Maxwell $\vec{A} \cdot \vec{k} = 0$.

$\Rightarrow$ If we choose $\vec{k}$ along $\hat{z}$ we only need $\vec{E}_x$ & $\vec{E}_y$ to describe $\vec{E}$ $\Rightarrow$ Use Jones vector.

$$\Rightarrow E(x,t) = \begin{bmatrix} A_x e^{+j\omega t} \\ A_y e^{+j\omega t} \end{bmatrix}$$

> Represent amplitudes as a vector called Jones vector.

$$J = \begin{bmatrix} A_x \\ A_y \end{bmatrix} \rightarrow \text{Used to describe Polarization State}$$

> $J = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow$ Horizontal polarization.

> $$J=\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} E_x(x,t) \\ E_y(x,t) \end{bmatrix} = \begin{bmatrix} Re\{A_x(x,t)\, e^{+j\omega t}\} \\ Re\{A_y(x,t)\, e^{+j\omega t}\} \end{bmatrix} = \begin{bmatrix} 0 \\ Re\{e^{+j\omega t}\} \end{bmatrix} = \begin{bmatrix} 0 \\ \cos\omega t \end{bmatrix}$$

$\downarrow$ vertical!

> $$J = \begin{bmatrix} 1 \\ j \end{bmatrix} \Rightarrow \begin{bmatrix} E_x(x,t) \\ E_y(x,t) \end{bmatrix} = \begin{bmatrix} Re\{e^{+j\omega t}\} \\ Re\{j e^{j\omega t}\} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\omega t \\ -\sin\omega t \end{bmatrix} \Rightarrow \text{Circularly polarized.}$$

$\hookrightarrow$ Left hand. (IEEE)

$\Rightarrow J \rightarrow$ complex vector.

> A linear polarizer (horizontal) is a matrix operation applied to the Jones vector.

$$\begin{bmatrix} A_x \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{} \begin{bmatrix} A_x \\ A_y \end{bmatrix}$$

$\hookrightarrow$ Jones matrix of horizontal polarizer.

> Jones matrix transforms polarization of one type to another.

> $$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ j \end{bmatrix} = ? \qquad M = \begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix}$$

> <u>But how is this realized?</u>

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} A_x \\ A_y \end{bmatrix} e^{+jk \cdot z}$$

> ignoring ωt → phasors.
> wave travelling in $\hat{z}$.

> propagation by a distance ~~d~~ d ⇒ phase shift $= \frac{2\pi d}{\lambda} = \underline{\underline{kd}}$

> If the medium has a refractive index $n \Rightarrow \Delta\phi = nkd$.
since $\lambda \rightarrow \lambda/n$

> If the material has different n for x & y components.
This material is called "birefringent". Eg: Calcide.

⇒ $$\begin{bmatrix} A_x \\ A_y \end{bmatrix} \longrightarrow \begin{bmatrix} A_x e^{+jkn_x d} \\ A_y e^{+jkn_y d} \end{bmatrix} \underset{\text{as}}{\overset{\text{same}}{\Leftrightarrow}} \begin{bmatrix} e^{+jkn_x d} & 0 \\ 0 & e^{jkn_y d} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \end{bmatrix}$$

> Birefringent media act as a complex diagonal Jones matrix.

> Global variations in phase don't matter only phase difference does. Therefore,

$$\begin{bmatrix} A_x \\ A_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{+jk(n_y - n_x)d} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{+jk\Delta n d} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \end{bmatrix}$$

> Where Δn is called the birefringence.

> If $\Delta n d = \frac{\lambda}{4}$ we get $\begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix} \rightarrow$ quarter wave plate. j & $-j$ are same.

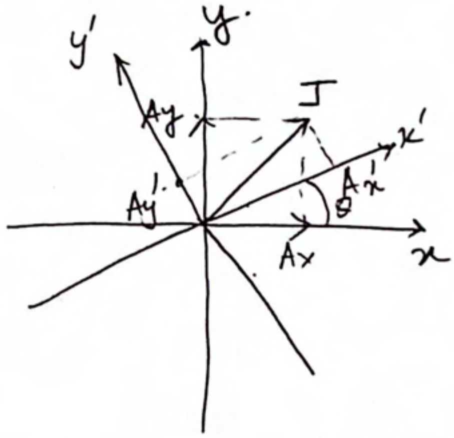
> Half wave plate $\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow E_y$ component is flipped in sine. Polarization is mirrored w.r.t X axis.

> If the plate itself is rotated, the polarization coming out is rotated $\Rightarrow$ It can convert a linearly polarized light source to any arbitrary orientation without any loss of intensity.

> What is a full wave plate? It does not transform the E field for a set of wavelengths $\Rightarrow$ Full wave plate followed by a linear polarizer is a filter!

> Rotation of an optical element affects the jones matrix

$J_{xy} = \begin{bmatrix} A_x \\ A_y \end{bmatrix}$ $\quad J_{x'y'} = \begin{bmatrix} A_x' \\ A_y' \end{bmatrix}$



$A_x' = \cos\theta \, A_x + \sin\theta \, A_y$

$A_y' = -\sin\theta \, A_x + \cos\theta \, A_y$.

$$\Rightarrow J_{x'y'} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} J_{xy}$$

$\hookrightarrow$ Denoted as $R_{+\theta}$ $\hookrightarrow$ by convention here θ is +ve for counterclockwise.

> Now we want to see how the rotated element acts on the vector $J_{x'y'}$. Which is just the Jones Matrix.

$$\overset{\text{Transmitted vector}}{J_{Tx'y'}} = M\, J_{x'y'}$$

> Going back to our original axis.

$$J_{Txy} = \overset{-\theta \text{ since rotating back.}}{R_{-\theta}} \cdot J_{Tx'y'}$$

$\Rightarrow$

$$\boxed{J_{Txy} = R_{-\theta} M R_{-\theta} J_{xy}}$$

$\Rightarrow$ Jones matrix of a rotated element is given by $\boxed{R_{-\theta} M R_{\theta}}$

$$\boxed{M_{\theta} = R_{-\theta} M R_{\theta}}$$

> If the wave is vertically polarized & horizontal polarizer kills it

> If it is RHCP, a LHCP element kills it.

$\Rightarrow$ Any field can be killed by applying the right filter?

> Not in reality. If the polarization varies rapidly,

> Then on average the wave energy cannot be completely attenuated.

> Called partial polarization.

> Unpolarized $\Rightarrow$ the polarization states take on every value. (Sunlight is unpolarized)

> For a single frequency, phase difference between E_x & E_y is constant $\Rightarrow$ it is polarized.

> Partially polarized waves can be defined using the correlation between E_x & E_y $\langle E_x(t)\, E_y(t)^* \rangle$

> Define coherency matrix. $\langle\ \rangle \rightarrow$ Time average.

$$G = \langle E_x(t)\, E_y(t)^* \rangle$$

$$= \begin{bmatrix} \langle |E_x(t)|^2 \rangle & \langle E_x(t)\, E_y(t)^* \rangle \\ \langle E_y(t)\, E_x(t)^* \rangle & \langle |E_y(t)|^2 \rangle \end{bmatrix}$$

$$= \left\langle \begin{bmatrix} E_x(t) \\ E_y(t) \end{bmatrix} \begin{bmatrix} E_x(t)^* & E_y(t)^* \end{bmatrix} \right\rangle$$

$$= \langle E(t)\, E(t)^{\dagger} \rangle$$ → conjugate transpose.

> Applying operations (Jones matrices) to $G \Rightarrow M \cdot G \cdot M^{\dagger}$

> Total intensity is given by the trace of the coherency matrix.

$$= \langle |E_x(t)^2| \rangle + \langle |E_y(t)|^2 \rangle = \mathrm{Tr}(G)$$

> $\boxed{G = J \cdot J^{\dagger}}$ $\Rightarrow$ If linearly polarized $J = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\Rightarrow G = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

> If $J_{norm} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ → normalised Jones vector $\Rightarrow G = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

> RCP $\Rightarrow J_{norm} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ j \end{bmatrix} \Rightarrow G = \frac{1}{2} \begin{bmatrix} 1 & -j \\ j & 1 \end{bmatrix}$

> For unpolarized light there is no Jones vector since E_x & E_y change rapidly. But a coherency matrix does exist.

$\langle E_x(t)\, E_y(t)^* \rangle = 0$

** 1) The correlation between E_x & E_y must be 0 for unpolarized light. Since they are completely random.

2) > Also observe, the time average intensity of E_x & E_y must be same if it is truly unpolarized ⇒ impartial

$$\Rightarrow \langle |E_x(t)|^2 \rangle = \langle |E_y(t)|^2 \rangle$$

⇒ If we normalize total intensity to be 1. The coherency matrix is given by $\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

⇒ Unpolarized ⇒ $G = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Horizontal ⇒ $G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

45° ⇒ $G = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

RHCP ⇒ $G = \frac{1}{2}\begin{bmatrix} 1 & -j \\ j & 1 \end{bmatrix}$

⇒ Another way to represent partially polarized light is just to use coefficients to give any coherency matrix as a linear combination of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ & $\begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix}$.

Therefore,

$$G = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + S_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + S_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + S_3 \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix}\right)$$

> S_1 & S_2 are coupled since the real parts of & must be same.
[∵ G is Hermitian ?]

> S_3 & S_1/S_2 give 2 degrees of freedom.

> Stokes Vector = $\begin{bmatrix} 1 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix}$ where S_1, S_2, S_3 are Stoke's parameter

> Unpolarized : $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$; $X \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$; $45° \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$; RCP $\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

<u>How do we measure S</u> ?

> Transmitted intensity: $I_1 = Tr\left(M G M^{+}\right) = \frac{1}{2}(1+S_1)$

if $M = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \; 0] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ → horizontally polarized.

⇒ Apply a horizontal polarizer, measure intensity & $I_1 = \frac{1}{2}(1+S_1)$

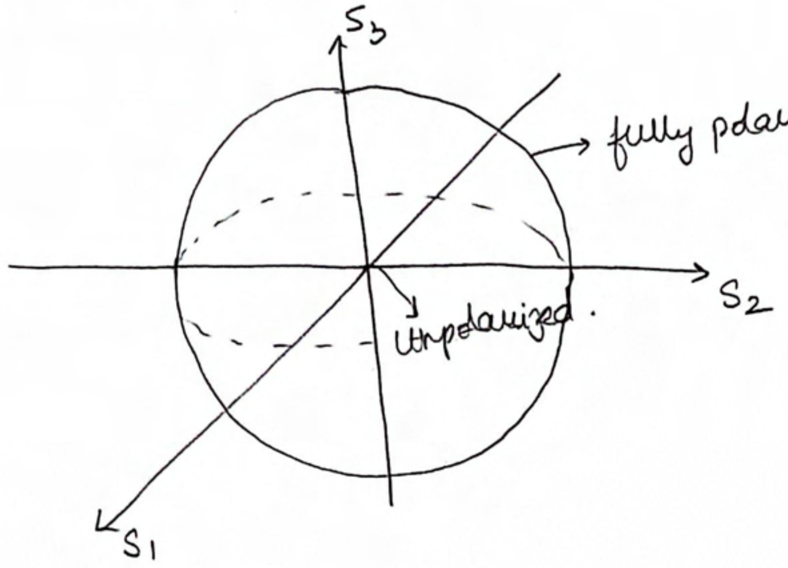
→ Diagonal polarizer ⇒ $I_{transmitted} = I_2 = \frac{1}{2}(1+S_2)$

→ R H C P ⇒ $I_{transmitted} = I_3 = \frac{1}{2}(1+S_3)$.

→ This is similar to S parameters!

Poincaré Sphere

* Visual representation of all the polarization states w.r.t Stokes parameters as the axes.



Fully polarized state with normalized intensity $\Rightarrow$ $G = \begin{bmatrix} |A_x| \\ |A_y| e^{-j\phi} \end{bmatrix} \begin{bmatrix} |A_x| & |A_y| e^{j\phi} \end{bmatrix}$

$$= \begin{bmatrix} |A_x|^2 & |A_x||A_y|(\cos\phi - j\sin\phi) \\ |A_x||A_y|(\cos\phi + j\sin\phi) & |A_y|^2 \end{bmatrix}$$

$$\Rightarrow S_1 = |A_x|^2 - |A_y|^2$$
$$S_2 = 2|A_x||A_y|\cos\phi$$
$$S_3 = 2|A_x||A_y|\sin\phi$$
notice $\sqrt{S_1^2 + S_2^2 + S_3^2} = 1$

$\Rightarrow$ Fully polarized waves lie on a sphere with radius 1.

$\Rightarrow$ For unpolarized light. $S_1, S_2, S_3 = 0 \Rightarrow$ origin.

$\Rightarrow$ Length of $\vec{S}$ indicates degree of polarization.

In general the intensity is not normalized to 1

and the Stokes vector is $S = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix}$

$S_0 = I_{tot}$

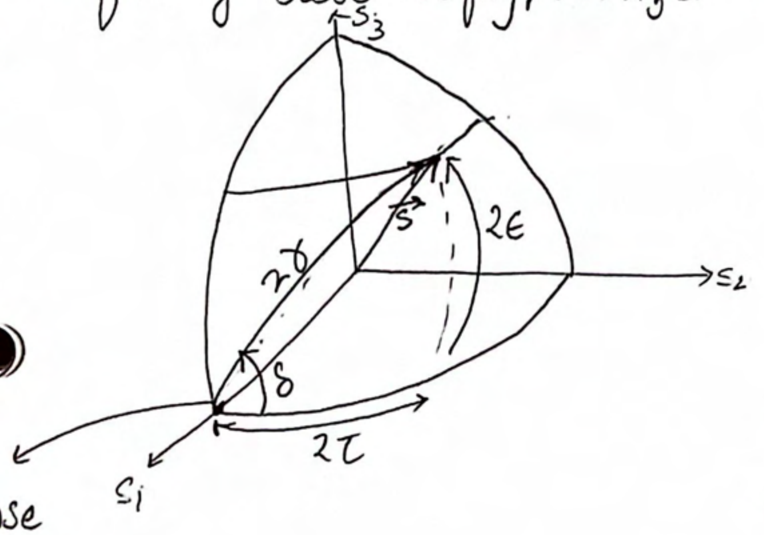
$S_1 = 2I_1 - I_{tot}$

$S_2 = 2I_2 - I_{tot}$

$S_3 = 2I_3 - I_{tot}$

$$|\vec{S}| = \frac{\sqrt{S_1^2 + S_2^S + S_3^2}}{I_{tot}}.$$

> To fully describe fully polarized light we need two angles since radius is = 1



Choose this as reference

One pair is 2τ & 2ϵ.

