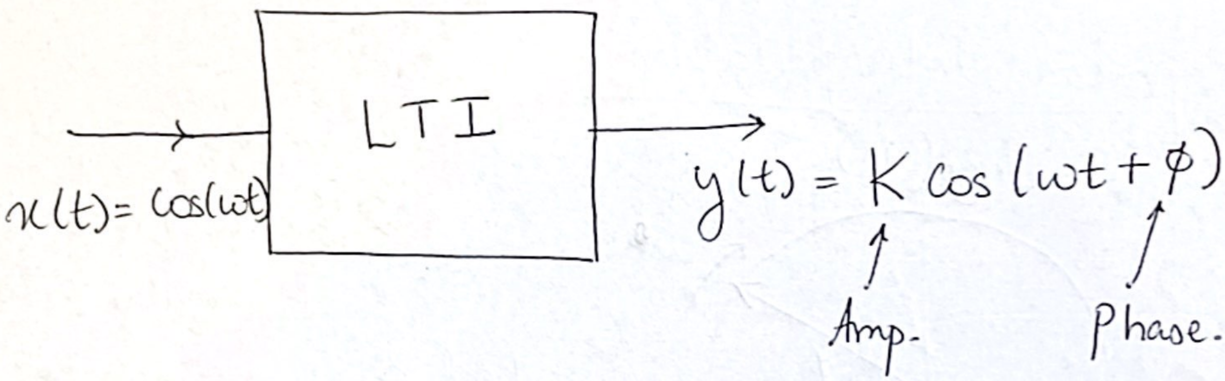
Lec 4

Frequency Response

> We most often care about the response of a circuit when the input is sinusoidal.

> **LTI** ⇒ Frequency remains the same, only amplitude & phase can change.
↳ Will prove later.



> Solving even a simple RC LPF is painful in time domain; algebra of sinusoidal functions is hairy. So we introduce complex exponentials → "phasors".

Phasor notation

> Since ω remains invariant in an LTI system we want a convenient representation that captures magnitude & phase & keeps track of it.

$$x(t) = A\cos(\omega t + \phi)$$

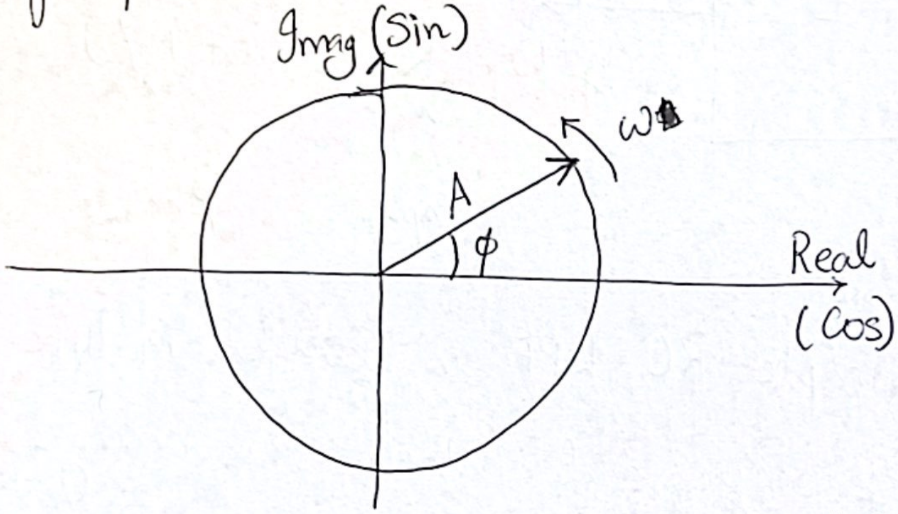
$$= A \, Re\left\{ e^{j(\omega t+\phi)} \right\}$$

Recall Euler
$$e^{jx} = \cos x + j\sin x$$

$$= Re\left\{ \underbrace{Ae^{j\phi}} \, e^{j\omega t} \right\}$$

PHASOR $\rightarrow \tilde{X} = Ae^{j\phi} \rightarrow$ Independent of ω & complex number.

