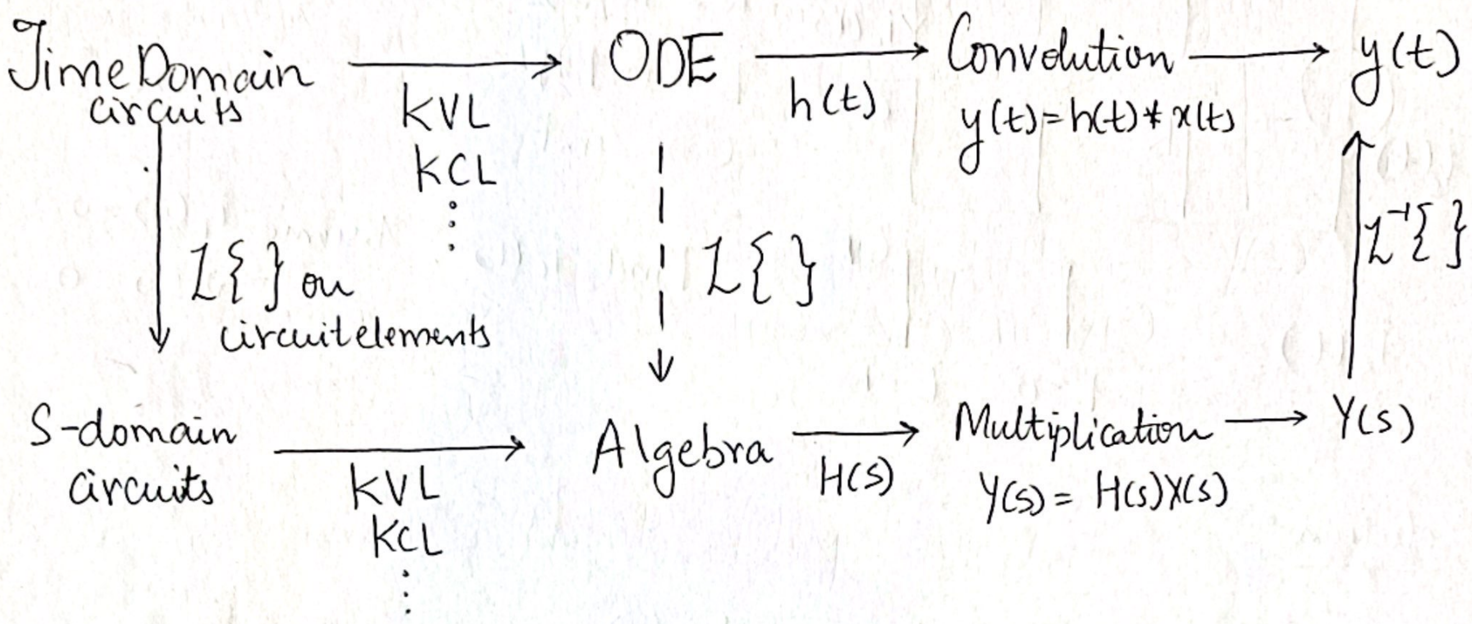
Lec 5 - Laplace Transforms & Freq. Response contd.