> longitude $= 2\tau$; $0° \leq \tau \leq 180°$

> latitude $= 2\epsilon$; $-45° \leq \epsilon \leq 45°$

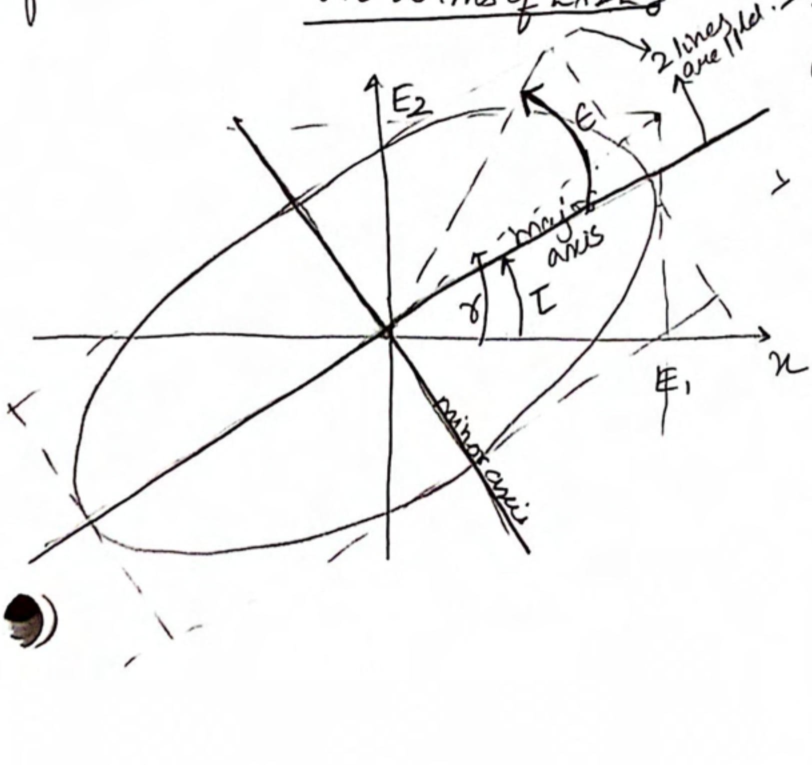
τ = tilt angle

$\epsilon = \cot^{-1}(AR)$ } On the actual polarization plot.

Another pair is 2γ & δ

> Choose a reference on the equator & draw a 'great circle' through $\vec{S}$.

In terms of Ex & Ey



> 2γ is angle subtended by $\vec{S}$ on the great circle with reference as origin.

> δ is angle b/w great circle & equator.

$\gamma = \tan^{-1}(E_2/E_1)$; $0° \leq \gamma \leq 90°$

δ = phase difference b/w E_1 & E_2

$-180° \leq \delta \leq 180°$

$$\cos 2\gamma = \cos 2\epsilon \cos 2\tau$$

$$\tan \delta = \frac{\tan 2\epsilon}{\sin 2\tau}$$

$$\tan 2\tau = \tan 2\gamma \cos \delta$$

$$\sin 2\epsilon = \sin 2\gamma \sin \delta$$

Case i $\delta = 0$ or $\delta = \pm 180°$ ⇒ E_x & E_y are in phase or out of phase ⇒ any point on equator is linearly polarized.

- At the reference origin ($\epsilon = \tau = 0$) horizontal polarization
- $90°$ from 'origin' tilt is $45°$, $180°$ ⇒ vertically polarized.

Case ii $\delta = \pm 90°$ ⇒ $E_2 = E_1$ ($2\gamma = 90°$ & $2\epsilon = \pm 90°$) ⇒ E_x & E_y are in phase quadrature ⇒ poles have circular polarization.

North pole ⇒ LHCP, South pole ⇒ RHCP according to IEEE.

→ Upper hemisphere is LHCP. Lower is RHCP.

** Voltage response of an antenna $V = K \cos \frac{M.M_a}{2}$

> Where MM_a is the angle subtended by great-circle line from polarization state M to M_a.

> Where M → polarization state of wave.
M_a → polarization state of antenna.
K → constant → depends on field strength & antenna size.

$MM_a = 0$ ⇒ Antenna is matched. & response is maximum.

$MM_a = 180°$ ⇒ no response eg: $M =$ (north pole) & $M_a =$ (south pole) ⇒ $MM_a = 180°$ ⇒ no response.

Power Theorem

"The total power radiated by the source is the integral over the surface of the sphere of the radial component S_r of the average poynting vector".

$$P = \oiint \vec{S} \cdot \vec{ds} = \oiint S_r \, ds$$

$ds = r^2 \sin\theta \, d\theta \, d\phi$

$S_r = |S|$

S_r has no θ or ϕ components for isotropic sources.

For Isotropic source

$$P = S_r \oiint ds = S_r 4\pi r^2$$

$$\Rightarrow \boxed{S_r = \frac{P}{4\pi r^2}} \quad W/m^2$$

Radiation intensity "U" $\longrightarrow$ "Power per unit solid angle".

$U = S_r \cdot r^2 \rightarrow$ in Watts per unit steradian. $\rightarrow$ dimensionless.

> Radiation intensity is independant of radius!

$$P = \iint U \sin\theta \, d\theta \, d\phi$$

$$\boxed{P = \iint U \, d\Omega}$$

"The total power radiated is given by the integral of the radiation intensity U over a solid angle of 4π".

For isotropic source

$$\boxed{P = 4\pi U_0}$$

Directivity

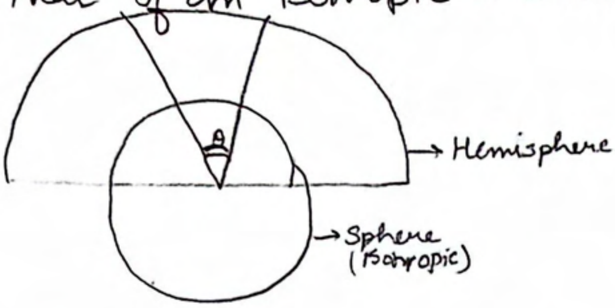
D = Ratio of maximum radiation intensity over it's average radiation intensity.

$$\boxed{D = \frac{U_m}{U_0}}$$

Hemisphere $\Rightarrow U_m \Rightarrow 2\pi U_m = P$.

$\Rightarrow$ If the same power is radiated by an isotropic source $P = 4\pi U_0$ we have $\frac{U_m}{U_0} = 2 = D$!

$\Rightarrow$ The power per unit solid angle of a hemispherical source is twice as that of an isotropic source if radiated powers are same.



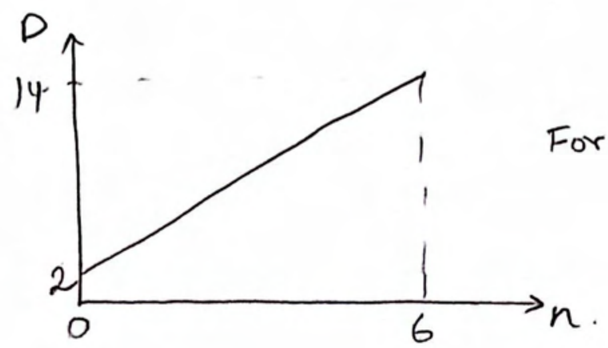
> Unidirectional cosine source $U = U_m \cos\theta \Rightarrow D = 4$!

> Bidirectional cosine $\Rightarrow D = 2$.

Bidirectional sine : $U = U_m \sin\theta \Rightarrow P = \pi^2 U_m \Rightarrow D = \frac{4}{\pi} = 1.27$

" Sine^2 : $U = U_m \sin^2\theta \Rightarrow P = \frac{8}{3}\pi U_m \Rightarrow D = 1.5$

↳ short dipole



For $U = U_m \cos^n\theta$ + unidirectional $\rightarrow \theta \in [0, \pi/2]$, $\phi \in [0, 2\pi]$

Directivity

$$D = \frac{4\pi}{\Omega_A} = \frac{4\pi}{\int_0^{2\pi}\int_0^{\pi} P_n(\theta)\sin\theta \, d\theta \, d\phi} \simeq \frac{41000}{\theta^\circ_{HP}\,\phi^\circ_{HP}}$$

Ω_A: Beam area

$\hookrightarrow$ (1) neglecting minorlobes
(2) Assuming $\Omega_A = \theta_{HP}\phi_{HP}$ for all antennas.

> Pencil beam $\Rightarrow$ symmetric around $\theta = 0°$ axis
> Beam area of isotropic source = 4π.

> $$S_r = \frac{1}{2}\frac{E^2}{Z}$$

$S_r \rightarrow$ average poynting vector.
$E \rightarrow \sqrt{E_\theta^2 + E_\phi^2} \rightarrow$ amplitude of total E field.
$Z \rightarrow 377\,\Omega$ for space.

> $$P_n = \frac{S_r}{S_{rm}} = \frac{U}{U_m} = \left(\frac{E}{E_m}\right)^2$$

P_n: relative power pattern
S_{rm}: max. Poynting vector magnitude.
U_m: max radiation intensity.
$\frac{E}{E_m}$: relative total field pattern

---

4 quantities to fully describe the far field.

1) Amplitude of the polar component E_θ as a fn of r, θ, ϕ
2) " " " azimuthal " E_ϕ " " " .
3) Phase lag δ of E_ϕ behind E_θ as a fn of θ & ϕ
4) Phase lag η of either field component behind its value at a reference point as a fn of r, θ & ϕ.
   $\hookrightarrow$ useful when adding field components from multiple sources

Chapter -3 - Point Sources.

> Far field is fully described by 4 quantities.

i) $E_\theta(\theta, \phi)$

ii) $E_\phi(\theta, \phi)$

iii) $\delta(\theta, \phi)$ - phase difference between E_θ & E_ϕ.

iv) $\eta(\theta, \phi, r)$ - phase lag of either E_θ or E_ϕ w.r.t some reference.

more imp for multiple sources

> In the far field $\bar{S}$ has only radial component $S_r \Rightarrow |\bar{S}| = S_r$

Simplified power theorem.

$$P = \oiint \bar{S} \cdot \overline{ds}$$

$$P = \oiint S_r \, ds$$

P - power radiated.

S_r - avg. poynting vector along $\bar{r} \rightarrow$ Watt/m^2

$ds = r^2 \sin\theta \, d\theta \, d\phi$

For isotropic source $S_r = \frac{P}{4\pi r^2}$

Radiation intensity

> Power per unit solidangle. $\boxed{U = r^2 S_r} \rightarrow$ Watt/steradian.

dimension less.

> Independent of radius. Whereas S_r is dependent on radius, (more power flows through a unit area closer to the source)

$$\therefore P = \iint U \sin\theta \, d\theta \, d\phi = \iint U \, d\Omega.$$

For an anisotropic source having a maximum radiation intensity of U_m. The directivity $D = \frac{U_m}{U_0} \rightarrow$ radiation intensity of an isotropic source radiating the same total power.

Eg: $U = U_m \cos\theta. \Rightarrow P = \int_0^{2\pi}\int_0^{\pi/2} U_m \cos\theta \sin\theta \, d\theta \, d\phi = \pi U_m$

$$\Rightarrow \pi U_m = 4\pi U_0 \Rightarrow \boxed{D = \frac{U_m}{U_0} = 4}$$

Directivity in detail

$D = \frac{U_m}{U_0}$; $U_0 = \frac{P}{4\pi}$

Let $U = U_a f(\theta, \phi) \rightarrow$ shape of radiation intensity.
$U_a \rightarrow$ constant

$\Rightarrow P = \iint U \, d\Omega = \iint U_a f(\theta, \phi) \, d\Omega$ & $U_m = U_a f(\theta, \phi)_{max}$

$\Rightarrow U_0 = \frac{\iint U_a f(\theta, \phi) \, d\Omega}{4\pi}$

$\Rightarrow \boxed{D = \frac{4\pi f(\theta, \phi)_{max}}{\iint f(\theta, \phi) \sin\theta \, d\theta \, d\phi}}$ or $D = \frac{4\pi}{\Omega_A}$

where, $\boxed{\Omega_A = \frac{\iint f(\theta, \phi) \, d\Omega}{f(\theta, \phi)_{max}}}$

$\rightarrow \Omega_A$ is the solid angle through which all the power radiated would stream if the power per unit solid angle equaled the maximum value U_m over the beam area.

Therefore, $\Omega_A \simeq \theta_{HP} \phi_{HP}$ for single lobe patterns.

$\Rightarrow D \simeq \frac{4\pi}{\theta_{HP} \phi_{HP}} \approx \frac{41000}{\theta^\circ_{HP} \phi^\circ_{HP}}$ → only works for small minor lobes & symmetric beams.

Field patterns.

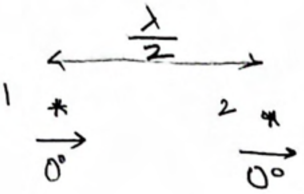
avg Poynting vector $S_r = \frac{1}{2} \frac{\sqrt{E_\theta^2 + E_\phi^2}}{Z_0}$

Chapter 4 Arrays of Point sources

- An intuitive approach to understanding how phases add & cancel.

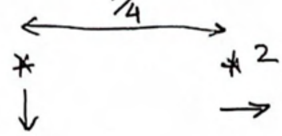
→ Assume phasor notation for sources. $\rightarrow 0°$ $\uparrow 90°$ $\leftarrow 180°$ $\downarrow 270°$ $\swarrow$ the rotation is counterclockwise.

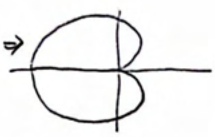
→ Eg:



> As wave from 1 reaches 2, it's phase is preserved like a surfer riding a wave. But in this time 2 has turned 180° due to $\frac{\lambda}{2}$ spacing. Therefore they cancel in the endfire direction. ⇒ 

→ Eg: add.



cancel. ⇒ 

- Pattern multiplication.

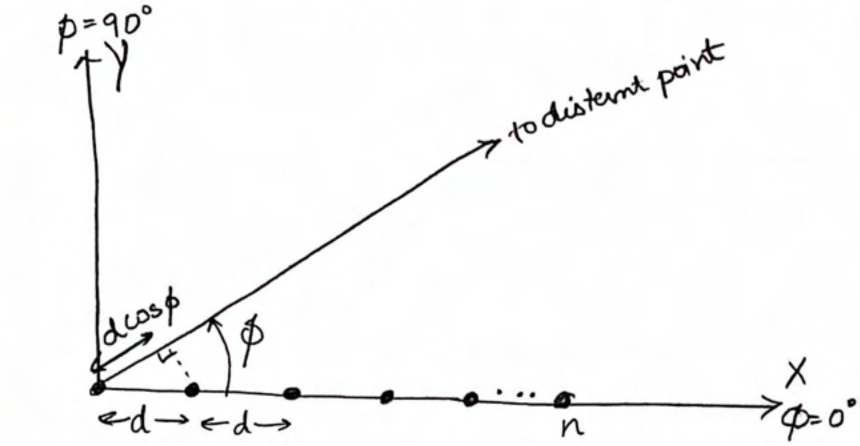
"The field pattern of an array of nonisotropic but similar point sources is the product of the pattern of the individual source and the pattern of an array of isotropic point sources, having the same locations, relative amplitudes and phases as the non isotropic point sources".