Graphical Interpretation



> This vector is rotating at angular velocity ω.
> Its projection onto the Real axis is the time domain signal

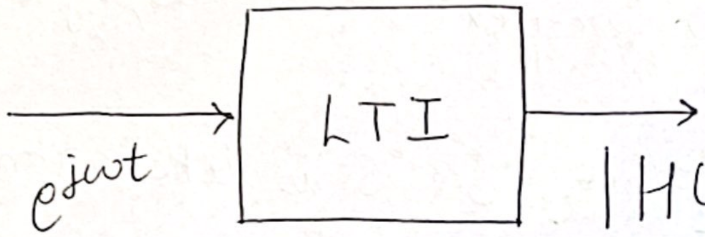
> All vectors in an LTI system rotate at ω so we ignore this rotation & treat just A & ϕ. The math works due to linearity! $e^{j\omega t}$ & $Re\{\,\}$ can be taken outside all the equations since they are common.

> Derivatives & Integrals also preserve $e^{j\omega t}$!

$$\frac{d}{dt}\left(A e^{j\phi} e^{j\omega t}\right) = j\omega\left(A e^{j\phi} e^{j\omega t}\right)$$

$$\int A e^{j\phi} e^{j\omega t}\, dt = \frac{1}{j\omega} A e^{j\phi} e^{j\omega t}$$

⇒ Calculus is reduced to algebra!

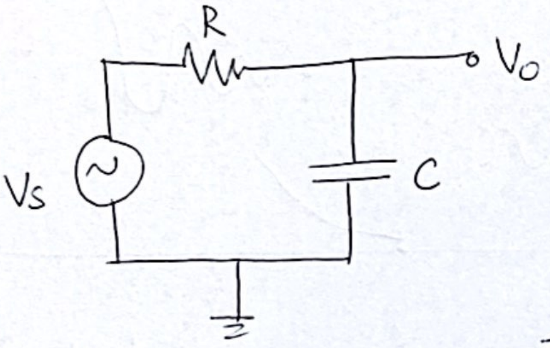


$$\underbrace{|H(j\omega)|\, e^{j\angle H(j\omega)}}\; e^{j\omega t}$$

$H(j\omega)$ → complex.

{ Frequency Response, aka Transfer function.

<u>Low Pass Filter.</u>



KVL ⇒ $V_o(t) = V_s(t) - \tau \frac{dV_o(t)}{dt}$

$V_s(t) = \tilde{V}_s e^{j\omega t}$

$V_o(t) = |V_o| e^{j\phi} e^{j\omega t} = \tilde{V}_o e^{j\omega t}$

⇒ $\tilde{V}_o + \tau j\omega \tilde{V}_o = \tilde{V}_s$

⇒ $\frac{\tilde{V}_o}{\tilde{V}_s} = \frac{1}{1+j\omega\tau}$ ⇒ $\boxed{H(j\omega) = \frac{1}{1+j\omega\tau}}$

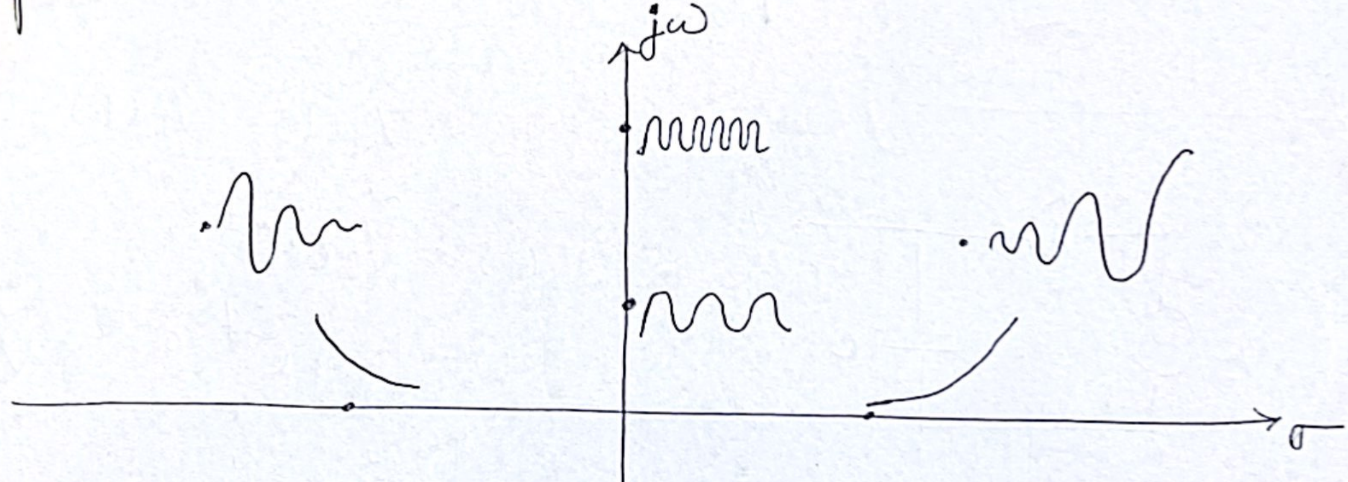
→ We will revisit Frequency Response again & go deeper (it is related to impulse response)