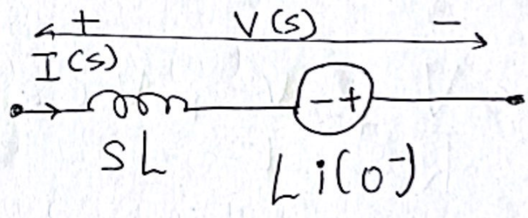


S-domain Circuits

Resistors : $v(t) = R\,i(t) \Rightarrow V(s) = R\,I(s)$

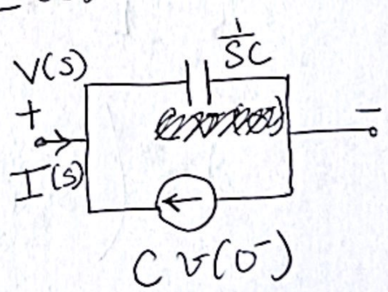
$Z(s) = R$

Inductors : $v(t) = L\dfrac{di(t)}{dt} \Rightarrow V(s) = sL\,I(s) - L\,i(0^-)$



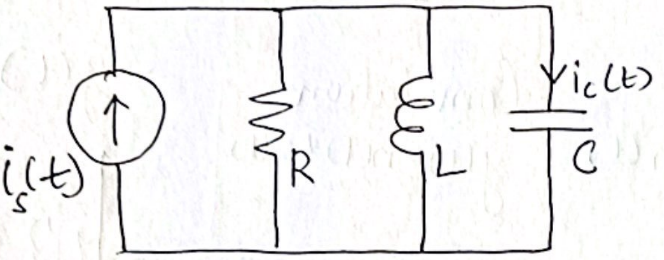
$Z(s) = sL$

Capacitors : $i(t) = C\dfrac{dv(t)}{dt} \Rightarrow I(s) = sC\,V(s) - C\,v(0^-)$



$Z(s) = \dfrac{1}{sC}$

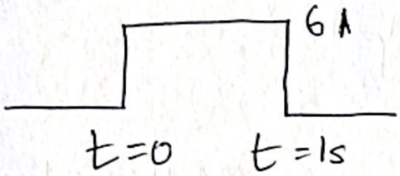
Example



$R = 125\,\Omega$

$L = 0.1\,H$

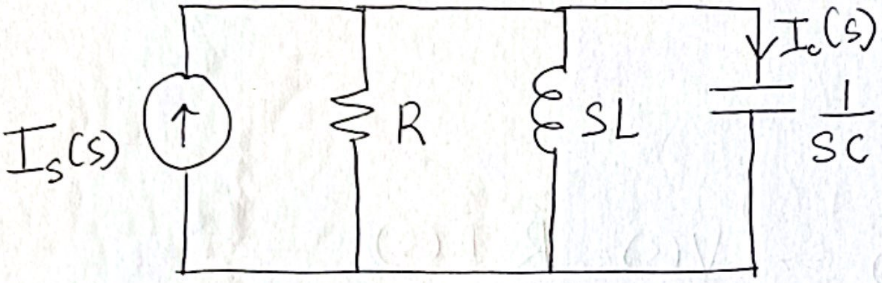
$C = 4\,mF$

$i_s(t) =$ 

Find $i_c(t)$. Assume $i_L(0^-) = 0$

$v_c(0^-) = 0$

Solution:



$I_s(s) = \mathcal{L}\{i_s(t)\}$

$i_s(t) = 6\,u(t) - 6\,u(t-1)$

$\Rightarrow I_s(s) = \frac{6}{s} - \frac{6}{s}e^{-s}$

KCL $\Rightarrow$ $\frac{V}{R} + \frac{V}{sL} + VsC = I_s$

$$I_c = \frac{V}{\frac{1}{sC}} = sC\left[I_s\left(\frac{1}{R} + \frac{1}{sL} + sC\right)^{-1}\right]$$

$$= I_s\left(\frac{s^2}{s^2 + \frac{s}{RC} + \frac{1}{LC}}\right)$$

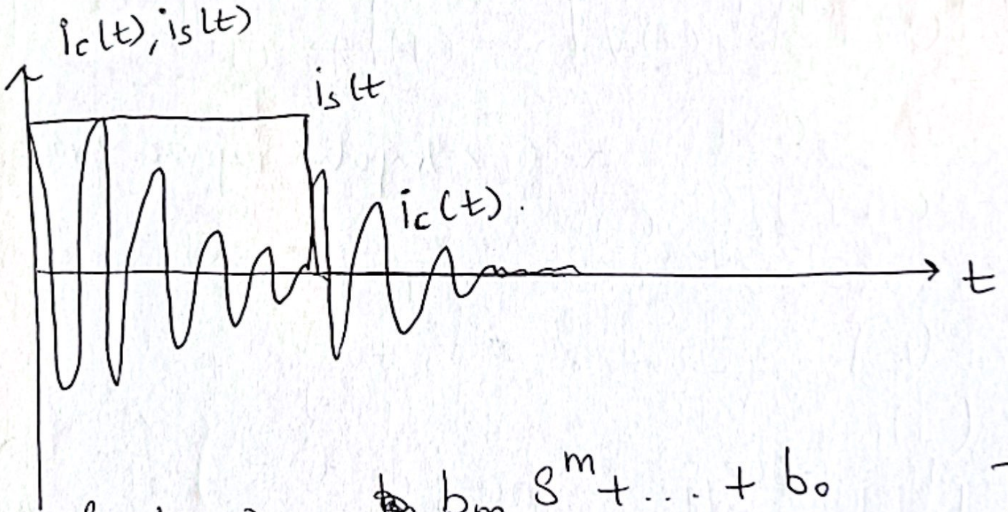
$$\Rightarrow I_c(s) = \overbrace{\left(\frac{s^2}{s^2 + \frac{s}{RC} + \frac{1}{LC}}\right)}^{H(s)} \left(\frac{6}{s} - \frac{6e^{-s}}{s}\right)$$