"The total phase pattern is the sum of the phase patterns of the individual source and the array of isotropic point sources".

$$\therefore E = \underbrace{f(\theta,\phi)}_{\substack{\text{field pattern}\\ \text{of individual}\\ \text{source}}} \underbrace{F(\theta,\phi)}_{\substack{\text{field pattern}\\ \text{of array of}\\ \text{isotropic sources.}}} \underline{/ f_p(\theta,\phi) + F_p(\theta,\phi)}$$

-

Array factor of linear arrays.



E = Electric field in the far field.

$$= 1 + e^{j\psi} + e^{2j\psi} + e^{3j\psi} + \dots + e^{(n-1)j\psi}$$

where $\boxed{\psi = \frac{2\pi}{\lambda} \cdot d\cos\phi + \delta}$

δ — phase diff of adjacent sources.

> Here source 1 is chosen as the phase reference.

> This is a geometric series.

$$\Rightarrow E = \frac{1 - e^{jn\psi}}{1 - e^{j\psi}}$$

$$= \frac{\sin(n\psi/2)}{\sin(\psi/2)} \cdot \angle \frac{n-1}{2}\psi$$

If the phase center is the centerpoint of the array,

$$E = \frac{\sin(n\psi/2)}{\sin(\psi/2)}$$

Normalizing to 1.

$$\boxed{AF = \frac{1}{n} \frac{\sin(n\psi/2)}{\sin(\psi/2)}}$$

The maximum radiation occurs in the direction that $\psi = 0$. This is not always possible since $\psi \neq 0$ for all ϕ in some cases.

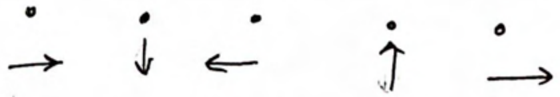
Case (1) Broadside array

Let $\delta = 0 \Rightarrow$ if $\phi = \frac{\pi}{2}$ & $\frac{3\pi}{2}$, $\psi = 0 \Rightarrow$ maximum occurs along y axis. Note that $d > \lambda$ gives side lobes equal in magnitude to the main lobe. These are called "grating lobes"

Case (2) Endfire array.

Make $\psi = 0$ & $\phi = 0 \Rightarrow \delta = -d_r$.

$\Rightarrow$ if $d_r = \frac{\lambda}{4} \Rightarrow$ $n=2$ should lag $n=1$ by $90°$



However for increased directivity it has been shown that $\boxed{\delta = -\left(d_r + \frac{\pi}{n}\right)}$ is better. However make sure to keep the spacing $< \frac{\pi}{2}$ for small back lobes.

Case (3) Arbitrary direction.

If ϕ_1 is the desired direction we only need to solve for.

$$\psi = 0 = d_r \cos\phi_1 + \delta.$$

$$\Rightarrow \quad \delta = -d_r \cos\phi_1$$

$d_r = \frac{\lambda}{2}$ & $\phi = 60°$

$$\Rightarrow \delta = -\frac{\lambda}{2} \times \frac{1}{2} = -\frac{\lambda}{4}$$

$$\Rightarrow \delta = \frac{2\pi}{\lambda} \times -\frac{\lambda}{4} = -\frac{\pi}{2}$$

Null directions

$$\phi_0 = \arccos\left[\left(\pm \frac{2k\pi}{n} - \delta\right)\frac{1}{d_r}\right]$$

> k must not be a multiple of n.

$\therefore$ FNBW for long broadside arrays is $\simeq \dfrac{2\lambda}{nd}$.

$(nd \gg k\lambda)$

$\therefore$ FNBW for long endfire arrays is $\simeq 2\sqrt{\dfrac{2\lambda}{nd}}$.

in radians

$\therefore$ FNBW for increased directivity endfire arrays is $\simeq 2\sqrt{\dfrac{\lambda}{nd}}$.

→ In general broadside antennas have much higher directivity for the same total area.

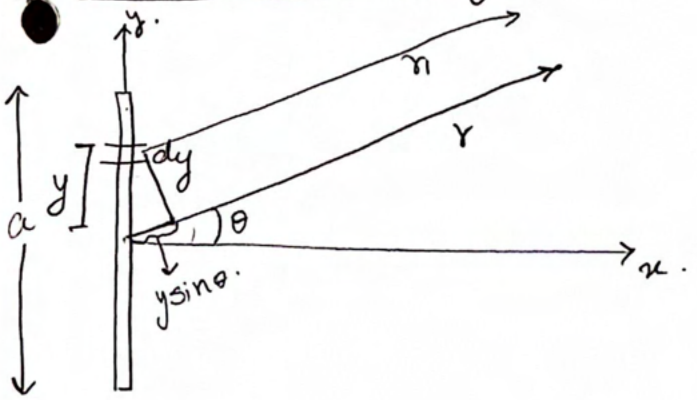
→ Binomial distribution of amplitudes gives 0 sidelobes but a broader main lobe.

→ The optimal tradeoff is given by the Dolph Tchebyscheff distribution. The binomial & edge distributions are limiting case of the DT distribution.

→ A uniform distribution  represents a square wave in analogous terms, this has higher harmonics in F.T terms & therefore there are several sidelobes. This fourier analogy is generally true.

→ The DT distribution is worked out in 4-12 in the book.

Continuous arrays.



$$dE = \frac{A}{r_1} e^{j\omega(t - (\frac{r_1}{c}))} dy$$

$$= \frac{A}{r_1} e^{j(\omega t - \beta r_1)} dy$$

$$\Rightarrow E = \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{A}{r_1} e^{j(\omega t - \beta r_1)} dy$$

if $r_1 \gg a \Rightarrow$ $$E = \frac{A e^{j\omega t}}{r_1} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-j\beta r_1} dy$$ & $r_1 = r - y\sin\theta$.

$$\Rightarrow E = A' \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{j\beta y \sin\theta} dy = \frac{A e^{j(\omega t - \beta r)}}{r_1} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{j\beta y \sin\theta} dy$$

$$\Rightarrow E = \frac{2A'}{\beta \sin\theta} \sin\left(\frac{\beta a}{2} \sin\theta\right)$$

Let $\psi' = \beta a \sin\theta = a_r \sin\theta$. where $a_r = \beta a = \frac{2\pi}{\lambda} a$ = array length, (rad.)

$$\Rightarrow E = a A' \frac{\sin(\psi'/2)}{(\psi'/2)}$$

Normalising;

$$\boxed{E = \frac{\sin(\psi'/2)}{\psi'/2}}$$ → same as large array, replace a with nd & vice versa for other results from arrays.

Huygen's Principle

"Each point on a primary wave front can be considered to be a new source of a secondary spherical wave and that a secondary wave front can be constructed as the envelope of these secondary waves".

→ The diffraction of light from a slot in an infinite sheet can be modelled by a continuum of isotropic point sources because of Huygen's principle & in fact the far field diffraction pattern is of the form $\frac{\sin(\psi'/2)}{(\psi'/2)}$ where, $\psi' = \frac{2\pi a \sin\theta}{\lambda}$

→ The horn antenna & a uniform current sheet can also be similarly modelled.

→ Huygen's theory ignores phase & also the currents excited on the slot or horn edges.

Ch 5 - Electric Dipole. [Assuming the dipole is end loaded to maintain constant current everywhere]

› Retarded vector potential of a Hertzian dipole.

$$A_z = \mu L I \frac{e^{-jkr}}{4\pi r} \cdot e^{j\omega t}$$

only has a z component in far field.

› Retarded scalar potential. of a Hertzian dipole.

$$V = \frac{I_0 L \cos\theta \cdot e^{j\omega t} e^{-jkr}}{4\pi \epsilon_0 c} \left(\frac{1}{r} + \frac{c}{j\omega} \frac{1}{r^2} \right)$$

$$\bar{E} = -j\omega \bar{A} - \nabla V$$

$$\bar{H} = \frac{1}{\mu} \nabla \times \bar{A}$$

$$A_r = A_z \cos\theta ; \quad A_\theta = -A_z \sin\theta$$

$$E_r = \frac{I_0 L \cos\theta \, e^{j\omega(t - r/c)}}{2\pi \epsilon_0} \left(\frac{1}{c r^2} + \frac{1}{j\omega r^3} \right)$$

$$E_\theta = \frac{I_0 L \sin\theta \, e^{j\omega(t - r/c)}}{4\pi \epsilon_0} \left(\frac{j\omega}{c^2 r} + \frac{1}{c r^2} + \frac{1}{j\omega r^3} \right).$$

$$H_\phi = \frac{I_0 L \sin\theta \, e^{j\omega(t - r/c)}}{4\pi} \left(\frac{j\omega}{c r} + \frac{1}{r^2} \right).$$

$H_r = H_\theta = E_\phi = 0.$

In the far field.

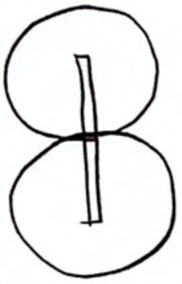
$$E_\theta = j \frac{I_0 \beta L}{4\pi \epsilon_0 c r} \sin\theta \; e^{j\omega(t - r/c)}.$$

$$H_\phi = j \frac{I_0 \beta L}{4\pi r} \sin\theta \; e^{j\omega(t - r/c)}$$

& $\frac{E_\theta}{H_\phi} = 377\Omega$

→ In the near field E & H are in time phase quadrature thus resembling a resonator.

→ At intermediate distances E_r & E_θ approach time phase quadrature, therefore ⇒ circular polarization but parallel to direction of propagation & hence a "cross field".

→  → Near field E_r pattern.

→ DC or quasi stationary case. $(\omega \to 0)$

$$E_r = \frac{q_0 L \cos\theta}{2\pi \epsilon_0 r^3} \; ; \; E_\theta = \frac{q_0 L \sin\theta}{4\pi \epsilon_0 r^3} \; ; \; H_\phi = \frac{I_0 L \sin\theta}{4\pi r^2} \rightarrow \text{Biot Savart law}$$

Electrostatic field from a stationary dipole.

→ The scalar potential does not contribute to the far field & therefore

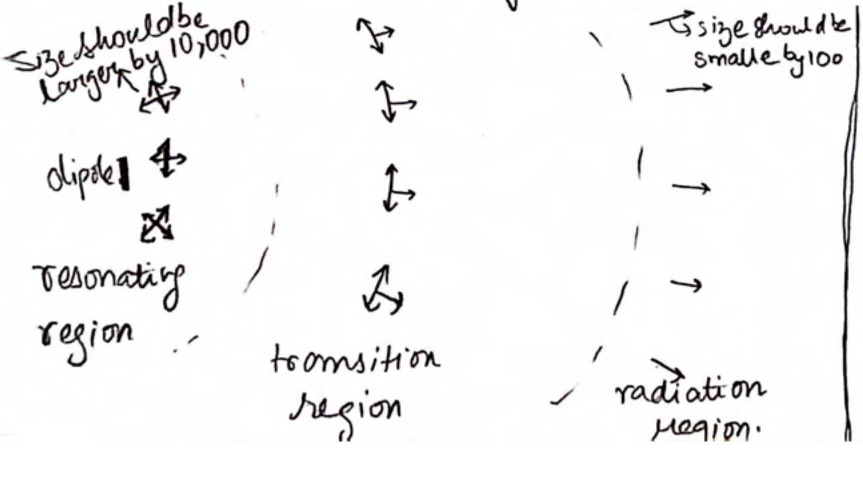
$$|\bar{E}| = E_\theta = -j\omega A_\theta$$

$$|\bar{H}| = H_\phi = \frac{E_\theta}{Z} = \frac{-j\omega}{Z} A_\theta \quad \text{or} \quad H_\phi = |\bar{H}| = \frac{1}{\mu_0} |\nabla \times \bar{A}|$$

$$\rightarrow S_r = \frac{1}{2} E_\theta H_\phi \cos(-45^\circ) = \frac{1}{2\sqrt{2}} E_\theta H_\phi \; ; \; S_\theta = 0$$

→ E_θ & H_ϕ are in phase in the far field.

→ Near the dipole $S_r = S_\theta = 0$ since energy is only stored & not radiated. S_r only becomes nonzero in the far field.



↔ ⇒ energy oscillation (reactive power)

→ ⇒ energy propagation.

Radiation resistance.

- > Integrate S_r over a large sphere for P_{rad}.
  > $P_{rad} = I_{rms}^2 R$. where I_{rms} = rms current on the dipole.

Average Poynting vector $\bar{S} = \frac{1}{2} \mathrm{Re}\left(\bar{E} \times \bar{H}^*\right)$

$$\Rightarrow S_r = \frac{1}{2} \mathrm{Re}\, E_\theta H_\phi^*$$

$$= \frac{1}{2} |H_\phi|^2 \sqrt{\frac{\mu}{\epsilon}}$$

$$\Rightarrow \quad P_{rad} = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}} \int_0^{2\pi}\int_0^{\pi} |H_\phi|^2 r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{32}\sqrt{\frac{\mu}{\epsilon}} \frac{\beta^2 I_o^2 L^2}{\pi^2} \int_0^{2\pi}\int_0^{\pi} \sin^3\theta \, d\theta \, d\phi$$

$$\boxed{P_{rad.} = \sqrt{\frac{\mu}{\epsilon}} \frac{\beta^2 I_o^2 L^2}{12\pi}}$$

$$\Rightarrow \quad R_r \times \left(\frac{I_o}{\sqrt{2}}\right)^2 = \sqrt{\frac{\mu}{\epsilon}} \frac{\beta^2 I_o^2 L^2}{12\pi}$$ → Assuming no loss in the antenna to heat.

$$\Rightarrow \boxed{R_r = \sqrt{\frac{\mu}{\epsilon}} \frac{\beta^2 L^2}{6\pi}}$$

If power is propagating in air $\Rightarrow \sqrt{\frac{\mu_o}{\epsilon_o}} = 377$

$$\Rightarrow \boxed{R_r = 80\pi^2 \left(\frac{L}{\lambda}\right)^2}$$ → length of small dipole!

- If $\frac{L}{\lambda} = 0.1 \Rightarrow \boxed{R_r = 7.9\Omega}$ → Very small even for not so small dipole.

not that small.

→ If the dipole is not end loaded but short, the current tapers almost linearly from the middle to the ends with an average value of $\frac{1}{2} I_{max}$.

$$\Rightarrow P_{rad} = \sqrt{\frac{\mu}{\epsilon}} \frac{\beta^2 I_0^2 L^2}{4 \times 12\pi}$$

$$\Rightarrow \boxed{R_r = 20\pi^2 \left(\frac{L}{\lambda}\right)^2}$$ → Even smaller than before.

