Lec 3 - Linear Time Invariant (LTI) Systems

> **System:** Something that has an input & an output.

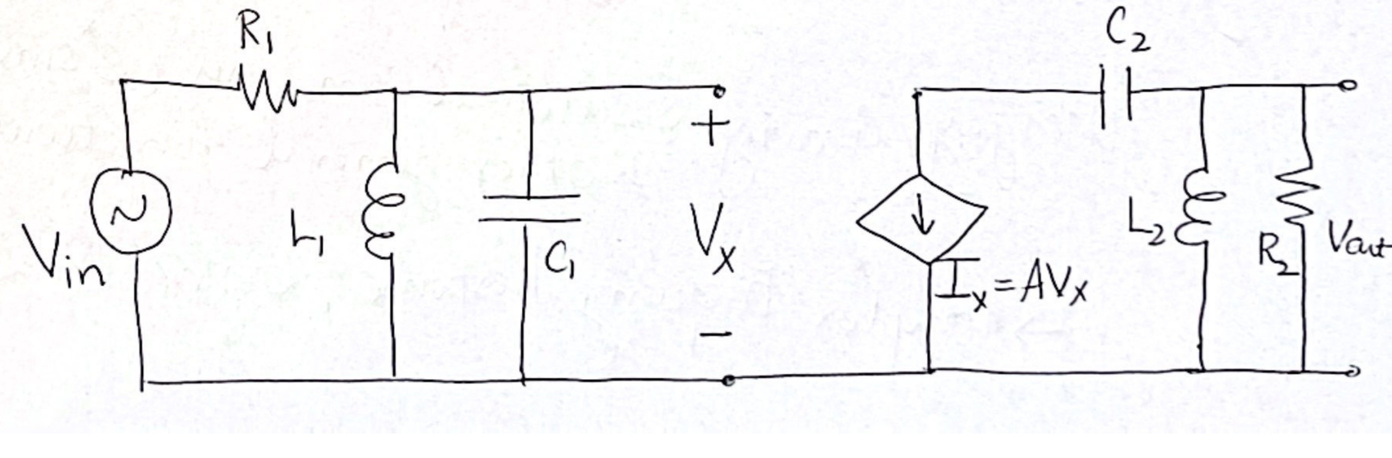
> **Linear:** $f(ax+by) = af(x) + bf(y)$ from Lec 2.

> **Time invariant:** $y(t) = f(x(t)) \Rightarrow y(t-\tau) = f(x(t-\tau))$.

> We model complex circuits as LTI systems because the analysis tools we have (Linear algebra, Fourier Transforms etc.) only work for LTI systems.

> In circuits the inputs to the system are sources and outputs are voltages and/or currents somewhere. A circuit is LTI if it has linear elements (resistors, inductors, capacitors, dep. sources) & their values are fixed in time.

Ex:



> Given an LTI system we want to study how to find $y(t)$ for a particular $x(t)$. There are different analysis techniques for different classes of inputs & they are all closely related.

> Classes of inputs $x(t)$

- Impulse function (Dirac Delta fn.) $\delta(t)$
  ↳ Impulse Response.

- Step function
  ↳ Step Response.

- Sinusoidal functions ($\sin x$, $\cos x$)
  ↳ Frequency Response (Fourier Transform)

- Decaying/Growing Sinusoidal functions ($e^{x}\sin x$, $e^{-x}\cos x$) & other general functions.
  ↳ Complex Frequency Response (Laplace Transform).