> What if we want to use decaying or growing sinusoids of the form $e^{\sigma t}\cos(\omega t)$? $(\sigma \in \mathbb{R})$.

$$e^{\sigma t}\cos(\omega t) = \text{Re}\{e^{\sigma t} e^{j\omega t}\} = \text{Re}\{e^{(\sigma + j\omega)t}\}$$

Let $S = \sigma + j\omega$. This is equivalent to taking $e^{j\omega t}$ & making ω "complex". $\omega = \omega_r + j\omega_i$

$\Rightarrow$ $e^{j\omega t} = e^{-\omega_i t} e^{j\omega_r t}$. So S is like a complex frequency. It captures both oscillation & decay in a compact form.



This is a "trivial" mapping from time to S-domain. We want a "useful" mapping.

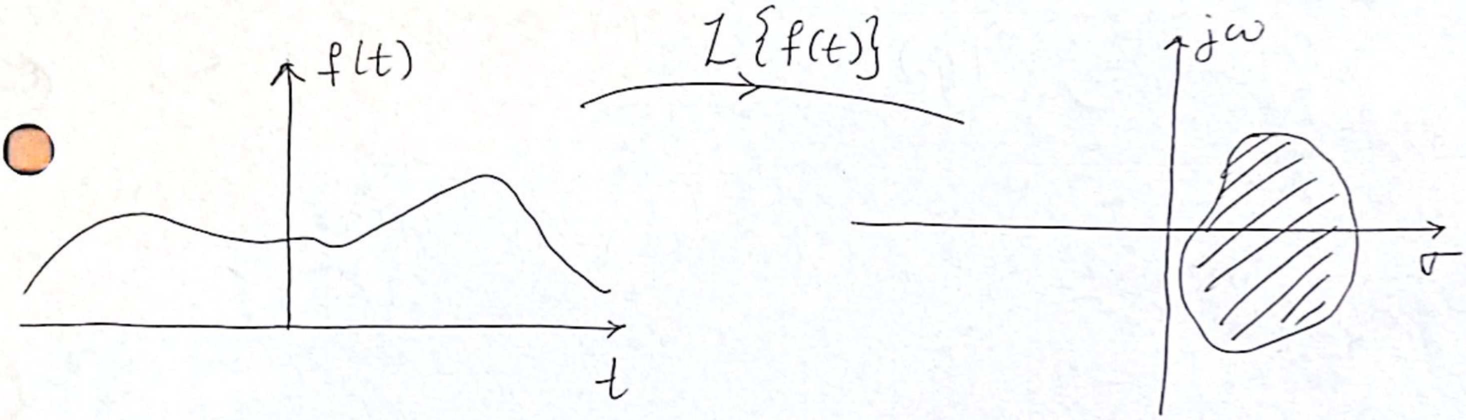
This gives intuition but to find actual representation of a signal in S-plane we need to define the Laplace Transform.

Laplace Transform

> A mapping from timedomain to S-domain that has very nice properties, which will be super useful in solving ODEs.

$$F(s) = \int_0^\infty f(t)\, e^{-St}\, dt$$

. Here $f(t) = 0 \ \forall\ t < 0 \rightarrow$ Causality of $h(t)$.



Recap: > Our goal was to study LTI systems with arbitrary input signals

> In time domain we need to use KVL, KCL, analysis to get $h(t)$ & then do a convolution with $x(t)$.