Thin linear antennas

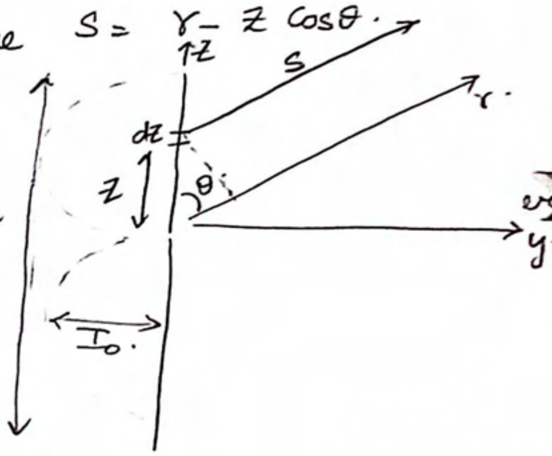
> If the wire diameter is $< \frac{\lambda}{100}$, the current distribution is sinusoidal.

> Retarded current $[I] = I_0 \sin\left[\frac{2\pi}{\lambda}\left(\frac{L}{2} \pm z\right)\right] e^{j\omega(t - r/c)}$.

> Break it down to ∞ small dipoles,

$$dE_\theta = \frac{j\, 60\pi\, [I] \sin\theta\, dz}{s\lambda}$$

where $s = r - z\cos\theta$.



$$H_\phi = \int_{-L/2}^{L/2} dH_\phi$$

Simplifying,

$$H_\phi = \frac{j[I_0]}{2\pi r}\left[\frac{\cos[(\beta L\cos\theta)/2] - \cos(\beta L/2)}{\sin\theta}\right]$$

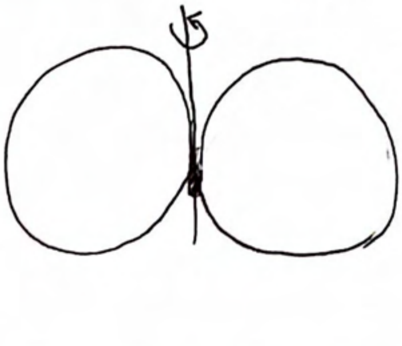
where $[I_0] = I_0 e^{j\omega(t - r/c)}$.

$$E_\theta = \frac{j60[I_0]}{r}\left[\frac{[\cos(\beta L\cos\theta)/2] - \cos(\beta L/2)}{\sin\theta}\right]$$

→ stuff in the bracket gives the pattern.

case (i) $L = \frac{\lambda}{2}$

$$\Rightarrow E_{pattern} = \frac{\cos\left[\frac{\pi}{2}\cdot\cos\theta\right]}{\sin\theta}$$

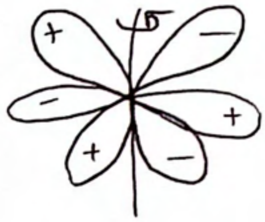


HPBW = 78°

Slightly more directional than a small dipole where HPBW = 90°

Case (ii) $L = \lambda$ $\quad E_{pattern} = \dfrac{\cos(\pi \cos\theta) + 1}{\sin\theta}$ $\quad$ HPBW = 47°.

Case (iii) $L = \frac{3\lambda}{2}$ $\quad E_{pattern} = \dfrac{\cos\left(\frac{3\pi}{2}\cos\theta\right)}{\sin\theta}$.



Radiation resistance of $\lambda/2$ dipole:

$$P_{rad} = 30 I_0^2 \int_0^{\pi} \frac{\{\cos[(\beta L/2)\cos\theta] - \cos(\beta L/2)\}^2}{\sin\theta}\, d\theta$$

$$R_0 = \frac{P_{rad}}{I_0^2/2}$$

$$\Rightarrow R_0 = 60 \int_0^{\pi} \frac{\{\cos[(\beta L/2)\cos\theta] - \cos(\beta L/2)\}^2}{\sin\theta}\, d\theta.$$

Remember that the R here is always referred to the point where the current maximum occurs. For $\frac{\lambda}{2}$ it occurs at the center ⇒ where the T line connects so it is convenient.

After a whole bunch of calculation involving some special functions we get $\boxed{R_0 = 73\,\Omega}$

→ In reality there is also a small inductance at the terminals, this can be removed by making it exactly resonant. This is done by making L slightly less than $\lambda/2$. However now R is slightly lower.

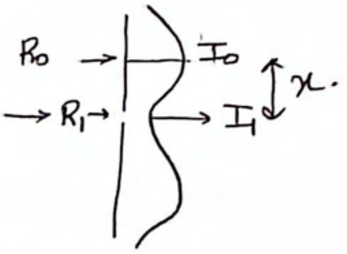
$$\boxed{R_0 = 73 + j42.5}$$

Radiation resistance of a point that is not the current maximum

→ R_0 is found at current maximum but for antennas whose feed is not at the current max. we need to transform R_0 to the feed point.

→ Can we feed at R_0? No because it would change the current distribution.

→ Power at I_0 & R_0 is same & power supplied at feed.



$$\Rightarrow \frac{I_0^2}{2} R_0 = \frac{I_1^2}{2} R_1$$

$$\Rightarrow R_1 = \left(\frac{I_0}{I_1}\right)^2 R_0.$$

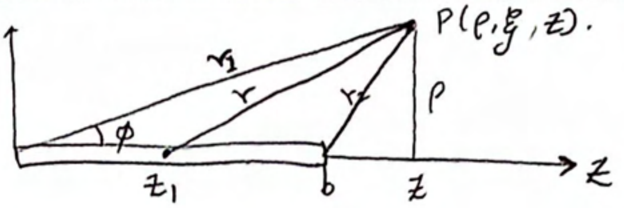
& $I_1 = I_0 \cos(\beta x)$.

$$\Rightarrow \boxed{R_1 = \frac{R_0}{\cos^2 \beta x}} \rightarrow \text{At current minimum (if } x = \tfrac{\lambda}{4}) \rightarrow \infty.$$

In reality the current is not zero since the conductor is not of 0 thickness but R_1 can be very large. $\simeq k\Omega$.

> These antennas so far are resonance based. However, there are several antennas that have a travelling wave inside them & therefore the above analysis does not work since current distributions are not sinusoidal but uniform.

Thin linear antenna with a uniform travelling wave



$$H_\xi = j\omega\epsilon(\nabla\times\overline{\Pi})_\xi = -j\omega\epsilon\frac{\partial\overline{\Pi}_z}{\partial\rho} \quad \& \quad \overline{\Pi}_z = \frac{1}{4\pi j\omega\epsilon}\int_0^b \frac{[I]}{r}\,dz_1$$

After a whole bunch of calculation, in the far field.

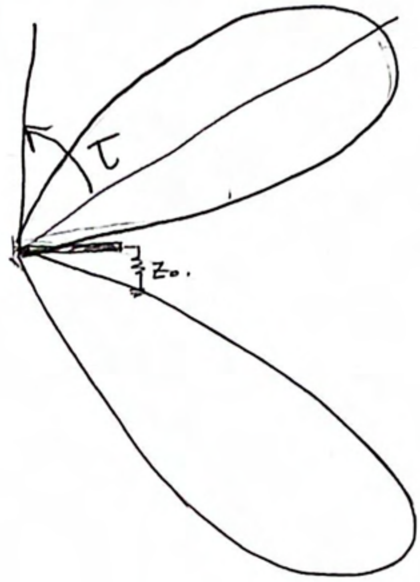
$$[I] = I_0\sin\omega\left(t-\frac{r}{c}-\frac{z_1}{v}\right).$$

$v = pc$ ↳ ratio of vel. along conductor to vel. of light.

$$H_\xi = \frac{I_0\,p}{2\pi r_1}\left\{\frac{\sin\phi}{1-p\cos\phi}\sin\left[\frac{\omega b}{2pc}(1-p\cos\phi)\right]\right\}\boxed{\left[\omega\left(t-\frac{r_1}{c}\right)-\frac{\omega b}{2pc}(1-p\cos\phi)\right]}$$

p here is the relative phase velocity.

Case (i) Linear λ/2 antenna. (p=1)



HPBW = 60° → was 78° for $\frac{\lambda}{2}$ resonant antenna.
τ = 25°

→ If p is reduced, τ↑ & HPBW↓ but sidelobes start appearing.

→ As l↑ in case ii & iii, τ↑↑ & sidelobes also ↑

→ The general ration patter is a conical shape along the direction of current.

Loop Antenna.

- > HW5 of 530 covers this topic
  - > Small loops of <u>same area</u> have the radiation pattern if they are squares or circles.
  - > Small loops produce fields that are in phase quadrature with small dipoles & hence can produce circular polarization when used in combination. Refer to HW5.
  - > Radiation resistance: $R_r \simeq 31171 \left(\frac{A}{\lambda^2}\right)^2$, $A = \pi a^2$. Works only for small loops. If n turns $\Rightarrow$ $R_r \simeq 31171 \left(n\frac{A}{\lambda^2}\right)^2$. This is pretty lo
  - > For large loops $R_r = 3720 \frac{a}{\lambda}$ but need to ensure current is in phase throughout the loop (use phase shifters). This is pretty high.

- <u>Note</u>: for an antenna with loss R_L & radiation resistance R_r. the radiation efficiency $k = \frac{R_r}{R_r+R_L}$ & Gain $= kD$.

$$\Rightarrow \boxed{G = \frac{R_r}{R_r+R_L} \cdot \frac{4\pi A_{em}}{\lambda^2}}$$

ignoring dielectric loss.

Skin depth $\delta = \frac{1}{\sqrt{f\pi\mu\sigma}}$

μ = permeability of medium H/m

σ = conductivity $Ohm^{-1} m^{-1}$

ferrite rod can affect μ.

$$\Rightarrow \boxed{R_{L\,loop} = \frac{L}{\sigma\pi d\delta}}$$

where L = length of loop
d = conductor diameter.

→ Loop antenna efficiency is fairly low. They are still used since multiple turns & a ferrite core can increase R_r. However this antenna is rarely used today.

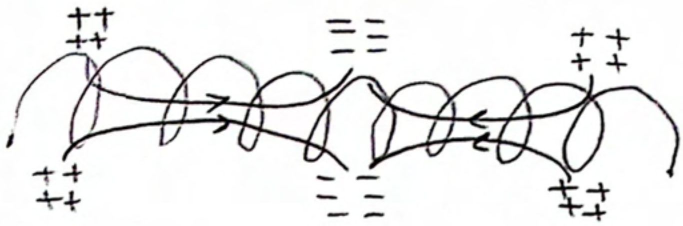
Chapter 7 Helical antenna.

- Circularly polarized.
- High gain
- Wideband R_r & wideband in terms of locking to the axial mode of radiation.
- Therefore used extensively for space comm.
- If diameter $\simeq \lambda \Rightarrow$ antenna. $\rightarrow$ radiation mode

  If diameter $<< \lambda \Rightarrow$ slow wave structure (TWT). $\rightarrow$ Transmission mode.

Transmission mode. (Low frequency) $C << \lambda$ where $C = \pi D$.

$\Rightarrow$ one lambda undergoes several turns.



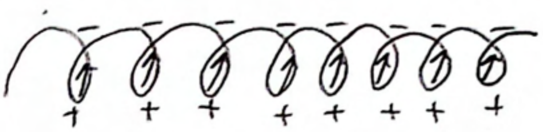
Fields are along the axis & therefore interact with electron beam in TWTs.

$\hookrightarrow$ T_0 mode. (may radiate) (but in Broadside direction & small R_r)

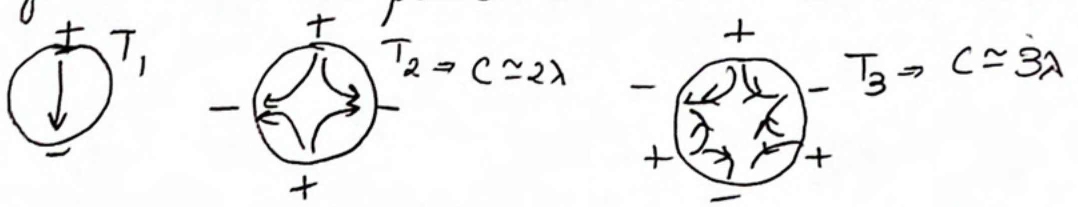
Radiation mode.

$C \simeq \lambda \longrightarrow T_1$ mode

(Radiates in endfire direction)



Higher modes are possible with distributions as shown,



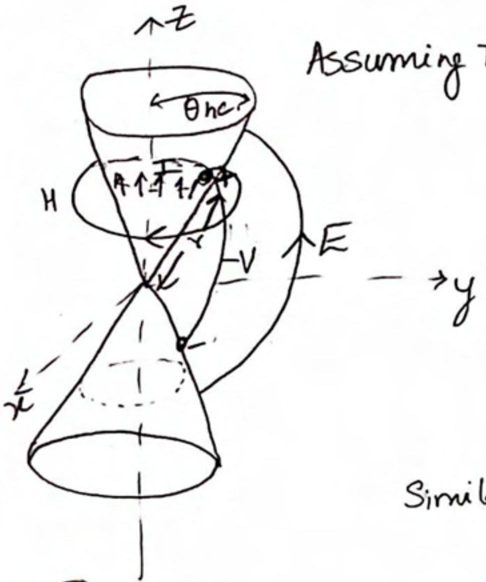
- The analysis is quasiempirical so just read the book. (Sec 7-14 is cool)

Ch 8 Biconical Antenna.

- **Characteristic impedance of infinite biconical antenna.**

› Transmission lines guide plane waves

› Biconical antennas guide spherical waves.



Assuming TEM mode.

$$\nabla \times \bar{E} = -j\omega\mu \bar{H}$$

$$\Rightarrow \frac{\hat{\phi}}{r}\frac{\partial (rE_\theta)}{\partial r} = -\hat{\phi}\, j\omega\mu H_\phi$$

$$\Rightarrow \frac{1}{r}\frac{\partial (rE_\theta)}{\partial r} = -j\omega\mu H_\phi \quad \text{—①}$$

Similary $\nabla \times \bar{H} = j\omega\epsilon \bar{E}$ gives,

$$\frac{\partial (rH_\phi)}{\partial r} = -j\omega\epsilon (rE_\theta) \quad \text{—②}$$

Combining ①&② $\Rightarrow$ $$\frac{\partial^2 (rH_\phi)}{\partial r^2} = -\omega^2\mu\epsilon\,(rH_\phi)$$

$$\Rightarrow H_\phi = \frac{1}{r\sin\theta} H_0 e^{-j\beta r}$$ where $\beta = \omega\sqrt{\mu\epsilon}$

$e^{-j\beta r}$ → outgoing travelling wave.

$$\Rightarrow E_\theta = \frac{Z_0}{r\sin\theta} H_0 e^{-j\beta r}$$

$$V(r) = \int_{\theta_{hc}}^{\pi-\theta_{hc}} E_\theta\, r\, d\theta = 2 Z_0 H_0 e^{-j\beta r} \ln\left[\cot\frac{\theta_{hc}}{2}\right]$$

Notice both V&I are also travelling waves

- $$I(r) = \int_0^{2\pi} H_\phi\, r\sin\theta\, d\phi = 2\pi r H_0 e^{-j\beta r}$$

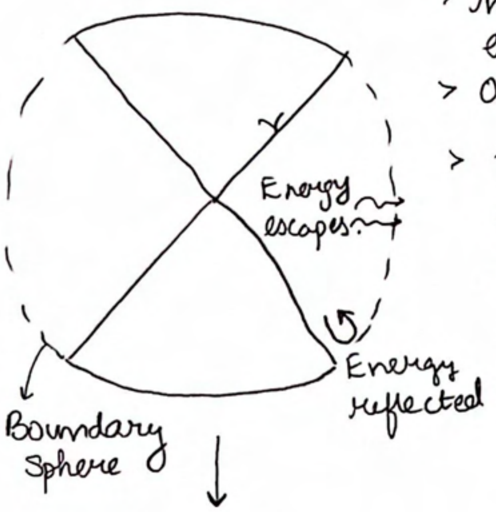
Characteric impedance $$\boxed{Z_K = \frac{Z_0}{\pi}\ln\cot\frac{\theta_{hc}}{2}} \simeq 120 \ln\frac{2}{\theta_{hc}}$$ (for air & $\theta_{hc} < 20^\circ$)

Uniform since it is independ...