Impulse Response

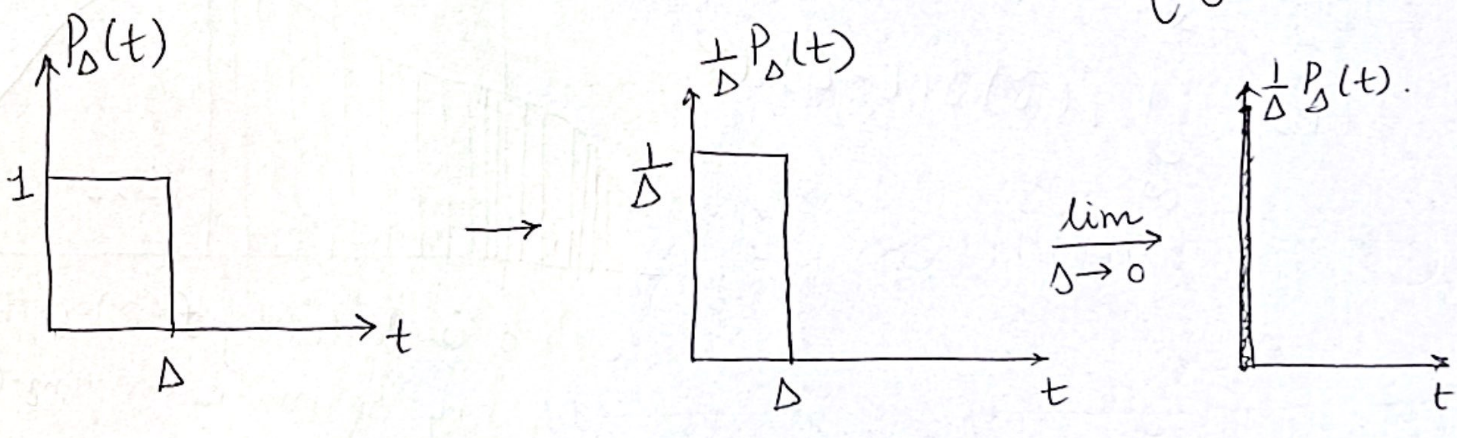
- Dirac Delta function

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undefined} & t = 0 \end{cases}$$

Such that $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

It can be constructed from a pulse function of area $= 1$ as the width $\to 0$ in the limit.

$$\delta(t) = \lim_{\Delta \to 0} \left(\frac{1}{\Delta} \cdot P_\Delta(t) \right) \quad \text{where} \quad P_\Delta(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t \le \Delta \\ 0 & t > \Delta \end{cases}$$

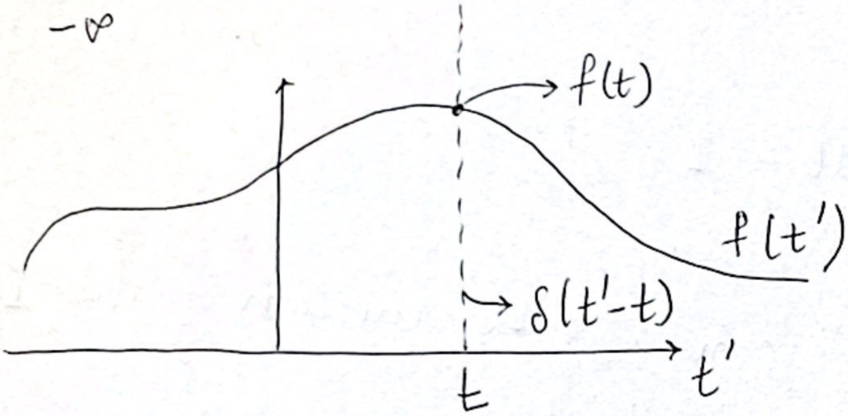


> Why is $\delta(t)$ useful? $\to$ Sifting Property!

Sifting Property

"Sifting" $\longrightarrow$ Extracting $\rightarrow$ $\delta(t)$ extracts the value of a function at time t.

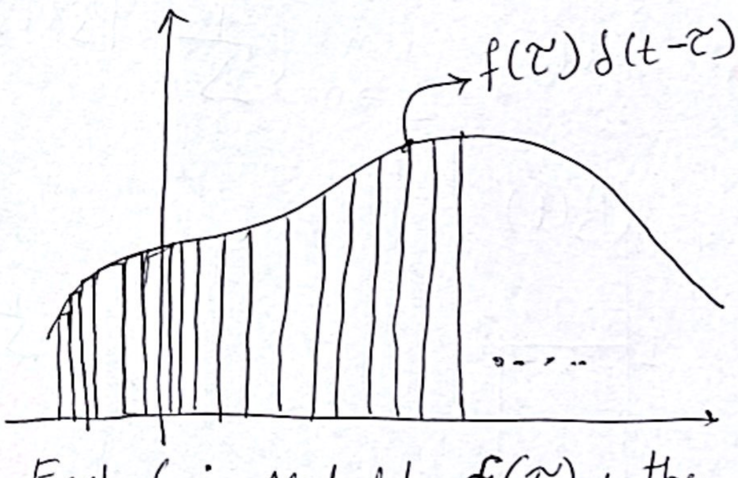
$$\int_{-\infty}^{\infty} f(t')\,\delta(t'-t)\,dt' = f(t).$$



The integral only *fires* when $t'=t$ because of $\delta(t'-t)$.