$$= \frac{6s}{s^2+2s+2500} - \frac{6s}{s^2+2s+2500}e^{-s}$$

$\downarrow$ Partial Fraction expansion & $\mathcal{L}^{-1}\{\}$

$$6e^{-t}\cos(50t + 1.15°)u(t)$$

$$\Rightarrow i_c(t) = 6\{e^{-t}\cos(50t + 1.15°)\}u(t)$$
$$- 6e^{-(t-1)}\cos(50(t-1) + 1.15°)u(t-1)$$



> In general $H(s) = \frac{b_m s^m + \dots + b_0}{a_n s^n + \dots + a_0} \rightarrow$ Transfer Function

Zeroes $\Rightarrow$ values of s where Num $= 0$ roots of num.

Poles $\Rightarrow$ values of s where Den $= 0$ roots of den.

$$H(s) = \frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

How do we interpret $H(s)$?

$$H(s) = \frac{Y(s)}{X(s)} \Rightarrow Y(s) = H(s)X(s) \xrightarrow{ILT}$$

$$\Rightarrow y(t) = \underbrace{h(t)}_{\text{impulse response}} * x(t)$$

$\Rightarrow$ $H(s)$ is the LT of $h(t)$.

> RHP pole $\Rightarrow$ unstable! $e^{-\alpha t} \longleftrightarrow \frac{1}{s+\alpha}$

$\alpha < 0 \Rightarrow$ Impulse response grows exponentially.

$$\Rightarrow H(s) = \int_0^{\infty} h(t)e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$$ since $h(t) = 0$ for $t < 0$ due to causality!

> When $s = j\omega$ we have $H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt$

⇒ The imaginary axis of the S-plane for $H(s)$ represents the Fourier Transform of $h(t)$. This axis is indeed special because it captures a very important feature of the frequency domain.

Eigenfunction excitations

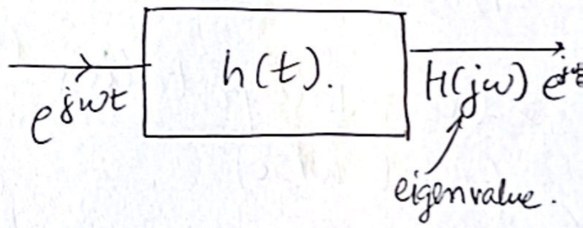
Let $x(t) = e^{j\omega t} \longrightarrow$ Eigenfunctions of LTI Systems.

$$\Rightarrow \quad y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega t} e^{-j\omega\tau}\, d\tau$$

$$= e^{j\omega t} H(j\omega)$$

$$= H(j\omega)\, e^{j\omega t}$$



$$\boxed{y(t) = H(j\omega)\, x(t)} \quad \text{when } x(t) = e^{j\omega t}$$

⇒ $H(j\omega)$ is the magnitude & phase added to a sinusoidal excitation at ω. → This is only true at $s = j\omega$.

Recap

$x(t) = \delta(t) \Rightarrow y(t) = h(t)$ impulse response.

$x(t) = e^{j\omega t} \Rightarrow y(t) = \underbrace{H(j\omega)}\, e^{j\omega t}$ → frequency response aka transfer function