→ Input impedance $Z_i = Z_K$ for infinite case.

→ If the lower cone is replaced by a ground plane the Z_K is halved since V is halved.

Finite Biconical antenna. (Similar to finite Tline).



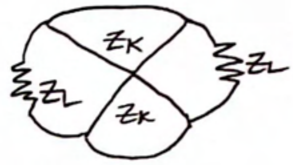
> Inside the sphere both TEM & Higher order modes (HOM) exist.

> Outside only H.O.M exist.

> TEM wave reflects at the boundary but not perfectly. Part of the reflection goes to H.O.M & part is transmitted as H.O.M.

> Energy escapes from the equatorial plane.

> The structure can now be modelled as a line terminated by Z_L which must be found.



→ Finding Z_K is shown only for very narrow cones in the book.

→ Far field pattern is similar to dipole when cone angle is small.

→ Small discussion on biconical & bowtie antennas at the end.

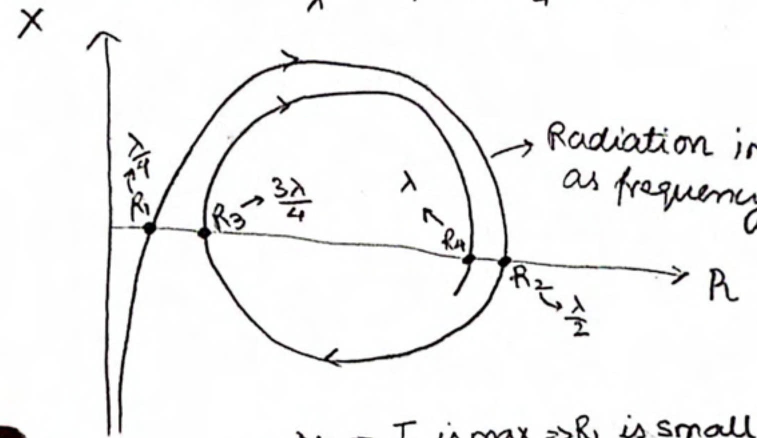
Ch 9 The Cylindrical Antenna. – Moment method

> Sinusoidal current distribution assumption is only valid for thin conductors. Let us consider thick ones.

Integral Equation Method. → Tedious so read the book.

> Thicker dipoles are more broadband but R is lower at resonances.

> Input impedance is purely real when antenna is resonant & this occurs at $\frac{l}{\lambda} = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \ldots$ where $2l = L$.



This curve is smaller as $\frac{l}{a}$ ↓ ⇒ wider cylinder.

$\lambda/4$ ⇒ I is max ⇒ R_1 is small.

$\frac{3\lambda}{4}$ ⇒ I is again max ⇒ R_3 is small.

$\frac{\lambda}{2}$ ⇒ I is min ⇒ R_1 is large

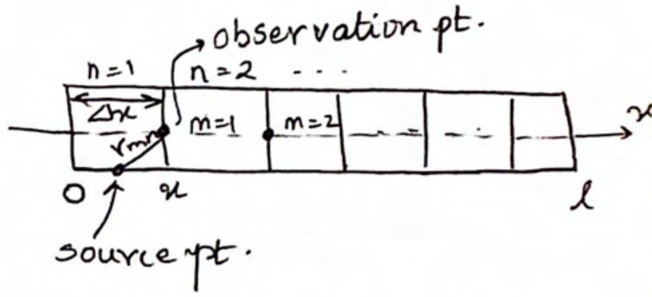
> Radiation pattern is very close to thin dipole even for relatively thick cylinders.

Method of Moments.

→ Casting an integral equation in the form of a matrix equation.

Example : What is the charge distribution on a conductor of length l?

Solution: Divide conductor into N segments & assume charge is localised in each segment.



→ Integral equation is

$$V = \frac{1}{4\pi\epsilon_0}\int_0^l \frac{\rho_L(x)}{r}\,dx. \quad (1)$$

Let $\overline{\rho_L(x)}_n$ be the average charge density on each segment Δx_n.

⇒ Total charge on segment n is $Q_n = \overline{\rho_L(x)}_n \cdot \Delta x_n$; $n = 1, 2, 3 \ldots N$.

⇒ Total charge on rod is $Q = \sum_{n=1}^{N} Q_n$.

⇒ (1) is $V = \int_0^l \frac{\rho_L(x)}{4\pi\epsilon_0 r}\,dx.$

⇒ Rewriting $\boxed{V_m = \sum_{n=1}^{N} l_{mn} Q_n}$ where $l_{mn} = \frac{1}{4\pi\epsilon r_{mn}}$

V_m is potential at m.

$$\Rightarrow \begin{bmatrix} l_{11} & l_{12} & l_{13} & \ldots & l_{1N} \\ l_{21} & l_{22} & l_{23} & \ldots & l_{2N} \\ \vdots & \vdots & \vdots & & \vdots \\ l_{M1} & l_{M2} & l_{M3} & \ldots & l_{MN} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_N \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_M \end{bmatrix}$$

$$\Rightarrow \quad \underbrace{[l_{mn}]}_{\text{Known.}}[Q_n] = [V_m]$$

$[V_m]$ → must satisfy boundary condition that $V_1 = V_2 = \ldots = V_M$ since potential on a conductor is constant. Also $M = \frac{N}{2}$ since symmetry across midpt. Therefore, we can now solve for $[Q_n]$

As N increases soln. converges to true value.

→ Applying Moment method to Antennas.

The integral equation for $\bar{E}$ in terms of the Green's fn is cast into a matrix form.

$$E(z) = \int_{-L/2}^{L/2} I(z')\, G(r_{mn})\, dz'$$

$$\Rightarrow \begin{bmatrix} E(z_1) \\ E(z_2) \\ \vdots \\ E(z_N) \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1N} \\ \vdots & & & \\ G_{N1} & - & - & G_{NN} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}$$

From boundary conditions if $\sigma = \infty$, $E(z) = -E(z')$. Multiplying both sides by Δz

$$\Rightarrow \Delta z \,[G_{mn}]\,[I_n] = \underbrace{-\Delta z\,[E_m]}$$

$$\Rightarrow [Z_{mn}]\,[I_n] = -\,[V_m]$$ ⇒ we can find impedance of a wire antenna.

→ The text offers several examples.

Ch 10: Self and Mutual Impedances.

- **Self impedance**:- Impedance looking into antenna terminals in the absence of nearby objects & in the absence of ohmic loss.

  Terminal / Driving point impedance:- Impedance in the presence of nearby objects such as ground plane & other scatterers/antennas.

  Radiation resistance:- Real part of self impedance.

Self impedance of a thin linear centerfed antenna that is an odd multiple of $\lambda/2$ long with sinusoidal current distribution.

$$Z_{11} = 30\left[\mathrm{Cin}(2\pi n) + j\, \mathrm{Si}(2\pi n)\right]$$

- where $\mathrm{Cin}(y) + j\,\mathrm{Si}(y) = \mathrm{Ein}(jy) = \int_0^{jy} \frac{1-e^{-w}}{w}\, dw$

$\Rightarrow R_{11} = 30\,\mathrm{Cin}(2\pi n)$; $X_{11} = 30\,\mathrm{Si}(2\pi n)$

For $\lambda/2$

$$\boxed{\begin{aligned} R_{11} &= 30\,\mathrm{Cin}(2\pi) = 73\,\Omega \\ X_{11} &= 30\,\mathrm{Si}(2\pi) = 42.5\,\Omega \end{aligned}}$$

Same as that in section 5.6

$\rightarrow$ Not zero $\Rightarrow \frac{\lambda}{2}$ antennas are not resonant. Need to shorten them to make them resonant

For $\frac{3\lambda}{2}$ $\quad Z_{11} = 105.5 + j\,45.5\,\Omega$

Could rewrite, $\boxed{Z_{11} = 30\left[0.577 + \ln(2\pi n) - \mathrm{Ci}(2\pi n)\right]}$

Where, $\mathrm{Ci}(x) = -\int_x^{\infty} \frac{\cos t}{t}\, dt$; $\mathrm{Cin}(x) = \int_0^x \frac{1-\cos t}{t}\, dt = \overset{1.781}{\gamma} + \ln x - \mathrm{Ci}(x)$

- $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt \qquad \mathrm{si}(x) = -\int_x^{\infty} \frac{\sin t}{t}\, dt$

As $n \to \infty$ $\quad R_{11} \to 30\left[0.577 + \ln(2\pi n)\right] \Rightarrow$ increases at a log rate to ∞

> If the antenna length is not restricted to odd multiples of $\frac{\lambda}{2}$ we can still find R_{11}.

Most general for thin wire antennas (center fed) with sinusoidal current.

$$\boxed{R_{11} = 30\left[\left(1 - \cot\frac{\beta L}{2}\right) Cin(2\beta L) + 4\cot^2\frac{\beta L}{2}\, Cin\, \beta L + 2\cot\left(\frac{\beta L}{2}\right)(S_i\, 2\beta L - 2 S_i\, \beta L)\right]}$$

For short dipoles,

$$\boxed{R_{11} \cong 5(\beta L)^2}$$

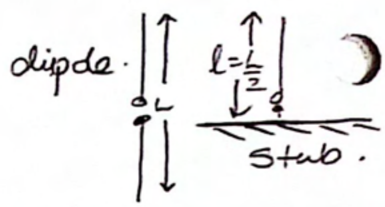
which is similar to Eq 2.20.3 which is

$$\boxed{R_r = 790\left(\frac{I_{av}}{I_0}\right)^2\left(\frac{L}{\lambda}\right)^2}$$

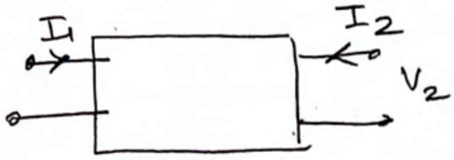
when $I_{av} = \frac{1}{2} I_0 \Rightarrow$ sinusoidal.

Here, $\beta = \frac{2\pi}{\lambda}$; L = length of antenna.

To find impedances for the case of stub antenna perp to ∞ ground plane just divide by $\frac{1}{2}$ & replace $L = 2l$.



Mutual impedance.



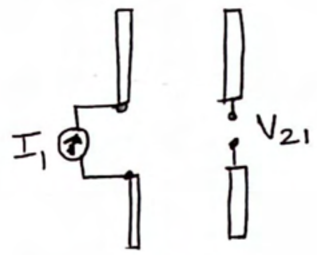
$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

$I_2 = 0 \Rightarrow$ port 2 is open, then $Z_{21} = \frac{V_2}{I_1}$

> This Z_{21} is the almost mutual impedance between port ① & ②. Remember port 2 must be open.

> Let us instead define mutual impedance as follows:

$$\boxed{Z_{21} = -\frac{V_{21}}{I_1}}$$ when port 2 is open. V_{21} is same as V_2.



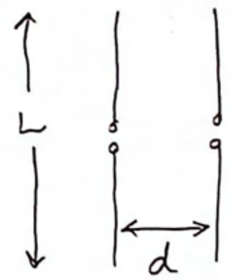
from reciprocity $Z_{12} = Z_{21} = -\frac{V_{21}}{I_1} = -\frac{V_{12}}{I_2}$

Case i Mutual impedance of parallel antennas.

$$Z_{21} = 30\left\{2\,Ei(-j\beta d) - Ei\left[-j\beta\left(\sqrt{d^2+L^2}+L\right)\right] - Ei\left[-j\beta\left(\sqrt{d^2+L^2}-L\right)\right]\right\}$$

where $Ei(\pm jy) = Ci(y) \pm j Si(y)$

[This is for L is odd multiple of $\frac{\lambda}{2}$]



→ For array calculations, $R_{11} - R_{21}$ is important & when d is small we can write.

$d \leq 0.05\lambda \Rightarrow 1\%$ error

$d \leq 0.1\lambda \Rightarrow 5\%$ error

$$\boxed{R_{11} - R_{21} \simeq 60\pi^2\left(\frac{d}{\lambda}\right)^2}$$

→ For the general case when L is any length refer to Eq 10.5.10.

→ Divide by 2 & replace $L = 2l$ for 2 stubs on ∞ ground plane.

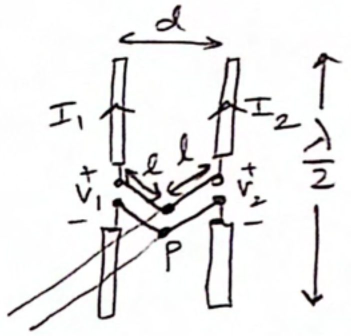
> Case (ii) is for 2 co-linear antennas ——•—— ——•—— refer to Section 10-6.

> Case (iii) is for 2 antennas in echelon ——•—— ——•—— refer to Section 10-7. This is the more general case.

> Section 10-8 gives Z_{12} for further generalisations.

Ch 11 Arrays of dipoles and apertures

- **Driving point Impedance (Broadside)**



→ Impedance at P for 2 $\frac{\lambda}{2}$ dipoles.

$V_1 = I_1 Z_{11} + I_2 Z_{12}$

self impedance of 1 — Mutual impedance of the 2 elements.

Note $I_1 = I_2$ ⇒ $V_1 = I_1 (Z_{11} + Z_{12})$

$V_2 = I_2 (Z_{22} + Z_{12})$

⇒ Terminal impedance at element 1 $Z_1 = \frac{V_1}{I_1} = Z_{11} + Z_{12}$

Similarly $Z_2 = Z_{22} + Z_{12}$

- Also, $Z_{11} = Z_{22}$ since the elements are identical.

⇒ $Z_1 = Z_2 = Z_{11} + Z_{12} = R_{11} + R_{12} + j(X_{11} + X_{12})$

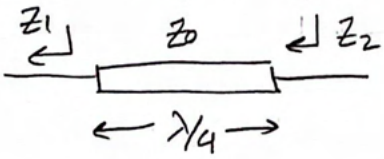
$= 73 - 13 + j(43 - 29)$ when spacing $= \frac{\lambda}{2}$

⇒ $\boxed{Z_1 = Z_2 = 60 + j14}$ → Could resonate it out with a series cap or shorten the dipole a bit

> If $l = \frac{\lambda}{2}$ ⇒ impedance at P is 30 Ω, if $j14$ is resonated out.