> Another way to look at this is as if $\delta(t)$ is "building" the function.

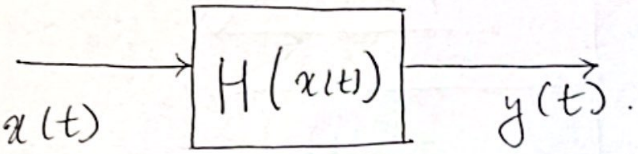
$$f(t) = \int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau$$



Each δ is scaled by $f(\tau)$ & then added up to make the function.

> Both the "sifting" & "building" viewpoints are equivalent & useful in different ways.

LTI Systems (Impulse Response)



Impulse Response : Response $y(t)$ of a system when the input is an impulse ($x(t) = \delta(t)$).

$$y(t) = H(x(t))$$

$$\Rightarrow h(t) = H(\delta(t)) \rightarrow \text{impulse response.}$$

Time invariance $\Rightarrow$ $h(t-\tau) = H(\delta(t-\tau))$

Since $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$

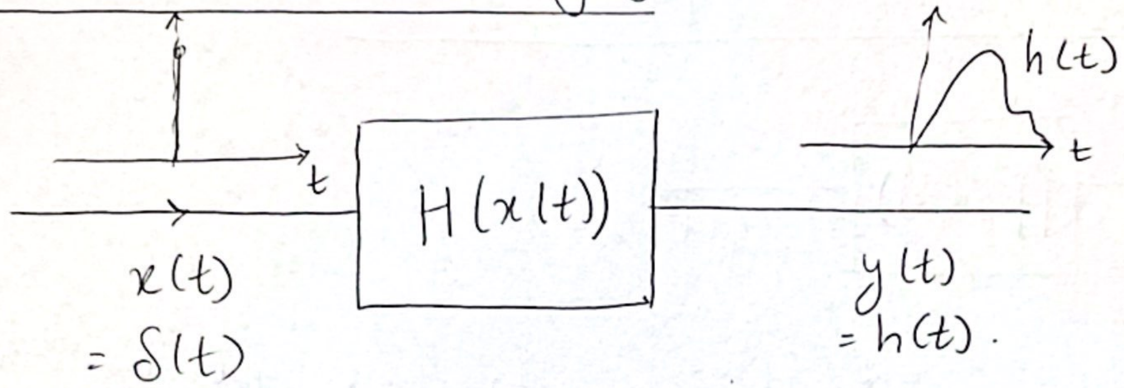
$$\Rightarrow y(t) = H\left(\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right)$$

Linearity $\Rightarrow$ H can be taken inside the integral.

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,H(\delta(t-\tau))\,d\tau$$ } H only acts on t. τ is a constant.

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau} \rightarrow \text{Convolution.}$$

What is convolution saying.



> For a $\delta(t)$ input I have some response $\rightarrow h(t)$.

> I can break up my input $x(t)$ in <u>scaled</u> & <u>shifted</u> deltas, find the response to each of them & then <u>add</u> them back up to get the response $y(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

add Scaled Shifted

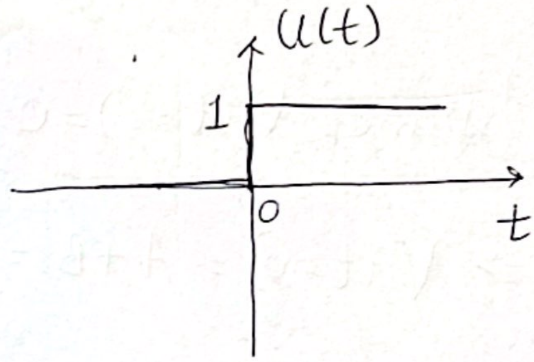
$$y(t) = x(t) * h(t)$$

$\Rightarrow$ For an LTI System the impulse response <u>fully characterizes the system</u>. From $h(t)$ you can find the response to any arbitrary input.