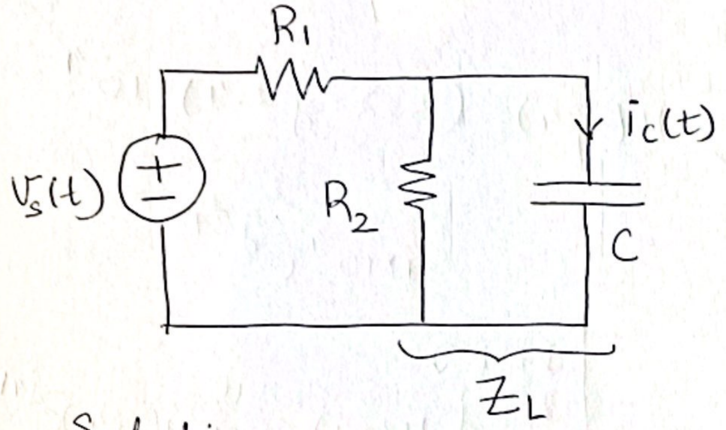
since $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)}$

$$\mathcal{L}\{h(t)\} = H(s) = H(j\omega)\Big|_{s=j\omega}$$

$$F\{h(t)\} = H(j\omega) \longrightarrow \text{Bode Plot approximates this.}$$

Fourier Domain $H(j\omega)$	Laplace Domain $H(s)$
> Steady state sinusoidal (or other) signals $e^{j\omega t}$ ⇒ <u>steady state response</u>.	> Transient signals ⇒ <u>transient response</u>.
> No initial conditions.	> Can have initial conditions.
> Clear interpretation (Bode Plots). (point-wise)	> More complex $\mathcal{L}\{h(t)\}$. (not pointwise).
> Assumes stability,	> Stability can be tested (RHP poles).
> Cannot enforce causality.	> Enforces causality.
> No ROC (implicit)	> ROC (explicit).

Example



$R_1 = 2k\Omega$, $R_2 = 4k\Omega$

$C = 2.5 \times 10^{-4}\,F$

$v_s(t) = 5\cos(4t)$

Find $i_c(t)$.

Solution

$$V_s(j\omega) = \mathcal{F}\{v_s(t)\} = 5\pi\left[\delta(\omega-4) + \delta(\omega+4)\right]$$

> **Recall**
> FT of $A\cos(\omega_0 t)$ is
> $\pi A\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$

- $$H(j\omega) = \frac{I_c(j\omega)}{V_s(j\omega)} = \frac{V_L}{V_s} \cdot \frac{I_c}{V_L}$$

$$= \frac{Z_L}{Z_L + R_1} \cdot j\omega C \quad , \quad Z_L = R_2 \parallel Z_c = \frac{R_2}{1 + j\omega C R_2}$$

$$H(j\omega) = \frac{j\omega C R_2}{R_2 + R_1(1 + j\omega C R_2)}$$

- $$I_c(j\omega) = H(j\omega)\, V_s(j\omega)$$

$$= H(j\omega)\left[5\pi\{\delta(\omega-4) + \delta(\omega+4)\}\right]$$

$$= \frac{5\pi}{\cancel{2\pi}}\left[H(j4)\,\delta(\omega-4) + H(-j4)\,\delta(\omega+4) \right]$$

$$H(j4) = (3.2 + j2.4)\times 10^{-4} \quad ; \quad H(-j4) = (3.2 - j2.4)\times 10^{-4}$$

$$= 4\times 10^{-4}\, e^{j37^\circ} \qquad\qquad = 4\times 10^{-4}\, e^{-j37^\circ}$$

$$\Rightarrow i_c(t) = FT^{-1}\{I_c(j\omega)\}$$

$$= 2\times 10^{-3}\cos(4t + 37^\circ)\ A$$

IFT ?

$$\frac{5\pi}{2\pi}\left[H(j4)\, e^{j4t} + H(-j4)\, e^{-j4t} \right]$$

$$= \frac{5}{\cancel{4}}\left[2\cancel{4}\times 10^{-4} e^{j4t+37^\circ} + 2\cancel{4}\times 10^{-4} e^{-j4t-37^\circ} \right]$$

$$= 20\times 10^{-4}\left[\frac{e^{j(4t+37^\circ)} + e^{-j(4t+37^\circ)}}{2} \right]$$

$$= 2mA \cdot \cos(4t + 37^\circ)$$

Using Eigenfunctions.

$$i_c(t) = H(j\omega)\, v_s(t)$$

$$= Re\{4\times 10^{-4} e^{j37^\circ} \cdot 5\times e^{j4t}\}$$

$$= Re\{2\times 10^{-3} e^{j(4t+37^\circ)}\}$$

$$= 2\times 10^{-3}\cos(4t + 37^\circ)\ A$$