> We could also design Z_0 of lines to get any impedance at P. if $l = \frac{\lambda}{4}$.

Recall,



⇒ $\boxed{Z_0 = \sqrt{Z_1 Z_2}}$

- **Driving point Impedance (Endfire)** ⇒ $I_1 = -I_2$.

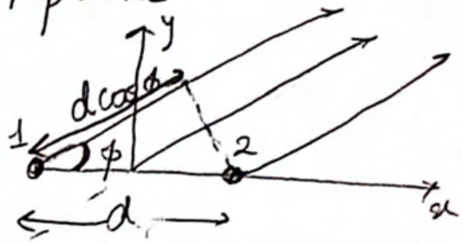
$Z_1 = Z_2 = R_{11} - R_{12} + j(X_{11} - X_{12})$

$\boxed{Z_1 = Z_2 = 86 + j72\ \Omega}$

Gain of 2 element array over single $\frac{\lambda}{2}$ dipole.

$G_f(\phi)\left[\frac{A}{HW}\right]$ → This is the gain in the H plane ⇒ XY plane.

- G_f → gain in field intensity (not power)
- (ϕ) → fn. of ϕ
- A → Array
- HW → Half wave dipole.



E Field in horizontal plane → $E(\phi) = 2kI_1 \cos\frac{\psi}{2}$, where $2k$ is constant

→ $\psi = d_r\cos\phi + \delta$; δ = phase diff. b/w currents ⇒ $I_2 = I_1\angle\delta$

→ ψ is phase difference b/w fields from the 2 elements in far field.

→ $d_r = \frac{2\pi d}{\lambda}$ distance b/w sources in radians

$$V_1 = I_1 Z_{11} + I_2 Z_{12} = I_1(Z_{11} + Z_{12}\angle\delta)$$

$$V_2 = I_2 Z_{22} + I_1 Z_{12} = I_2(Z_{22} + Z_{12}\angle{-\delta})$$

⇒ $Z_1 = \frac{V_1}{I_1} = Z_{11} + Z_{12}\angle\delta$; $Z_2 = Z_{22} + Z_{12}\angle{-\delta}$ → driving point impedances.

⇒ $R_1 = R_{11} + |Z_{12}|\cos(\tau+\delta)$ where $\tau = \arctan\frac{X_{12}}{R_{12}}$

$R_2 = R_{22} + |Z_{12}|\cos(\tau-\delta)$

⇒ Power in element 1 = $P_1 = |I_1|^2 R_1 = |I_1|^2[R_{11} + |Z_{12}|\cos(\tau+\delta)]$

$P_2 = |I_2|^2[R_{22} + |Z_{12}|\cos(\tau-\delta)]$ & $R_{11} = R_{22}$

$P = P_1 + P_2 = 2|I_1|^2(R_{11} + R_{12}\cos\delta)$ → Find I_1 in terms of P & do the same for single dipole; express E for both cases in terms of I & divide

⇒ Gain in horizontal plane
$$\boxed{G_f(\phi)\left[\frac{A}{HW}\right] = \sqrt{\frac{2R_{11}}{R_{11} + R_{12}\cos\delta}}\left|\cos\left(\frac{d_r\cos\phi + \delta}{2}\right)\right|}$$

Similarly,

Gain in vertical plane.
$$\boxed{G_f(\theta)\left[\frac{A}{HW}\right] = \sqrt{\frac{2R_{11}}{R_{11} + R_{12}\cos\delta}}\left|\cos\left(\frac{d_r\sin\theta + \delta}{2}\right)\right|}$$

Arrays with Parasitic Elements.

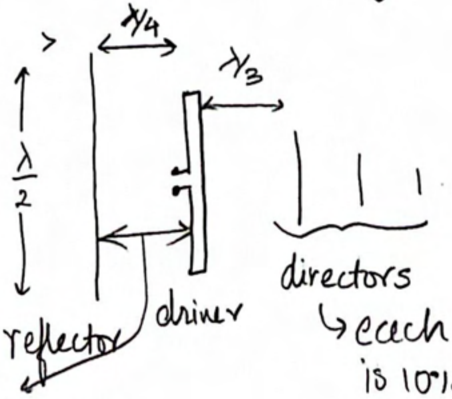
- Distance from driven element ~~must~~ desirable be less than $\frac{\lambda}{4}$.

$$V_1 = I_1 Z_{11} + I_2 Z_{12}$$

$$0 = I_2 Z_{22} + I_1 Z_{12}$$

Solve for I_1, I_2 & write $E(\phi)$.

- When the $\frac{\lambda}{2}$ parasitic dipole is inductive (longer than its resonant length) it acts as a reflector. When it is capacitive it acts as a director.
- Tuning for the desired X_2 (inductive or capacitive may be achieved by varying the length or adding inductors / capacitors at the centre joint in series, the former is preferred.



Yagi-Uda array.

each director is 10% smaller than the last.

Could be much smaller to leverage W8JK closely spaced arrays.

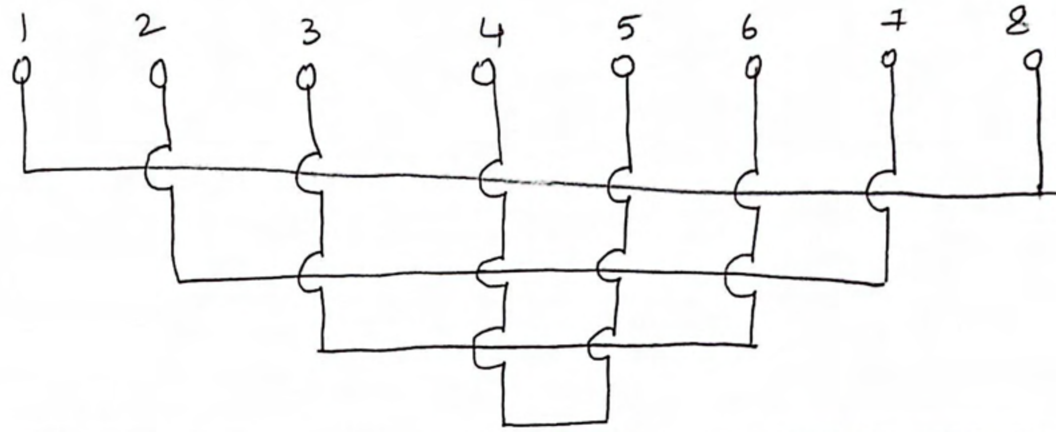
- The Landsdorfer Shaped dipole array gives maximum gain for such reflector-director arrays (Sec. 11.9e)

Retro array or Retro reflector

→ Reflects incoming wave back along same direction. Reradiated signal is the conjugate of incident signal.

→ Eg 1 :- Square corner reflector.

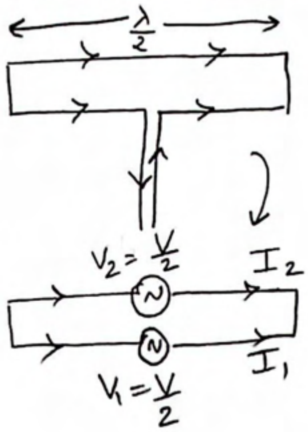
→ Eg 2 :- Van Atta retro array → pairs of (1,8) (2,7) (3,6) (4,5) are connected by equal length cables.



→ elements could be $\lambda/4$ dipoles.

→ All 4 cables are of equal length.

Folded dipole.



$d \simeq \frac{\lambda}{100}$

Assume, V is the emf applied at the terminals. This is equally split b/w the 2 "closely spaced dipoles".

$$\Rightarrow V_1 = Z_{11} I_1 + Z_{12} I_2$$

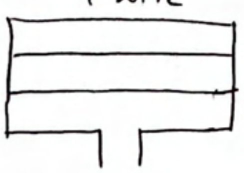
$$I = I_1 = I_2 \Rightarrow \frac{V}{2} = I \left(Z_{11} + Z_{12} \right)$$

$$\text{Since } d \simeq \frac{\lambda}{100} \Rightarrow Z_{11} = Z_{12} = Z_1$$

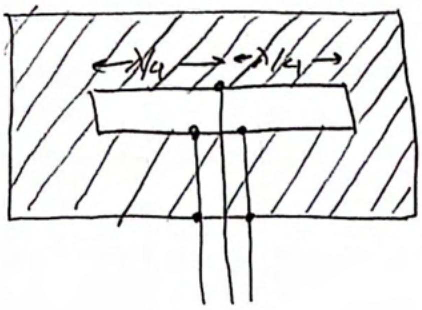
$$\Rightarrow \boxed{\frac{V}{I} = Z_{in} = 4 Z_1}$$

N-wire dipole $\Rightarrow$ $\boxed{Z_{in} = N^2 Z_1}$

4 wire



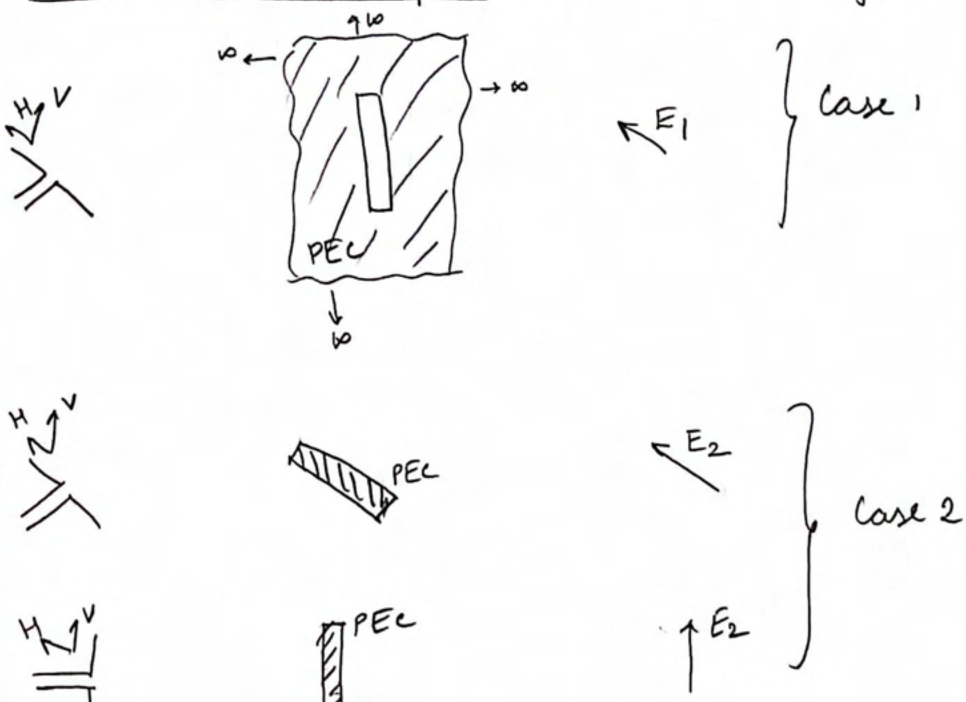
Ch 13 Slot, Horn and Complementary Antennas.



- Offset feed for better matching.
- Complementary to the dipole antenna formed with 2 strips (λ/4). 2 differences are that E & H are interchanged & the E & the E & H fields on the 2 sides of the sheet are discontinuous & the direction is reversed for both $\bar{E}$ & $\bar{H}$ on the 2 sides of the sheet.

→ When the sheet is finite in area, all fields along the plane of the sheet become nulls since fields on opposite sides cancel. The overall pattern also becomes wavy. As the sheet is made larger these waves become more numerous but smaller & tend to a circular shape as size → ∞.

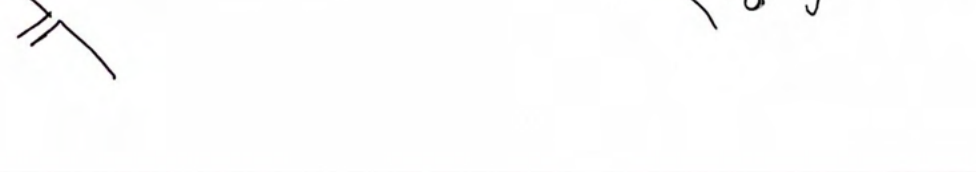
Babinet's Principle. → (Alternative approach from that in 530)



Replace surface with it's complement & exchange $\bar{E}$ & $\bar{H}$ fields. Then, the total field with no screen is the sum of the the 2 cases.

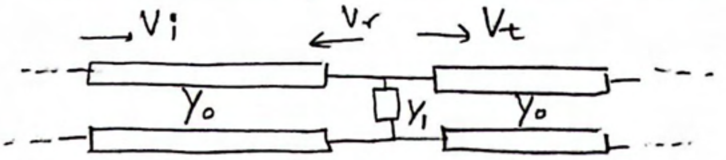
$$\Rightarrow \boxed{\bar{E}_1 + \bar{E}_2 = \bar{E}_0}$$

$$\Rightarrow \frac{\bar{E}_1}{\bar{E}_0} + \frac{\bar{E}_2}{\bar{E}_0} = 1$$

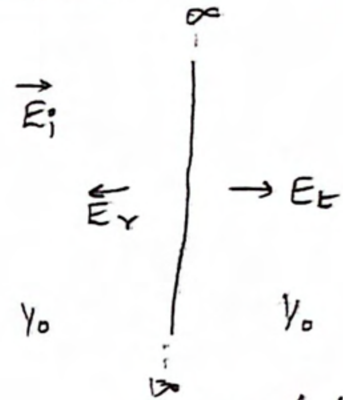


Impedance relation from Babinet's principle.

Proof using Transmission Lines.



$\approx$



$Y_0 = \frac{1}{377}\,\Omega$; $Y_0 = \frac{H_i}{E_i} = -\frac{H_r}{E_r} = \frac{H_t}{E_t}$

$Y_1 \longrightarrow$ Admittance of a square section of this sheet regardless of it size.

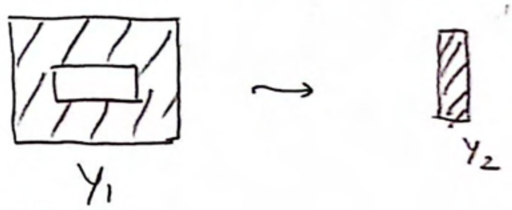
$$\tau_V = \frac{V_t}{V_i} = \frac{2Y_0}{2Y_0 + Y_1} \approx \tau_E = \frac{E_t}{E_i} = \frac{2Y_0}{2Y_0 + Y_1}$$

Say the screen is now replaced by it's complementary screen with Y_2

$$\tau_E' = \frac{E_t'}{E_i} = \frac{2Y_0}{2Y_0 + Y_2}$$ & from BP we have $\frac{E_t}{E_i} + \frac{E_t'}{E_i}$

$$\Rightarrow \frac{2Y_0}{2Y_0+Y_1} + \frac{2Y_0}{2Y_0+Y_2} = 1$$

→ Note: Here the complimentary screen should also be rotated.



$$\Rightarrow Y_1 Y_2 = 4Y_0^2$$

$$\Rightarrow \boxed{Z_1 Z_2 = \frac{Z_0^2}{4}} \Rightarrow \boxed{Z_0 = 2\sqrt{Z_1 Z_2}}$$

→ Impedance of inifinitesimal $\frac{\lambda}{2}$ dipole $\boxed{Z_d = 73 + j43.5}$ inductive

Complementary infinitesimal $\frac{\lambda}{2}$ slot $\boxed{Z_s = 363 - j211}$

→ To convert a cylindrical dipole with thickness D into a slot of width W, use $\boxed{W = 2D}$

→ A dipole with $L/D = 28$ & length $= 0.925\lambda$ has terminal impedance $Z_d = 710 + j0 \Rightarrow \boxed{Z_s = 50\,\Omega}$ for complimentary slot.