> Laplace ~~allow~~ allows ~~to~~ us to reduce the complexity from ODE + convolution $\longrightarrow$ algebra + multiplication!

ODEs $\longrightarrow$ Algebra

> $\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0)$, where $F(s) = \mathcal{L}\{f(t)\}$.

$= sF(s)$ when $f(0) = 0$

[Proof: Integration by parts $\int_0^\infty \underbrace{\frac{df}{dt}}_{dv} \underbrace{e^{-st}}_{u} dt = \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st}dt$]

$$\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} = s^n F(s) \quad \text{when } f(0) = 0$$

$\Rightarrow$ $a_n f^{(n)}(t) + \ldots + a_1 f'(t) + a_0 f(t) = x(t)$ & $x(0) = 0$, $f(0) = 0$

$\Rightarrow$ $F(s) = \dfrac{X(s)}{a_n s^n + \ldots + a_1 s + a_0}$.

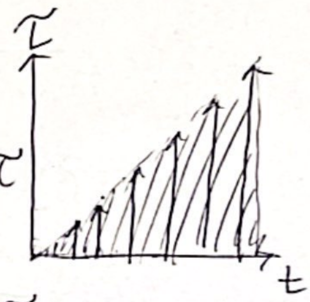
Convolution $\longrightarrow$ Multiplication

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

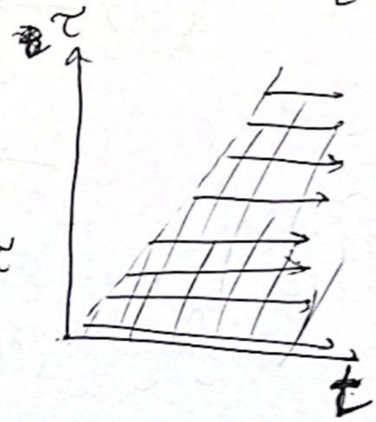
$$\mathcal{L}\{f * g\} = F(s)\,G(s)$$

<u>Proof</u>: $$\mathcal{L}\{f*g\} = \int_{t=0}^{\infty}\left[\int_{\tau=0}^{t} f(\tau)\, g(t-\tau)\, d\tau\right] e^{-st}\, dt$$

$$= \int_{\tau=0}^{\infty}\int_{t=\tau}^{\infty} f(\tau)\, g(t-\tau)\, e^{-st}\, dt\, d\tau$$



Let $u = t-\tau \Rightarrow dt = du$



$$= \int_{\tau=0}^{\infty}\int_{u=0}^{\infty} f(\tau)\, g(u)\, e^{-s(u+\tau)}\, du\, d\tau$$

$$= \int_{\tau=0}^{\infty} f(\tau)\, e^{-s\tau}\, d\tau \int_{u=0}^{\infty} g(u)\, e^{-su}\, du$$

$$= F(s)\, G(s). \quad \blacksquare$$

Properties of Laplace Transforms

$f(t) \to F(s)$

$f(at) \to \frac{1}{a} F\left(\frac{s}{a}\right)$

$e^{at} f(t) \to F(s-a)$

$f(t-a)\, u(t-a) \to e^{-as} F(s)$

$\int_0^t f(\tau)\, d\tau \longrightarrow \frac{1}{s} F(s)$

$t^n f(t) \longrightarrow (-1)^n F^{(n)}(s)$

$f^{(n)}(t) \longrightarrow s^n F(s)$
if $f(0) = 0$.

Laplace Transform Pairs (ignore ROC in this class)

$\delta(t) \longrightarrow 1$

$\delta(t-\tau) \longrightarrow e^{-\tau s}$

Time Shift

integrate

$u(t) \longrightarrow \frac{1}{s}$

integrate

$t^n u(t) \longrightarrow \frac{n!}{s^{n+1}}$

multiply by exponent

$t^n e^{-\alpha t} u(t) \longrightarrow \frac{n!}{(s+\alpha)^{n+1}}$

Set $n=0$ & make α complex.

$\sin(\omega t)\, u(t) \longrightarrow \frac{\omega}{s^2+\omega^2}$

$\cos(\omega t)\, u(t) \longrightarrow \frac{s}{s^2+\omega^2}$