Step Response

Step Function

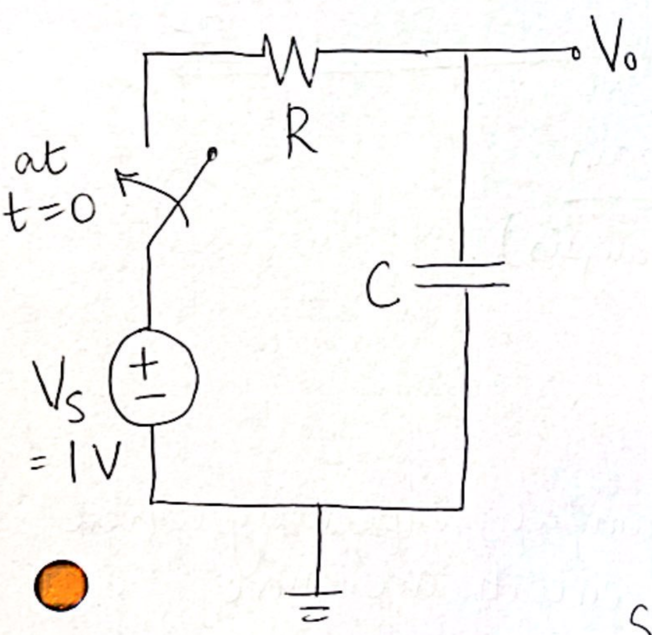
$$u(t) = \begin{cases} 0 & t < 0 \\ \text{undefined} & t = 0 \\ 1 & t > 0 \end{cases}$$



<u>Note</u>: $u(t) = \int_{-\infty}^{t} \delta(t)\,dt \Rightarrow \delta(t) = \frac{du(t)}{dt}$

> Step response also fully characterizes a system. In circuits it is useful in analyzing "switched" systems.

<u>Example</u> (LPF)



$$KVL \Rightarrow -V_s + iR + V_o = 0$$

$$\Rightarrow i = \frac{V_s - V_o}{R} = C\frac{dV_o}{dt}$$

$$\Rightarrow \boxed{\frac{dV_o}{dt} + \frac{V_o}{RC} = \frac{V_s}{RC}} \rightarrow \text{ODE of order 1.}$$

Solution is $V_o(t) = Ae^{Kt} + B \rightarrow$ Ansatz.

$$\Rightarrow \; Ak\, e^{kt} + \frac{A}{RC} e^{kt} + \frac{B}{RC} = \frac{u(t)}{RC} \longrightarrow \circledast$$

$u(t) \rightarrow V_s(t) = 1V$ for $t > 0$.

Assume $V_o(t=0) = 0 \Rightarrow$ uncharged capacitor.

$$\Rightarrow V_o(t=0) = A + B = 0 \Rightarrow A = -B.$$

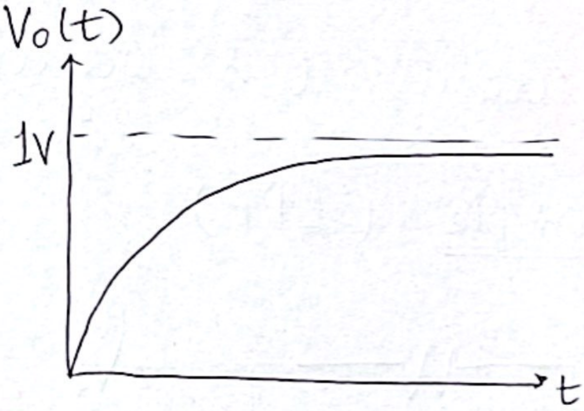
> For $t > 0$, $u(t) = 1 \Rightarrow B = 1$ since Eq $\circledast$ must be true for all t.

> Also $Ak + \frac{A}{RC} = 0$ for the same reason.

$$\Rightarrow k = -\frac{1}{RC}$$

$$\Rightarrow \boxed{V_o(t) = 1 - e^{\frac{-t}{RC}}}$$

$RC \rightarrow$ Time Constant τ



<u>Procedure for solving in time-domain</u>

1) Use KVL, KCL (nodal or mesh analysis).
2) Find ODE for desired output.
3) Solve using Ansatz.

> Solving ODEs in time domain is painful, especially when sinusoidal inputs are used & the circuits are more complex. So we prefer to work in the complex frequency domain $\longrightarrow$ Laplace Transform.