→ 50Ω slot antenna.

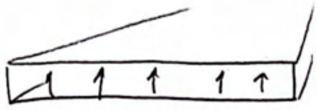
- $\boxed{L = 0.925\lambda}$ $\quad \frac{2L}{W} = 28 \Rightarrow W = \frac{2L}{28} = \frac{L}{14}$

$$\Rightarrow \boxed{W = 0.066\lambda}$$

→ Increasing D or W increases the bandwidth of the dipole/slot.
→ Make sure the sheet is atleast one λ wide on all sides.
→ Impedance also varies with the terminal connections.

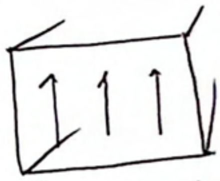
Horn Antennas.

→ Exponential taper ⇒ better matching. 
→ Sectoral transition ⇒ constant phase center. 

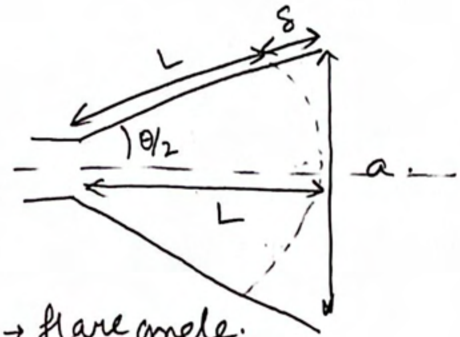


- Hplane sectoral.

Eplane Sectoral.



Pyramidal horn.



θ → flare angle.
a → aperture.

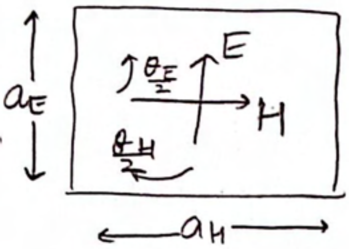
$$\cos\frac{\theta}{2} = \frac{L}{L+\delta} \qquad \sin\frac{\theta}{2} = \frac{a}{2(L+\delta)} \qquad \tan\frac{\theta}{2} = \frac{a}{2L}$$

$$(L+\delta)^2 = L^2 + \frac{a^2}{4}$$

$$\Rightarrow \cancel{L^2} + 2L\delta + \delta^2 = \cancel{L^2} + \frac{a^2}{4}$$

$$\Rightarrow L = \frac{a^2}{8\delta} - \frac{\delta}{2}$$

$$\simeq \boxed{L = \frac{a^2}{8\delta}}$$



In Eplane $\delta \leq 0.25\lambda$ & in H plane $\delta \leq 0.4\lambda$ since $E_{tan} \to 0$ in Hplane for TE_{10}

- > There is an optimum value of δ such that directivity is maximized with minimum side lobes. We dont want δ to add an additional 180° of phase to the field at the boundary. But ↑δ ⇒ a↑ ⇒ directivity ↑.

$\therefore \quad \delta_0 = \dfrac{L}{\cos(\theta/2)} - L$

optimum dimensions $\Rightarrow$ $$\boxed{L = \frac{\delta_0 \cos(\theta/2)}{1-\cos(\theta/2)}}$$

$0.1\lambda_{\text{free space}} \leq \delta_0 \leq 0.4\lambda_{\text{free space}}$.

→ A lens compensated horn increased the velocity of the wave around the boundary to negate the δ effect.

→ Also only TE_{10} mode must be excited ⇒ waveguide width should be b/w $\lambda/2$ & 1λ.

→ If a_E & $a_H > 1\lambda$. The E plane & H plane patterns become independent of each other. E plane pattern of an E plane horn & E plane pattern of pyramidal horn of same a_E & θ_E are the same.

→ Tolerance in path length for H plane is higher for TE_{10} mode since field → 0 near edges in H plane.

→ E plane directivity is higher, but side lobes are also higher since field is more uniform.

→ Directivity $D = \dfrac{4\pi A_e}{\lambda^2} = \dfrac{4\pi \epsilon_{ap} A_p}{\lambda^2}$

A_e – effective aperture area
ϵ_{ap} → aperture efficiency $\simeq 0.6$
A_p – physical aperture area

$A_p = a_E a_H$ or πr^2 ; $(a_E, a_H, r > 1\lambda)$

$D \cong \dfrac{7.5 A_p}{\lambda^2}$ ⇒ $\boxed{D \cong 10 \log(7.5\, a_{E\lambda}\, a_{H\lambda})}$

$a_{E\lambda}, a_{H\lambda}$ → aperture in λ.

HPBW

Empirical & apply when apertures are several wavelengths

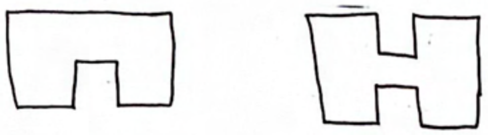
Uniformly illuminated rectangular horn ⇒ $\dfrac{51^\circ}{L_\lambda}$ → same as array

use these.
$\begin{cases} \text{optimum E plane} \Rightarrow \dfrac{56^\circ}{a_{E\lambda}} \\ \text{optimum H plane} \Rightarrow \dfrac{67^\circ}{a_{H\lambda}} \end{cases}$

Uniformly illuminated circular horn (conical) ⇒ $\dfrac{58^\circ}{D_\lambda}$

Ridge Horns.

> Waveguide cross section.



→ Extend this into the horn to get very high BW since cut off freq. of dominant mode drops.

Septum Horns.

> A septum plate is used to reduce E plane sidelobes. Not sure exactly how it looks.

→ Increasing $\theta \Rightarrow a\uparrow \Rightarrow D\uparrow$, but side lobes $\uparrow$, cost same

→ Increasing $L \Rightarrow a\uparrow \Rightarrow D\uparrow$, sidelobes same, cost $\uparrow$

Horn Antennas by Balanis.

Review of Rectangular waveguides.

Fields for TE modes

$$E_x = i\omega\epsilon\left(\frac{n\pi}{b}\right)\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{i\beta_{mn}z}$$

$$E_y = -i\omega\epsilon\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{i\beta_{mn}z}$$

$$H_x = -i\beta_{mn}\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{i\beta_{mn}z}$$

$$H_y = -i\beta_{mn}\left(\frac{n\pi}{b}\right)\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{i\beta_{mn}z}$$

$$H_z = \left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{i\beta_{mn}z}$$

$$\beta_{mn} = \sqrt{K^2-\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$$

$$f_{c_{TE}} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$$

Dominant mode TE10

$$E_x = 0 \;;\; H_y = 0 \;;\; E_z = 0$$

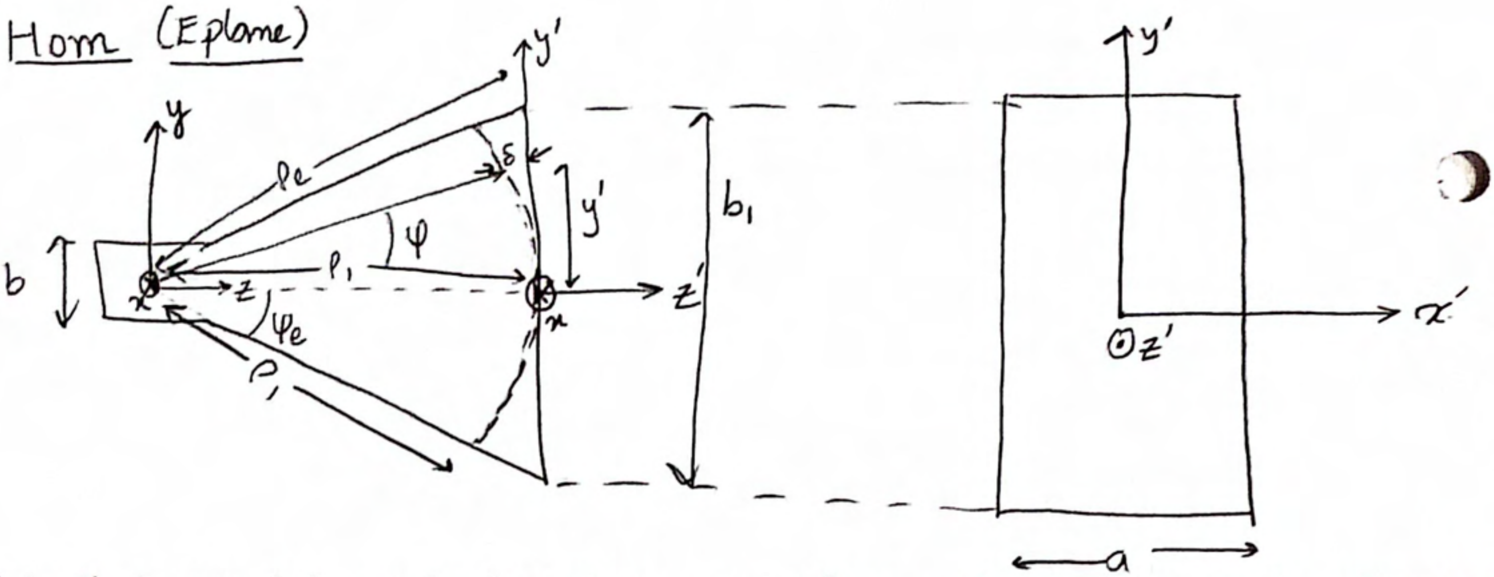
$$E_y = \frac{-i\omega\epsilon\pi}{a}\sin\left(\frac{\pi x}{a}\right)e^{i\beta_{mn}z}$$

$$H_x = \frac{-i\beta_{mn}\pi}{a}\sin\left(\frac{\pi x}{a}\right)e^{i\beta_{mn}z}$$

$$H_z = \left(\frac{\pi}{a}\right)^2\cos\left(\frac{\pi x}{a}\right)e^{i\beta_{mn}z}$$

$$f_{c_{TE10}} = \frac{1}{2\pi a\sqrt{\mu\epsilon}}$$

Horn (Eplane)



> Note that orientation & location of axis is different here than in the waveguide so field may look different but are identical except for exp. term.

$E_{z}' = E_{x}' = H_y' = 0$ & $\rho_1 = \rho_e \cos\varphi_e$.

$E_y' \cong E_1 \cos\left(\frac{\pi}{a}x'\right) e^{\frac{jky'^2}{2\rho_1}}$

$H_z' = jE_1\left(\frac{\pi}{ka\eta}\right) \sin\left(\frac{\pi x'}{a}\right) e^{\frac{jky'^2}{2\rho_1}}$

$H_x' = -\frac{E_1}{\eta} \cos\left(\frac{\pi x'}{a}\right) e^{\frac{jky'^2}{2\rho_1}}$

These are the fields on the aperture. The quadratic exponential term arises due to the δ phase difference of the wave on the aperture. These are not the fields propagating from the horn.

$(\delta(y') + \rho_1)^2 = \rho_1^2 + (y')^2$

$\Rightarrow \delta(y') = -\rho_1 + \rho_1\left[1 + \left(\frac{y'}{\rho_1}\right)^2\right]^{1/2}$ & using binomial expansion.

$\delta(y') = \frac{1}{2}\left(\frac{y'^2}{\rho_1}\right) \rightarrow$ phase correction is added.

→ To solve for radiated fields we need to use aperture concepts which are outlined in notes on Aperture Antennas.

$\bar{J}_s = \hat{n} \times \bar{H} \Rightarrow J_y = H_x' = -\frac{E_1}{\eta} \cos\left(\frac{\pi x'}{a}\right) e^{\frac{jky'^2}{2\rho_1}} \quad -a/2 \le x' \le a/2$

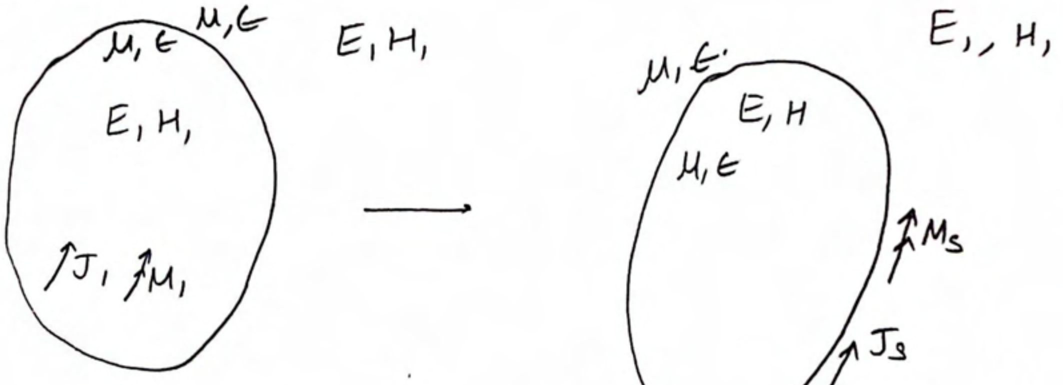
$\bar{M}_s = -\hat{n} \times \bar{E} = \bar{M}_x = E_y' = E_1 \cos\left(\frac{\pi x'}{a}\right) e^{\frac{jky'^2}{2\rho_1}} \quad -\frac{b_1}{2} \le y' \le b_1/2$

& $\bar{J}_s, \bar{M}_s$ are 0 outside the aperture.

Aperture Antennas - Balanis.

> Using uniqueness theorem & image theory, we can easily solve for radiated fields once we know the fields that are tangential on the aperture.

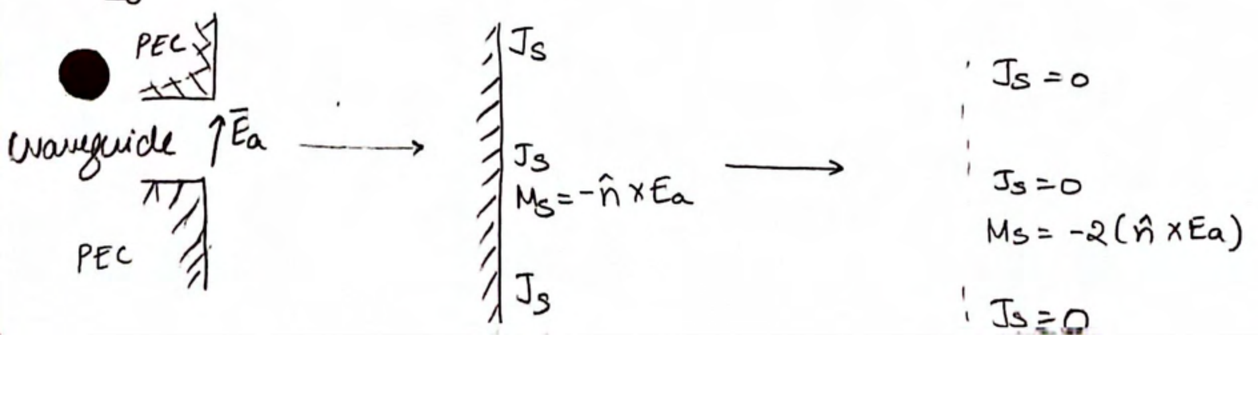
> Suppose sources J_1, M_1 radiate fields E_1, H_1 everywhere. We can replace fields inside a volume V_1 that contains the sources by $\bar{E}$ & $\bar{H}$ by placing surface currents J_s, M_s that satisfy the boundary conditions everywhere. We no longer need J_1, M_1.



$\bar{J}_s = \hat{n} \times [\bar{H}_1 - \bar{H}]$ $\quad$ $\bar{M}_s = -\hat{n} \times [\bar{E}_1 - \bar{E}]$

To make life easier, since E, H can be arbitrary we can make them 0 by making V_1 a PEC or PMC. Next, if the surface of V_1 is planar we can use image theory to get rid of it. This reduces either J_s (if we used PEC) or M_s (if we used PMC) to 0 & the other one would be doubled. Thereby reducing a complex current distribution J_1, M_1 to a simple one J_s or M_s.

Eg:



If we knew H_b instead of E_a we could use J_s instead of M_s

Recall: magnetic & electric vector potentials A, F.

$$\bar{A} = \frac{\mu}{4\pi} \int_{S'} \bar{J}_s(r') \frac{e^{ikr}}{r} ds'$$

$$\bar{A}_{FF} \simeq \mu \frac{e^{ikr}}{4\pi r} \int_{S'} \bar{J}_s(r') e^{-ikr'\cdot\hat{r}} ds'$$

$$\bar{F} = \frac{\epsilon}{4\pi} \int_{S'} \bar{M}_s(r') \frac{e^{ikr}}{r} ds'$$

$$\bar{F}_{FF} \simeq \epsilon \frac{e^{ikr}}{4\pi r} \int_{S'} \bar{M}_s(r') e^{-ikr'\cdot\hat{r}} ds'$$

Let $\bar{N} = \int_{S'} \bar{J}_s(r') e^{-ikr'\cdot\hat{r}} ds'$ & $\bar{L} = \int_{S'} \bar{M}_s(r') e^{-ikr'\cdot\hat{r}} ds'$

$$\Rightarrow \quad E_\theta \simeq \frac{ike^{ikr}}{4\pi r}(L_\phi + \eta N_\theta) \qquad H_\theta = -ik\frac{e^{ikr}}{4\pi r}\left(N_\phi - \frac{L_\theta}{\eta}\right)$$

$$E_\phi \simeq -ik\frac{e^{ikr}}{4\pi r}(L_\theta - \eta N_\phi) \qquad H_\phi \simeq ik\frac{e^{ikr}}{4\pi r}\left(N_\theta - \frac{L_\phi}{\eta}\right)$$

Where,

$$N_\theta = \iint_{S'} [J_x \cos\theta\cos\phi + J_y \cos\theta\sin\phi - J_z \sin\theta] e^{-ik\bar{r}'\cdot\hat{r}} ds'$$

$$N_\phi = \iint_{S'} [-J_x \sin\phi + J_y \cos\phi] e^{-ik\bar{r}'\cdot\hat{r}} ds'$$

$$L_\theta = \iint_{S'} [M_x \cos\theta\cos\phi + M_y \cos\theta\sin\phi - M_z \sin\theta] e^{-ik\bar{r}'\cdot\hat{r}} ds'$$

$$L_\phi = \iint_{S'} [-M_x \sin\phi + M_y \cos\phi] e^{-ik\bar{r}'\cdot\hat{r}} ds'$$

Going back to the Horn

$$N_\theta = \iint_{S'} J_y \cos\theta \sin\phi \, e^{-ik\bar{r}'\cdot\hat{r}} ds$$

$$= \int_{-\frac{b_1}{2}}^{\frac{b_1}{2}} \int_{-\frac{a}{2}}^{a/2} -\frac{E_1}{\eta} \cos\left(\frac{\pi x'}{a}\right) e^{ik\frac{y'^2}{2\rho_1}} . \cos\theta \sin\phi \, e^{-ik\bar{r}'\cdot\hat{r}} ds'.$$

$$\hat{r} = \sin\theta\cos\phi \, \hat{x} + \sin\theta \sin\phi \, \hat{y} + \cos\theta \, \hat{z}$$

$$\bar{r}' = x'\hat{x} + y'\hat{y}$$

$$\Rightarrow \hat{r}'\cdot\hat{r} = x'\sin\theta\cos\phi + y'\sin\theta\sin\phi$$

$$\Rightarrow N_\theta = \int_{-\frac{b_1}{2}}^{\frac{b_1}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} -\frac{E_1}{\eta} \cos\left(\frac{\pi x'}{a}\right) \cos\theta\sin\phi \, e^{ik\frac{y'^2}{2\rho_1}} e^{-ik(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)} ds'$$

$$= -\frac{E_1}{\eta}\cos\theta\sin\phi \underbrace{\int_{-\frac{b_1}{2}}^{\frac{b_1}{2}} e^{ik\left(\frac{y'^2}{2\rho_1} - y'\sin\theta\sin\phi\right)} dy'}_{I_2} \underbrace{\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos\left(\frac{\pi x'}{a}\right) e^{-ikx'\sin\theta\cos\phi} dx'}_{I_1}$$

$$\Rightarrow I_1 = -\left(\frac{\pi a}{2}\right)\left[\frac{\cos\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\left(\frac{ka}{2}\sin\theta\cos\phi\right)^2 - \left(\frac{\pi}{2}\right)^2}\right]$$

I_2 can be reduced using Fresnel integrals $C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt$

Refer to Balanis for the derivation. $S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt$

$$I_2 = \sqrt{\frac{\pi\rho_1}{k}} e^{-i\left(\frac{k_y^2\rho_1}{2k}\right)} \left\{\left[C(t_2) - C(t_1)\right] + i\left[S(t_2) - S(t_1)\right]\right\}$$

Where $k_y = k\sin\theta\sin\phi$

$$t_1 = \sqrt{\frac{1}{\pi k\rho_1}} \left(-\frac{kb_1}{2} - k_y\rho_1\right)$$

$$t_2 = \sqrt{\frac{1}{\pi k\rho_1}} \left(\frac{kb_1}{2} - k_y\rho_1\right)$$

Therefore,

$$N_\theta = E_1 \frac{\pi a}{2}\sqrt{\frac{\pi \rho_1}{K}}\, e^{-i\left(\frac{k_y^2 \rho_1}{2K}\right)} \left\{ \frac{\cos\theta \sin\phi}{\eta} \left[\frac{\cos\left(\frac{K_x a}{2}\right)}{\left(\frac{K_x a}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2} \right] F(t_1, t_2) \right\}$$

where, $K_x = K\sin\theta\cos\phi \quad k_y = K\sin\theta \sin\phi$

$$\& \quad F(t_1; t_2) = [C(t_2) - C(t_1)] + i[S(t_2) - S(t_1)]$$

Similarly,

$$N_\phi = \frac{E_1 \pi a}{2}\sqrt{\frac{\pi \rho_1}{k}}\, e^{-i\left(\frac{k_y^2 \rho_1}{2K}\right)} \left\{ \frac{\cos\phi}{\eta} \left[\frac{\cos\left(\frac{k_x a}{2}\right)}{\left(\frac{k_x a}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2} \right] F(t_1, t_2) \right\}$$

$$L_\theta = \frac{E_1 \pi a}{2}\sqrt{\frac{\pi \rho_1}{K}}\, e^{-i\left(\frac{k_y^2 \rho_1}{2K}\right)} \left\{ -\cos\theta\cos\phi \left[\frac{\cos\left(\frac{k_x a}{2}\right)}{\left(\frac{k_x a}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2} \right] F(t_1, t_2) \right\}$$

$$L_\phi = \frac{E_1 \pi a}{2}\sqrt{\frac{\pi \rho_1}{k}}\, e^{-i\left(\frac{k_y^2 \rho_1}{2K}\right)} \left\{ \sin\phi \left[\frac{\cos\left(\frac{k_x a}{2}\right)}{\left(\frac{k_x a}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2} \right] F(t_1, t_2) \right\}$$

Therefore,

$$E_\theta = \frac{i a\sqrt{\pi k \rho_1}\, E_1 e^{ikr}}{8r} \left\{ e^{-i\left(\frac{k_y^2 \rho_1}{2k}\right)} \sin\phi\,(1+\cos\theta) \left[\frac{\cos\left(\frac{k_x a}{2}\right)}{\left(\frac{k_x a}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2} \right] F(t_1, t_2) \right\}$$

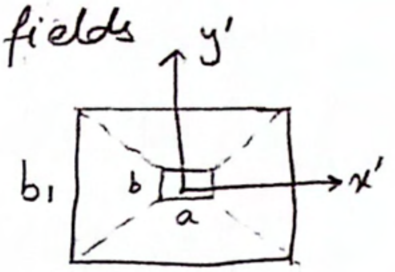
$$E_\phi = \frac{i a\sqrt{\pi k \rho_1}\, E_1 e^{ikr}}{8r} \left\{ e^{-i\frac{(k_y^2 \rho_1)}{2k}} \cos\phi\,(1+\cos\theta) \left[\frac{\cos\left(\frac{k_x a}{2}\right)}{\left(\frac{k_x a}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2} \right] F(t_1, t_2) \right\}$$

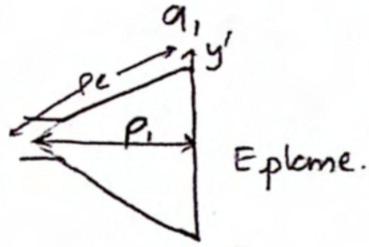
Pyramidal Horn

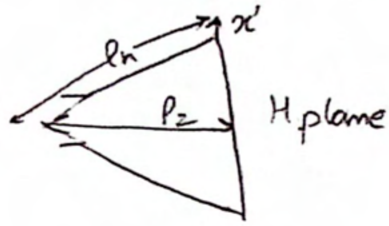
- $E_y' = E_0 \cos\left(\frac{\pi}{a_1}x'\right) e^{i\frac{k}{2}\left(\frac{x'^2}{\rho_2}+\frac{y'^2}{\rho_1}\right)}$

$H_x' = -\frac{E_0}{\eta}\cos\left(\frac{\pi}{a_1}x'\right) e^{i\frac{k}{2}\left(\frac{x'^2}{\rho_2}+\frac{y'^2}{\rho_1}\right)}$

} Aperture fields







$$\Rightarrow J_y(x',y') = -\frac{E_0}{\eta}\cos\left(\frac{\pi}{a_1}x'\right) e^{\frac{ik}{2}\left(\frac{x'^2}{\rho_2}+\frac{y'^2}{\rho_1}\right)}$$

$$M_x(x',y') = E_0 \cos\left(\frac{\pi}{a_1}x'\right) e^{\frac{ik}{2}\left(\frac{x'^2}{\rho_2}+\frac{y'^2}{\rho_1}\right)}$$

$$N_\theta = -\frac{E_0}{\eta}\cos\theta\sin\phi\, I_1 I_2$$

$$N_\phi = -\frac{E_0}{\eta}\cos\phi\, I_1 I_2$$

$$L_\theta = E_0\cos\theta\cos\phi\, I_1 I_2$$

- $L_\phi = -E_0\sin\phi\, I_1 I_2$

where $I_1 = \frac{1}{2}\sqrt{\frac{\pi\rho_2}{k}}\left(e^{-i\frac{k_x'^2\rho_2}{2k}}\left\{[C(t_2')-C(t_1')]+i[S(t_2')-S(t_1')]\right\} + e^{-i\frac{k_x''^2\rho_2}{2k}}\left\{[C(t_2'')-C(t_1'')]+i[S(t_2'')-S(t_1'')]\right\}\right)$

$$I_2 = \sqrt{\frac{\pi\rho_1}{k}}\, e^{-i\left(\frac{k_y^2\rho_1}{2k}\right)}\left\{[C(t_2)-C(t_1)]+i[S(t_2)-S(t_1)]\right\}.$$

$t_1' = \sqrt{\frac{1}{\pi k\rho_2}}\left(-\frac{ka_1}{2}-k_x'\rho_2\right)$; $t_2' = \sqrt{\frac{1}{\pi k\rho_2}}\left(\frac{ka_1}{2}-k_x'\rho_2\right)$; $k_x' = k\sin\theta\cos\phi+\frac{\pi}{a_1}$

$t_1'' = \sqrt{\frac{1}{\pi k\rho_2}}\left(-\frac{ka_1}{2}-k_x''\rho_2\right)$; $t_2'' = \sqrt{\frac{1}{\pi k\rho_2}}\left(\frac{ka_1}{2}-k_x''\rho_2\right)$; $k_x'' = k\sin\theta\cos\phi-\frac{\pi}{a_1}$

$k_y = k\sin\theta\sin\phi$; $t_1 = \sqrt{\frac{1}{\pi k\rho_1}}\left(-\frac{kb_1}{2}-k_y\rho_1\right)$; $t_2 = \sqrt{\frac{1}{\pi k\rho_1}}\left(\frac{kb_1}{2}-k_y\rho_1\right)$

- $C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt$; $S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt$

Far field E&H fields

$$E_\theta = -ik \frac{e^{ikr}}{4\pi r} [L_\phi + \eta N_\theta]$$

$$E_\theta = -ikE_0 \frac{e^{ikr}}{4\pi r} [\sin\phi (1+\cos\theta) I_1 I_2]$$

$$E_\phi = -ik \frac{e^{ikr}}{4\pi r} [L_\theta - \eta N_\phi]$$

$$E_\phi = -ikE_0 \cdot \frac{e^{ikr}}{4\pi r} [\cos\phi (\cos\theta+1) I_1 I_2]$$

$$H_\theta = -ik \frac{e^{ikr}}{4\pi r} \left[N_\phi - \frac{L_\theta}{\eta}\right]$$

$$H_\theta = \frac{-ikE_0}{\eta} \frac{e^{ikr}}{4\pi r} [\cos\phi (1+\cos\theta)] I_1 I_2$$

$$H_\phi = \frac{ikE_0}{\eta} \frac{e^{ikr}}{4\pi r} [\sin\phi (1+\cos\theta)] I_1 I_2$$